

Chapter 2

Plane Elasticity Theory

2.1 The Fundamental Equations

The state of stress in a plane elastic body is determined by three components of stress: σ_{xx} , σ_{xy} , σ_{yy} .

Recall that σ_{xx} , σ_{xy} are the components, in the x - and y -directions, of the force per unit area exerted, at (x, y) , on the plane normal to the x -axis, applied from the $x+$ side to the $x-$ side, as shown in Figure 2.1.1.

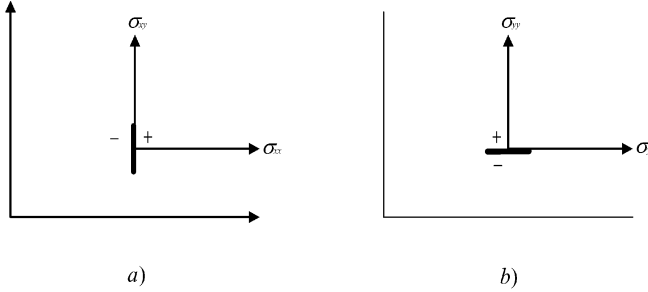


Fig. 2.1.1 (a) Stress $\sigma_{xx}\hat{i} + \sigma_{xy}\hat{j}$ acts on a plane with normal \hat{i} . (b) Stress $\sigma_{xy}\hat{i} + \sigma_{yy}\hat{j}$ acts on a plane with normal \hat{j} .

Similarly, $\sigma_{yx} \equiv \sigma_{xy}$, and σ_{yy} are the components of the force per unit area on the plane normal to the y -axis, again applied from the side $y+$ to the side $y-$.

They satisfy the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}. \quad (2.1.1)$$

The deformation of the elastic body can be expressed by the relative extensions ε_{xx} , ε_{yy} in the directions of the x - and y -axes, and by the angular rotation ε_{xy} .

The *elastic strains* $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$ are related to the components u, v of the *elastic displacement vector* (u, v) by the equations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}. \quad (2.1.2)$$

Since the *three* strain components are expressed in terms of *two* displacement components, there should be some relationship between them. This is called the *strain compatibility condition*:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{2\partial^2 \varepsilon_{xy}}{\partial x \partial y}. \quad (2.1.3)$$

There are two variants of plane elastic problems: plane strain and plane stress. Plane strain is an idealisation of the elastic state in an infinitely long cylinder with its axis being the z -axis, and acted upon by forces in the x, y -plane that are independent of z . In plane strain, u and v are functions of x, y only, while w , the elastic displacement in the z -direction, is zero:

$$u = u(x, y), \quad v = v(x, y), \quad w = 0 \quad (2.1.4)$$

so that the strains

$$\varepsilon_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad \varepsilon_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad (2.1.5)$$

are all zero.

Plane stress is an idealisation of the state of stress in a thin plate acted on by forces in its plane; in plane stress $\sigma_{xz} = 0 = \sigma_{yz} = \sigma_{zz}$.

In plane strain, the components of strain are related to the components of stress by the stress-strain equations

$$\sigma_{xx} = \lambda\theta + 2\mu\varepsilon_{xx}, \quad \sigma_{yy} = \lambda\theta + 2\mu\varepsilon_{yy}, \quad \sigma_{xy} = 2\mu\varepsilon_{xy}. \quad (2.1.6)$$

Here $\theta = \varepsilon_{xx} + \varepsilon_{yy}$ is the relative increase of volume, or *dilatation*, and λ, μ are *Lamé's constants*. The strains, in their turn, may be related to the stresses by the equations

$$2\mu\varepsilon_{xx} = \sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy}), \quad 2\mu\varepsilon_{yy} = \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy}), \quad (2.1.7)$$

$$2\mu\varepsilon_{xy} = \sigma_{xy}$$

The *Young's modulus*, E , and *Poisson's ratio*, ν , are related to λ and μ by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}, \quad (2.1.8)$$

and the inverse equations

$$\lambda = \frac{Ev}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \quad (2.1.9)$$

In plane stress we start with the full stress-strain equations for an isotropic elastic body:

$$\begin{aligned} \sigma_{xx} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{xx}, & \sigma_{yy} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{yy}, \\ \sigma_{zz} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz}, \\ \sigma_{xz} &= 2\mu\varepsilon_{xz}, & \sigma_{yz} &= 2\mu\varepsilon_{yz}, & \sigma_{xy} &= 2\mu\varepsilon_{xy}. \end{aligned} \quad (2.1.10)$$

If $\sigma_{zz} = 0$, then $\lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz} = 0$. Thus $\varepsilon_{zz} = -\lambda(\varepsilon_{xx} + \varepsilon_{yy})/(\lambda + 2\mu)$ and $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = [1 - \lambda/(\lambda + 2\mu)](\varepsilon_{xx} + \varepsilon_{yy})$. Thus

$$\sigma_{xx} = \lambda^*\theta + 2\mu\varepsilon_{xx}, \quad \sigma_{yy} = \lambda^*\theta + 2\mu\varepsilon_{yy}, \quad \sigma_{xy} = 2\mu\varepsilon_{xy}, \quad (2.1.11)$$

where

$$\lambda^* = \lambda[1 - \lambda/(\lambda + 2\mu)] = 2\mu\lambda/(\lambda + 2\mu). \quad (2.1.12)$$

The effective Poisson's ratio for plane stress is therefore

$$\nu^* = \frac{\lambda^*}{2(\lambda^* + \mu)} = \frac{\nu}{1 + \nu}. \quad (2.1.13)$$

We conclude that the equations for plane stress may be derived from those of plane strain by replacing μ, λ by μ, λ^ ; or equivalently, μ, ν by μ, ν^* .*

If the expressions (2.1.7) are substituted in the compatibility equation (2.1.3), we find

$$\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \frac{\nu \partial^2}{\partial y^2}(\sigma_{xx} + \sigma_{yy}) + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \frac{\nu \partial^2}{\partial x^2}(\sigma_{xx} + \sigma_{yy}) = \frac{2\partial^2 \sigma_{xy}}{\partial x \partial y}.$$

The equilibrium equations (2.1.1) yield

$$\frac{2\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2}$$

so that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = 0. \quad (2.1.14)$$

We write this as

$$\Delta(\sigma_{xx} + \sigma_{yy}) = 0. \quad (2.1.15)$$

The equilibrium equations (2.1.1) are satisfied if

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \quad (2.1.16)$$

where $U(x, y)$ is called the *Airy stress function*. Substituting (2.1.16) into (2.1.15) we find that

$$\frac{\partial^4 U}{\partial x^4} + \frac{2\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0. \quad (2.1.17)$$

(Galin uses Φ for the Airy stress function, and then later uses Φ for one of the complex potentials. We use U here to avoid confusion.) This equation, called the *biharmonic equation*, is written

$$\Delta^2 U = 0; \quad (2.1.18)$$

We say that U is *biharmonic*.

In solving plane contact problems we shall make use of *complex variable methods*.

We now give a brief account of a method of formulating the basic equations for plane problems in the theory of elasticity. The detailed exposition of these matters may be found in Muskhelishvili (1953) or Gladwell (1980).

If we substitute the expressions (2.1.6) for $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ into the equations (2.1.16), and express $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$ in terms of u, v we find

$$\lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial u}{\partial x} = \frac{\partial^2 U}{\partial y^2}, \quad (2.1.19)$$

$$\lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y} = \frac{\partial^2 U}{\partial x^2}, \quad (2.1.20)$$

$$\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = -\frac{\partial^2 U}{\partial x \partial y}. \quad (2.1.21)$$

We may determine $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ from the first two equations. Introducing the notation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \Delta U = P \quad (2.1.22)$$

we find

$$2\mu \frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} P, \quad (2.1.23)$$

$$2\mu \frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} P, \quad (2.1.24)$$

since U is biharmonic, $P = \Delta U$ is harmonic.

$P(x, y)$ and $Q(x, y)$ are said to be *conjugate* if they satisfy the Cauchy–Riemann equations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}. \quad (2.1.25)$$

If P is harmonic, so is Q , because

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = -\frac{\partial^2 P}{\partial x \partial y} + \frac{\partial^2 P}{\partial y \partial x} = 0.$$

Moreover, $P + iQ$ is a function of the complex variable $z = x + iy$. Put $\bar{z} = x - iy$ and $f(x, y) = P(x, y) + iQ(x, y)$, then $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$ and we can write f as a function of z and \bar{z} ; $\partial x / \partial \bar{z} = 1/2$, $\partial y / \partial \bar{z} = -1/2i = i/2$. Thus

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + \frac{i \partial f}{\partial y} \right) \\ &= \frac{1}{2} \left(\frac{\partial P}{\partial x} + \frac{i \partial Q}{\partial x} + \frac{i \partial P}{\partial y} - \frac{\partial Q}{\partial y} \right) = 0. \end{aligned}$$

We conclude that f is a function of z only.

Thus, if Q is conjugate to P then

$$P(x, y) + iQ(x, y) = f(z). \quad (2.1.26)$$

Let us now find the expressions for the displacements u, v . Introduce two more conjugate harmonic functions p, q , where

$$\phi(z) = p + iq = \frac{1}{4} \int f(z) dz, \quad (2.1.27)$$

then since

$$\frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = -\frac{i}{2},$$

we have

$$\begin{aligned} \phi'(z) &= \frac{\partial}{\partial z}(p + iq) = \frac{1}{2} \frac{\partial}{\partial x}(p + iq) - \frac{i}{2} \frac{\partial}{\partial y}(p + iq), \\ &= \frac{1}{2} \frac{\partial p}{\partial x} + \frac{i}{2} \frac{\partial q}{\partial x} - \frac{i \partial p}{2 \partial y} + \frac{1}{2} \frac{\partial q}{\partial y}. \end{aligned}$$

The Cauchy–Riemann, equations are

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}$$

so that

$$\phi'(z) = \frac{\partial p}{\partial x} + \frac{i \partial q}{\partial x} = \frac{1}{4}(P + iQ),$$

and

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{1}{4}P, \quad \frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y} = \frac{1}{4}Q. \quad (2.1.28)$$

This means that equations (2.1.23), (2.1.24) may be written

$$2\mu \frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} \frac{\partial p}{\partial x},$$

$$2\mu \frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} \frac{\partial q}{\partial y}.$$

After integration, we get

$$2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} p + f_1(y), \quad (2.1.29)$$

$$2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} q + f_2(x), \quad (2.1.30)$$

and after substitution into (2.1.21) we find

$$f_1'(y) + f_2'(x) = 0.$$

Since each term in this equation must be constant, we find

$$f_1'(y) = 2\mu\gamma, \quad f_2'(x) = -2\mu\gamma$$

so that

$$f_1(y) = 2\mu(\gamma y + \alpha), \quad f_2(x) = 2\mu(-\gamma x + \beta).$$

They correspond to a rigid body displacement and rotation:

$$u = \gamma y + \alpha, \quad v = -\gamma x + \beta$$

and we can omit them.

Let us now form the function $U - px - qy$. It is harmonic because equation (2.1.28) gives

$$\Delta(U - px - qy) = P - \frac{2\partial p}{\partial x} - \frac{2\partial q}{\partial y} = 0.$$

Since $U - px - qy$ is harmonic, it is the real part of a function $\chi(z)$, i.e.,

$$2(U - px - qy) = \chi(z) + \overline{\chi(z)}.$$

But

$$\begin{aligned} 2(px + qy) &= (x - iy)(p + iq) + (x + iy)(p - iq) \\ &= \bar{z}\phi(z) + z\overline{\phi(z)} \end{aligned}$$

so that

$$2U = \bar{z}\phi(z) + z\overline{\phi(z)} + \chi(z) + \overline{\chi(z)} \quad (2.1.31)$$

and

$$\frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} = \frac{2\partial U}{2\bar{z}} = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \quad (2.1.32)$$

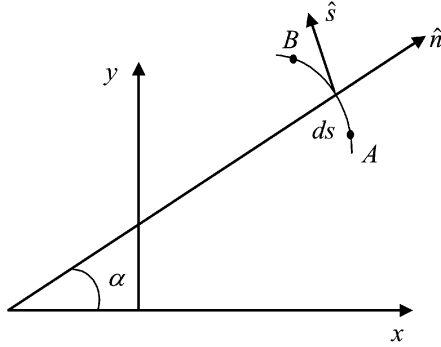


Fig. 2.1.2 The directions \hat{n} and \hat{s} .

where

$$\psi(z) = \chi'(z). \quad (2.1.33)$$

Now, using (2.1.29), (2.1.30) we find

$$2\mu(u + iv) = -\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) + 2\frac{(\lambda + 2\mu)}{\lambda + \mu}(p + iq);$$

we have neglected $f_1(y)$ and $f_2(x)$. Substituting from (2.1.27), (2.1.32) we find

$$2\mu(u + iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \quad (2.1.34)$$

where

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\nu. \quad (2.1.35)$$

Note that for plane stress,

$$\kappa^* = 3 - 4\nu^* = (3 - \nu)/(1 + \nu). \quad (2.1.36)$$

If $\nu = 0.3$, then $\kappa = 1.8$, $\kappa^* = 2.08$.

We now express σ_{xx} , σ_{xy} , σ_{yy} , and also certain complex combinations of these quantities, in terms of $\phi(z)$ and $\psi(z)$.

Consider the arc AB situated in the region occupied by the elastic body (Figure 2.1.2) and denote the length of the arc measured in the positive direction from A to B by ds .

We denote the normal to the arc AB by \hat{n} ; we take as positive the direction along the normal lying to the right of an observer moving along the arc from A to B . We denote the components of force acting on ds from the direction of the outside normal, i.e., in Figure 2.1.2, from the upper right to the lower left, by $X_n ds$ and $Y_n ds$. In terms of the stress components we have

$$\sigma_{xx} = X_x, \quad \sigma_{xy} = X_y = Y_x, \quad \sigma_{yy} = Y_y. \quad (2.1.37)$$

The components X_n, Y_n are

$$X_n = \sigma_{xx} \cos \alpha + \sigma_{xy} \sin \alpha, \quad Y_n = \sigma_{xy} \cos \alpha + \sigma_{yy} \sin \alpha,$$

so that on introducing the Airy stress function U and noting that

$$\frac{d}{ds} = (\hat{s} \cdot \nabla) = -\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y},$$

we find

$$\begin{aligned} X_n &= \cos \alpha \frac{\partial^2 U}{\partial y^2} - \sin \alpha \frac{\partial^2 U}{\partial x \partial y} = \frac{d}{ds} \left(\frac{\partial U}{\partial y} \right), \\ Y_n &= -\cos \alpha \frac{\partial^2 U}{\partial x \partial y} + \sin \alpha \frac{\partial^2 U}{\partial x^2} = -\frac{d}{ds} \left(\frac{\partial U}{\partial x} \right), \end{aligned}$$

so that

$$(X_n + iY_n)ds = \frac{d}{ds} \left(\frac{\partial U}{\partial y} - i \frac{\partial U}{\partial x} \right) ds = -i \frac{d}{ds} \left(\frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} \right) ds.$$

Substituting from (2.1.32) we have

$$(X_n + iY_n)ds = -i \frac{d}{ds} \left(\phi(z) + z\bar{\phi}'(\bar{z}) + \bar{\psi}(z) \right) ds. \quad (2.1.38)$$

Take \hat{n} in the direction \hat{j} , then $\alpha = \frac{\pi}{2}$, and $X_n = \sigma_{xy}$, $Y_n = \sigma_{yy}$,

$$\frac{d}{ds} = -\frac{\partial}{\partial x} = -\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}},$$

and

$$\sigma_{xy} + i\sigma_{yy} = i(\phi'(z) + \bar{\phi}'(\bar{z}) + z\bar{\phi}''(\bar{z}) + \bar{\psi}'(\bar{z})),$$

or

$$\sigma_{yy} - i\sigma_{xy} = \phi'(z) + \bar{\phi}'(\bar{z}) + z\bar{\phi}''(\bar{z}) + \bar{\psi}'(\bar{z}).$$

Similarly, taking \hat{n} on the \hat{i} direction, so that $\alpha = 0$,

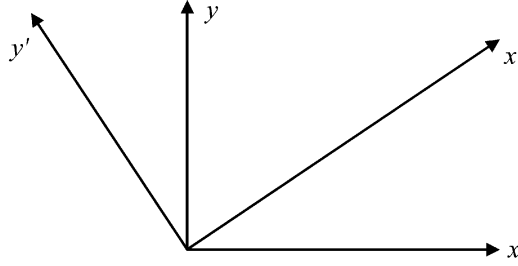
$$\sigma_{xx} + i\sigma_{xy} = \phi'(z) + \bar{\phi}'(\bar{z}) - z\bar{\phi}''(\bar{z}) - \bar{\psi}'(\bar{z}).$$

Introduce the notation

$$\phi'(z) = \Phi(z), \quad \psi'(z) = \Psi(z) \quad (2.1.39)$$

then these equations may be written

$$\sigma_{xx} + i\sigma_{xy} = \Phi(z) + \bar{\Phi}(\bar{z}) - z\bar{\Phi}'(\bar{z}) - \bar{\Psi}(\bar{z}), \quad (2.1.40)$$

Fig. 2.1.3 The axes x' , y' .

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \bar{\Phi}(\bar{z}) + z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z}). \quad (2.1.41)$$

These may be combined to give

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \bar{\Phi}(\bar{z})], \quad (2.1.42)$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)]. \quad (2.1.43)$$

The three stress components σ_{xx} , σ_{yy} , σ_{xy} are (the only three non-zero) components of the stress tensor; they are the components σ_{11} , σ_{22} , σ_{12} of the rank-two symmetric tensor with components σ_{ij} , $i, j = 1, 2, 3$. Under a change of axes, the components change according to the usual tensor law. In particular, if x' , y' are axes as shown in Figure 2.1.3, then

$$\sigma_{x'x'} = \cos^2 \alpha \sigma_{xx} + \sin^2 \alpha \sigma_{yy} + 2 \cos \alpha \sin \alpha \sigma_{xy} \quad (2.1.44)$$

$$\sigma_{y'y'} = \sin^2 \alpha \sigma_{xx} + \cos^2 \alpha \sigma_{yy} - 2 \cos \alpha \sin \alpha \sigma_{xy} \quad (2.1.45)$$

$$\sigma_{x'y'} = -\cos \alpha \sin \alpha (\sigma_{xx} - \sigma_{yy}) + (\cos^2 \alpha - \sin^2 \alpha) \sigma_{xy}. \quad (2.1.46)$$

These may be rewritten as

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \cos 2\alpha \left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right) + \sin 2\alpha \sigma_{xy} \quad (2.1.47)$$

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \cos 2\alpha \left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right) - \sin 2\alpha \sigma_{xy} \quad (2.1.48)$$

$$\sigma_{x'y'} = -\sin 2\alpha \left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right) + \cos 2\alpha \sigma_{xy}. \quad (2.1.49)$$

The combinations $\sigma_{xx} + \sigma_{yy}$ and $\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}$ are convenient for the investigation of the state of stress in an elastic body. The sum of the stresses, $\sigma_{xx} + \sigma_{yy}$, is an invariant: equations (2.1.47), (2.1.48) show that

$$\sigma_{xx} + \sigma_{yy} = \sigma_{x'x'} + \sigma_{y'y'}. \quad (2.1.50)$$

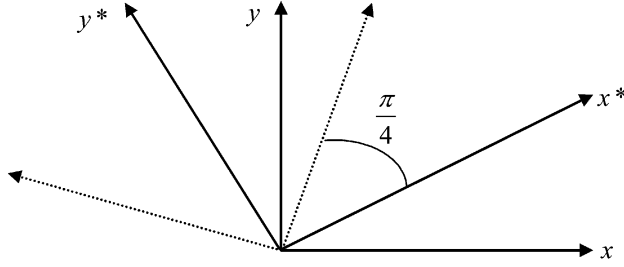


Fig. 2.1.4 The principal axes x^* , y^* and the directions corresponding to maximum shearing stress.

The principal directions of stress at a given point, are those for which $\sigma_{x'y'} = 0$. Equation (2.1.49) shows that these are given by

$$\sin 2\alpha \left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right) = \cos 2\alpha \sigma_{xy}. \quad (2.1.51)$$

We may choose two roots of this equation: α and $\alpha + \frac{\pi}{2}$. Denote these two directions by x^* , y^* as in Figure 2.1.4, then

$$\cos 2\alpha = \frac{\sigma_{xx} - \sigma_{yy}}{2\tau}, \quad \sin 2\alpha = \frac{\sigma_{xy}}{\tau} \quad (2.1.52)$$

where

$$\tau = \left\{ \left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right\}^{\frac{1}{2}}. \quad (2.1.53)$$

When α satisfies (2.1.51), then

$$\cos 2\alpha \left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right) + \sin 2\alpha \sigma_{xy} = \tau$$

so that, from equations (2.1.47) and (2.1.48)

$$\begin{aligned} \sigma_{x^*x^*} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \tau \\ \sigma_{y^*y^*} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} - \tau \end{aligned}$$

The maximum shearing stress at the point occurs for the angles β given by

$$\cos 2\beta = \frac{\sigma_{xy}}{\tau}, \quad \sin 2\beta = - \left(\frac{\sigma_{xx} - \sigma_{yy}}{2\tau} \right). \quad (2.1.54)$$

Clearly, combining (2.1.52), (2.1.54) we find

$$\cos 2\alpha \cos 2\beta + \sin 2\alpha \sin 2\beta = 0$$

so that $\cos(2\alpha - 2\beta) = 0$, or $\alpha - \beta = \pm \frac{\pi}{4}$. This means that the directions of maximum shearing stress bisect the angles between the principal directions, as shown in Figure 2.1.4. The maximum shearing stress is

$$\sigma_{x'y'} = -\sin 2\beta \left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right) + \cos 2\beta \sigma_{xy} = \tau :$$

the maximum shearing stress is τ .

The magnitude of $\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}$ is twice the maximum shearing stress, τ , at the given point. The principal stresses at the point are (the values of $\sigma_{x^*x^*}$, $\sigma_{y^*y^*}$ respectively)

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \tau, \quad (2.1.55)$$

$$\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \tau. \quad (2.1.56)$$

Now equations (2.1.42), (2.1.43) give

$$\tau = |\bar{z}\Phi'(z) + \Psi(z)| \quad (2.1.57)$$

$$\sigma_1 = \Phi(z) + \bar{\Phi}(\bar{z}) + |\bar{z}\Phi'(z) + \Psi(z)|, \quad (2.1.58)$$

$$\sigma_2 = \Phi(z) + \bar{\Phi}(\bar{z}) - |\bar{z}\Phi'(z) + \Psi(z)|. \quad (2.1.59)$$

2.2 Stresses and Displacements in a Semi-Infinite Elastic Plane

Usually, the linear dimensions of the area of contact are small compared with the radii of curvature of the touching bodies. Therefore, we assume when considering plane contact problems, that the elastic body which is subjected to the pressure of the punch is semi-infinite. For plane problems, we assume that the elastic body occupies a semi-infinite plane. This assumption somewhat distorts the picture of the state of stress. However, this distortion is appreciable only fairly far away from the contact region.

In this chapter we give the solutions for a number of plane contact problems. Some results appear for the first time, others were given earlier, in particular, in the third edition of Muskhelishvili (1953).

However, we employ a slightly different method for solving these problems. In Muskhelishvili (1953), the problem is reduced to the determination of the functions $\Phi(z)$ and $\Psi(z)$ in (2.1.39). In this section, we introduce the functions $w_1(z)$ and $w_2(z)$ which are integrals of Cauchy type, whose densities are the normal pressure and tangential load acting on the boundary. $\Phi(z)$ and $\Psi(z)$, from which the state of stress in an elastic half-plane can be found, are easily determined from $w_1(z)$ and $w_2(z)$.

The functions $w_1(z)$ and $w_2(z)$ have many advantages: anisotropic contact problems, problems for a moving punch, and also more complicated problems (those with zones of various types on the contact region) can be reduced to mixed boundary value problems for these functions.

Thus, we shall proceed to determine the stresses and displacements in a half-plane on whose boundary normal pressure is applied and tangential stress is distributed.

We shall make use of equation (2.1.41):

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \bar{\Phi}(\bar{z}) + z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z}).$$

We consider this complex combination of stresses for the half-plane under the assumption that the stresses tend to zero at infinity. This implies the following behaviour at infinity, i.e., for large values of $|z|$:

$$\begin{aligned}\Phi(z) &= \frac{\gamma_1}{z} + o\left(\frac{1}{z}\right), & \Psi(z) &= \frac{\gamma_2}{z} + o\left(\frac{1}{z}\right) \\ \Phi'(z) &= -\frac{\gamma_1}{z^2} + o\left(\frac{1}{z^2}\right)\end{aligned}$$

We recall the elements of the theory of Cauchy integrals, see Gladwell (1980) for a fuller version.

We start with the definition of a holomorphic function of a complex variable z . The function $f(z)$ is said to be holomorphic (sometimes the term regular is used) in a finite region S of the complex plane if it is single-valued in S , and its complex derivative $f'(z)$ exists at every point in S . The condition that $f(z)$ have a complex derivative is so strong that it may be proved that if $f(z)$ is holomorphic in S , then it will possess complex derivatives $f^{(n)}(z)$ of any finite order at every point in S , so that each such derivative will also be holomorphic in S . (Note the contrast with functions of a real variable, where the existence of $f''(x)$ by no means follows from the existence of $f'(x)$.) Further, it may be expanded in a series

$$f(x) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

about any point $z_0 \in S$. If the region S is infinite, then $f(z)$ is said to be holomorphic at infinity if $f(1/z)$ is holomorphic at the origin. This means that, for large $|z|$, $f(z)$ may be expanded in the form

$$f(z) = \sum_{n=0}^{\infty} b_n z^{-n}.$$

If $f(z)$ is holomorphic in the entire complex plane, except the point at infinity, then it must be a polynomial in z . If, in addition it is holomorphic at infinity, then it must be a constant.

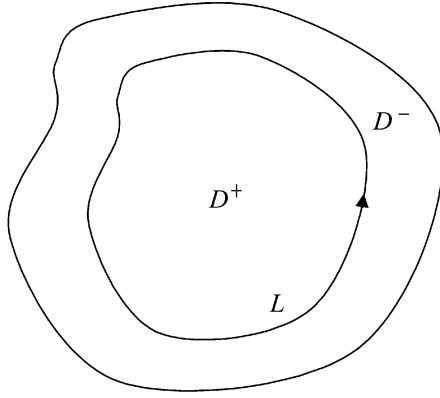


Fig. 2.2.1 The contour L divides S into D^+ and D^- .

Now we introduce

Theorem 1 (Cauchy's Theorem). *If L is a simple closed contour lying wholly in a region S in which the function $f(z)$ is holomorphic, then*

$$\int_L f(t)dt = 0, \quad (2.2.1)$$

where we shall use t to denote the generic point of the contour L .

The contour L divides S into two parts, D^+ lying to the left, the inside of L , and D^- on the right, the outside, as shown in Figure 2.2.1.

Apply this theorem to the function $f(z) = 1/(z - z_0)$, which is holomorphic in any region excluding z_0 . If z lies outside L , i.e., in D^- , then Cauchy's Theorem gives

$$\int_L \frac{dt}{t - z_0} = 0 \text{ for } z_0 \in D^-.$$

If z_0 lies inside L , i.e., $z_0 \in D^+$, then we construct the contour $L + C_1 + C_\varepsilon + C_2$, as shown in Figure 2.2.2, so that again z_0 lies outside the contour, and

$$\left\{ \int_L + \int_{C_1} + \int_{C_\varepsilon} + \int_{C_2} \right\} \frac{dt}{t - z_0} = 0.$$

But the integrals along C_1 , C_2 are equal and opposite, and the integral around C_ε may be evaluated by writing $t = z_0 + \varepsilon \exp(i\theta)$, $dt = i\varepsilon \exp(i\theta)$ so that, since C_ε is traversed clockwise,

$$-\int_{C_\varepsilon} \frac{dt}{t - z_0} = \int_0^{2\pi} \frac{i\varepsilon \exp(i\theta)}{\varepsilon \exp(i\theta)} d\theta = 2\pi i,$$

and therefore

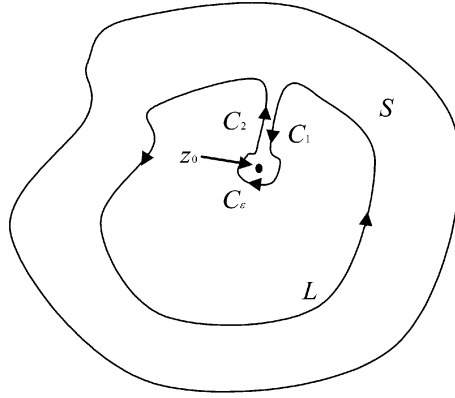


Fig. 2.2.2 The point z_0 lies outside the contour $L + C_1 + C_2 + C_\varepsilon$.

$$\frac{1}{2\pi i} \int_L \frac{dt}{t - z_0} = 1 \text{ for } z_0 \in D^+.$$

Now write

$$\int_L \frac{f(t)dt}{t - z_0} = \int_L \frac{f(t) - f(z_0)}{t - z_0} dt + f(z_0) \int_L \frac{dt}{t - z_0}.$$

If L is a closed contour lying in a region in which $f(z)$ is holomorphic, then $(f(z) - f(z_0))/(z - z_0)$ will also be holomorphic, so that the first integral will be zero, giving

$$\frac{1}{2\pi i} \int_L \frac{f(t)dt}{t - z_0} = \begin{cases} f(z_0), & \text{if } z_0 \in D^+ \\ 0, & \text{if } z_0 \in D^- \end{cases} \quad (2.2.2)$$

We emphasize that this equation holds provided that $f(z)$ is holomorphic in S .

Now let L again be a simple closed contour, and let $f(t)$ be a function given and continuous on L ; it need be defined only on L , not as a function in S . The equation

$$F(z) = \frac{1}{2\pi i} \int_L \frac{f(t)dt}{t - z} \quad (2.2.3)$$

defines a function which may easily be shown to be holomorphic everywhere except on L . Such a function is called a Cauchy integral. If $f(t)$ happens to be the boundary value of a function $f(z)$ holomorphic in S then, according to (2.2.2),

$$F(z) = \begin{cases} f(z), & \text{if } z \in D^+ \\ 0, & \text{if } z \in D^- \end{cases} \quad (2.2.4)$$

Note, however, that $F(z)$ may be defined by (2.2.3) provided only that $f(t)$ is continuous on L . (Even this condition may be relaxed.)

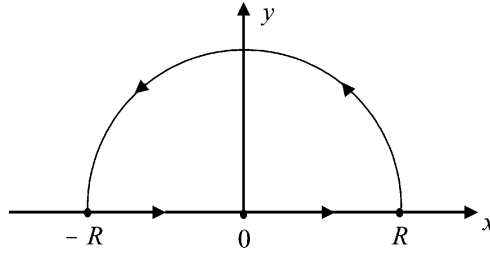


Fig. 2.2.3 A semi-circle in the upper half-plane.

We need to extend these results to the case in which L is the whole x -axis. Consider the contour shown in Figure 2.2.3 consisting of a semi-circle of radius R and the segment $(-R, R)$. If $f(z)$ is holomorphic in the upper half plane and

$$f(z) = \frac{\gamma}{z} + o\left(\frac{1}{z}\right)$$

at infinity, then equation (2.2.2) gives

$$\frac{1}{2\pi i} \int_{C_R} \frac{f(t)dt}{t-z} + \frac{1}{2\pi i} \int_{-R}^R \frac{f(t)dt}{t-z} = \begin{cases} f(z), & \text{if } z \in D^+ \\ 0, & \text{if } z \in D^- \end{cases}.$$

For large R , we write

$$\frac{1}{t-z} = \frac{1}{t} + \frac{z}{t^2} + \dots$$

and evaluate the integral around C_R . Here $t = R \exp(i\theta)$, $dt = iR \exp(i\theta)d\theta$, so that the leading term in the expansion has the form

$$\gamma \int_{C_R} \frac{dt}{t^2} = \gamma \int_0^\pi \frac{iR \exp(i\theta)}{R^2 \exp(2i\theta)} d\theta = O\left(\frac{1}{R}\right).$$

Thus, letting $R \rightarrow \infty$, we find that the integral around C_R vanishes, and so

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)dt}{t-z} = \begin{cases} f(z), & \text{if } z \in S^+ \\ 0, & \text{if } z \in S^- \end{cases} \quad (2.2.5)$$

where, in the limit, S^+ and S^- are the upper and lower half-planes respectively.

If the point $z = x + iy$ is in the upper half-plane, i.e., $y > 0$, then $\bar{z} = x - iy$ is in the lower half-plane and $\bar{f}(\bar{z})$ is holomorphic in the lower half-plane. Thus, applying the second line of (2.2.5) to the lower half-plane, we deduce that

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\bar{f}(t)dt}{t-z} = 0, \quad z \in S^+ \quad (2.2.6)$$

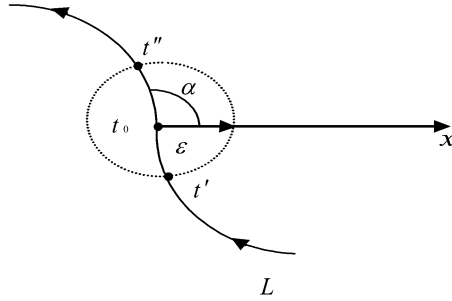


Fig. 2.2.4 The points t', t'' are on L , equidistant from t_0 .

where again S^+ denotes the upper half-plane.

Now return to (2.2.3), and assume that $f(t)$ is defined and continuous on L . $F(z)$ is holomorphic everywhere except on L . We compute the limiting values $F^+(t)$ and $F^-(t)$ as z approaches a point t of L from D^+ or D^- respectively.

To do this, we assume that, in addition to being continuous on L , $f(t)$ satisfies a so-called Hölder condition. The function $f(t)$ is said to satisfy a Hölder condition on L if there exist parameters A, λ , where $0 < \lambda < 1$ such that, for every two points t_1, t_2 of L we have

$$|f(t_2) - f(t_1)| < A|t_2 - t_1|^\lambda. \quad (2.2.7)$$

The function $f(t)$ is said to satisfy a Hölder condition in the neighbourhood of a point $t_0 \in L$ if (2.2.6) holds for all t_1, t_2 sufficiently near t_0 . Under this condition, we shall show that $F(t)$ in (2.2.3) may be given a meaning when $z \in L$, and $F(z)$ tends to definite limits $F^+(t), F^-(t)$ as $z \rightarrow t \in L$ from D^+ or D^- .

Let $t_0 \in L$, and suppose $f(t)$ satisfies a Hölder condition in the neighbourhood of t_0 . Let t', t'' be two points on L on either side of t_0 , such that

$$|t_0 - t'| = |t_0 - t''| = \varepsilon$$

as shown in Figure 2.2.4. The Cauchy Principal Value of the integral (2.2.3) at t_0 is defined to be

$$\frac{1}{2\pi i} \oint_L \frac{f(t)dt}{t - t_0} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{L-\ell} \frac{f(t)dt}{t - t_0}$$

where ℓ is the arc $t't''$. The integral may be written

$$\frac{1}{2\pi i} \int_{L-\ell} \frac{f(t) - f(t_0)}{t - t_0} dt + \frac{f(t_0)}{2\pi i} \int_{L-\ell} \frac{dt}{t - t_0}.$$

Since $\frac{|f(t) - f(t_0)|}{|t - t_0|} < A|t - t_0|^{\lambda-1}$, the limit of the first integral exists in the ordinary sense, i.e., provided only that t', t'' tend to t_0 ; it is not necessary for $|t_0 - t'|, |t_0 - t''|$ to be equal. The second integral is

$$\int_{L-\ell} \frac{dt}{t-t_0} = [\ell n(t-t_0)]'_{t''}$$

where we have taken a branch of $\ell n z$ that is continuous on $L-\ell$. Now

$$t' = t_0 + \varepsilon \exp[i(\alpha + \pi)], \quad t'' = t_0 + \varepsilon \exp(i\alpha)$$

so that

$$\ell n(t' - t_0) - \ell n(t'' - t_0) = i\pi$$

and the Cauchy Principal Value of the integral is

$$\frac{1}{2\pi i} \oint_L \frac{f(t)dt}{t-t_0} = \frac{1}{2\pi i} \int_L \frac{f(t) - f(t_0)}{t-t_0} dt + \frac{1}{2} \cdot f(t_0).$$

This is the meaning that will be attached to the integral (2.2.3) when $z \in L$; thus

$$F(t_0) = \frac{1}{2\pi i} \oint_L \frac{f(t)}{t-t_0} = \frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int_L \frac{f(t) - f(t_0)}{t-t_0} dt. \quad (2.2.8)$$

Now return to equation (2.2.3) and write

$$F(z) = \frac{1}{2\pi i} \int_L \frac{f(t) - f(t_0)}{t-z} dt + \frac{f(t_0)}{2\pi i} \int_L \frac{dt}{t-z}$$

where $t_0 \in L$. It may be proved that the first integral tends to

$$\frac{1}{2\pi i} \int_L \frac{f(t) - f(t_0)}{t-t_0} dt$$

as $z \rightarrow t_0$, from whichever side of L . The second integral has, by the argument used before, the values

$$\frac{f(t_0)}{2\pi i} \int_L \frac{dt}{t-z} = \begin{cases} f(t_0), & \text{if } z \in D^+ \\ 0, & \text{if } z \in D^- \end{cases}.$$

Thus, the limits of $F(z)$ as $z \rightarrow t_0$, from D^+ and D^- are respectively

$$\begin{aligned} F^+(t_0) &= \frac{1}{2\pi i} \int_L \frac{f(t) - f(t_0)}{t-t_0} dt + f(t_0) \\ F^-(t_0) &= \frac{1}{2\pi i} \int_L \frac{f(t) - f(t_0)}{t-t_0} dt. \end{aligned}$$

Now, returning to the definition of the Cauchy Principal Value of the integral in (2.2.3) we have

$$F^+(t_0) = \frac{1}{2} f(t_0) + \frac{1}{2\pi i} \oint_L \frac{f(t)}{t-t_0} dt \quad (2.2.9)$$

$$F^-(t_0) = -\frac{1}{2}f(t_0) + \frac{1}{2\pi i} \oint_L \frac{f(t)}{t - t_0} dt \quad (2.2.10)$$

These equations, called the Plemelj formulae, are often written in the form

$$F^+(t_0) - F^-(t_0) = f(t_0), \quad t_0 \in L \quad (2.2.11)$$

$$F^+(t_0) + F^-(t_0) = \frac{1}{\pi i} \oint_L \frac{dt}{t - t_0}, \quad t_0 \in L. \quad (2.2.12)$$

When L is the real axis, then these results still hold if $f(t)$ is finite and integrable along any finite part of the axis, and satisfies the condition

$$f(t) = f(\infty) + O(|t|^{-\lambda}) \quad \lambda > 0$$

for large $|t|$. We then define the Cauchy integral (2.2.3) as

$$F(z) = \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{-N}^N \frac{f(t)}{t - z} dt$$

and find

$$F(z) = \pm \frac{1}{2}f(\infty) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t) - f(\infty)}{t - z} dt$$

where the sign is \pm according to whether $z \in S^+$ or $z \in S^-$. Further details may be found in Muskhelishvili (1953) or Gladwell (1980).

We now return to the text. Galin assumes that the elastic body occupies the lower half-plane. While this is perhaps appealing to an engineer – a punch is pressed down on a medium, it complicates the mathematics. Also, this section in the original version is made complicated by the chosen notation; we have therefore changed the notation and rearranged the analysis.

Suppose that the elastic body, occupying the upper half-plane, is subject to normal and shear stresses

$$\sigma_{yy}(x, 0) = -p(x), \quad \sigma_{xy}(x, 0) = -q(x) \quad (2.2.13)$$

as shown in Figure 2.2.5. Remember the convention regarding these stresses shown in Figure 2.1.1.

Equation (2.1.41) gives

$$-p(x) + iq(x) = \{\Phi(z) + \bar{\Phi}(\bar{z}) + \bar{z}\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})\}|_{y=0}, \quad (2.2.14)$$

where in the third term on the right, we have replaced z by \bar{z} ($z = \bar{z}$ on the x -axis). Taking the complex conjugate of this equation, we find

$$-p(x) - iq(x) = \{\bar{\Phi}(\bar{z}) + \Phi(z) + z\Phi'(z) + \Psi(z)\}|_{y=0}. \quad (2.2.15)$$

Multiply each of these equations by $1/(2\pi i(x - z))$ and integrate over $(-\infty, \infty)$, using equations (2.2.5), (2.2.6) and making use of the fact that both $\Phi(z)$ and $\Phi(z) +$

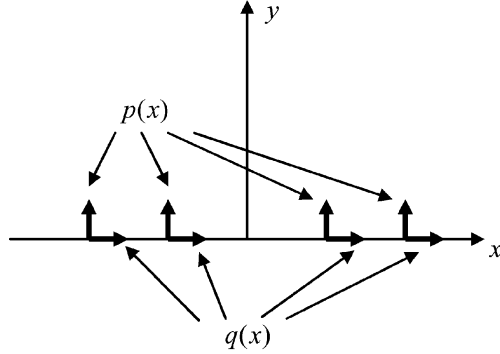


Fig. 2.2.5 The upper half-plane is subjected to distributed forces on the boundary.

$z\Phi'(z) + \Psi(z)$ are holomorphic in the upper half-plane. We find

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{-p(x) + iq(x)}{x - z} dx = \Phi(z), \quad (2.2.16)$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{-p(x) - iq(x)}{x - z} dx = \Phi(z) + z\Phi'(z) + \Psi(z). \quad (2.2.17)$$

Now turn to equation (2.1.34) for the displacements

$$2\mu(u(x, 0) + iv(x, 0)) = \kappa\phi(z) - \bar{z}\bar{\phi}'(\bar{z}) - \bar{\psi}(\bar{z})|_{y=0}$$

where again, in the second term, we have replaced z by \bar{z} . Differentiating w.r.t. x and using (2.1.39), we find

$$2\mu(u'(x, 0) + iv'(x, 0)) = \kappa\Phi(z)|_{y=0} - \{\bar{\Phi}(\bar{z}) + \bar{z}\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})\}|_{y=0}. \quad (2.2.18)$$

Now $\Phi(z)$ is given by (2.2.16) and $\Phi(z) + z\Phi'(z) + \Psi(z)$ by (2.2.17). Thus, according to (2.2.9), the value of $\Phi^+(x)$ is

$$\Phi^+(x) = -\frac{1}{2}(p(x) - iq(x)) - \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{p(t) - iq(t)}{t - x} dt$$

and similarly

$$\Phi(x) + x\Phi'(x) + \Psi(x)|^+ = -\frac{1}{2}(p(x) + iq(x)) - \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{p(t) + iq(t)}{t - x} dt$$

where these integrals are interpreted as Cauchy principal values. Inserting these into (2.2.18), we find

$$2\mu(u'(x, 0) + iv'(x, 0)) = -\frac{\kappa - 1}{2}(p(x) - iq(x)) - \frac{\kappa + 1}{2\pi i} \oint_{-\infty}^{\infty} \frac{p(t) - iq(t)}{t - x} dt.$$

Separating the real and imaginary parts, we find

$$2\mu u'(x, 0) = -\frac{\kappa - 1}{2}p(x) + \frac{\kappa + 1}{2\pi} \oint_{-\infty}^{\infty} \frac{q(t)dt}{t - x}, \quad (2.2.19)$$

$$2\mu v'(x, 0) = \frac{\kappa - 1}{2}q(x) + \frac{\kappa + 1}{2\pi} \oint_{-\infty}^{\infty} \frac{p(t)dt}{t - x}. \quad (2.2.20)$$

Introducing the parameters

$$\beta = \frac{\kappa - 1}{\kappa + 1}, \quad \vartheta = \frac{\kappa + 1}{4\mu}, \quad (2.2.21)$$

we may write

$$\frac{u'(x)}{\vartheta} = -\beta p(x) + \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{q(t)dt}{t - x}, \quad (2.2.22)$$

$$\frac{v'(x)}{\vartheta} = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{p(t)dt}{t - x} + \beta q(x). \quad (2.2.23)$$

Note that the integrals must be interpreted as Cauchy principal values. If the stresses are applied over a finite interval $(-a, b)$, then the integrals will have limits $-a$ and b .

Suppose the stresses act over a finite interval $(-a, b)$, then we may integrate (2.2.22), (2.2.23) w.r.t. x and find

$$\frac{u(x, 0)}{\vartheta} = -\beta \int_{-a}^x p(t)dt - \frac{1}{\pi} \oint_{-a}^b q(t) \ell n|t - x|dt + C_1 \quad (2.2.24)$$

$$\frac{v(x, 0)}{\vartheta} = -\frac{1}{\pi} \oint_{-a}^b p(t) \ell n|t - x|dt + \beta \int_{-a}^x q(t)dt + C_2 \quad (2.2.25)$$

where C_1, C_2 are arbitrary constants. The equations are due to Muskhelishvili (1953).

If we use Young's modulus, E , and Poisson's ratio ν , instead of $\mu, \kappa + 1$ and β , we have

$$\kappa = 3 - 4\nu, \quad 2\mu = E/(1 + \nu) \quad (2.2.26)$$

so that

$$\frac{1}{\vartheta} = \frac{4\mu}{\kappa + 1} = \frac{2E}{(1 + \nu)(4 - 4\nu)} = \frac{E}{2(1 - \nu^2)}, \quad \beta = \frac{1 - 2\nu}{2(1 - \nu)}. \quad (2.2.27)$$

We now introduce two functions holomorphic in the upper half-plane:

$$w_1(z) = \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{p(t)dt}{t-z} = u_1 + i v_1, \quad (2.2.28)$$

$$w_2(z) = \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{q(t)dt}{t-z} = u_2 + i v_2. \quad (2.2.29)$$

(Note that Galin omits the factor $1/(2\pi i)$ in the definitions of w_1 and w_2 . The analysis is neater if it is included.) Using equation (2.2.9), we see that the upper boundary values of these functions are

$$w_1^+(x) = \frac{1}{2}p(x) + \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{p(t)dt}{t-x} = u_1^+(x) + i v_1^+(x), \quad (2.2.30)$$

$$w_2^+(x) = \frac{1}{2}q(x) + \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{q(t)dt}{t-x} = u_2^+(x) + i v_2^+(x), \quad (2.2.31)$$

so that

$$u_1^+(x) = \frac{1}{2}p(x), \quad v_1^+(x) = -\frac{1}{2\pi} \oint_{-\infty}^{\infty} \frac{p(t)dt}{t-x}, \quad (2.2.32)$$

$$u_2^+(x) = \frac{1}{2}q(x), \quad v_2^+(x) = -\frac{1}{2\pi} \oint_{-\infty}^{\infty} \frac{q(t)dt}{t-x}, \quad (2.2.33)$$

and we may write equations (2.2.22), (2.2.23) as

$$\frac{u'(x, 0)}{2\vartheta} = -\beta u_1^+(x) - v_2^+(x), \quad (2.2.34)$$

$$\frac{v'(x, 0)}{2\vartheta} = -v_1^+(x) + \beta u_2^+(x). \quad (2.2.35)$$

We now establish certain properties of the functions $w_1(z)$ and $w_2(z)$. Equations (2.2.32), (2.2.33) show that the real parts of these functions are related to the normal pressure and shear stress acting on the surface $y = 0$. These quantities can become infinite at certain points. We now investigate the character of the singularities that $w_1(z)$ and $w_2(z)$ can have.

If a concentrated force is applied to the boundary of the half-plane, this can be pictured as the transmission of pressure (and shear stress) by means of an extremely narrow punch. In this case, the functions $w_1(z)$ and $w_2(z)$ possess poles of the first order.

When, on the other hand, the pressure and shear stress is transmitted by means of a punch of finite width, there can be no concentrated forces under the punch, even at the ends. It follows that the real parts of $w_1(z)$ and $w_2(z)$ can have only integrable singularities on the real axis. This condition is satisfied if the functions of $w_1(z)$ and $w_2(z)$, which are integrals of Cauchy type, have singularities of the form $(z-c)^{-\theta}$, where $0 < \theta < 1$.

To obtain the limiting forms of $w_1(z)$, $w_2(z)$ as $z \rightarrow \infty$, we return to equations (2.2.28), (2.2.29):

$$w_1(z) \rightarrow \frac{iP}{2\pi z}, \quad w_2(z) \rightarrow \frac{iQ}{2\pi z} \quad (2.2.36)$$

where

$$P = \int_{-a}^b p(t)dt, \quad Q = \int_{-a}^b q(t)dt \quad (2.2.37)$$

are the resultants of the forces applied by the punch. If the normal pressure and shear are distributed over a finite number of intervals of finite length, then $w_1(z)$, $w_2(z)$ will still have the form (2.2.36) at infinity. In the contact problems discussed in this book, $w_1(z)$ and $w_2(z)$ will always possess these properties.

We now express the functions $\phi'(z) \equiv \Phi(z)$ and $\psi'(z) \equiv \Psi(z)$, which serve as the basis for determining the stresses, in terms of $w_1(z)$ and $w_2(z)$. Equations (2.2.16), (2.2.17) give

$$\Phi(z) = -w_1(z) + iw_2(z), \quad (2.2.38)$$

$$\Psi(z) = -2iw_2(z) + zw_1'(z) - izw_2'(z), \quad (2.2.39)$$

from which the stresses σ_{xx} , σ_{xy} , σ_{yy} may be found by using (2.1.42), (2.1.43).

Contact Problems

The legacy of L.A. Galin

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