Preface

Many dynamical systems that occur naturally in the description of physical processes are piecewise-smooth. That is, their motion is characterized by periods of smooth evolutions interrupted by instantaneous events. Traditional analysis of dynamical systems has restricted its attention to smooth problems, thus preventing the investigation of non-smooth processes such as impact, switching, sliding and other discrete state transitions. These phenomena arise, for example, in any application involving friction, collision, intermittently constrained systems or processes with switching components.

Literature that draws attention to piecewise-smooth systems includes the comprehensive work of Brogliato [38, 39], the detailed analysis of Kunze [165], the books on bifurcations in discontinuous systems [193, 177] and various related edited volumes [268, 35]. These books contain many examples largely drawn from mechanics and control. Also there is a significant literature in the control and electronics communities; see for example the book [193], which has many beautiful examples of chaotic dynamics induced by non-smooth phenomena. Earlier studies of non-smooth dynamics appeared in the Eastern European literature; for instance the pioneering work of Andronov *et al.* on non-smooth equilibrium bifurcations [5], Feigin [98, 80] on C-bifurcations, Peterka [216] and Babitskii [19] on impact oscillators, and Filippov [100] on sliding motion. Delving into this and other literature, one finds that piecewise-smooth systems can feature rich and complex dynamics.

In one sense, jumps and switches in a system's state represent the grossest form of nonlinearity. On the other hand, many examples appear benign at first glance since they are composed of pieces of purely linear systems, which are solvable closed form. However, this solvability is in general an illusion since one does not know *a priori* the times at which the switches occur. Nevertheless, the analysis of such dynamics is not intractable, and indeed, many tools of traditional bifurcation theory may be applied. However, it has become increasingly clear that there are distinctive phenomena unique to discontinuous systems, which can be analyzed mathematically but fall outside the usual methodology for smooth dynamical systems.

Indeed, for smooth systems, governed by ordinary differential equations, there is now a well established qualitative, topological theory of dynamical systems that was pioneered by Poincaré, Andronov and Kolmogorov among others. This theory has led to a mature understanding of bifurcations and routes to chaos—see, for example the books by Kuznetsov [168], Wiggins [273], Arrowsmith & Place [9], Guckenheimer & Holmes [124] and Seydel [232]. The key step in the analysis is to use topological equivalence, Poincaré maps, center manifolds and normal forms to reduce all possible transitions under parameter variation to a number of previously analyzed cases. These ideas have also informed modern techniques for the numerical analysis of dynamical systems and, via time-series analysis, techniques for the analysis of experimental data from nonlinear systems. The bifurcation theory methodology has shown remarkable success in describing dynamics observed in many areas of application including, via center-manifold and other reduction techniques, spatially extended systems. However, most of these successes are predicated on the dynamical system being smooth.

The purpose of this book then is to introduce a similar qualitative theory for non-smooth systems. In particular we shall propose general techniques for analyzing the bifurcations that are unique to non-smooth dynamical systems, so-called *discontinuity-induced* bifurcations (DIBs for short). This we propose as a general term for all transitions in dynamics specifically brought about through interaction of invariant sets of the system ('attractors') with a boundary in phase space across which the system has some kind of discontinuity. First and foremost, we shall give a consistent classification of all known DIBs for piecewise-smooth continuous-time dynamical systems (flows), including such diverse phenomena as sliding, chattering, grazing and corner collision. We will then describe a unified analytical framework for reducing the analysis of each such bifurcation involving a periodic orbit to that of an appropriately defined Poincaré map. This process is based on the construction of so-called *discontinuity mappings* [198, 64], which are analytical corrections made to account for crossing or tangency with discontinuity boundaries. We introduce the notion of the *degree of smoothness* depending on whether the state, the vector field or one of its derivatives has a jump across a discontinuity boundary. We show how standard examples such as impact oscillators, friction systems and relay controllers can be put into this framework, and show how to construct discontinuity mappings for tangency of each kind of system with a discontinuity boundary.

The analysis is completed by a classification of the dynamics of the Poincaré maps so-obtained. Thus we provide a link between the theory of bifurcations in piecewise-smooth flows and that associated with discontinuity crossings of fixed points of piecewise-smooth maps—so-called *border-collision* bifurcations [207, 21], which are just particular examples of a DIB. The presentation is structured in such a manner to make it possible for a reader to follow a series of steps to take a non-smooth dynamical system arising in an application from an outline description to a consistent mathematical characterization.

Throughout, the account will be motivated and illustrated by copious examples drawn from several areas of applied science, medicine and engineering; from mechanical impact and friction oscillators, through power electronic and control systems with switches, to neuronal and cardiac and models. In each case, the theory is compared with the results of a numerical analysis or, in some cases, with data from laboratory experiments. More general issues concerning the numerical and experimental investigation of piecewise-smooth systems are also discussed.

The manner of discourse will rely heavily on geometric intuition through the use of sketch figures. Nevertheless, care will be taken to single out as theorems those results that do have a rigorous proof, and where the proof is not presented, a reference will be given to the appropriate literature.

The level of mathematics assumed will be kept to a minimum: nothing more advanced than multivariable calculus, differential equations and linear algebra traditionally taught at undergraduate level on mathematics, engineering or applied science degree programs. A familiarity with the basic concepts of nonlinear dynamics would also be useful. Thus, although the book is aimed primarily at postgraduates and researchers in any discipline that impinges on nonlinear science, it should also be accessible to many final-year undergraduates.

We now give a brief outline each chapters.

Chapter 1. Introduction. This serves as a non-technical motivation for the rest of the book. It can in fact be read in isolation and is intended as a primer for the non-specialist. After a brief motivation of why piecewise-smooth systems are worthy of study, the main thrust of the chapter is to immerse the reader in the kind of dynamics that are unique to piecewise-smooth systems via a series of case studies. The first case study is the single-degree-of-freedom impact oscillator. The notion of grazing bifurcation is introduced along with the dynamical complexity that can result from this seemingly innocuous event. Agreement is shown among theory, numerics and physical experiment. After brief consideration of bi-linear oscillators, we then consider two mathematically related systems that can exhibit recurrent sliding motion: a relay controller and a stick-slip friction system. The next case study concerns a well-used electronic circuit with a switch, the so-called DC–DC converter. Finally, we consider one-dimensional maps that arise through the study of these flows, including a simple model of heart attack prediction. Here we introduce the ubiquitous *period-adding cascade* that is unique to non-smooth systems.

Chapter 2. Qualitative theory of non-smooth dynamical systems. The aim here is to set out concisely the mathematical and notational framework of the book. We present a brief introduction to the qualitative theory of dynamical systems for smooth systems, including a brief review of standard bifurcations, stressing which of these also makes sense for piecewise-smooth systems. The formalism of piecewise-smooth systems is introduced, although no specific attempt is made to develop an existence and uniqueness theory. However, a brief introduction is given to the extensive literature on other more rigorous mathematical formulations for non-smooth dynamics, such as differential inclusions, complementarity systems and hybrid dynamical systems. A working definition of discontinuity-induced bifurcation is given from a topological point of view, which motivates a brief list of the kinds of discontinuity-induced bifurcations that are likely to occur as a single parameter is varied. The notion of discontinuity mapping is introduced, and such a map is carefully derived in the case of transverse crossing of a discontinuity boundary. The chapter ends with a discussion on numerical techniques, both direct and indirect, that will be used throughout the rest of the book for investigating the dynamics of example systems and calculating the appropriate bifurcation diagrams.

Chapter 3. Border collision in piecewise-smooth continuous maps. This chapter contains results on the dynamics of discrete-time continuous maps that are locally composed of two linear pieces. First border-collision bifurcations are analyzed whereby a simple fixed point passes through the boundary between the two map pieces. General criteria are established for the existence and stability of simple period-one and -two fixed points created or destroyed in such transitions, by using information only on the characteristic polynomial of the matrix representation of the two sections of the map. Analogs of simple fold and period-doubling bifurcations are shown to occur, albeit where the bifurcating branch has a non-smooth rather than quadratic character. The cases of one and two dimensions are considered in detail. Here, more precise information can be established such as conditions for the existence of period-adding, and cascades of such as another parameter (representing the slope of one of the linear pieces) is varied. Finally, we consider maps that are noninvertible in one part of their domain. For such maps, conditions can be found for the creation of robust chaos, which has no embedded periodic windows.

Chapter 4. Bifurcations in general piecewise-smooth maps. Here the analysis of the previous chapter is generalized to deal with maps that crop up as normal forms of the grazing and other non-smooth bifurcations analyzed in subsequent chapters, and which change their form across a discontinuity boundary. First, we treat maps that are piecewise-linear but discontinuous. We then proceed to study continuous maps that are a combination of a linear and a square-root map, and finally maps that combine a linear map with an $\mathcal{O}(3/2)$ or a quadratic map. In each case we study the existence of both periodic and chaotic behavior and look at the transitions between these states. Of particular interest will be the identification of *period-adding* behavior in which, under the variation of a parameter, the period of a periodic state increases in arithmetic progression, accumulating onto a chaotic solution.

Chapter 5. Boundary equilibrium bifurcations in flows. This chapter collects and reviews various results on the global consequences of an equilibrium point encountering the boundary between two smooth regions of phase space in a piecewise-smooth flow. Cases are treated where the vector field is continuous across the boundary and where it is not (and indeed where the boundary may itself be attracting—the Filippov case). In two dimensions, a more or less complete theory is possible since the most complex attractor is a limit cycle, which may be born in a non-smooth analog of a Hopf bifurcation. In the Filippov case, so-called pseudo-equilibria that lie inside the sliding region can be created or destroyed on the boundary, as they can for impacting systems.

Chapter 6. Limit cycle bifurcations in impacting systems. We return to the one-degree-of-freedom impact oscillator from the Introduction, stressing a more geometrical approach to understanding the broad features of its dynamics. Within this approach, grazing events are thought of as leading to singularities in the phase space of certain Poincaré maps. These singularities are shown to organize the shape of strange attractors and also the basins of attraction of competing attractors. An attempt is made to generalize such geometrical considerations to general n-dimensional hybrid systems of a certain class. The narrative then switches to dealing with grazing bifurcations of limit cycles within this general class. The discontinuity mapping idea is used to derive normal form maps that have a square-root singularity. The technique is shown to work on several example systems. The chapter also includes a treatment of chattering (a countably infinite sequence of impacts in a finite time) and multiple impacts, including a simple example of a triple collision.

Chapter 7. Limit cycle bifurcations in piecewise-smooth flows. This chapter treats the general case of non-Filippov flows and two specific kinds of bifurcation event where a periodic orbit grazes with a discontinuity surface. In the first kind the periodic orbit becomes tangent to a smooth surface. In the second kind the periodic orbit passes through a non-smooth junction between two surfaces. For both kinds, discontinuity mappings are calculated and normal form mappings derived that can be analyzed using the techniques of the earlier chapters. Examples of the theory are given including general bilinear oscillators, a certain stick-slip system and the DC–DC convertor introduced in Chapter 1.

Chapter 8. Sliding bifurcations in Filippov systems. The technique of discontinuity mappings is now applied to the situations where flows can slide along the attracting portion of a discontinuity set in the case where the vector fields are discontinuous. Four non-generic ways that periodic orbits can undergo sliding are identified that lead to four bifurcation events. Each event involves the fundamental orbit involved in the bifurcation gaining or losing a sliding portion. The mappings derived at these events typically have the property of being non-invertible due to the loss of initial condition information inherent in sliding. So a new version of the theory of Chapters 3 and 4 has to be derived, dealing with this added complication. Examples of relay controllers and friction oscillators introduced in Chapter 1 are given further treatment in the light of this analysis.

Chapter 9. Further applications and extensions. This chapter contains a series of additional case study applications that serve to illustrate further bifurcations and dynamical features, a detailed analysis of which would be beyond the scope of this book. Each application arises from trying to understand or model some experimental or in service engineered or naturally occurring system. The further issues covered include the notion of parameter fitting to experimental data, grazing bifurcations of invariant tori and examples of codimension-two bifurcations.

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