Cumulant Control Systems: The Cost-Variance, Discrete-Time Case

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Summary. The expected value of a random cost may be viewed either as its first moment or as its first cumulant. Recently, the Kalman control gain formulas have been generalized to finite linear combinations of cost cumulants, when the systems are described in continuous time. This paper initiates the investigation of cost cumulant control for discrete-time systems. The cost variance is minimized, subject to a cost mean constraint. A new version of Bellman's optimal cost recursion equation is obtained and solved for the case of full-state measurement. Application is made to the First Generation Structural Benchmark for seismically excited buildings.

1 Introduction

The 1960s saw a burst of controls research whose impact upon theory and application has continued to this day, without much measurable lessening. Pivotal in this burst was the pioneering work of R. E. Kalman, embracing concepts such as linear-quadratic-Gaussion (LQG) control, with linear dynamical systems, quadratic costs, and Gaussian noises. Kalman considered both discrete-time and continuous-time linear system models, and imported the ideas of Lyapunov analysis to incorporate notions of uniform controllability, uniform observability, and uniform asymptotic stability. The separation principle, Kalman–Bucy and Kalman filters, and the Kalman optimal control gain formulae, have become commonplace.

The approach of Kalman to LQG problems was, of course, based upon minimizing the average cost. We remark that the average cost is the first entry in two famous sequences of random cost statistics. The first sequence is that of the cost moments; the second sequence is that of the cost cumulants.

Without more information it would not be possible to surmise whether Kalman's formulae derived their efficacy from the average cost being a moment or from the average cost being a cumulant. Recently, however, K. D. Pham, [Pha04],

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[PSL02-1], [PSL02-2], has shown that the Kalman separation principle, filter, and optimal control gains generalize naturally to optimal control problems based upon finite linear combinations of cost cumulants. This suggests strongly that the successful operative methods in the Kalman advances were cost-cumulant enabled. Khanh's work was in continuous-time.

Moreover, the cost-cumulant control strategy families studied by Pham also display many of the same desirable features known to LQG designers. Indeed, Pham has carried out very promising applications of these algorithms to cable-stayed bridges [PSL04], structures excited by wind [PJSSL04], and buildings shaken by earthquakes [PSL02-3].

In view of these developments, it is both natural and desirable to examine the corresponding research issues for the other family of systems studied by Kalman, those in discrete time.

This paper initiates such investigations. Cost variance is minimized, subject to cost mean constraint. A new version of Bellman's optimal cost recursion equation is obtained, and solved for the case of full-state measurement. The theory is based upon the dissertation by Cosenza [Cos69]. Application is made to the First Generation Structural Benchmark for seismically excited buildings [SDD98].

2 Problem Definition

Let I be a subset of the integers and $\mathbb{R}^1$ be the 1-fold product of the real line. Consider then the systems whose behavior is governed by the following stochastic difference equations:

$$x(j+1) = f(j,x(j),u(j),w(j)), \qquad x(n_0) = x_0, \tag{1}$$
$$y(j) = g(j,x(j),v(j)), \tag{2}$$

where $x(j) \in \mathbb{R}^n$ is the system state, $u(j) \in \mathbb{R}^m$ is the control input, $w(j) \in \mathbb{R}^p$ is the actuation noise, $y(j) \in \mathbb{R}^q$ is the system output, and $v(j) \in \mathbb{R}^r$ is the measurement noise, $j \in I$. The initial condition of equation (1) is given by $x(n_0)$, where n_0 is the smallest element in I. Let $f : I \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}^n$ and $g : I \times \mathbb{R}^n \times \mathbb{R}^r \mapsto \mathbb{R}^q$ be Borel measurable, with the probability density functions of $w(j)$, $v(j)$, and x_0 given, $j \in I$.

Define $U(j) \triangleq \{u(n_0), u(n_0+1), \ldots, u(j)\}$, with a similar definition made for the remaining variables of equations (1) and (2), and let $Z(j) \triangleq \{Y(j), U(j-1)\}$, $n_0 < j$, with $Z(n_0) = y(n_0)$. It is then possible to denote the unique solution of equation (1) satisfying the initial condition $x(n_0) = x_0$ by $\theta(j)$, where $\theta(j) \triangleq \theta(j; n_0, x_0; U(j-1), W(j-1))$, $j \in I$, and to specify that the control laws be of the form $k(j) \triangleq k(j, Z(j))$, where $k(j,\cdot,\cdot) : \mathbb{R}^{q(j-n_0+1)} \times \mathbb{R}^{m(j-n_0)} \mapsto \mathbb{R}^m$, $j \in I$. Observe that $Z(j)$ contains all the information available to the controller at time j, and that the form chosen for $k(j)$, together with a boundedness requirement, contributes to the definition of the class of admissible controls.

With the definitions and notation recently introduced it is now possible to formulate a performance index as $J(n_0) \triangleq J(n_0, x_0; U(N-1), W(N-1))$, where

$$J(n_0) = \sum_{j=n_0+1}^{N} L(j, \theta(j), u(j-1)), \tag{3}$$

and where the loss function $L : I \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^+$ (the nonnegative real line) is Borel measurable. Since $f(\cdot,\cdot,\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$ are all Borel measurable, the performance index is a random variable and consequently one of its statistical moments must be selected for optimization. In this investigation it is desired to minimize the variance of $J(n_0)$ while its mean is forced to obey a constraint. In mathematical parlance, it is desired to find $k(j, Z(j))$, $n_0 \le j \le N-1$, such that

$$E\left\{J^2(n_0)|Z(n_0)\right\} - E^2\left\{J(n_0)|Z(n_0)\right\} \tag{4}$$

is minimized, while

$$E\{J(n_0)|Z(n_0)\} = h(n_0, Z(n_0)), \tag{5}$$

where $E\{\cdot|\cdot\}$ denotes the conditional expectation operator. The form of the function $h : I \times \mathbb{R}^q \mapsto \mathbb{R}^+$ is selected *a priori* based on practical considerations, such as desired response, permissible deviations from the desired response, complexity of the controller, etc.

Observe that the choice of $h(n_0, Z(n_0))$ is not entirely arbitrary, for if

$$\alpha(n_0, Z(n_0)) = \inf_{U(N-1)} E\{J(n_0)|Z(n_0)\}, \tag{6}$$

then $h(n_0, Z(n_0))$ must always be greater than $\alpha(n_0, Z(n_0))$. This constraint on h, together with equation (5), completes the definition of the class of admissible controls.

3 Recursion Equation

In this section, a recursion equation for the optimal variance cost is derived. The procedure employed is the standard procedure for this type of problem; first, the constraint equation is appended to the expression to be minimized by means of a Lagrange multiplier, $\mu(n_0)$, and then the resulting equation is imbedded into the more general class of problems where n_0 is a variable rather than a fixed initial time. It is clear that the solution of the more general problem leads trivially to the solution of the problem posed in Section 2. Consequently, it is desired to find $\mu(j)$ and $k(i, Z(i))$, $j \le i \le N-1$, $j \in I$, such that

$$E\left\{J^2(j)|Z(j)\right\} - E^2\left\{J(j)|Z(j)\right\} + 4\mu(j)\left[E\{J(j)|Z(j)\} - h(j, Z(j))\right] \tag{7}$$

is minimized, where $\mu(j) \in \mathbb{R}$ is a Lagrange multiplier, and where the 4 premultiplying $\mu(j)$ has been introduced just for convenience.

Before proceeding with the development of the recursion equation, however, let $k_j \triangleq \{k(j), k(j+1), \ldots, k(N-1)\}$, $j \in I$, and let

$$VC(j,Z(j)|k_j) = E\left\{J^2(j)|Z(j)\right\} - E^2\{J(j)|Z(j)\} + 4\mu(j)\left[E\{J(j)|Z(j)\} - h(j,Z(j))\right], \quad j \in I, \tag{8}$$

where VC signifies "variance cost."

Define $VC_0(N-1,Z(N-1))$ to be the optimal value of $VC(N-1,Z(N-1)|k_{N-1})$, that is,

$$VC_0(N-1,Z(N-1)) = \min_{k(N-1),\mu(N-1)} \Big\{E\left\{L^2(N)|Z(N-1)\right\} - E^2\{L(N)|Z(N-1)\} + 4\mu(N-1)[E\{L(N)|Z(N-1)\} - h(N-1,Z(N-1))]\Big\}, \tag{9}$$

where $L(j) \triangleq L(j,\theta(j),k(j-1))$. Note in particular that if $k_0(N-1)$ is the control law which leads to $VC_0(N-1,Z(N-1))$, then

$$E\{L(N,\theta_0(N),k_0(N-1))|Z(N-1)\} = h(N,Z(N-1)), \tag{10}$$

where

$$\theta_0(N) = f(N-1,\theta(N-1),k_0(N-1),w(N-1)), \tag{11}$$

and therefore, combining equations (10) and (9) it follows that

$$VC_0(N-1,Z(N-1)) = E\left\{L^2(N,\theta_0(N),k_0(N-1))|Z(N-1)\right\} - E^2\{L(N,\theta_0(N),k_0(N-1))|Z(N-1)\}. \tag{12}$$

Similarly,

$$VC_0(N-2,Z(N-2)) = \min_{k_{N-2},\mu(N-2)} \Big\{E\left\{(L(N)+L(N-1))^2|Z(N-2)\right\} - E^2\{L(N)+L(N-1)|Z(N-2)\} + 4\mu(N-2)[E\{L(N)+L(N-1)|Z(N-2)\} - h(N-2,Z(N-2))]\Big\}, \tag{13}$$

which after some manipulation may be written as

$$VC_0(N-2,Z(N-2)) = \min_{k_{N-2},\mu(N-2)} \Big(\Gamma(N-2) - E^2\{L(N)|Z(N-2)\} + E\left\{E\left\{L^2(N)|Z(N-1)\right\}|Z(N-2)\right\} + 2E\{L(N)L(N-1)|Z(N-2)\} - 2E\{L(N)|Z(N-2)\}E\{L(N-1)|Z(N-2)\}\Big), \tag{14}$$

where

$$\Gamma(N-2) = E\left\{L^2(N-1)|Z(N-2)\right\} - E^2\{L(N-1)|Z(N-2)\}$$
$$+ 4\mu(N-2)[E\{L(N)+L(N-1)|Z(N-2)\}$$
$$- h(N-2,Z(N-2))]\,. \tag{15}$$

If now $E\left\{E^2\{L(N)|Z(N-1)\}|Z(N-2)\right\}$ is added and subtracted from equation (14), then

$$VC_0(N-2,Z(N-2)) = \min_{k_{N-2},\mu(N-2)} \Big\{\Gamma(N-2) + E\{E\left\{L^2(N)|Z(N-1)\right\}$$
$$+ 2E\{L(N)L(N-1)|Z(N-2)\}$$
$$- 2E\{L(N)|Z(N-2)\}E\{L(N-1)|Z(N-2)\}$$
$$+ E\{E^2\{L(N)|Z(N-1)\}|Z(N-2)\}$$
$$- E^2\{L(N)|Z(N-2)\}$$
$$- E^2\{L(N)|Z(N-1)\}|Z(N-2)\}\Big\}. \tag{16}$$

However, since the process under consideration is a multistage decision process, the principle of optimality may be applied to it, and equation (16) then becomes

$$VC_0(N-2,Z(N-2)) = \min_{k(N-2),\mu(N-2)} \Big\{\Gamma_0(N-2)$$
$$+2E\{L_0(N)L(N-1)|Z(N-2)\}$$
$$-2E\{L_0(N)|Z(N-2)\}E\{L(N-1)|Z(N-2)\}$$
$$+E\{E\{L_0^2(N)|Z(N-1)\}$$
$$-E^2\{L_0(N)|Z(N-1)\}|Z(N-2)\}$$
$$+E\{E^2\{L_0(N)|Z(N-1)\}|Z(N-2)\}$$
$$-E^2\{L_0(N)|Z(N-2)\}\Big\}, \tag{17}$$

where

$$\Gamma_0(N-2) = E\{L^2(N-1)|Z(N-2)\} - E^2\{L(N-1)|Z(N-2)\}$$
$$+ 4\mu(N-2)[E\{L_0(N)+L(N-1)|Z(N-2)\}$$
$$- h(N-2,Z(N-2))]\,, \tag{18}$$

and $L_0(N) = L(N,\theta_0(N),k_0(N-1))$.

Furthermore, if equations (17) and (12) are combined, then

$$\begin{aligned} VC_0(N-2, Z(N-2)) = \min_{k(N-2),\mu(N-2)} \Big\{ & \Gamma_0(N-2) \\ & + 2E\{L_0(N)L(N-1)|Z(N-2)\} \\ & - 2E\{L_0(N)|Z(N-2)\}E\{L(N-1)|Z(N-2)\} \\ & + E\{E^2\{L_0(N)|Z(N-1)\}|Z(N-2)\} \\ & - E^2\{L_0(N)|Z(N-2)\} \\ & + E\{VC_0(N-1, Z(N-1))|Z(N-2)\}\Big\}. \end{aligned} \tag{19}$$

Proceeding by induction, it follows that

$$\begin{aligned} VC_0(i, Z(i)) = \min_{k(i),\mu(i)} \Bigg\{ & \Gamma_0(i) + 2E\left\{ \sum_{j=i+2}^{N} L_0(j)L(i+1) \middle| Z(i) \right\} \\ & - 2E\left\{ \sum_{j=i+2}^{N} L_0(j) \middle| Z(i) \right\} E\{L(i+1)|Z(i)\} \\ & + E\left\{ E^2\left\{ \sum_{j=i+2}^{N} L_0(j) \middle| Z(i+1) \right\} \middle| Z(i) \right\} \\ & - E^2\left\{ E\left\{ \sum_{j=i+2}^{N} L_0(j) \middle| Z(i+1) \right\} \middle| Z(i) \right\} \\ & + E\{VC_0(i+1, Z(i+1))|Z(i)\} \Bigg\}, \end{aligned} \tag{20}$$

where $n_0 \le i \le N-2$, $VC_0(N-1, Z(N-1))$ is as given by equation (9) and where

$$\begin{aligned} \Gamma_0(i) = {} & E\{L^2(i+1)|Z(i)\} - E^2\{L(i+1)|Z(i)\} \\ & + 4\mu(i)\left[E\left\{ \sum_{j=i+2}^{N} L_0(j) + L(i+1) \middle| Z(i) \right\} - h(i, Z(i)) \right]. \end{aligned} \tag{21}$$

Theorem 1. *Consider the nonlinear problem given in (1) and (3). A solution k^* is the optimal minimum cost variance (MCV) strategy, if there exists a solution $VC_0(i, Z(i))$ to*

$$VC_0(i, Z(i)) = \min_{k(i)} \Bigg\{ E\{VC_0(i+1, Z(i+1))|Z(i)\} + E\left\{L^2(i+1)|Z(i)\right\} \tag{22}$$

$$- E^2\{L(i+1)|Z(i)\} + E\left\{ E^2\left\{ \sum_{j=i+2}^{N} L_0(j) \middle| Z(i+1) \right\} \middle| Z(i) \right\}$$

$$+E^2\left\{E\left\{\sum_{j=i+2}^{N} L_0(j)\middle|Z(i+1)\right\}\middle|Z(i)\right\}$$

$$+2E\left\{\left(\sum_{j=i+2}^{N} L_0(j)\right)L(i+1)\middle|Z(i)\right\}$$

$$-2E\left\{\sum_{j=i+2}^{N} L_0(j)\middle|Z(i)\right\}E\left\{L(i+1)|Z(i)\right\}$$

$$+4\mu(i)\left[E\left\{\sum_{j=i+2}^{N} L_0(j)+L(i+1)\middle|Z(i)\right\}-M(i,Z(i))\right]\Bigg\},$$

where $\gamma(k)$ is a Lagrange multiplier, $L_0(j)=L(j,x(j),k^*(j-1))$, and k^* is the minimizing argument of (22).

Proof. From the one-step analysis, we see that the variance cost is minimized. We need to prove by the method of induction that (22) holds. We shall assume that (22) holds for time $i+1$. We now will need to show that with this assumption, equation (22) is valid for time i. By the definition of $VC_0(i,Z(i))$ we have

$$VC_0(i,Z(i)) = \min_{k(i),\cdots,k(N-1)} \big\{E\{J^2(i,x(i);k)|Z(i)\} - E^2\{J(i,x(i);k)|Z(i)\}$$
$$+4\mu(i)\left[E\{J(i,x(i);k)|Z(i)\} - M(i,Z(i))\right]\big\},$$

which by substitution gives

$$V(i,Z(i)) = \min_{k(i),\cdots,k(N-1)} \big\{E\{(L(i+1)+J(i+1,x(i+1);k))^2|Z(i)\}$$
$$-E^2\{L(i+1)+J(i+1,x(i+1);k)|Z(i)\}$$
$$+4\mu(i)\left[E\{J(i+1,x(i+1);k)|Z(i)\}-M(i,Z(i))\right]\big\}$$
$$= \min_{k(i),\cdots,k(N-1)} \big\{E\{L^2(i+1)|Z(i)\}$$
$$+2E\{L(i+1)J(i+1,x(i+1);k)|Z(i)\}$$
$$+E\{J^2(i+1,x(i+1);k)|Z(i)\}-E^2\{L(i+1)|Z(i)\}$$
$$-E^2\{J(i+1,x(i+1);k)|Z(i)\}$$
$$-2E\{L(i+1)|Z(i)\}E\{J(i+1,x(i+1);k)|Z(i)\}$$
$$+4\mu(i)\left[E\{J(i+1,x(i+1);k)|Z(i)\}-M(i,Z(i))\right]\big\}$$

where J is given in (3) with i in place of 0. Now by using the principle of optimality we have

$$VC_0(i,Z(i)) = \min_{k(i)}\Big\{E\{L^2(i+1)|Z(i)\}-E^2\{L(i+1)|Z(i)\}$$

$$
\begin{aligned}
&+2E\left\{\left(\sum_{j=i+2}^{N} L(j)_0\right) L(i+1)|Z(i)\right\} \\
&-2E\left\{\sum_{j=i+2}^{N} L_0(j)|Z(i)\right\} E\left\{L(i+1)|Z(i)\right\} \\
&+E\left\{E\left\{\left(\sum_{j=i+2}^{N} L_0(j)\right)^2 |Z(i+1)\right\}|Z(i)\right\} \\
&-E\left\{E^2\left\{\sum_{j=i+2}^{N} L_0(j)|Z(i+1)\right\}|Z(i)\right\} \\
&+E\left\{E^2\left\{\sum_{j=i+2}^{N} L_0(j)|Z(i+1)\right\}|Z(i)\right\} \\
&-E^2\left\{E\left\{\sum_{j=i+2}^{N} L_0(j)|Z(i+1)\right\}|Z(i)\right\} \\
&+4\mu(i)\left[E\{\sum_{j=i+2}^{N} L_0(j)+L(i+1)|Z(i)\}-M(i,Z(i))\right]\Bigg\},
\end{aligned}
$$

where we still only have the mean constraint for time i. But for time $i+1$, the mean constraint is satisfied if the optimal solution $k^*(i+1,x(i+1))$ is played. Therefore equation (22) is satisfied for time i. □

With this result we can now turn our attention to solving the special case when the system is linear and the cost is quadratic. We apply the nonlinear, nonquadratic cost results and get a recursion equation for this case. We then determine the optimal MCV strategy for full-state feedback information.

4 Linear Quadratic Case

Let I again denote a subset of the integers with n_0 as its smallest element and introduce $\mathbb{R}^{m\times n}$ and $\mathbb{S}^{n\times n}$ where $\mathbb{R}^{m\times n}$ represents the linear space of $m\times n$ real matrices and $\mathbb{S}^{n\times n}$ the real linear space of $n\times n$ symmetric matrices. Consider then the controllable system described by the following stochastic difference equations:

$$
\begin{aligned}
x(j+1) &= A(j)x(j)+B(j)u(j)+w(j)\,, \qquad x(n_0)=x_0\,, && (23)\\
y(j) &= x(j)\,, && (24)
\end{aligned}
$$

where $A(j)\in\mathbb{R}^{n\times n}$ is bounded and nonsingular, and $B(j)\in\mathbb{R}^{n\times m}$ is bounded, $j\in I$. The actuation noise sequence, $w(j)$, is a sequence of identically distributed, zero mean, independent Gaussian variables with covariance matrix given by

$$
E\{w(j) >< w(j)\} = Q_W\,, \tag{25}
$$

where $Q_W \in \mathbb{S}^{n\times n}$ is a time-invariant diagonal matrix, and where $\cdot >< \cdot : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ is the dyad.

The loss function is given by

$$L(j,\theta(j),k(j-1)) = \langle \theta(j), R(j-1)\theta \rangle + \langle k(j-1), P(j-1)k(j-1) \rangle \,, \quad j \in I\,, \tag{26}$$

where $< \cdot,\cdot >: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ is the Euclidean inner product and $\theta(j)$ is the unique solution of equation (23) satisfying the initial condition $x(n_0) = x_0$. $R(j)$ and $P(j)$ are positive definite,[5] bounded, and symmetric for all j, $j \in I$.

Similarly, the mean value constraint is given by

$$h(n_0, Z(n_0)) = m(n_0) + \langle \theta(n_0), M(n_0)\theta(n_0) \rangle \,, \tag{27}$$

where $m(n_0) \in \mathbb{R}^+$ and the matrix $M(n_0) \in \mathbb{S}^{n\times n}$ must be bounded and positive definite. Both $m(n_0)$ and $M(n_0)$ must be selected such that

$$h(n_0, Z(n_0)) > \alpha(n_0, Z(n_0))\,, \tag{28}$$

where $\alpha(n_0, Z(n_0))$ is as given by equation (6).

The assumption of linear control laws leads naturally to quadratic optimal costs, that is, for linear control laws it is always possible to write

$$VC_0(i+1, Z(i+1)) = v_0(i+1) + \langle \theta(i), V_0(i+1)\theta(i) \rangle \,, \tag{29}$$

where $v_0(i) \in \mathbb{R}^+$ and $V_0(i) \in \mathbb{S}^{n\times n}$ is nonnegative definite, and where $n_0 \le i \le N-1$. Therefore,

$$E\{VC_0(i+1, Z(i+1)) | Z(i)\} = v_0(i+1) + \langle \beta(i), V_0(i+1)\beta(i) \rangle + \mathrm{Tr}\{V_0(i+1)Q_W\}\,. \tag{30}$$

If the following definition is introduced, $R_M(i) = R(i) + M(i+1)$ for $n_0 \le i \le N-1$, and the terminal conditions are given as $m(N) = 0$, $M(N) = 0$, $v_0(N) = 0$, and $V_0(N) = 0$, then with some mathematical manipulations we have

$$\begin{aligned} VC_0(i, Z(i)) = \min_{k(i),\mu(i)} \Big\{ & 4\langle \beta(i), R_M(i)Q_W R_M(i)\beta(i) \rangle \\ & + E\left\{ \langle w(i), R_M(i)w(i) \rangle^2 \right\} - \mathrm{Tr}^2\{R_M(i)Q_W\} + v_0(i+1) \\ & + \langle \beta(i), V_0(i+1)\beta(i) \rangle + \mathrm{Tr}\{V_0(i+1)Q_W\} + 4\mu(i)\Big[m(i+1) \\ & + \langle k(i), P(i)k(i) \rangle + \langle \beta(i), R_M(i)\beta(i) \rangle + \mathrm{Tr}\{R_M(i)Q_W\} \\ & - m(i) - \langle \theta(i), M(i)\theta(i) \rangle \Big] \Big\}, \quad n_0 \le i \le N-1\,, \end{aligned} \tag{31}$$

[5] The assumptions that Q_W be diagonal and $R(j)$ be positive definite have been made for convenience only.

where $\beta(i) = A(i)\theta(i) + B(i)k(i)$. Performing the minimization with respect to $k(i)$, the optimal MCV controller is given as

$$k_0(i) = K_0(i)\theta(i) = -\left[B^T(i)\Lambda(i)B(i) + \mu(i)P(i)\right]^{-1} B^T(i)\Lambda(i)A(i)\theta(i), \tag{32}$$

where

$$\Lambda(i) = R_M(i)Q_W R_M(i) + V_0(i+1)/4 + \mu(i)R_M(i) \tag{33}$$

for $n_0 \leq i \leq N-1$. Using this optimal controller and performing the minimization in terms of $\mu(i)$ we have the mean constraint

$$\begin{aligned} M(i) &= K_0^T(i)P(i)K_0(i) + A_0^T(i)R_M(i)A_0(i) \\ m(i) &= m(i+1) + \mathrm{Tr}\{R_M(i)Q_W\} \end{aligned} \tag{34}$$

and we also have the variance

$$\begin{aligned} V_0(i) &= A_0^T(i)\left[4R_M(i)Q_W R_M(i) + V_0(i+1)\right]A_0(i) \\ v_0(i) &= v_0(i+1) + \mathrm{Tr}\{V_0(i+1)Q_W\} + E\left\{\langle w(i), R_M(i)w(i)\rangle^2\right\} - \mathrm{Tr}^2\{R_M(i)Q_W\}, \end{aligned} \tag{35}$$

where $A_0(i) = A(i) + B(i)K_0(i)$ is the closed loop A matrix and $n_0 \leq i \leq N-1$.

It is important to understand the differences between the recursion equations of a minimum mean problem and those of a minimum cost variance problem. In a minimum mean problem, the solution of the recursion equations leads to the minimum of the expected value of a performance index and to its corresponding control law. In a minimum cost variance problem, subsequent to the selection of $\mu(i)$, $n_0 \leq i \leq N-1$, solution of the recursion equations leads to a mean value of the performance index together with its corresponding minimum cost variance and optimal control law. By properly altering $\mu(i)$, $n_0 \leq i \leq N-1$, several such sets of expected values, minimum cost variances, and optimal control laws may be obtained. Clearly then, the amount of information which the optimization procedure herein employed furnishes concerning the performance index far exceeds that supplied by its mean value counterpart. Furthermore, observe that the minimum mean problem is a particular case of the problem herein solved, namely, it is the solution of the recursion equations in the limit as $\mu(i)$ approaches infinity, $n_0 \leq i \leq N-1$. Similarly, it may be shown that when it is possible to set $\mu(i)$ equal to zero, $n_0 \leq i \leq N-1$, then one obtains the solution of a MCV problem with no constraint on the mean value of the performance index. Generally speaking, such "pure" cost variance minimizations are not available in continuous time.

It is of interest to observe that the minimum cost variance corresponding to the smallest mean is finite. More interesting, however, is the fact that under certain conditions there exists a finite mean value whose corresponding $V_0(i)$ is zero, that is, there exists a finite mean value whose corresponding minimum cost variance is independent of the initial conditions. To prove this assertion, suppose Q_W and $B(i)$ are nonsingular, $n_0 \leq i \leq N-1$. Then, replacing $\mu(i)$ by zero in the recursion equations, it follows that

$$K_0(i) = B^{-1}(i)A(i), \qquad n_0 \le i \le N-1, \tag{36}$$

which, from equation (35), implies that $V_0(i)$ is zero, $n_0 \le i \le N-1$.

In the preceding paragraph it was hinted that it is not always possible to replace $\mu(i)$ by zero, the reason being that the solution of the recursion equations is contingent upon the nonsingularity of the matrix $B^T(i)\Lambda(i)B(i) + \mu(i)P(i)$, $n_0 \le i \le N-1$.

5 Application to First Generation Structural Benchmark for Earthquakes

With the theory now well established, the control algorithm discussed is applied to the First Generation Structural Benchmark under seismic excitation. The structure under consideration is a three-story building excited by an earthquake. For control purposes, the building has an active mass driver on the third floor. The benchmark problem has a 28-state evaluation model. In the interest of control, a 10-state design model is used. For more details on the building, models, and the discussion of the performance criteria, the reader is encouraged to refer to [SDD98]. The benchmark control design model is a continuous-time model, so to apply the results in this paper,

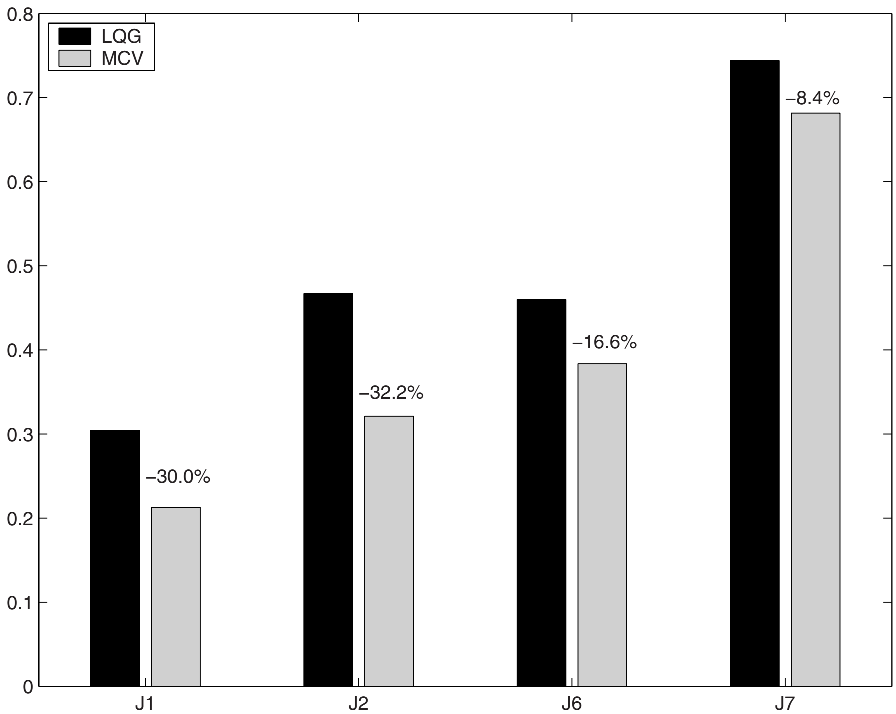

Fig. 1. Building performance.

the model is discretized. Furthermore the state and control weighting matrices, $R(j)$ and $P(j)$, are respectively selected to be $0.1I_{10}$ and 50, where I_{10} is the 10 by 10 identity matrix. For the MCV control, the parameter, μ, is selected to be 1.3×10^6. Simulation results appear in Figure 1 and Figure 2.

The results in Figure 1 represent the performance of the building. It is seen that there is a significant reduction for each of these performance criteria. For the root-mean-square criteria, J_1 and J_2, there is about a 30% reduction in the MCV case from the LQG controller results. For peak response of the building, there is also a notable decrease in the performance criteria. There is about a 16% reduction for J_6 and an 8.4% reduction for J_7.

With this reduction in the civil engineering criteria that deal with the building performance, the question becomes: What about the criteria that deal with the control effort? As would seem likely, the increase in performance corresponds with an increase in control effort, as seen in Figure 2. Despite this increase over the LQG case, the control is still within the constraint imposed on the control in the benchmark problem. This suggests that the MCV control makes more efficient use of the control resources available.

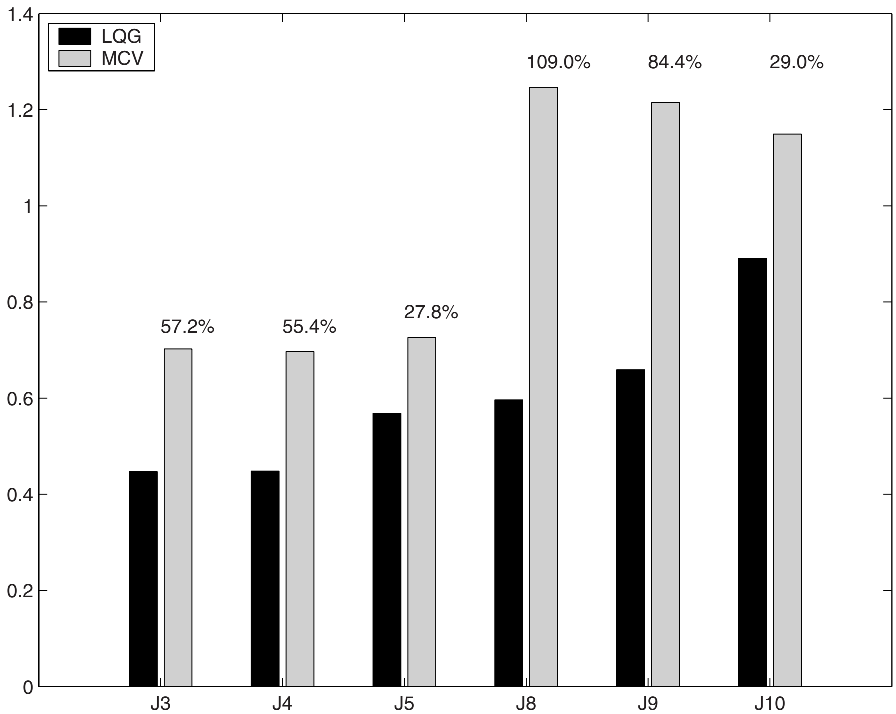

Fig. 2. Control effort.

6 Conclusion

A new version of the Bellman recursion equation for optimal cost variance has been obtained for the problem of minimizing the variance of a cost, given a constraint upon the cost mean. Although emphasis has been placed upon linear dynamical systems in discrete time, some of the steps were carried out for nonlinear, nonquadratic cases. A complete solution of the recursion has been obtained for the case of full-state measurements. However, the general recursion has been derived for the case of noisy measurements, and the next step of the research is to complete the solution for that case. The MCV controller was then applied to the First Generation Benchmark for seismically excited structures. The results were compared to those of the LQG control. The MCV controller showed substantial improvement over the LQG results, while observing the given control constraints.

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