2

Introduction to the Theory of Anisotropic and Inhomogeneous Materials

In this chapter, a very brief introduction to composite materials, the evaluation of material properties and their homogenization techniques are given. Also, a brief introduction is given to smart composites and the basic constitutive models for composites, where a few standard smart materials can be embedded. Towards the end of the chapter a brief description of how to obtain constitutive models for inhomogeneous materials, such as functionally graded materials, is presented.

2.1 Introduction to Composite Materials

As the name suggests, composite materials are obtained by combining two or more materials at the macroscale to obtain a useful structural material. Although these materials at the microscopic scale can be inhomogeneous, they can be considered homogeneous at the macroscopic level. These materials possess the qualities of each of the constituents and the choice of constituents depends on the application for which these materials are required. These materials are normally preferred due to their light weight, high strength, and high corrosion resistance properties. The two normal constituents of a composite material are the *Fiber* and the *Matrix*. Depending upon how they are bound together, different types of composite materials can be obtained. Owing to the difference in the constitutive behavior of these two constituent materials, the constitutive model of the compound material is normally anisotropic. Composites can be classified into three different categories, namely ***fibrous composites***, ***particulate composites*** and ***laminated composites***.

The fibrous composites consist of fibers or whiskers dispersed in a matrix to form a structural element. The fibers are normally expected to take all the load and all the fibers in the structural element are bound together by the matrix. In addition to binding, the matrix helps in stress transfer and also to protect the fibers from harmful environmental effects. The matrix material normally has low stiffness, density and strength compared to the fibers. Some of the commonly used fibers are made of carbon, graphite, boron, E-glass *etc.* while the most commonly used matrix material is epoxy, which is essentially a polymer material.

The particulate composite consists of particles of one or more materials suspended in a matrix of a different material. The particles and the matrix can be metallic or non-metallic. Concrete is a very good example of a particulate composite wherein the sand and the granite are bound by a matrix material (cement). Here the particles are non-metallic. Use of mica in glass is yet another example of a particulate composite, which is used extensively as an insulating material in electrical applications. For spacecraft, rocket propellents are used extensively as a fuel. These propellents consist of aluminum powder and perchlorate oxidizers mixed in an organic binder. The normal binder material is polyurethene. This is an example of metallic particles in a non-metallic composite.

For structural applications, the above two forms of composite are seldom used; here the common type is the laminated composite. Hence an entire section is devoted to this form of composite.

2.2 Theory of Laminated Composites

Laminated composites have found extensive use as aircraft structural materials due to their high strength to weight and stiffness to weight ratios. Their popularity stems from the fact that they are extremely light-weight and the laminate construction enables the designer to tailor the strength of the structure in any required direction depending upon the loading directions to which the structure is subjected. In addition to aircraft structures, they have found their way into many automobile and building structures. Apart from having better strength, stiffness and lower weight properties, they have better corrosion resistance, thermal and acoustic insulation properties than metallic structures.

The laminated composite structure consists of many laminas (plies) stacked together to form the structure. The number of plies or laminas depends on the strength that the structure is required to sustain. Each lamina contains fibers oriented in the direction where the maximum strength is required. These fibers are bound together by a matrix material. The laminated composite structure derives its strength from the fibers. The commonly used fibers are made of carbon, glass, Kevler and boron. The most commonly used matrix material is epoxy resin. These materials are orthotropic at the lamina level while at the laminate level, they exhibit highly anisotropic properties. The anisotropic behavior results in stiffness coupling, such as bending axial shear coupling in beams and plates, bending axial torsion coupling in aircraft thin-walled structures, *etc.* These coupling effects make the analysis of laminated composite structures very complex.

2.2.1 Micromechanical Analysis of a Lamina

A lamina is a basic element of a laminated composite structure, constructed from fibers that are bound together by the matrix resin. The strength of the lamina, and hence the laminate, depends on the type of fiber, their orientation and also the volume fraction of fiber in relation to the overall volume of lamina. Since the lamina is a heterogeneous mixture of fibers dispersed in a matrix, determination of the material properties of the lamina, which is assumed to be orthotropic in character, is a very involved process. The method used in the determination of lamina material properties is micromechanical analysis. According to Jones [11], micromechanics is the study of composite material behavior, wherein the interaction of the constituent materials is examined in detail as part of the definition of the behavior of the heterogeneous composite material.

Hence, the objective of micromechanics is to determine the elastic moduli of a composite material in terms of the elastic moduli of the constituent materials, namely the fibers and the matrix. Thus, the property of a lamina can be expressed as

$$Q_{ij} = Q_{ij}(E_f, E_m, \nu_f, \nu_m, V_f, V_m)\,, \tag{2.1}$$

where E, ν and V are the elastic moduli, Poisson's ratio and the volume fraction respectively, and f and m subscripts denote the fiber and the matrix, respectively. The volume fraction of fiber is determined from the expression: V_f = (volume of fiber)/(total volume of lamina) and $V_m = 1 - V_f$.

There are two basic approaches for the determination of material properties of the lamina. They can be grouped under the following heads: (1) the strength of materials approach and (2) the theory of elasticity approach. The first method gives an experimental way of determining the elastic moduli. The second method gives upper and lower bounds on the elastic moduli and not their actual values. In fact, there are many papers available in the literature that deal with the theory of elasticity approach to determine the elastic moduli of a composite. In this section, only the first method is presented. There are many classic textbooks on composites such as Jones [11] and Tsai [12] that cover this in detail.

2.2.2 Strength of Materials Approach to Determination of Elastic Moduli

The material properties of a lamina are determined by making some assumptions concerning the behavior of its constituents. The fundamental assumption is that the fiber is the strong constituent of a composite lamina and hence is the main load bearing member, and the matrix is weak and its main function is to protect the fibers from severe environmental effects. Also, the strains in the matrix and the fiber are assumed to be the same. Hence, a plane section before the application of bending stress remains plane after bending. In the present analysis, we consider a unidirectional, orthotropic composite lamina to derive expressions for the elastic moduli. In doing so, we limit our analysis to a small volume element, small enough to show the microscopic structural details, yet large enough to represent the overall behavior of the composite lamina. Such a volume is called the representative volume (RV). A simple RV is a fiber surrounded by matrix as shown in Figure 2.1.

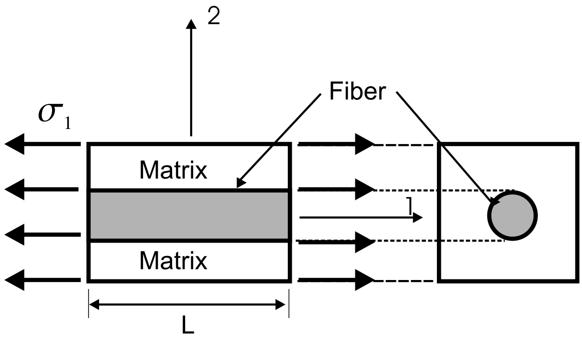

Fig. 2.1. RV for the determination of longitudinal material properties

First, the procedure for determining the elastic modulus E_1 is given. In Figure 2.1, the strain in the 1-direction is given by $\epsilon_1 = \Delta L/L$, where this strain is felt both by the matrix and the fiber, according to our basic assumption. The corresponding stresses in the fiber and the matrix are given by

$$\sigma_f = E_f \epsilon_1 \,, \quad \sigma_m = E_m \epsilon_1 \,. \tag{2.2}$$

Here E_f and E_m are the elastic modulus of the fiber and the matrix respectively. The cross-sectional area A of the RV is made up of the area of the fiber A_f and the area of the matrix A_m. If the total stress acting on the cross-section of the RV is σ_1, then the total load acting on the cross-section is

$$P = \sigma_1 A = E_1 \epsilon_1 A = \sigma_f A_f + \sigma_m A_m \,. \tag{2.3}$$

From the above expression, we can write the elastic moduli in the 1-direction as

$$E_1 = E_f \frac{A_f}{A} + E_m \frac{A_m}{A} \,. \tag{2.4}$$

The volume fraction of the fiber and the matrix can be expressed in terms of areas of the fiber and the matrix as

$$V_f = A_f/A \,, \quad V_m = A_m/A \,. \tag{2.5}$$

Using Equation (2.5) in Equation (2.4), we can write the modulus in the 1-direction as

$$E_1 = E_f V_f + E_m V_m \, . \tag{2.6}$$

Equation 2.6 is the well known rule of mixtures for obtaining the equivalent modulus of the lamina in the direction of the fibers.

The equivalent modulus E_2 of the lamina is determined by subjecting the RV to a stress σ_2 perpendicular to the direction of the fiber as shown in Figure 2.2. This stress is assumed to be the same in both the matrix and the fiber.

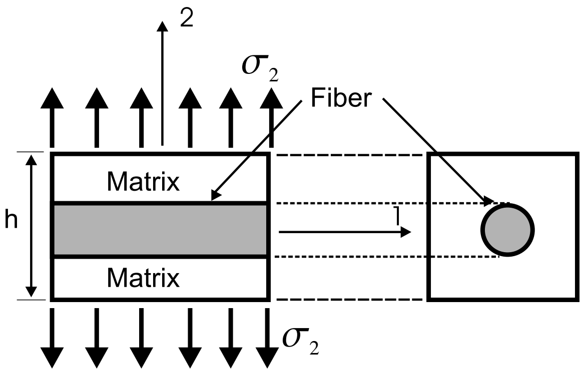

Fig. 2.2. RV for determination of transverse material property

The strains in the fiber and matrix due to this stress are given by

$$\epsilon_f = \sigma_2 / E_f \, , \quad \epsilon_m = \sigma_2 / E_m \, . \tag{2.7}$$

If h is the depth of the RV (see Figure 2.2), then this total strain ϵ_2 is distributed as a function of the volume fraction as

$$\epsilon_2 h = (V_f \epsilon_f + V_m \epsilon_m) h \, . \tag{2.8}$$

Substituting Equation (2.7) in Equation (2.8), we get

$$\epsilon_2 = V_f \frac{\sigma_2}{E_f} + V_m \frac{\sigma_2}{E_m} \, . \tag{2.9}$$

However, we have

$$\sigma_2 = E_2 \epsilon_2 = E_2 \left(V_f \frac{\sigma_2}{E_f} + V_m \frac{\sigma_2}{E_m} \right) \, . \tag{2.10}$$

From the above relation, the equivalent modulus in the transverse direction is given by

$$E_2 = \frac{E_f E_m}{V_f E_m + V_m E_f} \, . \tag{2.11}$$

The major Poissons ratio ν_{12} is determined as follows. If the RV of width W and depth h is loaded in the direction of the fiber, then both strains ϵ_1 and ϵ_2 will be induced in the 1 and 2 directions. The total transverse deformation δ_h is the sum of the transverse deformation in the matrix and the fiber and is given by

$$\delta_h = \delta_{hf} + \delta_{hm} \,. \tag{2.12}$$

The major Poissons ratio is also defined as the ratio of the transverse strain to the longitudinal strain and expressed as

$$\nu_{12} = -\epsilon_2/\epsilon_1 \,. \tag{2.13}$$

The total transverse deformation can also be expressed in terms of the depth h as

$$\delta_h = -h\epsilon_2 = h\nu_{12}\epsilon_1 \,. \tag{2.14}$$

Following the procedure adopted for the determination of the transverse modulus, the transverse displacement in the matrix and the fiber can be expressed in terms of its respective volume fraction and the Poissons ratio as

$$\delta_{hf} = hV_f\nu_f\epsilon_1 \,, \delta_{hm} = hV_m\nu_m\epsilon_1 \,. \tag{2.15}$$

Using Equations (2.14) and (2.15) in Equation (2.12), we can write the expression for the major Poissons ratio as

$$\nu_{12} = \nu_f V_f + \nu_m V_m \,. \tag{2.16}$$

By adopting a similar procedure to that used in the determination of the transverse modulus, we can write the shear modulus in terms of the constituent properties as

$$G_{12} = \frac{G_f G_m}{V_f G_m + V_m G_f} \,. \tag{2.17}$$

The next important property of the composite that requires determination is the density. For this, we begin with the total mass of the lamina, which is the sum of the masses of the fiber and the matrix. That is, the total mass M can be expressed in terms of the densities (ρ_f and ρ_m) and the volume fractions (V_f and V_m) as

$$M = M_f + M_m = \rho_f V_f + \rho_m V_m \,. \tag{2.18}$$

The density of the composite can then be expressed as

$$\rho = \frac{M}{V} = \frac{\rho_f V_f + \rho_m V_m}{V} \,. \tag{2.19}$$

Once the properties of the lamina are determined, then one can proceed to a macromechanical analysis of the lamina to characterize the constitutive model of the laminate.

2.2.3 Stress–Strain Relations for a Lamina

Determination of the overall constitutive model for a lamina of a laminated composite constitutes the macromechanical study of composites. Unlike the micromechanical study, where the composite is treated as a heterogeneous mixture, here the composite is presumed to be homogeneous and the effects of the constituent materials are accounted for only as an averaged apparent property of the composite. The following are the basic assumptions used in deriving the constitutive relations:

- The composite material is assumed to behave in a linear (elastic) manner. That is, Hookes law and the principle of superposition are valid.
- At the lamina level, the composite material is assumed to be homogeneous and orthotropic. Hence the material has two planes of symmetry, one coinciding with the fiber direction and the other perpendicular to the fiber direction.
- The state of the stress in a lamina is predominantly plane stress

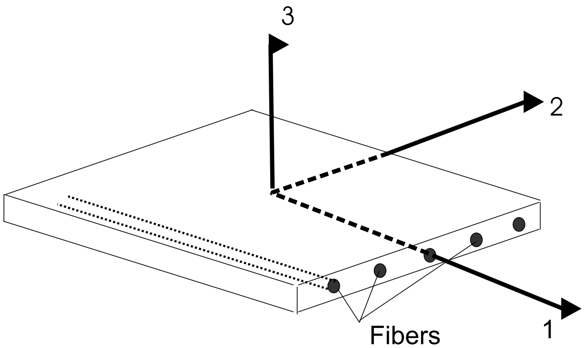

Fig. 2.3. Principal axes of a lamina

Consider the lamina shown in Figure 2.3 with its principal axes, which we denote the 1-2-3 axes. That is, axis 1 corresponds to the direction of the fiber and axis 2 is the axis transverse to the fiber. The lamina is assumed to be in a 3-D state of stress with six stress components given by $\{\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{23}, \tau_{13}, \tau_{12}\}$. For an orthotropic material in the 3-D state of stress, nine engineering constants require to be determined. The macromechanical analysis will begin from here. The stress–strain relationship for an orthotropic material is given by [11]

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} . \tag{2.20}$$

Here, S_{ij} are the material compliances. Their relationship with the engineering constants is given in Reference [11]. ν_{ij} is Poissons ratio for the transverse strain in the jth direction when the stress is applied in the ith direction, and is given by

$$\nu_{ij} = -\epsilon_{jj}/\epsilon ii \, . \tag{2.21}$$

The above condition is for $\sigma_{jj} = \sigma$ and all other stresses equal to zero. Since the stiffness coefficients $Q_{ij} = Q_{ji}$, it follows that the compliance matrix is also symmetrical, that is, $S_{ij} = S_{ji}$. This condition enforces the following relationship among Poissons ratio:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \, . \tag{2.22}$$

Hence, for a lamina under a 3-D state of stress, only three Poissons ratios namely ν_{12}, ν_{23} and ν_{31}, are required to be determined. Other Poissons ratio can be obtained from Equation (2.22).

For most of our analysis, we assume the condition of plane stress. Here, we derive the equations assuming that conditions of plane stress exist in the $1-2$ plane (see Figure 2.3). However, if one has to do an analysis of a laminated composite beam, which is essentially a 1-D member, the condition of plane stress will exist in the $1-3$ plane and a similar procedure could be followed.

For the plane stress condition in the 1-2 plane, we set the following stresses equal to zero in Equation (2.20), $\sigma_{33} = \tau_{23} = \tau_{13} = 0$. The resulting constitutive model under plane stress conditions can be written as

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & 0 \\ -\nu_{21}/E_2 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} . \tag{2.23}$$

Note that the strain ϵ_{33} also exists, which can be obtained from the third constitutive equation

$$\epsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22} \, . \tag{2.24}$$

This equation indicates that Poissons ratios ν_{13} and ν_{23} should also exist. Inverting Equation (2.23), we can express the stresses in terms of the strains:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} , \tag{2.25}$$

where Q_{ij} are the reduced stiffness coefficients, which can be expressed in terms of the elastic constants as

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}\,,\; Q_{12} = \nu_{21}Q_{11}\,,\; Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}\,, Q_{66} = G_{12}\,. \quad (2.26)$$

2.2.4 Stress–Strain Relation for a Lamina with Arbitrary Orientation of Fibers

In most cases, the orientation of the global axes, which we call the $x - y$ axes and are geometrically natural for the solution of the problem, do not coincide with the lamina principle axes, which we have already designated as 1–2 axes. The lamina principal axes and the global axes are shown in Figure 2.4. A small element in the lamina of area dA is taken and the free body diagram (FBD) is shown in Figure 2.5.

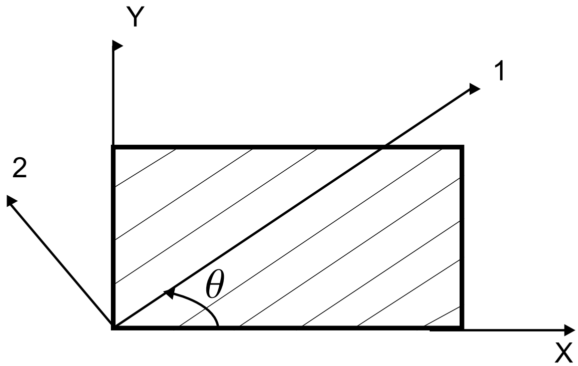

Fig. 2.4. Principal material axes of a lamina and the global $x - y$ axes

Consider the free body A. Summing all the forces in the 1-axis direction, we get

$$\begin{aligned}\sigma_{11}dA &- \sigma_{xx}(\cos\theta dA)(\cos\theta) - \sigma_{yy}(\sin\theta dA)(\sin\theta)\\ &- \tau_{xy}(\sin\theta dA)(\cos\theta) - \tau_{xy}(\cos\theta dA)(\sin\theta) = 0\,. \end{aligned} \quad (2.27)$$

On simplification, the above equation can be written as

$$\sigma_{11} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta\,. \quad (2.28)$$

Similarly, by summing all the forces along the 2-axis (free body A), we get

$$\begin{aligned}\tau_{12}dA &- \sigma_{xx}(\cos\theta dA)(\sin\theta) - \sigma_{yy}(\sin\theta dA)(\cos\theta)\\ &- \tau_{xy}(\sin\theta dA)(\sin\theta) - \tau_{xy}(\cos\theta dA)(\cos\theta) = 0\,. \end{aligned} \quad (2.29)$$

Simplifying the above equation, we get

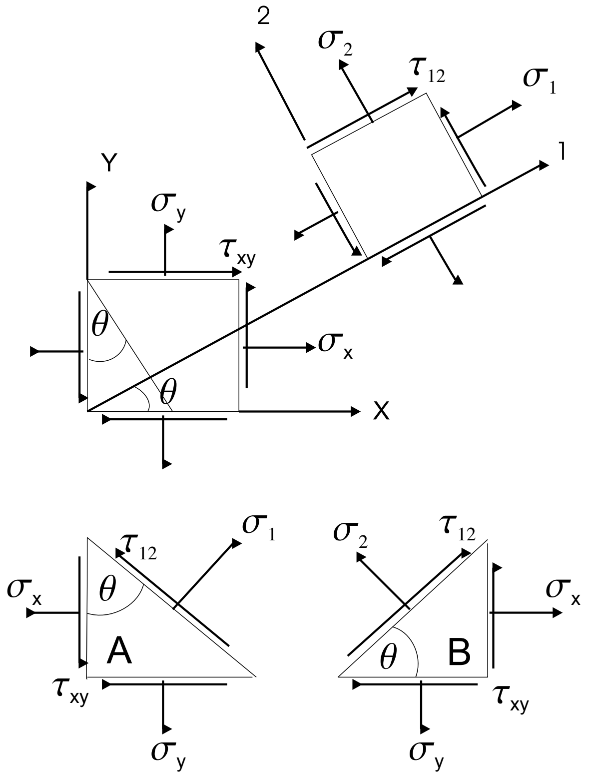

Fig. 2.5. Lamina and laminate coordinate system and FBD of a stressed element

$$\tau_{12} = -\sigma_{xx} \sin\theta \cos\theta + \sigma_{yy} \sin\theta \cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) . \tag{2.30}$$

Following the same procedure and summing all the forces in the 2-direction in the free body B, we can write

$$\sigma_{22} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta . \tag{2.31}$$

Equations (2.28), (2.31) and (2.30) can be written in matrix form as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C^2 & S^2 & 2CS \\ S^2 & C^2 & -2CS \\ -CS & CS & (C^2 - S^2) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} , \; C = \cos\theta , \; S = \sin\theta \tag{2.32}$$

or

$$\{\sigma\}_{1-2} = [T]\{\sigma\}_{x-y} .$$

In a similar manner, the strains at the 1–2 axis, can be transformed to the x–y axis by a similar transformation. Note that to have the same transformation, the shear strains are divided by 2. They can be written as

$$\left\{\begin{matrix}\epsilon_{11}\\ \epsilon_{22}\\ \frac{\gamma_{12}}{2}\end{matrix}\right\} = \begin{bmatrix} C^2 & S^2 & 2CS \\ S^2 & C^2 & -2CS \\ -CS & CS & (C^2-S^2)\end{bmatrix} \left\{\begin{matrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \frac{\gamma_{xy}}{2}\end{matrix}\right\} \text{ or } \{\bar{\epsilon}\}_{1-2} = [T]\{\bar{\epsilon}\}_{x-y}\,. \quad (2.33)$$

Inverting Equations (2.32) and (2.33), we can express the stresses and strains in global coordinates as

$$\left\{\begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{matrix}\right\} = \begin{bmatrix} C^2 & S^2 & -2CS \\ S^2 & C^2 & 2CS \\ CS & -CS & (C^2-S^2)\end{bmatrix} \left\{\begin{matrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{matrix}\right\},\ \{\sigma\}_{x-y} = [T]^{-1}\{\sigma\}_{1-2}\,. \quad (2.34)$$

$$\left\{\begin{matrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \frac{\gamma_{xy}}{2}\end{matrix}\right\} = \begin{bmatrix} C^2 & S^2 & -2CS \\ S^2 & C^2 & 2CS \\ CS & -CS & (C^2-S^2)\end{bmatrix} \left\{\begin{matrix}\epsilon_{11}\\ \epsilon_{22}\\ \frac{\gamma_{12}}{2}\end{matrix}\right\},\ or,\ \{\bar{\epsilon}\}_{x-y} = [T]^{-1}\{\bar{\epsilon}\}_{1-2}\,. \quad (2.35)$$

Actual strain vectors in both 1–2 and x–y axes $\{\epsilon\}_{1-2}$ and $\{\epsilon\}_{x-y}$ are related to $\{\bar{\epsilon}\}_{1-2}$ and $\{\bar{\epsilon}\}_{x-y}$ through a transformation matrix as

$$\left\{\begin{matrix}\epsilon_{11}\\ \epsilon_{22}\\ \gamma_{12}\end{matrix}\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\end{bmatrix} \left\{\begin{matrix}\bar{\epsilon}_{11}\\ \bar{\epsilon}_{22}\\ \frac{\bar{\gamma}_{12}}{2}\end{matrix}\right\} \text{ and } \left\{\begin{matrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{matrix}\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\end{bmatrix} \left\{\begin{matrix}\bar{\epsilon}_{xx}\\ \bar{\epsilon}_{yy}\\ \frac{\bar{\gamma}_{xy}}{2}\end{matrix}\right\}, \quad (2.36)$$

$$\{\epsilon\}_{1-2} = [R]\{\bar{\epsilon}\}_{1-2}\,, \{\epsilon\}_{x-y} = [R]\{\bar{\epsilon}\}_{x-y}\,.$$

Now the constitutive equation of a lamina in its principal directions (Equation (2.25)) can be written as

$$\{\sigma\}_{1-2} = [Q]\{\epsilon\}_{1-2}\,. \quad (2.37)$$

Substituting Equations (2.32), (2.33) and (2.36) in Equation (2.37), we get

$$[T]\{\sigma\}_{x-y} = [Q][R]\{\bar{\epsilon}\}_{1-2} = [Q][R][T]\{\bar{\epsilon}\}_{x-y} = [Q][R][T][R]^{-1}\{\epsilon\}_{x-y}\,. \quad (2.38)$$

Hence the constitutive relation in the global x–y axes can now be written as

$$\{\sigma\}_{x-y} = [\bar{Q}]\{\epsilon\}_{x-y} = [T]^{-1}[Q][R][T][R]^{-1}\{\epsilon\}_{x-y}\,. \quad (2.39)$$

Here the matrix $[\bar{Q}]$ is fully populated. Hence, although the lamina in its own principal direction is orthotropic, in the transformed coordinate, it represents complete anisotropic behavior, that is the normal stresses are coupled to the shear strains and vice versa. The elements of $[\bar{Q}]$ are given by

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}C^4 + 2(Q_{12}+2Q_{66})S^2C^2 + Q_{22}S^4\,,\\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})S^2C^2 + Q_{12}(S^4+C^4)\,,\\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})SC^3 + (Q_{12}-Q_{22}+2Q_{66})S^3C\,,\\
\bar{Q}_{22} &= Q_{11}S^4 + 2(Q_{12}+2Q_{66})S^2C^2 + Q_{22}C^4\,,\\
\bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})S^3C + (Q_{12}-Q_{22}+2Q_{66})SC^3\,,\\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})S^2C^2 + Q_{66}(S^4+C^4)\,,
\end{aligned} \quad (2.40)$$

which gives the constitutive equation of a lamina under plane stress in the 1–2 plane.

2.3 Introduction to Smart Composites

Laminated composites provide numerous opportunities to tailor the strength in the required direction and enable placement of embedded sensors and actuators at any critical location to monitor the performance of the structure. This facility is not available in conventional metallic structures. Since many smart materials are available in thin-film or powder form, embedding them in a laminated composite structure does not pose any serious problem. Those composites that have an embedded smart material patch are called smart composite structures. Figure 2.6 shows how a piezoelectric material can be embedded in a laminated composite.

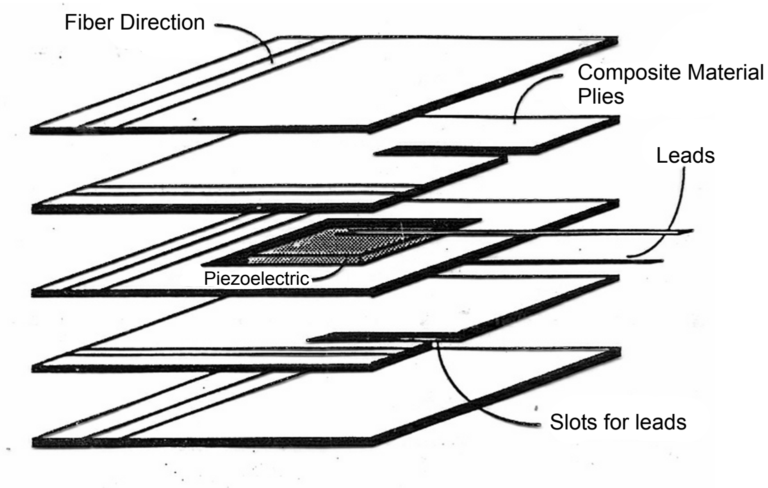

Fig. 2.6. Construction of a smart composite

Modeling systems with structures having embedded smart sensors and actuators is very similar to modeling conventional composite structures, wherein numerical techniques such as FEM or spectral techniques can be used. However, the modeling has to take care of the additional complexities arising due to the material properties of the smart materials. These are reflected in the constitutive law in the form of electromechanical coupling as in the case of piezo-ceramic or poly-vinylidine di-fluoride (PVDF) sensors or magneto-mechanical coupling as in the case of magnetostrictive sensors/actuators such as TERFENOL-D. From the modeling point of view, these complexities lead to additional matrices in the FEM/SFEM approach.

Piezoelectric or magnetostrictive materials have two constitutive laws, one of which is used for sensing and the other for actuation applications. For 2-D problems, the constitutive model for piezoelectric material is of the form

$$
\begin{aligned}
\{\sigma\}_{3x1} &= [Q]^{(E)}_{3x3}\{\varepsilon\}_{3x1} - [e]_{3x2}\{E\}_{2x1} \\
\{D\}_{2x1} &= [e]^{T}_{2x3}\{\varepsilon\}_{3x1} + [\mu]^{(\sigma)}_{2x2}\{E\}_{2x1}
\end{aligned}
\tag{2.41}
$$

The first part of this constitutive law is called the actuation law, while the second is called the sensing law. Here, $\{\sigma\}^T = \left\{ \sigma_{xx}\ \sigma_{yy}\ \tau_{xy} \right\}$ is the stress vector, $\{\varepsilon\}^T = \left\{ \varepsilon_{xx}\ \varepsilon_{yy}\ \gamma_{xy} \right\}$ is the strain vector, $[e]$ is the matrix of piezoelectric coefficients of size 3×2, which has units of N/V-mm, $\{E\}^T = \left\{ E_x\ E_y \right\} = \left\{ V_x/t\ V_y/t \right\}$ is the applied field in the two coordinate directions. It has units of V/mm. $[\mu]$ is the permittivity matrix of size 2×2, measured at constant stress and has units of N/V/V and $\{D\}^T = \left\{ D_x\ D_y \right\}$ is the vector of electric displacement in the two coordinate directions. This has units of N/V-mm. $[Q]$ is the mechanical constitutive matrix measure at constant electric field. Normally, Equation (2.41) is written in the form

$$
\{\varepsilon\} = [S]\{\sigma\} + [d]\{E\}\,. \tag{2.42}
$$

In the above expression, $[S]$ is the compliance matrix, which is the inverse of the mechanical material matrix $[Q]$ and $[d] = [Q]^{-1}[e]$ is the electromechanical coupling matrix, where the elements of this matrix have units mm/V and are direction dependent. In most analyses, it will be assumed that the mechanical properties will change very little with the change in the electric field and as a result, the actuation law (Equation (2.41) can be assumed to behave linearly with the electric field, while the sensing law (Equation (2.41))) can be assumed to behave linearly with the stress. This assumption considerably simplifies the analysis process.

The first part of Equation (2.41) represents the stresses developed due to a mechanical load, while the second part of the same equation gives the stresses due to a voltage input. From these equations, it is clear that the structure will be stressed due to the application of an electric field even in the absence of mechanical load. Alternatively, when the mechanical structure is loaded, it generates an electric field even in the absence of an applied electric field. In other words, the above constitutive law demonstrates the electromechanical coupling, which can be exploited for a variety of structural applications such as vibration control, noise control, shape control or structural health monitoring. Actuation using piezoelectric materials can be demonstrated using a plate of length L, width W and thickness t. Thin piezoelectric electrodes are placed on the top and bottom surface of the plate as shown in Figure 2.7. Such a plate is called a bimorph plate. When a voltage is passed between the electrodes as shown in the figure (which is normally referred to as the poling direction), deformation in the length, width and thickness directions is given by

$$
\delta L = d_{31}E_1 L = \frac{d_{31}VL}{t}, \quad \delta W = d_{31}E_2 W = \frac{d_{31}VW}{t}, \quad \delta t = d_{33}V\,. \tag{2.43}
$$

Here, d_{31} and d_{33} are the electromechanical coupling coefficients in the directions 1 and 3 respectively. Conversely, if a force F is applied in any of

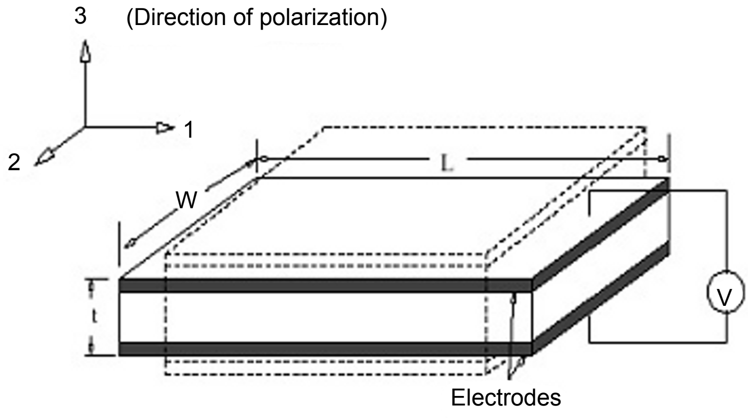

Fig. 2.7. Illustration of actuation effect in a piezoelectric plate

the length, width or thickness directions, the voltage V developed across the electrodes in the thickness direction is given by

$$V = \frac{d_{31}F}{\mu L} \quad \text{or}\, V = \frac{d_{31}F}{\mu W} \quad \text{or}\, V = \frac{d_{33}F}{\mu LW}\,. \tag{2.44}$$

Here μ is the dielectric permitivity of the material. The reversibility between strain and voltage makes piezoelectric materials ideal for both sensing and actuation.

There are different types of piezoelectric material that are used for many structural applications. The most commonly used material is PZT (lead zirconate titanate) material, which is extensively used as bulk actuator material as it has a high electromechanical coupling factor. On the other hand, due to the low electromechanical coupling factor, piezo-polymers (PVDF) are used only as sensor material. More recently, a new form of materials called piezo-fiber composite (PFC) has been found to be a very effective actuator material for use in vibration/noise control applications.

The constitutive laws (both actuation and sensing) for a magnetostrictive material such as TERFENOL-D are much more complex than those for piezoelectric materials. They are highly non-linear in behavior although they have a similar form to the piezoelectric material, which is given by

$$\{\varepsilon\} = [S]^{(H)}\{\sigma\} + [d]^T\{H\}\,, \tag{2.45}$$

$$\{B\} = \{d\}\{\sigma\} + [\mu]^{(\sigma)}\{H\}\,. \tag{2.46}$$

Here, $[S]$ is the compliance matrix measured at a constant magnetic field H, d is the magneto-mechanical coupling matrix, the elements of which have units of m/A, B is the vector of magnetic flux density in the two coordinate directions. It has units tesla, equal to weber/m^3. H is the magnetic field intensity vector in the two coordinate directions and has unit oersted, equal to A/m. It is related to the AC current $(I(t))$ through the relation $H = nI$, where n is the number of turns in the actuator. $[\mu]$ is the matrix of magnetic permeability measured at constant stress and has units of weber/A-m. As in the case of piezoelectric material, the first equation (Equation (2.46)) is the actuation constitutive law, while the second equation is the sensing law. The stress strain relations are different for different magnetic field intensities. The strain is proportional to stress only for small H. For higher magnetic field intensities, both sensing and actuator equations require to be simultaneously solved to arrive at the correct stress–strain relation. This is because changes in the magnetic field cause changes in the stress, which in turn changes the magnetic permeability. Hence, the characterization of the material properties of TERFENOL-D is more difficult than for piezoelectric material.

In Chapter 11, we deal with the modeling of smart composites where we will use these constitutive models extensively. However, only linear behavior is assumed for most examples reported in this book. The constitutive model for smart composites is obtained in a similar manner to that for laminated composites, where the smart patches are also considered as a lamina to obtain the averaged properties.

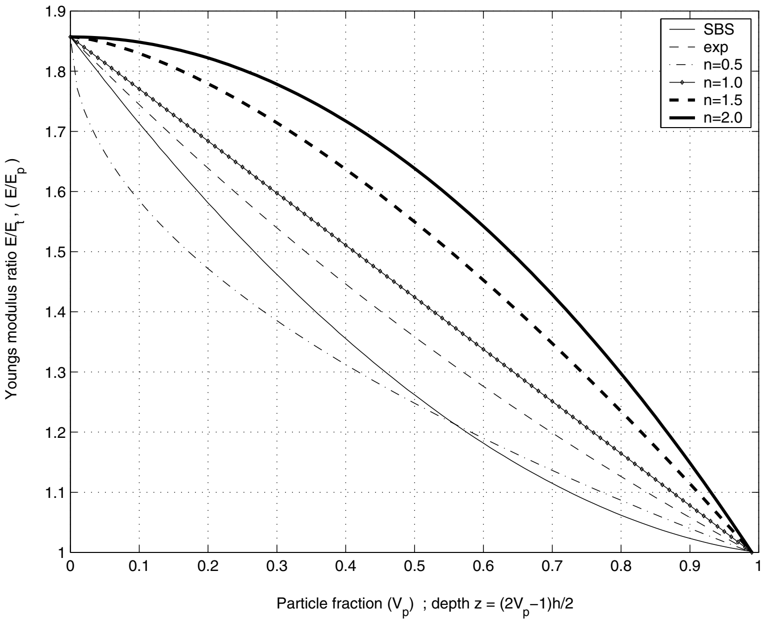

Fig. 2.8. Variation of Young's modulus for different models

2.4 Modeling Inhomogeneous Materials

Several analytical and computational models are available in the literature (see References [13] and [14]) that discuss the issue of finding suitable functions for approximating the modulus variation in an inhomogeneous material. There are several criteria for selecting them. They are desired to be continuous, simple and should have the ability to exhibit curvature, both "concave upward" and "concave downward" [14]. Here, two types of variations are considered, which generally cover all the existing analytical models. The exponential law, which is more common in fracture studies of functionally graded materials (FGM) (see References [15] and [16]), and does not show curvature in both directions, is given by

$$\mathcal{P}(z) = \mathcal{P}_t \, \exp(-\delta(1-2z/h)) , \qquad \delta = \frac{1}{2}\log\left(\frac{\mathcal{P}_t}{\mathcal{P}_b}\right) . \tag{2.47}$$

The power law, for commonly adopted Voight-type estimates [14], having all the desired properties and introduced by Wakashima *et al.* [17], is given by

$$\mathcal{P}(z) = (\mathcal{P}_t - \mathcal{P}_b)\left(\frac{z}{h}+\frac{1}{2}\right)^n + \mathcal{P}_b , \tag{2.48}$$

where $\mathcal{P}(z)$ denotes a typical material property (E, G, α, ϱ). $\mathcal{P}_t$ and $\mathcal{P}_b$ denote values of the variables at the topmost and bottommost layer of the structure, respectively, and n is a parameter, the magnitude of which determines the curvature. The working range of n is taken as $1/3$ to 3, as any value outside this range will produce an inhomogeneous material having too much of one phase (see [18]).

Another way of estimating material properties is by the rule-of-mixtures, which is generally employed for composite materials. A summary of this method can be found in References [11] and [12]. The concept of equivalent homogeneity results in different methods, namely, the composite sphere model, the three phase model, the composite cylinder model and the self-consistent scheme [19]. The composite sphere and cylinder models can be further improved by the step-by-step (SBS) method as given in Reference [20]. The method given for the particle reinforced composite material is best suited for use in the present context. The details are omitted here. In short, inhomogeneous materials such as FGM can be treated as a matrix particle mixture of different particle volume fractions, which vary smoothly vary throughout the depth of the structure. The two different materials at the top and bottom of the beam play the role of matrix and particle.

These different models for material property variations are compared in Figure 2.8, where the variation of the Young's modulus throughout the depth is plotted. Top and bottom materials (particle and matrix, respectively for the SBS method) are taken as steel and ceramic with a Young's modulus ratio of 1.857. The figure clearly shows the different trends of distribution for different models. In the SBS method, "constant area composition" is used and the particle volume fraction V_{p1}, is taken as 0.001. Since, the SBS method predicts only the elastic and thermal properties, in calculations the inertial properties are evaluated using the power law model with a suitable value for the exponent n.

In this chapter, we have presented a detailed introduction to the constitutive laws of fiber reinforced composite laminate and a brief description of the theory of smart composite. More detailed discussions on smart composite will be given in Chapter 11. Further, the popular choices of the functional form of material property variations for inhomogeneous materials are also provided.

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