2

Some One-Dimensional Examples

2.1 Introduction

Many of the strategies and techniques which are used in solving quite demanding scattering problems have been inspired by the methods used and results obtained when investigating wave motions on strings. In this connection we gather together in this chapter, for convenience and completeness, some of these details. The material is essentially well known and, consequently, is presented here in a largely formal manner. Full details can be found in the references cited in the Commentary.

Wave scattering phenomena involve three ingredients, an incoming or incident wave, an interaction zone in which the incident wave is perturbed in some manner and an outgoing wave which arises as a consequence of the perturbation. The waves in the interaction zone have, almost always, a very complicated structure. We shall see that this particular difficulty can be avoided, to a large extent, if we concentrate on the *consequence*s of the interaction rather than on the interaction itself. This we shall do by developing relationships between the incoming and outgoing processes, that is, we shall construct a scattering theory. To do this we first need to know how waves propagate in the absence of perturbations, that is, we need to study the FP. When details of the solutions to FP are well understood we can then turn to an investigation of the more demanding PP which can embrace such features as boundary conditions, forcing terms, variable coefficients and so on.

In the following sections we work through a number of specific problems associated with waves on an infinite string.

2.2 Free Problems

It is well known that the small amplitude transverse wave motion of a string is governed by an equation of the form [2] [7]

$$\{\partial_t^2 - c^2\partial_x^2\}u(x, t) = f(x, t), \quad (x, t) \in \mathbf{R} \times \mathbf{R} \tag{2.1}$$

where f characterises a force applied to the string, $u(x, t)$ denotes the transverse displacement of the string at a point x at time t and c represents the velocity of a wave which might have been generated in the string. Throughout, we use the notation ∂_t^n to denote the nth partial derivative with respect to the variable t and similarly for other variables. Also, the subscript notation will be used to denote differentiation of dependent variables.

We shall see later that it is quite sufficient for most of our purposes to study only the homogeneous form of (2.1). Specifically, we shall see in Section 2.8 that if we can solve the homogeneous equation then we will also be able to solve the inhomogeneous equation by using Green's function techniques and Duhamel's Principle. Consequently, in this chapter, unless otherwise stated, we take as a prototype equation

$$\{\partial_t^2 - c^2\partial_x^2\}u(x, t) = 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R} \tag{2.2}$$

Solutions, $u(x, t)$, of this equation will, in general, be required to satisfy appropriate initial conditions to control the variations with respect to the time variable t and, similarly, certain boundary conditions to control the variations in the displacements with respect to the space variable x. However, in most cases of practical interest at all those points x that are a long way away from any boundary the effect of the boundary will be minimal since it could take quite a time, depending on the wave velocity, c, for the boundary influences to have any substantial effect at these points x. Thus for large values of t the wave motion is largely unaffected by the boundaries, that is the waves are (virtually) free of the boundary influence.

In this chapter we shall take as our prototype free problem (FP) the Initial Value problem (IVP)

$$\{\partial_t^2 - c^2\partial_x^2\}u(x, t) = 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R} \tag{2.3}$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbf{R} \tag{2.4}$$

Associated with this FP are a variety of perturbed problems (PP) which could involve, for example, either forcing terms or variable coefficients or boundary conditions or combinations of these. Consequently, in any study of waves and their echoes there are three things that have to be done before anything else.

- Determine the general form of solutions to the equation governing the wave motion.
- Investigate initial value problems IVPs associated with the defining equation governing the wave motion and develop, in the absence of any perturbation, constructive methods of solution. This is taken as the underlying FP.
- Investigate PPs associated with the above FP and develop for them constructive methods of solution.

Once these three matters have been satisfactorily addressed then we will be well placed actually to compare solutions of the FP and the PPs and so develop a scattering theory. By means of such a scattering theory, which sometimes might appear to have a very abstract structure, we will see that we will be able to analyse, in an efficient and thoroughly constructive manner, the echo signals arising from perturbations of an otherwise free system. Furthermore, we shall see in later chapters that many of these one-dimensional techniques and strategies can be extended to cater for much more complicated systems than those dealing with waves on strings.

2.3 Solutions of the Wave Equation

In this section we obtain the general form of solutions of the one-dimensional wave equation

$$\{\partial_t^2 - c^2\partial_x^2\}u(x, t) = 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R} \tag{2.5}$$

To this end introduce new variables, the so-called **characteristic coordinates**,

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct \tag{2.6}$$

We remark that the lines $\xi =$ constant and $\eta =$ constant are called **characteristic lines** for (2.5) [9]. Transforming (2.5) to the new variables ξ, η we have

$$2x = \eta + \xi \quad \text{and} \quad 2ct = \eta - \xi \tag{2.7}$$

and we will write

$$u(x, t) = v(\xi, \eta) \tag{2.8}$$

Consequently, using the chain rule we obtain

$$\begin{aligned} 2cv_\xi &= cu_x - u_t \\ 4c^2 v_{\xi\eta} &= c^2 u_{xx} + cu_{xt} - cu_{tx} - u_{tt} = \mathrm{c}^2 u_{xx} - u_{tt} \end{aligned} \tag{2.9}$$

Thus the wave equation (2.5) transforms under (2.6) into

$$v_{\xi\eta}(\xi, \eta) = 0 \tag{2.10}$$

The equation (2.10) has a general solution of the form

$$v(\xi, \eta) = f(\xi) + g(\eta) \tag{2.11}$$

where f, g are arbitrary, but sufficiently differentiable functions which are determined in terms of imposed conditions to be satisfied by a solution of interest.

Returning to the original variables we have

$$u(x, t) = f(x - ct) + g(x + ct) \tag{2.12}$$

In order that (2.12) should be a solution of (2.5), in the classical sense, then the functions f and g must be twice continuously differentiable functions of their arguments. We shall see later that we can relax this requirement.

In (2.12) the function f characterises a wave travelling to the right, unchanging in shape and moving with a velocity $c > 0$. To see this consider the f-wave when it is at position x_0 at time $t = 0$. Then in this case the wave has a shape (profile) given by $f(x_0)$. At some future time $t \neq 0$ the wave will have reached a point $x = x_0 + ct$. Consequently,

$$f(x - ct) = f(x_0)$$

which indicates that the shape of the wave is the same at the point (x, t) as it is at the point (x_0, t). Clearly, since $x > x_0$ we see that $f(x - ct)$ represents a wave travelling to the right with velocity c and which is unchanging in shape.

Similarly, $g(x + ct)$ represents a wave travelling to the left with velocity c and which is unchanging in shape.

We notice that since

$$\{c\partial_x + \partial_t\}f(x - ct) = 0, \quad \{c\partial_x - \partial_t\}g(x + ct) = 0 \tag{2.13}$$

then both $f(x - ct)$ and $g(x + ct)$ individually satisfy the wave equation

$$\{\partial_t^2 - c^2\partial_x^2\}w(x, t) = 0 \tag{2.14}$$

It will be convenient at this stage to introduce some notation. This we can do quite conveniently by considering the following particular solution of (2.14)

$$w(x, t) = a\cos(kx - \omega t - \varepsilon), \quad a > 0, \quad \omega > 0 \tag{2.15}$$

This is a **harmonic wave** defined in terms of the quantities

k = wave (propagation) number
ω = angular frequency
a = amplitude
ε = phase angle.

A number of perhaps more familiar wave features can be defined in terms of these quantities; specifically,

$c = \omega/k =$ **wave velocity**. The wave travelling in the positive direction if $k > 0$.
$\lambda = 2\pi/k =$ **wave length**
$f = \omega/2\pi =$ **frequency** of the (harmonic) oscillation
$T = f^{-1} = 2\pi/\omega =$ **period** of the oscillation
$\theta(x, t) = kx - \omega t - \varepsilon =$ the **phase** of the wave.

It will often be convenient to define the corresponding **complex harmonic wave**

$$\psi(x, t) = a\exp\{i\theta(x, t)\} = C\exp\{i(kx - \omega t)\} \tag{2.16}$$

where $w(x, t) = \mathrm{Re}(\psi(x, t))$ with $a = |C|$ and $\varepsilon = -\arg C$.

If a depends on either x or t then w is referred to as an **amplitude modulated wave**.

If $\theta(x, t)$ is nonlinear in either x or t then w is referred to as a **phase modulated wave**.

A wave of the form

$$w(x, t) = e^{-pt}\cos(kx - \omega t - \varepsilon), \quad p > 0 \tag{2.17}$$

is a **damped harmonic wave**.

A wave of the form

$$w(x, t) = e^{-qx}\cos(kx - \omega t - \varepsilon), \quad p > 0 \tag{2.18}$$

is an **attenuated harmonic wave**.

Solutions of (2.14) that have the specific form

$$w(x, t) = X(x)T(t) \tag{2.19}$$

are known as **separable solutions**. A typical example of such a solution is

$$w(x, t) = \sin(\pi x)\cos(\pi ct) \tag{2.20}$$

Direct substitution of (2.20) into (2.14) readily shows that (2.20) is indeed a solution of the wave equation. A general feature of waves such as (2.20) is that they are constant in time, that is they are **stationary or non-propagating waves**. To see this notice that the **nodes** of (2.20), that is, those points x at which $w(x, t) = 0$ and the **antinodes** of (2.20), that is those points x at which $w_x(x, t) = 0$ maintain permanent positions, $x = \ldots -1, 0, 1, 2 \ldots$ and $x = \ldots -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \ldots$ respectively, for all time t. We notice that (2.20) can be written in the form

$$w(x,t) = \sin(\pi x)\cos(\pi ct) = \frac{1}{2}\sin(\pi(x-ct)) + \frac{1}{2}\sin(\pi(x+ct)) \tag{2.21}$$

Thus the **stationary wave** (2.20) is seen to be a superposition of two travelling waves, travelling in opposite directions with velocity c.

We have seen that solutions of (2.14) have the general form

$$w(x,t) = f(x - ct) + g(x + ct) \tag{2.22}$$

We will often be interested in solutions (2.22) that have a particular time dependence. This can mean that we might look for solutions that have, for example, the following specific and separated form

$$w(x,t) = X(x)\exp\{i\omega t\} \tag{2.23}$$

Substituting (2.23) into (2.14) we find that

$$\{\partial_x^2 + k^2\}X(x) = 0, \quad k = \omega/c \tag{2.24}$$

This equation has the solution

$$X(x) = Ae^{ikx} + Be^{-ikx} \tag{2.25}$$

and we thus obtain, using (2.23) and (2.25)

$$w(x,t) = A\exp\{i(kx - \omega t)\} + B\exp\{-i(kx + \omega t)\} \tag{2.26}$$

Thus, comparing (2.22) and (2.26) we arrive at the following sign convention.

For waves with a time dependence $\exp\{-i\omega t\}$, then
$\exp\{ikx\}$ characterises a wave travelling to the right (increasing x)
$\exp\{-ikx\}$ characterises a wave travelling to the left (decreasing x).

2.4 Solutions of Initial Value Problems

In this section we study IVP of the form (2.3), (2.4) with $|x| < \infty$ and $t > 0$.

We have seen that the general solution of (2.3) has the form

$$u(x,t) = f(x - ct) + g(x + ct) \tag{2.27}$$

Substituting the initial conditions (2.4) into (2.27) we obtain

$$\varphi(x) = f(x) + g(x) \tag{2.28}$$

$$\psi(x) = -cf'(x) + cg'(x) \tag{2.29}$$

We notice that since f and g are assumed, at this stage, to be twice continuously differentiable it follows that the initial conditions must be such that φ is twice continuously differentiable and ψ is once continuously differentiable.

Integrate (2.29) and obtain

$$\frac{1}{c}\int_{x_0}^{x} \psi(s)ds = -f(x) + g(x) \tag{2.30}$$

where x_0 is an arbitrary constant.

From (2.28) and (2.30) we obtain

$$f(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_{x_0}^{x} \psi(s)ds \tag{2.31}$$

$$g(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_{x_0}^{x} \psi(s)ds \tag{2.32}$$

Substituting (2.31), (2.32) into (2.27) we obtain

$$u(x,t) = \frac{1}{2}\{\varphi(x-ct) + \varphi(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \tag{2.33}$$

This is the celebrated **d'Alembert solution** of the one-dimensional wave equation. The interval $[x_1 - ct_1, x_1 + ct_1]$, $t > 0$ of the x-axis is the **domain of dependence** of the point (x_1, t_1). The reason for this name is that (2.33) indicates that $u(x_1, t_1)$ depends only on the values of φ taken at the ends of this interval and the values of ψ at all points of this interval. The region, D, for which $t > 0$, $x - ct \le x_1$ and $x + ct \ge x_1$ is known as the **domain of influence** of the point x_1. This is because the value of φ at x_1 influences the solution $u(x, t)$ on the boundary of D whilst the value of ψ at x_1 influences $u(x, t)$ throughout D.

It will be instructive to consider the d'Alembert solution (2.33) in the following two particular cases.

Example 2.1. Assume

- Initial velocity of the string is everywhere zero.
- Initial displacement of the string is only nonzero in the interval (x, t).

In this case (2.33) reduces to

$$u(x,t) = \frac{1}{2}\{\varphi(x-ct) + \varphi(x+ct)\} \tag{2.34}$$

Thus we see that the forward wave $\varphi(x - ct)$ and the backward wave $\varphi(x + ct)$ each travel with a velocity c and have initial amplitude $\frac{1}{2}\varphi(x)$, that is with amplitude one half of the original (initial) amplitude $u(x, t)$. To find the solution, $u(x, t)$, at some other time we displace the graph $\varphi(x)$ by an amount ct in opposite directions. For instance, suppose the initial displacement was a triangle with base (α, β). Then as we have seen, this initial waveform will be the sum of two triangular waveforms each having half the original amplitude. At any other time, t, the displacement waveform will again be the sum of two triangular waveforms, each of half the original amplitude, one travelling to the right, the other to the left. The displacement waveform is obtained by summing the ordinates of the two displaced graphs. This can turn out to be a very complicated process. Furthermore, to write down the formula for $u(x, t)$ at any point (x, t) can often be very difficult. In the relatively simple case of waves on strings an indication of the complicated nature of the displacement waveform can be obtained by graphical means. To see this consider the specific case when the initial displacement is the triangular waveform

$$\varphi(x) = u(x,0) = \begin{cases} 0; & x \leq 0 \\ \dfrac{2u_0(x-\alpha)}{\beta-\alpha}; & \alpha \leq x \leq \dfrac{1}{2}(\alpha+\beta) \\ \dfrac{2u_0(\beta-x)}{\beta-\alpha}; & \dfrac{1}{2}(\alpha+\beta) \leq x \leq \beta \\ 0; & x \geq \beta \end{cases} \tag{2.35}$$

Displacing this waveform in the manner mentioned above and plotting the results on a graph of u against x yields a series of graphs which are snapshots of the displacement waveform at a fixed time. Carrying this through for time steps of duration

$$\frac{m(\beta-\alpha)}{10c}, \quad m = 0, 1, 2, \ldots$$

we will readily notice the following behaviour:

For $t = 0$ to $t = \dfrac{2(\beta-\alpha)}{5c}$

In this interval the backward and forward waves interact and produce a very complicated graph for the displacement waveform.

For $t \geq \dfrac{3(\beta-\alpha)}{5c}$

The backward and forward waves would seem to have "passed through" each other and exhibit no evidence of the "interaction" which was seen earlier. However, simple as the graph would now appear to be we do know that these two waves will have interacted and will "contain" information to that effect. We would like to obtain this information without going through all the complexities which would arise when investigating the interaction zone. This is a principal goal of scattering theory. We shall demonstrate how this can be achieved in the chapters that follow.

Example 2.2. Assume

- Initial velocity of the string is only nonzero in the interval (α, β).
- Initial displacement of the string is everywhere zero.

In this case (2.33) reduces to

$$\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \tag{2.36}$$

We argue in much the same way as in Example 2.1. The difference now is that ψ rather than φ is prescribed. As an illustration we shall consider the case when $\varphi(x)$ has the triangular waveform given in (2.35). Once again we confine attention to a graphical method and plot "snapshots" of the displacement $u(x, t)$ at various fixed times. We quickly see that there are five regions of interest,

1. $x + ct \le \alpha$
2. $a \le x + ct \le \beta$
3. $x - ct \le \alpha$ and $x + ct \ge \beta$
4. $\alpha \le x - ct \le \beta$
5. $x - ct \ge \beta$

In Region 1 we have $x + ct \le \alpha$ and hence the entire range of integration in (2.36) is outside the interval (α, β). Consequently, $u(x, t) = 0$ in Region 1.

In a similar manner $u(x, t) = 0$ in Region 5.

In Region 2 we have $\alpha \le x + ct \le \beta$. Consequently

$$x - ct = x + ct - 2ct \le \beta - 2ct < \alpha \tag{2.37}$$

The last inequality will only hold for

$$t > (\beta - \alpha)/2c \tag{2.38}$$

Therefore, in Region 2,

$$u(x,t) = \frac{1}{2c}\int_{\alpha}^{x+ct}\psi(s)ds \tag{2.39}$$

Similarly, in Region 4, together with (2.38) we obtain

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{\beta}\psi(s)ds \tag{2.40}$$

Finally, in Region 3 since $x - ct \le \alpha$ and $x + ct \ge \beta$ it follows that

$$u(x,t) = \frac{1}{2c}\int_{\alpha}^{\beta}\psi(s)ds \tag{2.41}$$

and we conclude, since α and β, are constants, that $u(x, t)$ is a constant in Region 3.

In the case when

$$t < \frac{\beta - \alpha}{2c} \tag{2.42}$$

then arguing as before we find that in Region 3 we now have

$$\frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \tag{2.43}$$

which is not a constant.

These two examples illustrate quite clearly how complicated the wave structure can be. The situation can, of course, be expected to be even worse if any form of perturbation, such as a boundary condition for instance, is involved. In the following two sections we introduce some methods that will ease these difficulties. Whilst these methods may seem, at first sight, to be a little heavy-weight for use when discussing waves on strings nevertheless they do ease considerably the difficulties we have so far mentioned and more importantly they offer good prospects for dealing with more complicated problems than waves on strings.

Finally, in this section we return to (2.33) the d'Alembert solution form. We notice that if we introduce

$$u_+(x,t) := \begin{cases} \frac{1}{2}\varphi(x-ct) + \frac{1}{2c}\int_{x-ct}^{\infty} \psi(s) ds, & x > 0 \\ \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_{-\infty}^{x+ct} \psi(s) ds, & x < 0 \end{cases} \tag{2.44}$$

$$u_-(x,t) := \begin{cases} \frac{1}{2}\varphi(x+ct) - \frac{1}{2c}\int_{x+ct}^{\infty} \psi(s) ds, & x > 0 \\ \frac{1}{2}\varphi(x-ct) - \frac{1}{2c}\int_{-\infty}^{x-ct} \psi(s) ds, & x < 0 \end{cases} \tag{2.45}$$

then (2.33) can be written in the alternative form

$$u(x, t) = u_+(x, t) + u_-(x, t) \tag{2.46}$$

This form will be useful when we try to learn more about the behaviour of the solution, $u(x, t)$, for large t. We notice that $u_+(x, t)$ is a function of $(x - ct)$ for $x > 0$ and a function of $(x + ct)$ for $x < 0$. Consequently, we can write more compactly

$$u_+(x, t) = f_+(|x| - ct), \quad x \in \mathbf{R} \tag{2.47}$$

Similarly, we have

$$u_-(x,t) = f_-(|x| + ct), \quad x \in \mathbf{R} \tag{2.48}$$

We also notice that in the region $x > 0$ there are two "waves", $u_+(x,t)$ travelling to the right (increasing x) and $u_-(x,t)$ travelling to the left (decreasing x). Thus, with respect to the origin $x = 0$ the wave $u_+(x,t)$ is **outgoing** whilst the wave $u_-(x,t)$ is **incoming**.

Similarly, in the region $x < 0$ the wave $u_+(x,t)$ is **outgoing** (increasing negative x) with respect to the origin whilst $u_-(x,t)$ is **incoming** (decreasing negative x).

The concepts of incoming and outgoing waves are of crucial importance in scattering theory. They will be discussed in more detail in Chapter 6 and the Commentary.

2.5 Integral Transform Methods

In this and the following section we introduce some alternative methods of constructing solutions to FP for wave problems on strings. These methods have the virtue that they generalise quite readily when we need to deal with more complicated and demanding problems than waves on strings. Furthermore, we shall see that they also provide an efficient means for developing robust constructive methods for solving quite difficult problems.

An explicit method for constructing solutions to IVP for the wave equation is provided by the Plancherel theory of the Fourier transform [6], [8], [10]. Specifically we have the following basic formulae in $\mathbf{R}^n$.

$$(Ff)(p) =: \hat{f}(p) = \lim_{M\to\infty} \frac{1}{(2\pi)^{n/2}} \int_{|x|\le M} \exp(-ixp) f(x)dx \tag{2.49}$$

$$f(x) = (F^* \hat{f}) = \lim_{M\to\infty} \frac{1}{(2\pi)^{n/2}} \int_{|p|\le M} \exp(ixp) \hat{f}(p)dp \tag{2.50}$$

where $x = (x_1, x_2, \ldots, x_n)$, $p = (p_1, p_2, \ldots, p_n)$ and $x \cdot p = \Sigma_{j=1}^n x_j p_j$. Here F^* denotes the inverse of the transform F. We would emphasise that the integrals in (2.49), (2.50) are improper integrals and care must be taken when interpreting the limits in (2.49), (2.50). We return to these points in detail in Chapter 6. With this understanding we shall refer to (2.49), (2.50) as a **Fourier inversion theorem**.

This inversion theorem can be used to provide a representation, a so-called **spectral representation**, of differential expressions with constant coefficients. Such a representation will often reduce the complexities and inherent difficulties of a given problem. This is a consequence of the relation

$$(F(D_j f))(p) = ip_j(Ff)(p) \tag{2.51}$$

where $D_j = \partial/\partial x_j$, $j = 1, 2, \ldots, n$. For example if we write

$$A := -\Delta = -\sum_{j=1}^{n} \partial^2/\partial x^2$$

and if Φ is a "sufficiently nice" function then using (2.51) we can obtain the representation

$$(\Phi(A)f)(x) = \lim_{M\to\infty} \frac{1}{(2\pi)^{n/2}} \int_{|p|\le M} \exp(ixp)\Phi(|p|^2)\hat{f}(p)dp \tag{2.52}$$

In later chapters we shall refer to the three results (2.49), (2.50), (2.52) collectively either as a (**generalised**) **eigenfunction expansion theorem** or as a **spectral representation theorem** (with respect to A).

To illustrate the use of the above Fourier transforms we consider again the following IVP governing waves on a string

$$\{\partial_t^2 - c^2\partial_x^2\}u(x, t) = 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R} \tag{2.53}$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbf{R} \tag{2.54}$$

We now only need consider the case when $n = 1$ and then the inversion theorem (2.49), (2.50) can be conveniently written in the form

$$(Ff)(p) =: \hat{f}(p) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbf{R}} \exp(-ixp)f(x)dx = \int_{\mathbf{R}} \overline{w(x, p)}f(x)dx \tag{2.55}$$

$$f(x) = (F^* \hat{f}) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbf{R}} \exp(ixp)\hat{f}(p)dp = \int_{\mathbf{R}} w(x, p)\hat{f}(p)dp \tag{2.56}$$

where it is understood that the improper integrals appearing in (2.55), (2.56) are interpreted as limits as indicated above. We notice that the Fourier kernel

$$w(x, p) = \frac{1}{(2\pi)^{1/2}} \exp(ixp) \tag{2.57}$$

satisfies

$$\{\partial_x^2 + p^2\}w(x, p) = 0 \tag{2.58}$$

If we take the Fourier transform, (2.55), of the IVP for the partial differential equation (2.53) we obtain the following IVP for an ordinary differential equation

$$\{d_t^2 + c^2p^2\}\,\hat{u}(p,t) = 0 \tag{2.59}$$

$$\hat{u}(p,0) = \hat{\varphi}(p), \quad \hat{u}_t\,(p,0) = \hat{\psi}(p) \tag{2.60}$$

This IVP is easier to solve than that for the partial differential equation (2.53). Indeed, we see immediately that the solution is

$$\hat{u}(p,t) = (\cos(pct))\hat{\varphi}(p) + \frac{1}{pc}(\sin(pct))\hat{\psi}(p) \tag{2.61}$$

If we now apply the inverse Fourier transform, (2.56) to (2.61), then we obtain the required solution of the IVP (2.53), (2.54) in the form

$$\begin{aligned} u(x,t) &= \int_R w(x,p)\left\{(\cos(pct))\hat{\varphi}(p) + \frac{1}{pc}(\sin(pct))\hat{\psi}(p)\right\}dp \\ &= \{F * \hat{u}(\cdot,t)\}(x) \end{aligned} \tag{2.62}$$

To see how this form relates to that obtained earlier we first expand $\cos(pct)$ in the form

$$\cos(pct) = \frac{1}{2}\{e^{ipct} + e^{-ipct}\}$$

and use the result [3], [4, vol II]

$$(F(f(x-L)))(p) = e^{-ipL}\hat{f}(p) \tag{2.63}$$

It is then a straightforward matter to show that

$$F*(\hat{\varphi}(p)\cos(pct)) = \frac{1}{2}\{\varphi(x+ct) + \varphi(x-ct)\} \tag{2.64}$$

Similarly

$$\begin{aligned} F*\left(\frac{1}{pc}\hat{\varphi}(p)\sin(pct)\right) &= \frac{1}{2i\sqrt{2\pi}}\int_{\mathbf{R}}\hat{\psi}(p)\frac{1}{pc}\{e^{ip(x+ct)} - e^{ip(x-ct)}\}dp \\ &= \frac{1}{2c\sqrt{2\pi}}\int_{\mathbf{R}}\hat{\psi}(p)\left\{\int_{x-ct}^{x+ct} e^{ips}ds\right\}dp \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \end{aligned} \tag{2.65}$$

Combining (2.62), (2.64) and (2.65) we obtain

$$u(x,t) = \frac{1}{2}\{\varphi(x+ct) + \varphi(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \tag{2.66}$$

which is the familiar d'Alembert solution obtained earlier. We remark again that the Fourier transform of the given IVP for a *partial* differential equation yields, in this instance, an IVP for an *ordinary* differential equation. Whilst the ordinary differential equation is more readily solved than the partial differential equation there will remain the matter of inversion of the Fourier transform. Thus, three questions will always have to be addressed if we choose to adopt the integral transform approach.

- First, what is the most appropriate integral transform for use in reducing the given partial differential equation to an equivalent ordinary differential equation?
- Second, is there an inversion theorem of the form (2.49) available for use in dealing with the given IVP?
- Third, is there available a (spectral) representation theorem of the form (2.49), (2.50), (2.52) for use in dealing with the given IVP?

We emphasise that in dealing with our present FP we have been very lucky because if we use the Fourier integral transform then the Fourier Plancherel theory is available quite independently of any scattering requirements and we can answer the last two questions above in the affirmative. However, for a perturbation of this FP and indeed for more general problems than waves on a string we must always **prove** the availability of a representation theorem of the form (2.52). We return to this matter in more detail in Chapter 6 and subsequent chapters.

Finally, in this section we remark that we could have obtained (2.62) and hence (2.66) in another way. It turns out that this other approach will offer potentially powerful means of addressing a wide range of physically realistic problems.

Essentially, the method rests on how the partial differential equation for the problem of interest is cast into the form of an equivalent ordinary differential equation. Again for the purposes of illustration we consider the IVP (2.53), (2.54). We start by setting $A = -c^2\partial_x^2$ and then make what seems to be an outrageous assumption namely that for our immediate purposes A can be treated as a constant! This being done we arrive at the following IVP for an ordinary differential equation

$$\{d_t^2 + A\}u(x,t) = 0, \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{2.67}$$

This IVP has a solution which can be written in the form

$$u(x,t) = \left(\cos\left(tA^{1/2}\right)\right)\varphi(x) + A^{-1/2}\left(\sin\left(tA^{1/2}\right)\right)\psi(x) \tag{2.68}$$

It now remains to interpret such quantities as $\cos tA^{1/2}$.

From the standard theory of Fourier transforms [6]

$$(F(Af))(p) = (F(-c^2\partial_x^2 f))(p) = c^2p^2\hat{f}(p) \tag{2.69}$$

It then follows, because of the particularly simple form of A that we are using here, that for a "sufficiently nice" function Φ we have

$$(F(\Phi(A)f))(p) = \Phi(c^2p^2)\hat{f}(p) \tag{2.70}$$

Consequently, combining (2.70) and (2.68) we obtain

$$u(x,t) = \int_R w(x,p)\left\{(\cos(pct))\hat{\varphi}(p) + \frac{1}{pc}(\sin(pct))\hat{\psi}(p)\right\}dp \tag{2.71}$$

which is the same as (2.62) obtained by other means. We shall see that in this particular method the "outrageous assumption" can be justified, thus making the approach mathematically respectable.

Finally, we notice that the Fourier kernel $w(x, p)$, given by (2.57), satisfies

$$(A + c^2p^2)w(x, p) = 0 \tag{2.72}$$

Consequently, $w(x, p)$ would appear to be, in some sense, an **eigenfunction** of A with **eigenvalue** $(-c^2p^2)$. (See Chapter 4 for more details on this aspect.)

2.6 On the Reduction to a First Order System

An alternative method frequently used when discussing wave motions governed by an IVP of the form (2.53), (2.54) is to replace the IVP for the partial differential equation by an equivalent problem for a first order system. This approach has a number of advantages. Existence and uniqueness results can be readily obtained and, furthermore, energy considerations can be included quite automatically. We shall illustrate this approach here in an entirely formal manner. Precise analytical details will be provided in later chapters.

The initial value problem (2.53), (2.54) can be written in the form

$$\begin{bmatrix} u \\ u_t \end{bmatrix}(x,t) = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}\begin{bmatrix} u \\ u_t \end{bmatrix}(x,t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.73}$$

$$\begin{bmatrix} u \\ u_t \end{bmatrix}(x,0) = \begin{bmatrix} \varphi \\ \psi \end{bmatrix}(x) \tag{2.74}$$

where, as before $A = -c^2\partial_x^2$.

These equations can be conveniently written in the form

$$\{\partial_t - iM\}\Psi(x, t) = 0, \quad \Psi(x, 0) = \Psi_0(x) \tag{2.75}$$

where

$$\Psi(x,t)=\begin{bmatrix} u \\ u_t \end{bmatrix}(x,t), \quad \Psi_0(x)=\begin{bmatrix} \varphi \\ \psi \end{bmatrix}(x) \tag{2.76}$$

$$-iM=\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} \tag{2.77}$$

If we again make the "outrageous" assumption that A is a constant then it will follow that M is a constant and hence (2.75) can be reformulated as an IVP for an ordinary differential equation of the form

$$\{d_t - iM\}\Psi(t) = 0, \quad \Psi(0) = \Psi_0 \tag{2.78}$$

where we have used the notation

$$\Psi(x, t) = \Psi(\cdot, t)(x) =: \Psi(t)(x) \tag{2.79}$$

The solution of (2.78) can be obtained, by using an integrating factor technique, in the form

$$\Psi(t) = \exp(itM)\Psi(0) \tag{2.80}$$

Writing the exponential term in a series form we obtain

$$\begin{aligned} e^{itM} &= \sum_{n=0}^{\infty}\frac{(itM)^n}{n!} = \left\{\sum_{n=\text{even}} + \sum_{n=\text{odd}}\right\}\frac{(itM)^n}{n!} \\ &= \left\{I - \frac{t^2M^2}{2!} + \frac{t^4M^4}{4!} - \ldots\right\} + i\left\{tM - \frac{t^3M^3}{3!} + \frac{t^5M^5}{5!} - \ldots\right\} \end{aligned}$$

Now, using,

$$M = i\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}, \quad M^2 = A\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.81}$$

and recalling the series expansions for $\sin x$ and $\cos x$ we obtain

$$\begin{aligned} \exp(it\,M) &= \left\{I - \frac{t^2A}{2!} + \frac{t^4A^2}{4!} - \ldots\right\}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &\quad - A^{-1/2}\left\{tA^{1/2} - \frac{t^3A^{3/2}}{3!} + \frac{t^5A^{5/2}}{5!} - \ldots\right\}\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} = \cos\left(tA^{1/2}\right)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &\quad - A^{-1/2}\sin\left(tA^{1/2}\right)\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} = \begin{bmatrix} \cos\left(tA^{1/2}\right) & A^{1/2}\sin\left(tA^{1/2}\right) \\ -A^{1/2}\sin\left(tA^{1/2}\right) & \cos\left(tA^{1/2}\right) \end{bmatrix} \end{aligned} \tag{2.82}$$

If we substitute this expression into (2.79) it is clear that the first component of the solution (2.80) yields the same solution of the given IVP as obtained earlier.

This approach can be given a rigorous mathematical development as we shall see. We shall make considerable use of it in this book since, on the one hand, it provides a relatively easy means of settling questions of existence and uniqueness and, on the other, it offers good prospects for developing constructive methods.

So far we have only been discussing IVP for the one-dimensional wave equation, that is, the FP for waves on a string. In the next few sections we turn our attention to some PPs associated with this FP and indicate how the various methods discussed so far are either inadequate or need to be modified in certain ways.

2.7 Waves on Sectionally Homogeneous Strings

In our investigations, so far, of waves on strings we have considered the string to have uniform density throughout. This generated what we came to call the Free Problem (FP) associated with the classical wave equation. Associated with this FP there is a whole hierarchy of Perturbed Problems (PPs). Perhaps the simplest PP, in this case, would arise when we investigate waves on a string that has a piecewise uniform density. In this section we shall consider two particular cases.

2.7.1 A Two-Part String

Consider two semi-infinite strings Ω_1 and Ω_2 of (linear) density ρ_1 and ρ_2 respectively that are joined at the point $x = r$ and stretched at tension T with Ω_1 occupying the region $x < r$ and Ω_2 the region $x > r$. As the two strings have different (linear) densities then it follows that their associated wave speeds c_1, c_2 are also different.

We shall see that this problem is a one-dimensional version of the more general interface problems. In this latter problem a (given) incident wave travels in a homogeneous medium which terminates at an interface with another, different, homogeneous medium in which the wave can also travel. Examples of such problems are given, for instance, by electromagnetic waves travelling in air meeting the surface of a dielectric and by acoustic waves travelling in air meeting an obstacle. In this class of problems the interface "scatters" the given incident wave and gives rise to reflected and transmitted waves. When all these waves combine the resulting wave fields are readily seen to be quite different to those occurring in the related FP. Such problems are examples of so-called **target scattering** problems. We illustrate some of the features of such problems by considering the following one-dimensional problem.

The governing wave equation is

$$\{\partial_t^2 - c^2(x)\partial_x^2\}u(x, t) = 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R} \tag{2.83}$$

$$c(x) = \begin{cases} c_1, & x < r, \quad x \in \mathbf{R} \\ c_2, & x > r, \quad x \in \mathbf{R} \end{cases} \tag{2.84}$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \tag{2.85}$$

At the interface we shall require continuity of the displacement and of the transverse forces. This leads to boundary conditions of the form

$$u(r^-, t) = u(r^+, t), \quad u_x(r^-, t) = u_x(r^+, t) \tag{2.86}$$

We remark that we assume here, unless otherwise stated, that $r > 0$. This is not just to increase the complexity of the presentation. It is simply because in many later illustrations we shall find it a convenient means of keeping track of the target (i.e. interface).

Let $f(x - c_1t)$ denote a given incident wave in Ω_1. We assume that Ω_2 is initially at rest so that $u(x, 0) = 0$ and $u_t(x, 0) = 0$ for $x > r$.

The wave field, $u(x, t)$ which must satisfy (2.83), (2.84), (2.85) has the general form

$$u(x, t) = \begin{cases} f(x - c_1t) + g(x + c_1t), & x < r \\ h(x - c_2t) + H(x + c_2t), & x > r \end{cases} \tag{2.87}$$

However, since Ω_2 is initially at rest we must have

$$h'(\zeta) = 0 \quad \text{and} \quad H'(\zeta) = 0 \quad \text{for} \quad \zeta > 0 \tag{2.88}$$

Hence $H(x + c_2t)$ is a constant for $t > 0$ and therefore may be discarded. The appropriate wavefield is thus

$$u(x, t) = \begin{cases} f(x - c_1t) + g(x + c_1t), & x < r, \quad t > 0 \\ h(x - c_2t), & x > r, \quad t > 0 \end{cases} \tag{2.89}$$

We also notice that

$$\begin{aligned} f(\zeta) = h(\zeta) = 0, &\quad \zeta > r \\ g(\zeta) = 0, &\quad \zeta < r \end{aligned} \tag{2.90}$$

At the interface certain boundary conditions will always have to be satisfied by solutions of (2.83). The most immediate conditions are, as we have already mentioned, continuity of displacement:

$$u(r^-, t) = u(r^+, t) \tag{2.91}$$

continuity of transverse force:

$$u_x(r^-, t) = u_x(r^+, t) \tag{2.92}$$

where

$$u(r^-, t) = \lim_{\substack{x\to r\\x<r}} u(x, t), \quad u(r^+, t) = \lim_{\substack{x\to r\\x>r}} u(x, t) \tag{2.93}$$

and similarly for the derivatives.

Substitute (2.89) into (2.91), (2.92) to obtain

$$f(r - c_1t) + g(r + c_1t) = h(r - c_2t) \tag{2.94}$$

$$f'(r - c_1t) + g'(r + c_1t) = h'(r - c_2t) \tag{2.95}$$

where the primes denote differentiation with respect to the argument.

Integrate (2.95) to obtain

$$\frac{f(r-c_1t)}{-c_1} + \frac{g(r+c_1t)}{c_1} = \frac{h(r-c_2t)}{-c_2} \tag{2.96}$$

where the integration constant has been set to zero in order to ensure (2.90) is satisfied.

Solving (2.94), (2.96) for the unknowns g and h we obtain

$$g(x + c_1t) = \left(\frac{c_2 - c_1}{c_1 + c_2}\right) f(2r - x - c_1t), \quad x < r \tag{2.97}$$

$$h(x - c_2t) = \left(\frac{2c_2}{c_1 + c_2}\right) f\left(r\left(1 - \frac{c_1}{c_2}\right) + \frac{c_1}{c_2}x - c_{1t}\right), \quad x > r \tag{2.98}$$

It is worth noticing a number of interesting features of the solutions represented by (2.97) and (2.98). For convenience of presentation and without any loss of generality, at this stage, we shall assume that $r = 0$.

(i) When $c_2 = 0$ there is no transmitted wave, the reflected wave has the form

$$g(x + c_1t) = -f(-x - c_1t)$$

and the required solution is

$$u(x, t) = f(x - c_1t) - f(-x - c_1t) \tag{2.99}$$

Thus as expected the reflected wave travels in the opposite direction to the incident wave. However although the incident and reflected waves have the same shape they are seen to have opposite signs.

Finally we notice from (2.99) that $u(0, t) = 0$ for all t. Thus, the solution (2.99) describes waves on a semi-infinite string with a fixed point at $x = 0$.

(ii) When $c_1 = c_2$ there is no reflected wave; again as would be expected.

(iii) When $c_2 > c_1$ then the incident and reflected waves are seen to travel in opposite directions with the same profile but no change in sign.

Thus the reflected and incident waves are in phase at the junction (interface) provided $c_2 > c_1$ and are otherwise totally out of phase.

We will be able to obtain more detailed information about the wave field once we have introduced the notions of eigenfunction expansions and Green's functions.

Finally, in this section, we consider the case when the incident wave is a simple harmonic wave of angular frequency ω. In this case we will have an incident wave of the form

$$\begin{aligned} u_i(x, t) &= f(x - ct) = \exp\{ik(x - ct)\} = \exp\{i(kx - \omega t)\} \\ &= \exp\{ikx\} \exp\{-i\omega t\} \end{aligned} \tag{2.100}$$

where $\omega = kc$.

We notice that the incident wave separates into the product of two components, one only dependent on x, the other only dependent on t. It is natural to expect that this will be the case for the complete wave field. Consequently, for the nonhomogeneous string problem we are considering we can expect the complete wave field to be separable and to have the form

$$u(x,t) = \begin{cases} e^{-i\omega_1 t} v_1(x), & \omega_1 = c_1 k_1, \quad x < 0 \\ e^{-i\omega_2 t} v_2(x), & \omega_2 = c_2 k_2, \quad x > 0 \end{cases} \tag{2.101}$$

Therefore, bearing in mind the sign convention introduced just after (2.26) the space-dependent component of the wave field, $v(x)$, can be written in the form

$$v(x) = \begin{cases} e^{ik_1 x} + Re^{-ik_1 x}, & x < 0 \\ Te^{ik_2 x}, & x > 0 \end{cases} \tag{2.102}$$

If we now apply the boundary conditions (2.91), (2.92) which must hold for all $t > 0$ then we readily find

$$R = \frac{k_2 - k_1}{k_1 + k_2}, \quad T = \frac{2k_2}{k_1 + k_2} \exp\{i(\omega_2 - \omega_1)t\} \tag{2.103}$$

Here R and T are known as the **reflection** and **transmission coefficients** respectively. In the case when we are only interested in solutions that have the same frequency then these coefficients assume the simpler form

$$R = \frac{c_2 - c_1}{c_1 + c_2}, \quad T = \frac{2c_2}{c_1 + c_2}$$

Combining (2.101), (2.102) and (2.103) we see that we recover the solutions (2.97), (2.98) in the case when $r = 0$.

2.7.2 A Three-Part String

We shall assume in this section that a portion of a homogeneous string is subjected to an elastic restoring force $E(x)$ per unit length of the string. Newton's laws of motion then indicate that the equation governing wave motion on the string is

$$\{\partial_t^2 - c^2\partial_x^2\}u(x, t) - c^2\mu^2(x)u(x, t) = 0 \tag{2.104}$$

where $\mu^2(x) = E(x)/T_s$ and T_s denotes the string tension.

Equation (2.104) is of a form which is typical when investigating **potential scattering** problems. Here $c^2\mu^2(x)$ can be viewed as the potential term. We shall only be interested here in the case when $E(x)$ is localised. That is, $E(x)$ will be assumed either to vanish outside a finite region of the string or to decay exponentially away from some fixed reference point.

A wave incident on the elastic region (the "potential") will be partly reflected and partly transmitted. However, even if $\mu(x)$ has a constant value the solution of (2.104) is not as easy to obtain as the solutions (2.83). To see this consider the case when $\mu(x)$ has the constant value μ_0 and we seek solutions of (2.104) that have the one angular frequency ω. When this is the situation we assume a solution of the form

$$u(x, t) = w(x, \omega)e^{-i\omega t} \tag{2.105}$$

and, by direct substitution into (2.104), we find that $w(x, \omega)$ must satisfy

$$\left\{d_x^2 + \left(\frac{\omega}{\nu}\right)^2\right\}w(x, \omega) = 0 \tag{2.106}$$

where

$$\frac{1}{\nu^2} = \frac{1}{c^2} - \left(\frac{\mu_0}{\omega}\right)^2 \tag{2.107}$$

We then obtain the string displacement $u(x, t)$, by (2.105) in the form

$$u(x, t) = a\exp\left\{-i\omega\left(t - \frac{x}{\nu}\right)\right\} + b\exp\left\{-i\omega\left(t + \frac{x}{\nu}\right)\right\} \tag{2.108}$$

We see that, in (2.108), v is the phase velocity of the wave. Furthermore, we notice that the wave motion represented by (2.108) is dispersive since by (2.107) the phase velocity, v, of the wave is frequency dependent. It follows that distortionless propagation of the wave as described by f and g in the general solution of the classical wave equation, is no longer possible.

We also notice that there is a "cut off" frequency associated with the wave motions generated in this system. According to (2.106), (2.107) frequencies that are less than $\mu_0 c$ lead to an imaginary propagation constant. These low frequency disturbances do not propagate as waves, they merely move the string up and down in phase. Thus it is possible that localised wave motion might be excited on a nonhomogeneous string.

To illustrate the scattering of an incident wave by the elastic region (potential) consider a string with a segment that has a constant elastic restoring force so that we have

$$\mu(x) = \begin{cases} 0, & |x| > r \\ \mu_0, & |x| < r \end{cases} \tag{2.109}$$

If a wave of frequency ω and unit magnitude is incident on this region then the resulting spatial part of the wavefield may be written, as in the previous section, in the form

$$v(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x < -r \\ Te^{ik_2 x}, & x > +r \\ Ae^{i\alpha x} + Be^{-i\alpha x}, & |x| < r \end{cases} \tag{2.110}$$

where $k = \omega/c$ and $\alpha = \sqrt{k^2 - \mu_0^2}$. It is clear that a wave will not propagate in the elastic region unless $k \geq \mu_0$.

The reflection and transmission coefficients, R and T respectively, together with the constants A and B are determined by requiring continuity of displacement and slope at $x = \pm r$. It is a straightforward but rather lengthy matter to show that

$$R \exp(2ikr) = \frac{\mu_0}{D} \sin(2\alpha r) \tag{2.111}$$

$$T \exp(2ikr) = \frac{2i\alpha k}{D} \tag{2.112}$$

where

$$D = (k^2 + \alpha^2)\sin(2\alpha r) + 2ikr\cos(2\alpha r) \tag{2.113}$$

We see, from (2.111), that perfect transmission (i.e. $R = 0$) occurs when $\sin(2\alpha r) = 0$, that is, whenever an integral number of half-wavelengths of the wave on the elastic region fit into that region.

We also notice that R and T become unbounded at zeros of the denominator D. These will be identified as so-called **resonances** of the system.

2.8 Duhamel's Principle

So far, we have only been dealing with the homogeneous wave equation. We shall now show that this is really sufficient for many of our immediate purposes. That is, we shall show that the results we obtain when investigating the homogeneous equation can be used to generate solutions for the nonhomogeneous wave equation. As an illustration we consider in this section the IVP

$$\{\partial_t^2 - c^2\partial_x^2\}u(x, t) = f(x, t), \ (x, t) \in \mathbf{R} \times \mathbf{R} \tag{2.114}$$

$$u(x, 0) = \varphi(x), \ u_t(x, 0) = \psi(x) \tag{2.115}$$

For convenience of presentation we shall again write $A = -c^2\partial_x^2$.

We shall assume that the solution of (2.114), (2.115) has the form

$$u(x, t) = v(x, t) + w(x, t) \tag{2.116}$$

We now proceed as before by interpreting (2.114) as an ordinary differential equation. To this end we shall understand that u, the function defining the displacement $u(x, t)$, has the interpretation

$$u : t \to u(\cdot, t) =: u(t) \tag{2.117}$$

and similarly for v and w in (2.116).

With this in mind we assume that

$$\{d_t^2 + A\}v(t) = 0, \quad v(0) = \varphi, \quad v_t(0) = \psi \tag{2.118}$$

and that

$$\{d_t^2 + A\}w(t) = f(t), \quad w(0) = 0, \quad w_t(0) = 0 \tag{2.119}$$

We have already seen that the problem (2.118) has a solution of the form

$$v(t) = \left(\cos\left(tA^{1/2}\right)\right)\varphi + A^{-1/2}\left(\sin\left(tA^{1/2}\right)\right)\psi$$

which leads to

$$v(x, t) = \left(\cos\left(tA^{1/2}\right)\right)\varphi(x) + A^{-1/2}\left(\sin\left(tA^{1/2}\right)\right)\psi(x) \tag{2.120}$$

To obtain $w(x, t)$ in (2.119) we set

$$w(t) = \int_0^t \eta(t, \tau) d\tau \tag{2.121}$$

where $\eta(t, \tau)$ is assumed to satisfy

$$\{d_t^2 + A\}\eta(t, \tau) = 0, \ \eta(\tau, \tau) = 0, \ \eta_t(\tau, \tau) = f(\tau) \tag{2.122}$$

A straightforward calculation then shows that $w(t)$ defined by (2.121) is a solution of the IVP (2.119). Therefore, combining these results we obtain the required solution in the form

$$\begin{aligned} u(x,t) &= \left(\cos\left(tA^{1/2}\right)\right)\varphi(x) + A^{-1/2}\left(\sin\left(tA^{1/2}\right)\right)\psi(x) + \int_0^t \eta(t,\tau) d\tau \\ &= (B_t(t)\varphi)(x) + (B(t)\psi)(x) + \int_0^t \eta(t,\tau) d\tau \end{aligned} \tag{2.123}$$

where

$$B(t) := A^{-1/2}\left(\sin\left(tA^{1/2}\right)\right) \tag{2.124}$$

Bearing in mind how the solution form (2.120) was obtained and applying the same techniques to the IVP (2.122), where in this case the initial conditions are imposed at $t = \tau$ rather than $t = 0$, we obtain

$$\eta(t,\tau) = A^{-1/2}\left(\sin\left((t-\tau)A^{1/2}\right)\right)f(\tau) = B(t-\tau)f(\tau) \tag{2.125}$$

Hence

$$w(t) = \int_0^t \eta(t,\tau) d\tau = \int_0^t B(t-\tau)f(\tau) d\tau \tag{2.126}$$

and consequently

$$u(x,t) = (B_t(t)\varphi)(x) + (B(t)\psi)(x) + \int_0^t B(t-\tau)f(x,\tau) d\tau \tag{2.127}$$

Thus we see that if we can solve the homogeneous equation, that is, if we can interpret (2.124) in a practical, constructive manner, then we can solve the inhomogeneous equation in the form (2.127). This is known as **Duhamel's Principle** and (2.126) is said to define the **Duhamel Integral** for the inhomogeneous equation. With this section in mind we shall concentrate our attention, almost entirely, for the remainder of the book on the associated homogeneous equations.

2.9 On the Far Field Behaviour of Solutions

In this section we give a first indication of how solutions of FPs and PPs might be considered as being AE. This we shall do by considering the following IVP. Determine a quantity $w(x, t)$ which satisfies

$$\{\partial_t^2 + L(x)\}w(x, t) = 0 \tag{2.128}$$

$$w(x, s) = \varphi_s(x), \quad w_t(x, s) = \psi_s(x) \tag{2.129}$$

where $s \in \mathbf{R}$ is a fixed initial time and in (2.129) $\varphi_s(x)$ and $\psi_s(x)$ are given initial data. In (2.128)

$$L(x) = -c^2\partial_x^2 + V(x) \tag{2.130}$$

where c is the wave speed and V is a function characterising a perturbation of the (one-dimensional) Laplacian ∂_x^2.

Problems of the form (2.128)–(2.130) are typical of those which arise when investigating potential scattering, an example of which was given above when discussing wave motions on an elastically braced string. We would remark that (2.128) and (2.130) together provide a perturbation of the classical wave equation and such an equation is frequently referred to as the plasma wave equation.

When $V(x) \equiv 0$ we refer to (2.128)–(2.130) as a Free Problem (FP). We notice that this FP is governed by the familiar wave equation. When $V(x) \neq 0$ everywhere then (2.128)–(2.130) is referred to as a Perturbed Problem (PP).

We have seen in the previous sections that the wave equation, which governs the FP in this case, has solutions which can be written in the form

$$w(x, t) = f(x - ct) + g(x + ct) \tag{2.131}$$

where f and g are arbitrary functions which characterise waves, of constant profile, travelling with velocity c from left to right and right to left respectively.

In the particular case when both waves have the same time dependency, $\exp(-i\omega t)$, then the familiar separation of variables technique indicates that the wave equation has solutions, denoted $w(x, t, k)$, which we can write in the form

$$w(x, t, k) = \exp(-i\omega t)\{u_+(x, k) + u_-(x, k)\} \tag{2.132}$$

where the quantities $u_\pm(x, k)$ must satisfy

$$\left\{\frac{d^2}{dx^2} + k^2\right\}u_\pm(x, k) = 0, \quad k^2 = \frac{\omega^2}{c^2}, \quad x \in \mathbf{R} \tag{2.133}$$

However, although the $u_\pm(x, k)$ both satisfy (2.133) nevertheless they have different properties. This can be seen by first noticing that (2.133) yields

$$u_+(x,k) = \exp(ikx) \quad \text{and} \quad u_-(x,k) = \exp(-ikx) \tag{2.134}$$

Combining (2.132) and (2.134) we obtain

$$w(x,t,k) = \exp\left(-i\omega\left(t - \frac{x}{c}\right)\right) + \exp\left(-i\omega\left(t + \frac{x}{c}\right)\right) \tag{2.135}$$

Thus, recalling the form (2.131) we see that u_+ characterises a wave moving from left to right and u_- characterises a wave moving from right to left, both waves having the same time dependency $\exp(-i\omega t)$. Equivalently, we say that u_+ represents an **outgoing wave**, since in $\mathbf{R}$, for $x > 0$, it is moving away from the origin whilst u_- represents an **incoming wave** since, in $\mathbf{R}$, for $x < 0$, it is moving towards the origin. This feature of the wave motion leads quite naturally to the following definition which caters for more general situations [6].

Definition 2.3. Solutions $v(x,k)$ of the equation

$$\{\Delta + k^2\}v(x,k) = 0, \quad x \in \mathbf{R}, \quad n \geqslant 1 \tag{2.136}$$

are said to satisfy the **Sommerfeld radiation condition** if and only if for all k as $r = |x| \to \infty$

$$\left\{\frac{\partial}{\partial r} \mp k\right\} v(x,k) = o\left(\frac{1}{r}\right) \tag{2.137}$$

$$v(x,k) = O\left(\frac{1}{r^{1/2(n-1)}}\right) \tag{2.138}$$

The estimates (2.137), (2.138) are understood to hold uniformly with respect to the direction $x/|x|$. The estimate (2.137) taken with the minus (plus) sign is called the Sommerfeld outgoing (incoming) radiation condition.

With $u_\pm$ defined as in (2.134) it is clear that $u_+(x,k)$ is an outgoing solution and $u_-(x,k)$ is an incoming solution in the sense of Definition 2.3. A derivation of these radiation conditions can be found in the text cited in the Commentary.

In later sections we shall see that the $u_\pm(x,k)$ have the important property that *any* solution of (2.136) can be expressed as a linear combination of the $u_\pm(x,k)$ [1]. As a consequence we shall refer to the $u_\pm(x,k)$ as **fundamental solutions** of (2.133).

We now turn our attention to the case when the potential $V(x) \neq 0$, that is to the PP which is a perturbation of the FP we have just been discussing. A natural first step when dealing with this PP is to see if the time dependency in the problem can again be separated out. If we assume that this separation is possible then instead of (2.132) and (2.133) we now have

$$w(x,t,k)=\exp(-i\omega t)\{w_{+}(x,k)+w_{-}(x,k)\},\quad k^{2}=\frac{\omega^{2}}{c^{2}} \tag{2.139}$$

where

$$\left\{\frac{d^{2}}{dx^{2}}+k^{2}-Q(x)\right\}w_{\pm}(x,k)=0,\quad k^{2}=\frac{\omega^{2}}{c^{2}},\quad x\in\mathbf{R} \tag{2.140}$$

and

$$Q(x)=\frac{V(x)}{c^{2}} \tag{2.141}$$

Since (2.140) is clearly a perturbation of (2.133), which has certain fundamental solutions, $u_{\pm}(x,k)$, defined in (2.134), then it is natural to ask if (2.140) also has fundamental solutions and, if so, can they be regarded as perturbations of the $u_{\pm}(x,k)$? Furthermore, if such fundamental solutions of (2.140) exist then are they in some sense AE to the fundamental solutions, $u_{\pm}(x,k)$, of (2.133)? It turns out that for (2.140) there is a family of solutions, parameterised by k, for which the answer to both these questions is in the affirmative. These are the so-called Jost solutions.

2.9.1 Jost Solutions

Bearing in mind the remarks at the end of the last section the Jost solutions, whenever they exist, are those solutions of (2.140), denoted by $\psi^{\pm}$ and $\varphi^{\pm}$, which have the following uniform asymptotic behaviour:

as $x\to+\propto$

$$\psi^{\pm}(x,k)=u_{\pm}(x,k)\{1+o(1)\}=\exp(\pm ikx)\{1+o(1)\} \tag{2.142}$$

$$\psi^{\pm}_{x}(x,k)=\frac{\partial u_{\pm}(x,k)}{\partial x}\{1+o(1)\}=\exp(\pm ikx)\{\pm ik+o(1)\} \tag{2.143}$$

as $x\to-\infty$

$$\varphi^{\pm}(x,k)=u_{\mp}(x,k)\{1+o(1)\}=\exp(\mp ikx)\{1+o(1)\} \tag{2.144}$$

$$\varphi^{\pm}_{x}(x,k)=\frac{\partial u_{\mp}(x,k)}{\partial x}\{1+o(1)\}=\exp(\mp ikx)\{\pm ik+o(1)\} \tag{2.145}$$

Alternatively, these can be written in the more compact forms:

$$\lim_{x\to\infty}\{\exp(\mp ikx)\psi^{\pm}(x,k)\}=1 \tag{2.146}$$

$$\lim_{x\to\infty}\{\exp(\mp ikx)\psi^{\pm}_{x}(x,k)\}=\pm ik \tag{2.147}$$

$$\lim_{x\to-\infty}\{\exp(\pm ikx)\varphi^{\pm}(x,k)\}=1 \tag{2.148}$$

$$\lim_{x\to-\infty}\{\exp(\pm ikx)\varphi_x^{\pm}(x,k)\}=\mp ik \tag{2.149}$$

Thus we see that as $x\to\pm\infty$ the Jost solutions will have the behaviour of plane waves. Consequently, we can expect the perturbed equation (2.140) to have solutions which can be interpreted as **distorted plane waves** which are AE to solutions of the FP.

Perhaps one of the more convenient ways of settling the questions of existence and uniqueness of the Jost solutions is to represent them as solutions of certain integral equations. This will have the added bonus of yielding a constructive method. To this end we recall that solutions of an equation of the form

$$y''(x)+\{k^2-q(x)\}y(x)=0 \tag{2.150}$$

can be obtained by the variation of parameters method. Specifically, assume that (2.150) has a solution that can be written in the form

$$y(x)=A(x)\exp(ikx)+B(x)\exp(-ikx) \tag{2.151}$$

Substituting (2.151) into (2.150) it is natural to set

$$(2ik)A'(x)\exp(ikx)=q(x)y(x) \tag{2.152}$$

and so obtain

$$A(x)=\frac{1}{2ik}\int_0^x q(v)y(v)\exp(-ikv)dv+C_1 \tag{2.153}$$

Now substitute (2.152), (2.153) into (2.150) and obtain

$$B(x)=\frac{-1}{2ik}\int_0^x q(v)y(v)\exp(+ikv)dv+C_2 \tag{2.154}$$

The integration constants C_1 and C_2 have now to be chosen to ensure that asymptotic behaviours of the forms (2.142)–(2.145) are obtained as required. To this end, if we identify y in (2.150) with the Jost solutions ψ^+ and φ^+ and consider the form of $\exp(-ikx)\psi^+$and $\exp(+ikx)\varphi^+$ as $x\to\infty$ and $x\to-\infty$ respectively then the required asymptotic behaviour will be obtained if for ψ^+ we set

$$C_1=1-\frac{1}{2ik}=\int_0^{\infty} q(v)y(v)\exp(-ikv)dv \tag{2.155}$$

and

$$C_2 = \frac{1}{2ik} = \int_0^{\infty} q(v)y(v)\exp(+ikv)dv \tag{2.156}$$

For φ^+ we set

$$C_1 = \frac{1}{2ik} = \int_{-\infty}^{0} q(v)y(v)\exp(-ikv)dv \tag{2.157}$$

$$C_2 = 1 - \frac{1}{2ik} = \int_{-\infty}^{0} q(v)y(v)\exp(+ikv)dv \tag{2.158}$$

to obtain the required asymptotic behaviour.

Combining these several results we obtain

$$\psi^{\pm}(x,k) = \exp(\pm ikx) + \frac{1}{k}\int_x^{\infty} q(v)\sin(k(v-x))\psi^{\pm}(v,k)dv \tag{2.159}$$

$$\varphi^{\pm}(x,k) = \exp(\mp ikx) - \frac{1}{k}\int_{-\infty}^{x} q(v)\sin(k(v-x))\varphi^{\pm}(v,k)dv \tag{2.160}$$

which are Volterra integral equations of the second kind for the Jost solutions $\psi^{\pm}(x,k)$ and $\varphi^{\pm}(x,k)$.

Solutions of (2.159), (2.160) can be obtained by successive approximations in the form

$$\psi^{\pm}(x,k) = \exp(\pm ikx)\sum_{j=0}^{\infty} h_j^{\pm}(x,k) \tag{2.161}$$

$$\varphi^{\pm}(x,k) = \exp(\mp ikx)\sum_{j=0}^{\infty} g_j^{\pm}(x,k) \tag{2.162}$$

where, for example, with Im $k \geqslant 0$

$$\begin{aligned} g_0^{\pm}(x,k) &= 1 \\ g_{j+1}^{\pm}(x,k) &= -\frac{1}{k}\int_x^{\infty} q(v)\exp(\mp ik(v-x))\sin(k(v-x))g_j(v,k)dv \end{aligned} \tag{2.163}$$

Similar results for the $h_j^{\pm}$ can be obtained.

These Jost solutions have been extensively studied and their principal features are conveniently summarised in the following theorem.

Theorem 2.4. *The Schrödinger equation*

$$\left\{\frac{\partial^2}{\partial x^2} + (k^2 - Q(x))\right\} y(x,k) = 0, \quad x \in \mathbf{R}$$

has Jost solutions $\psi^{\pm}$ and $\varphi^{\pm}$ which satisfy the Volterra integral equations

$$\psi^{\pm}(x,k) = \exp(\pm ikx) + \frac{1}{k}\int_{x}^{\infty} Q(v)\sin(k(v-x))\psi^{\pm}(v,k)dv \qquad (2.164)$$

$$\varphi^{\pm}(x,k) = \exp(\mp ikx) - \frac{1}{k}\int_{-\infty}^{x} Q(v)\sin(k(v-x))\varphi^{\pm}(v,k)dv \qquad (2.165)$$

provided

$$\int_{-\infty}^{\infty} \{1+v\}Q(v)dv < \infty \qquad (2.166)$$

Furthermore

(i) The Jost solutions $\psi^{\pm}(x, k)$ and $\varphi^{\pm}(x, k)$ are unique. Successive approximations to these solutions can be obtained in the forms (2.161) and (2.162).

(ii) For every $x \in \mathbf{R}$ the Jost solutions $\psi^{+}(x, k)$ and $\varphi^{+}(x, k)$ and their derivatives $\psi_x^{+}(x, k)$ and $\varphi_x^{+}(x, k)$ are

(a) continuous with respect to k for Im $k \geqslant 0$

(b) analytic with respect to k for Im $k > 0$.

Analogous properties for $\psi^{-}(x, k)$ and $\varphi^{-}(x, k)$ hold for Im $k \leq 0$.

(iii) The Jost solutions $\psi^{\pm}(x, k)$ and $\varphi^{\pm}(x, k)$ are inter-related as follows.

$$\psi^{-}(x, k) = \psi^{+}(x, k^*) = \psi^*(x, k^*)$$

$$\varphi^{-}(x, k) = \varphi^{+}(x, -k^*) = \varphi^*(x, k^*)$$

2.9.2 Some Scattering Aspects

The Wronskian of two solutions, $y_1(x, k)$ and $y_2(x, k)$, of the Schrödinger equation (2.150) is denoted $W(y_1, y_2)$ and defined by

$$W(y_1, y_2) = y_1(x, k)y_2'(x, k) - y_1'(x, k)y_2(x, k) \qquad (2.167)$$

where the prime denotes differentiation with respect to x. A fundamental property of the Wronskian is given by the following theorem.

Theorem 2.5. *Two solutions of the Schrödinger equation* (2.150) *are linearly independent if and only if their Wronskian is non-zero.*

For the Jost solutions $\psi^{\pm}$ and $\varphi^{\pm}$ we find that as $x \to \infty$

$$W(\psi^{+}, \psi^{-}) = -2ik + o(1) \qquad (2.168)$$

and as $x \to -\infty$

$$W(\varphi^{+}, \varphi^{-}) = +2ik + o(1) \qquad (2.169)$$

Consequently Theorem 2.5 indicates that (ψ^+, ψ^-) and (φ^+, φ^-) are two linearly independent pairs of solutions of (2.140).

Since any linear combination of the pair (φ^+, φ^-) and the pair (φ^+, φ^-) will also yield a solution of (2.140) we see that we can, in particular, write

$$\varphi^+(x, k) = c_{11}(k)\psi^+(x, k) + c_{12}(k)\psi^-(x, k) \tag{2.170}$$

$$\psi^+(x, k) = c_{21}(k)\varphi^-(x, k) + c_{22}(k)\varphi^+(x, k) \tag{2.171}$$

where the coefficients c_{ij} have yet to be determined.

We now recall that in the case when $Q(x) = 0$ then the Schrödinger equation (2.140) reduces to the familiar wave equation which has the associated solutions $\psi_0^\pm(x, k)$ and $\varphi_0^\pm(x, k)$ where

$$\psi_0^\pm(x, k) = \exp(\pm ikx), \quad \varphi_0^\pm(x, k) = \exp(\mp ikx) \tag{2.172}$$

In this particular case the coefficients c_{ij} in (2.170), (2.171) are such that

$$c_{11}(k) = c_{22}(k) = 0$$

Furthermore, we notice that ψ^+ and φ^- characterise plane waves moving from left to right whilst ψ^- and φ^+ characterise plane waves moving from right to left.

When $Q(x) \neq 0$ for all x then we obtain (2.170), (2.171). The limiting behaviour of the Jost solutions given in (2.142)–(2.145) indicates that (2.170) represents a solution of the Schrödinger equation which, by the properties of the left-hand side, reduces to $\exp(-ikx)$ as $x \to -\infty$ whilst the right-hand side reduces to $c_{11}(k)$ $\exp(ikx) + c_{12}(k) \exp(-ikx)$ as $x \to \infty$. Consequently (2.170) is a solution of the Schrödinger equation (2.140) which represents the scattering, by the potential $Q(x)$, of a plane wave of amplitude $c_{12}(k)$ incident from $x = +\infty$ and moving right to left. The scattering process gives rise to a reflected plane wave of amplitude c_{11} moving left to right towards $x = +\infty$ and to a transmitted wave with unit amplitude moving right to left towards $x = -\infty$. It is customary to normalise this process so that the incident wave has unit amplitude in which case (2.170) is rewritten in the form

$$T_R(k)\varphi^+(x, k) = -R_R(k)\psi^+(x, k) + \psi^-(x, k) \tag{2.173}$$

where

$$R_R(k) = -\frac{c_{11}(k)}{c_{12}(k)}, \quad T_R(k) = \frac{1}{c_{22}(k)} \tag{2.174}$$

and the subscript R refers to the fact that we are dealing with an incident wave from the right. The minus sign is included for later convenience.

Similarly, (2.171) can be interpreted as a solution of the Schrödinger equation which represents the scattering, by a potential $Q(x)$, of a plane wave incident from $x = -\infty$. The process gives rise as before to a scattered and transmitted wave and normalising as before we can write (2.171) in the form

$$T_L(k)\psi^+(x,k) = \varphi(x,k) + R_L(k)\varphi^+(x,k) \tag{2.175}$$

where

$$R_L(k) = -\frac{c_{22}(k)}{c_{21}(k)}, \quad T_L(k) = \frac{1}{c_{21}(k)} \tag{2.176}$$

and the subscript L denotes that we are dealing with a wave incident from the left. Again the minus signs are included for later convenience.

The equations (2.173) and (2.175) can be written conveniently in the matrix form

$$\Phi^-(k) = S(k)\Phi^+(k) \tag{2.177}$$

where

$$\Phi^-(k) = \begin{bmatrix} \varphi^- \\ \psi^- \end{bmatrix}(k), \quad \Phi^+(k) = \begin{bmatrix} \varphi^+ \\ \psi^+ \end{bmatrix}(k) \tag{2.178}$$

and

$$S(k) = \begin{bmatrix} R_L & T_L \\ T_R & R_R \end{bmatrix}(k) \tag{2.179}$$

The matrix $S(k)$ is called the **scattering matrix** for the problem. Its role in scattering theory will be discussed in later sections. For the time being we simply note that it provides a connection between the incident fields and the scattered fields.

2.10 Concluding Remarks

It turns out that the various techniques and strategies we have outlined so far can be extended to cater efficiently and constructively with more general problems than those dealing simply with wave motions on strings. These generalisations can be made in a relatively easy manner if we choose to work with the actual functions involved rather than with the numerical values of the functions. To be able to do this requires that we should work within a mathematical structure which generates an easily solvable, but abstract, version of the given physical problem and yet is one which always ensures that there is an easy path back to the required physical (numerical) results. We indicate in the following chapter a way of achieving this.

Essentially the main steps are the following.

1. Identify a collection of elements which contains those elements which can be used.

   Characterise, for example, wave motions which have some particular property such as finite energy. Call this collection X.

   The elements of X are abstract quantities which can be thought of, for example, as functions themselves rather than their numerical values.
2. Endow X with a set of rules which will allow the elements of X to be manipulated algebraically and geometrically. These rules would parallel and extend the familiar processes which are used in the Euclidean space $\mathbf{R}^n$, $n < \infty$ The collection X taken together with the structure defined in terms of these algebraic and geometric rules we shall call a **space** and denote it here, for the time being, by H.
3. Introduce the notion of an **operator** which, in its simplest form, maps (transforms) one element of H into some other element of H.
4. Use these several notions to represent (realise) a given physical problem, which is essentially a problem involving numerical values of functions, in the space H. This will yield an abstract problem involving the functions themselves rather than their numerical values.
5. Investigate the availability of associated inverse operators as a means of solving the abstract problem.
6. Settle questions of the existence and uniqueness of solutions to the abstract problem. That is, examine the well-posed nature of the abstract problem.
7. Solve the abstract problem.
8. Interpret the solution of the abstract problem in a manner which will allow the recovery of the required physical results.

The first four steps can be made by introducing the notion of a so-called Hilbert space structure and using the properties of (linear) operators on such spaces.

Step 5 can be made using results from the spectral theory of (linear) operators on a Hilbert space.

Step 6 can be made efficiently and constructively using results from the elegant theory of semigroups of operators.

Step 7 can be achieved using results from the theory of ordinary differential equations but it must be remembered that the work is in the abstract space, H, rather than $\mathbf{R}^n$.

The final step here will be made by introducing the notion of a generalised Fourier transform and proving generalised Fourier inversion theorems modelled on the Fourier–Plancherel theory which can be established quite independently of any scattering phenomena.

With this preparation we will then be well placed to develop a scattering theory which would highlight constructive methods for analysing echo field phenomena. This we shall demonstrate by investigating in the following chapters a number of specific, physically relevant, problems. However, before embarking on this we shall first introduce, in the next chapter, a number of basic ideas and results from general mathematical analysis which will be sufficient to create a suitable mathematical structure in which to work and settle items 1–8 above.

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