2 Elementary Algebra

2.1 Introduction

The evolution of algebra can be traced back to a treatise called *Al-Kitab al-Jabr wa-l-Muqabala* written by the Persian mathematician, Muhammed ibn Mūsā al-Khowārizmī [circa 780–850]. An English translation of the title is *The Compendious Book on Calculation by Completion and Balancing*. It is highly probable that the word *algebra* descended from the middle words of the title *al-Jabr*.

It took a few centuries of experimentation before today's notation emerged. For instance, Robert Recorde [1] [1510–1558] was an academic at Oxford University and seems to be the first person to employ the word 'algebra' in his book *Pathway to Knowledge*. He also introduced the equality sign '=' in 1557 in his book *Whetstone of Witte*.

The German mathematician, Johannes Widman [2] [1462–1498], is credited with using the symbols '+' and '−' for the first time in his book on arithmetic in 1489. The English mathematician, William Oughtred [3] [1574–1660] introduced the '×' symbol to represent multiplication, not to mention his invention of an early form of the slide rule.

Today, algebra is a major branch of mathematics and allows us to write mathematical statements in the form:

$$\langle\text{LHS}\rangle = \langle\text{RHS}\rangle$$

where the left-hand-side (LHS) and right-hand-side (RHS) expressions are manipulated to resolve the value of some variable. For example, given

$$ax^2 + bx + c = 0 \tag{2.1}$$

a solution for x is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.2}$$

which is derived by applying the axioms of elementary algebra.

Let us remind ourselves of these rules.

2.2 Numbers, variables and arithmetic operators

To begin with, we acknowledge the existence of different sets of numbers such as natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, reals $\mathbb{R}$, rationals $\mathbb{Q}$, irrationals, etc., and the role of *variables* such as $x, y, z, \ldots$ etc. to stand in for numbers whose values are not stated explicitly, such as

$$x^2 = 16 \tag{2.3}$$

which implies that

$$x = \pm 4. \tag{2.4}$$

Next, we introduce the binary operators $+, -, \times$ and / to represent addition, subtraction, multiplication and division respectively.

2.3 Closure

Closure is a property that relates the result of a binary operation to the original operands. For example, if we add an integer to an integer the result is an integer, which secures closure for this operation. However, if we divide an integer by an integer, the result is not necessarily an integer, and closure fails. For instance,

$$3/4 = 0.75 \tag{2.5}$$

for although the numerator and denominator are integers, the result is a real quantity. This is not a problem that should worry us — simply something of which we should be aware.

2.4 Identity element

The *identity element* is a useful feature of any algebra and helps simplify algebraic expressions or invoke an analytical strategy. So let's find the identity elements for the four binary operators.

If λ (lambda) is the identity element and Ω (omega) is a binary operator then the following two rules must be satisfied:

$$x \Omega \lambda = x \tag{2.6}$$

and

$$\lambda \Omega x = x. \tag{2.7}$$

In the case of addition $\lambda = 0$ because

$$x + 0 = x \tag{2.8}$$

and

$$0 + x = x. \tag{2.9}$$

But in the case of subtraction, if $\lambda = 0$

$$x - 0 = x \tag{2.10}$$

but

$$0 - x = -x \tag{2.11}$$

which fails.

In the case of multiplication, $\lambda = 1$ because

$$x \times 1 = x \tag{2.12}$$

and

$$1 \times x = x. \tag{2.13}$$

But in the case of division, if $\lambda = 1$

$$x/1 = x \tag{2.14}$$

but

$$1/x \neq x \tag{2.15}$$

which fails. Similarly, if $\lambda = 0$ then

$$x/0 = \infty \tag{2.16}$$

which fails, and

$$0/x = 0 \tag{2.17}$$

which also fails.

2.5 Inverse element

The *inverse element* is a powerful analytical tool for solving equations and inverting functions. In the case of addition, the inverse of x is $-x$, whilst the inverse of $-x$ is x which ensures that

$$x + (-x) = 0 \tag{2.18}$$

and

$$-x + x = 0. \tag{2.19}$$

In the case of multiplication, the inverse of x is $1/x$, and the inverse of $1/x$ is x, which ensures that

$$x\left(\frac{1}{x}\right) = 1 \tag{2.20}$$

and

$$\left(\frac{1}{x}\right)x = 1. \tag{2.21}$$

The inverse element is related to the identity element as follows:

$$x \Omega x^{-1} = \lambda \tag{2.22}$$

where λ is the identity element. For addition $\lambda = 0$ and for multiplication $\lambda = 1$.

2.6 The associative law

When we add or multiply three or more numbers together the final result is independent of their order, which doesn't seem surprising. However, it is not true for subtraction or division. In general, the *associative law* is summarized as:

$$a + (b + c) = (a + b) + c \tag{2.23}$$

or

$$a(bc) = (ab)c. \tag{2.24}$$

We take for granted this last axiom where it is possible to associate different pairs of products, but it is extremely useful as its existence permits division within the algebra. Later on, we will discover that GA is also associative, which permits us to divide by vectors.

To see why subtraction and division fail, consider the following simple examples:

$$9 - (8 - 2) \neq (9 - 8) - 2 \tag{2.25}$$

and

$$9/(9/3) \neq (9/9)/3. \tag{2.26}$$

2.7 The commutative law

When we add or multiply two numbers the final result is independent of their sequence. This, too, may not seem surprising, but matrices and vectors *anticommute* when multiplied. The *commutative law* is summarized as:

$$a + b = b + a \tag{2.27}$$

and

$$ab = ba. \tag{2.28}$$

To see why subtraction and division fail, consider the following simple examples:

$$6 - 2 \neq 2 - 6 \tag{2.29}$$

and

$$9/3 \neq 3/9. \tag{2.30}$$

2.8 The distributive law

The *distributive law* of multiplication over addition is best summarized as:

$$a(b + c) = ab + ac \tag{2.31}$$

and

$$(b + c)a = ba + ca. \tag{2.32}$$

2.9 Summary

The above laws are the axioms for elementary algebra and are expressed formally as follows:

Given $$u, u_1, u_2, u_3 \in \mathbb{R}: \tag{2.33}$$

Closure

For all u_1 and u_2

$$\text{addition} \quad u_1 + u_2 \in \mathbb{R} \tag{2.34}$$

$$\text{multiplication} \quad u_1 u_2 \in \mathbb{R}. \tag{2.35}$$

Identity

For each u there is an identity element 0 and 1 such that

$$\text{addition} \quad u + 0 = 0 + u = u \tag{2.36}$$

$$\text{multiplication} \quad 1u = u. \tag{2.37}$$

Inverse

For each u there is an inverse element $-u$ and $1/u$ such that

$$\text{addition} \quad u + (-u) = -u + u = 0 \tag{2.38}$$

$$\text{multiplication} \quad u\left(\frac{1}{u}\right) = \left(\frac{1}{u}\right)u = 1 \quad (u \neq 0)\,. \tag{2.39}$$

Associativity

For all u_1, u_2 and u_3

$$\text{addition} \quad (u_1 + u_2) + u_3 = u_1 + (u_2 + u_3) \tag{2.40}$$

$$\text{multiplication} \quad u_1(u_2 u_3) = (u_1 u_2)u_3. \tag{2.41}$$

Commutativity

For all u_1 and u_2

$$\text{addition} \quad u_1 + u_2 = u_2 + u_1 \tag{2.42}$$

$$\text{multiplication} \quad u_1 u_2 = u_2 u_1. \tag{2.43}$$

Distributivity

For all u_1, u_2 and u_3

$$u_1(u_2 + u_3) = u_1u_2 + u_1u_3 \tag{2.44}$$

$$(u_1 + u_2)u_3 = u_1u_3 + u_2u_3. \tag{2.45}$$

This chapter on elementary algebra reminds us that a group of axioms are behind any algebra and later on we will discover that GA has its own set of axioms, some of which are familiar and others not so familiar. It is also important to know which aspects of an algebra are closed or not, so that we know what to expect when computing products. Finally, an algebra that is associative for products supports division, which is very useful for solving algebraic problems. The next chapter explores the algebra of complex numbers.

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