

## Game Theory: an Overview

### 2.1 Introduction

“Game theory is a branch of mathematics that is concerned with the actions of individuals who are conscious that their actions affect each other”. As such, game theory (hereafter GT) deals with interactive optimisation problems. While many economists in the past few centuries have worked on what can be considered game-theoretical (hereafter G-T) models, John von Neumann and Oskar Morgenstern are formally credited as the fathers of modern game theory. Their classic book *Theory of Games and Economic Behavior* [1] summarises the basic concepts existing at that time. GT has since enjoyed an explosion of developments, including the concept of equilibrium [2], games with imperfect information [3], cooperative games [4, 5], and auctions [6], to name just a few. Citing Shubik [7], “In the 50s ... game theory was looked upon as a curiosum not to be taken seriously by any behavioural scientist. By the late 1980s, game theory in the new industrial organisation has taken over: game theory has proved its success in many disciplines.”

GT is divided into two branches, called the *non-cooperative* and *cooperative* branches. The two branches of GT differ in how they formalise interdependence among the players. In the non-cooperative theory, a game is a detailed model of all the moves available to the players. By contrast, the cooperative theory abstracts away from this level of detail, and describes only the outcomes that result when the players come together in different combinations. Though standard, the terms non-cooperative and cooperative game theory are perhaps unfortunate. They might suggest that there is no place for cooperation in the former and no place for conflict, competition *etc.* in the latter. In fact, neither is the case. One part of the non-cooperative theory (the theory of repeated games) studies the possibility of cooperation in ongoing relationships. And the cooperative theory embodies not just cooperation among players, but also competition in a particularly strong, unfettered form. The non-cooperative theory might be better termed *procedural* game theory, the cooperative theory *combinatorial* game theory. This would indicate the real distinction between the two branches of the subject, namely that the first specifies

various actions that are available to the players while the second describes the outcomes that result when the players come together in different combinations.

The goal of this chapter is to give a brief overview about GT and, specifically, about G-T concepts and tools. Obviously, due to the need of short explanations, all proofs will be omitted, and we will only focus on the intuition behind the reported results.

## 2.2 Game Setup

To break the ground for next section on non-cooperative games, basic GT notation will be introduced: the reader can refer to Friedman [8] and Fudenberg and Tirole [9] if a more deep knowledge is required. A game in the normal form consists of: *players* (indexed by  $i = 1, 2, \dots, n$ ), a set of *strategies* (denoted by  $x_i, i = 1, 2, \dots, n$ ) available to each player and *payoffs* ( $\pi_i(x_1, x_2, \dots, x_n), i = 1, 2, \dots, n$ ) received by each player. Each strategy is defined on a set  $X_i, x_i \in X_i$ , so we call the Cartesian product  $X_1 \times X_2 \times \dots \times X_n$  the strategy space (typically the strategy space is  $R^n$ ). Each player may have a one-dimensional strategy or a multi-dimensional strategy. However, in simultaneous-move games each player's set of feasible strategies is independent from the strategies chosen by the other players, *i.e.*, the strategy choice of one player does not limit the feasible strategies of another player. A player's strategy can be thought of as the complete instruction for which actions to take in a game. For example, a player can give his or her strategy to a person that has absolutely no knowledge of the player's payoff or preferences and that person should be able to use the instructions contained in the strategy to choose the actions the player desires. Because each player's strategy is a complete guide to the actions that are to be taken, in the normal form the players choose their strategies simultaneously. Actions, which are adopted after strategies, are thus chosen and those actions correspond to the given strategies. The normal form can also be described as a static game, in contrast to the extensive form which is a dynamic game. If the strategy has no randomly determined choices, it is called a pure strategy; otherwise it is called a mixed strategy. There are situations in economics and marketing in which mixed strategies have been applied: *e.g.*, search models [10] and promotion models [11]. In a non-cooperative game the players are unable to make binding commitments regarding which strategy they will choose before they actually choose their strategies. In a cooperative game players are able to make these binding commitments. Hence, in a cooperative game players can make side-payments and form coalitions. The overview here reported starts with non-cooperative static games.

## 2.3 Non-cooperative Static Games

In non-cooperative static games the players choose strategies simultaneously and are thereafter committed to their chosen strategies. The solution concept for these games was formally introduced by John Nash [2] although some instances of using similar concepts date back to a couple of centuries. The concept is best described through best response functions.

**Definition 1.** *Given the  $n$ -player game, player  $i$ 's best response (function) to the strategies  $x_{-i}$  of the other players is the strategy  $x_i^*$  that maximizes player  $i$ 's payoff  $\pi_i(x_i, x_{-i})$ :  $x_i^*(x_{-i}) = \arg \max_{x_i} \pi_i(x_i, x_{-i})$ .*

If  $\pi_i$  is quasi-concave in  $x_i$  the best response is uniquely defined by the first-order conditions. Clearly, given the decisions of other players, the best response is the one that the best player  $i$  can hope for. Naturally, an outcome in which all players choose their best responses is a candidate for the non-cooperative solution. Such an outcome is called a Nash equilibrium (hereafter NE) of the game.

**Definition 2.** *An outcome  $(x_1^*, x_2^*, \dots, x_n^*)$  is a Nash equilibrium of the game if  $x_i^*$  is a best response to  $x_{-i}^*$  for all  $i = 1, 2, \dots, n$ .*

One way to think about an NE is as a fixed point of the best response mapping  $R^n \rightarrow R^n$ . Indeed, according to the definition, the NE must satisfy the system of equations  $\partial \pi_i / \partial x_i = 0$ , for all  $i$ . Recall that a fixed point  $x$  of mapping  $f(x)$ ,  $R^n \rightarrow R^n$  is any  $x$  such that  $f(x) = x$ . Define  $f_i(x_1, x_2, \dots, x_n) = \partial \pi_i / \partial x_i + x_i$ . By the definition of a fixed point,

$$f_i(x_1^*, x_2^*, \dots, x_n^*) = \partial \pi_i(x_1^*, \dots, x_n^*) / \partial x_i + x_i^* \rightarrow \partial \pi_i(x_1^*, \dots, x_n^*) / \partial x_i = 0, \text{ all } i.$$

Hence,  $x^*$  solves the first-order conditions if and only if it is a fixed point of mapping  $f(x)$  defined above. The concept of the NE is intuitively appealing. Indeed, it is a self-fulfilling prophecy. To explain, suppose a player is able to guess the strategies of the other players. A guess would be consistent with payoff maximisation (and therefore reasonable) only if it presumes that strategies are chosen to maximise every player's payoff given the chosen strategies. In other words, with any set of strategies that is not an NE there exists at least one player that is choosing a non payoff maximizing strategy. Moreover, the NE has a self-enforcing property: no player wants to unilaterally deviate from it since such behaviour would lead to lower payoffs. Hence the NE seems to be the necessary condition for the prediction of any rational behaviour by players.

Although attractive, numerous criticisms of the NE concept exist. Two particularly vexing problems are the non-existence of equilibrium and the multiplicity of equilibria. Without the existence of an equilibrium, little can be said regarding the likely outcome of the game. If there are multiple equilibria, then it is not clear which one will be the outcome. Indeed, it is possible the outcome is not even an equilibrium because the players may choose strategies from different equilibria. In some situations it is possible to rationalise away some equilibria via a refinement of the NE concept: e.g., trembling hand perfect equilibrium [12], sequential equilibrium [13] and proper equilibria [14]. In fact, it may even be

possible to use these refinements to the point that only a unique equilibrium remains.

An interesting feature of the NE concept is that the system optimal solution (a solution that maximises the total payoff to all players) need not be an NE. In fact, an NE may not even be on the Pareto frontier: the set of strategies such that each player can be made better off only if some other player is made worse off. A set of strategies is Pareto optimal if they are on the Pareto frontier; otherwise a set of strategies is Pareto inferior. Hence, an NE can be Pareto inferior. The Prisoner's Dilemma game is the classic example of this: only one pair of strategies is Pareto optimal (both "cooperate"), and the unique Nash equilibrium (both "defect") is Pareto inferior.

## 2.4 Existence of Equilibrium

An NE is a solution to a system of  $n$  equations (first-order conditions), so an equilibrium may not exist. Non-existence of an equilibrium is potentially a conceptual problem since in this case it is not clear what the outcome of the game will be. However, in many games an NE does exist and there are some reasonably simple ways to show that at least one NE exists. As already mentioned, an NE is a fixed point of the best response mapping. Hence fixed-point theorems can be used to establish the existence of an equilibrium. There are three key fixed point theorems, named after their creators: Brouwer, Kakutani and Tarski. (see [15] for details and references.) However, direct application of fixed-point theorems is somewhat inconvenient and hence generally not done (see [16] for existence proofs that are based on Brouwer's fixed-point theorem). Alternative methods, derived from these fixed-point theorems, have been developed. The simplest (and the most widely used) technique for demonstrating the existence of an NE is through verifying concavity of the players' payoffs, which implies continuous best response functions.

**Theorem 1.** [17]. *Suppose that for each player the strategy space is compact and convex and the payoff function is continuous and quasi-concave with respect to each player's own strategy. Then there exists at least one pure strategy NE in the game.*

If the game is symmetric (i.e., if the players' strategies and payoffs are identical), one would imagine that a symmetric solution should exist. This is indeed the case, as the next theorem ascertains.

**Theorem 2.** *Suppose that a game is symmetric and for each player the strategy space is compact and convex and the payoff function is continuous and quasi-concave with respect to each player's own strategy. Then there exists at least one symmetric pure strategy NE in the game.*

## 2.5 Multiple Equilibria

Many games are just not blessed with a unique equilibrium. The next best situation is to have a few equilibria. (The worst situation is either to have an infinite number of equilibria or no equilibrium at all.) The obvious problem with multiple equilibria is that the players may not know which equilibrium will prevail. Hence, it is entirely possible that a non-equilibrium outcome results because one player plays one equilibrium strategy while a second player chooses a strategy associated with another equilibrium. However, if a game is repeated, then it is possible that the players eventually find themselves in one particular equilibrium. Furthermore, that equilibrium may not be the most desirable one. If one does not want to acknowledge the possibility of multiple outcomes due to multiple equilibria, one could argue that one equilibrium is more reasonable than the others. For example, there may exist only one symmetric equilibrium and one may be willing to argue that a symmetric equilibrium is more focal than an asymmetric equilibrium. In addition, it is generally not too difficult to demonstrate the uniqueness of a symmetric equilibrium. If the players have one-dimensional strategies, then the system of  $n$  first-order conditions reduces to a single equation and one need only show that there is a unique solution to that equation to prove the symmetric equilibrium is unique. If the players have  $m$ -dimensional strategies,  $m > 1$ , then finding a symmetric equilibrium reduces to determining whether a system of  $m$  equations has a unique solution (easier than the original system, but still challenging).

## 2.6 Dynamic Games

The simplest possible dynamic game was introduced by Stackelberg [18]. In a Stackelberg duopoly model, player 1 chooses a strategy first (the Stackelberg leader) and then player 2 observes this decision and makes his own strategy choice (the Stackelberg follower). To find an equilibrium of a Stackelberg game (often called the Stackelberg equilibrium) we need to solve a dynamic two-period problem via backwards induction: first find the solution  $x_2^*(x_1)$  for the second player as a response to any decision made by the first player:

$$x_2^*(x_1) : \frac{\partial \pi_2(x_2, x_1)}{\partial x_2} = 0.$$

Next, find the solution for the first player anticipating the response by the second player:

$$\frac{d\pi_1(x_1, x_2^*(x_1))}{dx_1} = \frac{\partial \pi_1(x_1, x_2^*)}{\partial x_1} + \frac{\partial \pi_1(x_1, x_2)}{\partial x_2} \frac{\partial x_2^*}{\partial x_1} = 0.$$

Intuitively, the first player chooses the best possible point on the second player's best response function. Clearly, the first player can choose an NE, so the leader is always at least as well off as he would be in NE. Hence, if a player were allowed to choose between making moves simultaneously or being a leader in a game with complete information he would always prefer to be the leader.

## 2.7 Simultaneous Moves: Repeated and Stochastic Games

A different type of dynamic game arises when both players take actions in multiple periods. Two major types of this game exist: without and with time dependence. In the multi-period game without time dependence the exact same game is played over and over again (hence the term repeated games). The strategy for each player is now a sequence of actions taken in all periods. Consider one repeated game version of the competing newsvendor game in which the newsvendor chooses a stocking quantity at the start of each period, demand is realised and then leftover inventory is salvaged. In this case, there are no links between successive periods other than the players' memory about actions taken in all the previous periods. A fascinating feature of repeated games is that the set of equilibria is much larger than the set of equilibria in a static game and may include equilibria that are not possible in the static game. At first, one may assume that the equilibrium of the repeated game would be to play the same static NE strategy in each period. This is, indeed, an equilibrium but only one of many. Since in repeated games the players are able to condition their behaviour on the observed actions in the previous periods, they may employ so-called trigger strategies: the player will choose one strategy until the opponent changes his play, at which point the first player will change the strategy. This threat of reverting to a different strategy may even induce players to achieve the best possible outcome (*i.e.*, the centralised solution) which is called an implicit collusion. Many such threats are, however, non-credible in the sense that once a part of the game has been played, such a strategy is not an equilibrium anymore for the remainder of the game. To separate out credible threats from non-credible, Selten [19] introduced the subgame, a portion of the game (that is a game in itself) starting from some time period and a related notion of subgame-perfect equilibrium (this notion also applies in other types of games, not necessarily repeated), and equilibrium for every possible subgame (see Hall and Porteus [20] and van Mieghem and Dada [21] for solutions involving subgame-perfect equilibria in dynamic games).

## 2.8 Cooperative Games

The idea behind cooperative game theory has been expressed in this way: "Cooperative theory starts with a formalization of games that abstracts away altogether from procedures and concentrates, instead, on the possibilities for agreement. There are several reasons that explain why cooperative games came to be treated separately. One is that when one does build negotiation and enforcement procedures explicitly into the model, then the results of a non-cooperative analysis depend very strongly on the precise form of the procedures, on the order of making offers and counter-offers and so on. This may be appropriate in voting situations in which precise rules of parliamentary order prevail, where a good strategist can indeed carry the day. But problems of negotiation are usually more amorphous; it is difficult to pin down just what the procedures are. More fundamentally, there is a feeling that procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose

turn it is to speak. Detail distracts attention from essentials. Some things are seen better from a distance; the Roman camps around Metzada are indiscernible when one is in them, but easily visible from the top of the mountain” [22].

The subject of cooperative games first appeared in the seminal work of von Neumann and Morgenstern [1]. However, for a long time cooperative game theory did not enjoy as much attention in economics literature as non-cooperative GT. Cooperative GT involves a major shift in paradigms as compared to non-cooperative GT: the former focuses on the outcome of the game in terms of the value created through cooperation of (a subset of) players but does not specify the actions that each player will take, while the latter is more concerned with the specific actions of the players. Hence, cooperative GT allows us to model outcomes of complex business processes that otherwise might be too difficult to describe (*e.g.*, negotiations) and answers more general questions (*e.g.*, how well is the firm positioned against competition). In what follows, we will cover transferable utility cooperative games (including two solution concepts: the core of the game and the Shapley value).

## 2.9 *N*-Person Cooperative Games

Recall that the non-cooperative game consists of a set of players with their strategies and payoff functions. In contrast, in this case, although players are autonomous decision makers, they may have an interest in making binding agreements in order to have a bigger payoff at the end of the game. This agreement or partnership is the basic ingredient of the mathematical model of a cooperative game, and it is called a *coalition*. Mathematically, a coalition is a subset of the set of players  $N$  and we can denote it by  $S$ . To form a coalition  $S$ , it is required that agreements take place involving all players in the future coalition  $S$ . Whenever all players approve joining in a new entity called coalition, we can say that the new coalition is formed. Joining a coalition  $S$  also implies that there is no possible agreement between any member of  $S$  and any member not in  $S$  (set  $N \setminus S$ ). In short, the essential feature of a coalition is its foundational agreement that binds and reconstitutes the individuals as a coordinated entity. The *grand coalition* of all  $n$  players will be referred as coalition  $N$  (there is a total of  $2^n - 1$  possible coalitions); The *empty coalition* is a coalition made up of no members (the null set  $\emptyset$ ). A *coalition structure* is a means of describing how the players divide themselves into mutually exclusive coalitions. Any exhaustive partition of the players can be described by a set  $S \equiv \{S_1, S_2, \dots, S_m\}$  of the  $m$  coalitions that are formed. The set  $S$  is a partition of  $N$  that satisfies three conditions:

$$S_j \neq \emptyset, j = 1, \dots, m$$

$$S_i \cap S_j = \emptyset, \text{ for all } i \neq j, \text{ and } \cup S_j = N.$$

These conditions state that each player belongs to one and only one of the  $m$  non-empty coalitions within the coalition structure, and also specifies that none of

the players in any coalition  $m$  is connected to other players not in the coalition; finally, the mutually exclusive union of all coalitions  $m$  forms the grand coalition.

## 2.10 Characteristic Function and Imputation

von Neumann and Morgenstern [23] introduced the term *characteristic function* for the first time. More formally, we can define that:

**Definition 3.** For each subset  $S$  of  $N$ , the characteristic function  $\nu$  of a game gives the biggest amount  $\nu(S)$  that the members of  $S$  can be sure of receiving if they act together and form a coalition, without any help from other players not in  $S$ .

A restriction on this definition is that the value of the game to the empty coalition is zero, that is,  $\nu(\emptyset)$ . A further requirement that is generally made is called *superadditivity*. Superadditivity can be expressed as follows:

$$\nu(S \cup T) \geq \nu(S) + \nu(T) \text{ for all } S, T \subseteq N \text{ such that } S \cap T = \emptyset.$$

This means that the total payoff for the grand coalition is collectively rational, because the total payoff to the players is always as much as what they would get individually. This suggests the following definition.

**Definition 4.** A game in characteristic function form consists of a set of players, together with a function  $\nu$  defined for all subsets of  $N$ , such that  $\nu(S \cup T) \geq \nu(S) + \nu(T)$  whenever  $S$  and  $T$  are disjoint coalitions of players.

Games in which at least one possible coalition can increase the total payoff of its members are called *essential*, and those in which there is no coalition that improves the total payoff are called *inessential*. Mathematically, an *essential* game is one in which at least one of the superadditive inequalities  $\nu(S \cup T) \geq \nu(S) + \nu(T)$  is strict. The specific actions that players have to take to create this value are not specified: the characteristic function only defines the total value that can be created by utilising all players' resources. Hence, players are free to form any coalitions that are beneficial to them and no player is endowed with power of any sort. We will further restrict our attention to the transferable utility games in which the outcome of the game is described by real numbers  $\pi_i, i = 1, \dots, N$  showing how the total created value (or utility or pie)

$\pi(N) = \sum_{i=1}^N \pi_i$  was divided among players. Of course, one could offer a very simple rule prescribing division of the value; for example, a fixed fraction of the total pie can be allocated to each player. However, such rules are often too simplistic to be a good solution concept. A much more frequently used solution concept of the cooperative game theory is the *core* of the game. This concept can be compared to the NE for non-cooperative games:



**Definition 5.** The utility vector  $\pi_1, \dots, \pi_N$  is in the core (and will be called imputation) of the cooperative game if it satisfies  $\pi(N) = v(N)$ , group rationality, and  $x_i \geq v(\{i\})$ , individual rationality.

The core of the game, introduced by Gillies in 1953 [24], can be interpreted through the added-value principle. Define  $(N \setminus S)$  as a set of players excluding those in coalition  $S$  (coalition can include just one player). Then the contribution of a coalition  $S$  can be calculated as  $v(N) - v(N \setminus S)$ . Clearly, no coalition should be able to capture more than its contribution to the coalition (otherwise the remaining  $N \setminus S$  players would be better off without the coalition  $S$ ). Definition 5 clearly satisfies the added-value principle. Typically, when analysing a game, one has to calculate an added value from each player: if the value is zero, the player is not in the core of the game. If the core is non-empty, the added values of all players in the core comprise the total value that the players create. As is true for NE, the core of the game may not exist (*i.e.*, it may be empty) and the core is often not unique. When the core is non-empty, the cooperative demands of every coalition can be granted, but when the core is empty, at least one coalition will be dissatisfied. Shubik [7] noted that a game with a non-empty core is sociologically neutral, *i.e.* every cooperative demand by every coalition can be granted, and there is no need to resolve conflicts. On the other hand, in a coreless game, the coalitions are too strong for any mechanism to satisfy every coalitional demand. However, a core set with too many elements is not desirable, and it has little predictive power [25]. Imputations in the core, where they exist, have a certain stability, because no player or subset of players has any incentive to leave the grand coalition. But since many games have empty core, the core fails to provide a general solution for  $n$ -person games in characteristic form. von Neumann and Morgenstern [24] proposed a different solution concept more generally applicable than the core. That proposal is called the von Neumann Morgenstern solution or the stable set. The stable set is based on the concept of dominance, which is explained as follows. One imputation is said to dominate another if there is a subset of players who prefer the first to the second and can enforce it by forming a coalition.

## 2.11 Shapley Value

The concept of the core, though intuitively appealing, also possesses some unsatisfying properties. As we mentioned, the core might be empty or quite large or indeterministic. As it is desirable to have a unique NE in non-cooperative games, it is desirable to have a solution concept for cooperative games that results in a unique outcome and hence has a reasonable predictive power. Shapley [26] offered an axiomatic approach to the solution concept that is based on three rather intuitive axioms. First, the value of the player should not change due to permutations of players, *i.e.*, only the role of the player matters and not names or indices assigned to players. Second, if a player's added value to the coalition is zero then this player should not get any profit from the coalition, or in other words

only players generating added value should share the benefits. Finally, the third axiom requires additivity of payoffs: for any two characteristic functions  $v_1$  and  $v_2$  it must be that  $\pi(v_1 + v_2, N) = \pi(v_1, N) + \pi(v_2, N)$ .

The surprising result obtained by Shapley is that there is a unique equilibrium payoff (called the Shapley value) that satisfies all three axioms.

**Theorem 3.** *There is only one payoff function  $\pi$  that satisfies the three axioms. It is defined by the following expressions for  $\forall i \in N$  and all  $v$ :*

$$\pi_i(v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)).$$

The Shapley value assigns to each player his marginal contribution  $v(S \cup \{i\}) - v(S)$  when  $S$  is a random coalition of agents preceding  $i$  and the ordering is drawn randomly. To explain further, (see Myerson [14]), suppose players are picked randomly to enter into a coalition. There are  $|N|!$  different orderings for all players, and for each set  $S$  that does not contain player  $i$  there are  $|S|!(|N| - |S| - 1)!$  ways to order players so that all of the players in  $S$  are picked ahead of player  $i$ . If the orderings are equally likely, there is a probability of  $|S|!(|N| - |S| - 1)!/|N|!$  that when player  $i$  is picked he will find  $S$  players in the coalition already. The marginal contribution of adding player  $i$  to coalition  $S$  is  $v(S \cup \{i\}) - v(S)$ . Hence, the Shapley value is nothing more than a marginal (expected) contribution of adding player  $i$  to the coalition. Due to its uniqueness, the concept of the Shapley value has found numerous applications in economics and political sciences.

## 2.12 The Bargaining Game Model

To better understand the negotiational mechanism and theory, which will be shown more specifically in the next chapter, we here consider the former approach to this issue showing how to face the problem addressed by the bargaining in cooperative game theory. In order to do this, consider a group of two or more agents facing with a set of feasible outcomes, any one of which will be the result if it is accepted by unanimous agreement of all participants. In the event that no unanimous agreement is reached, a given disagreement outcome is the result. If the feasible outcomes are such that each participant can do better than the disagreement outcome, then there is an incentive to reach an agreement; however, so long as at least two of the participants differ over which outcome is most preferable, there is a need for bargaining and negotiation over which outcome should be agreed upon. Note that since unanimity is required, each participant has the ability to veto any outcome different from the disagreement outcome. To model this atomic negotiation process, we use the cooperative bargaining process initiated by Nash [27]. It is pertinent to mention that experimental bargaining theory indicates stronger empirical evidence of this bargaining theory than any others. Nash engaged in an axiomatic derivation of the bargaining solution. The solution refers

to the resulting payoff allocation that each of the participants unanimously agrees upon. The axiomatic approach requires that the resulting solution should possess a list of properties. The axioms do not reflect the rationale of the agents or the process in which an agreement is reached but only attempts to put restrictions on the resulting solution. Further, the axioms do not influence the properties of the feasible set. Before listing the axioms, we will now describe the construction of the feasible set of outcomes. Formally, Nash defined a two-person bargaining problem (which can be extended easily to more than two players) as consisting of a pair  $\langle F, d \rangle$  where  $F$  is a closed convex subset of  $R^2$ , and  $d = (d_1, d_2)$  is a vector in  $R^2$ .  $F$  is convex, closed, non-empty, and bounded. Here,  $F$ , the feasible set, represents the set of all feasible utility allocations and  $d$  represents the disagreement payoff allocation or the disagreement point. The disagreement point may capture the utility of the opportunity profit. Nash looked for a bargaining solution, *i.e.*, an outcome in the feasible set that satisfied a set of axioms. The axioms ensure that the solution is symmetric (identical players receive identical utility allocations), feasible (the sum of the allocations does not exceed the total pie), Pareto optimal (it is impossible for both players to improve their utilities over the bargaining solutions), the solution be preserved under linear transformations and be independent of “irrelevant” alternatives. Due to constraints on space, the reader can refer to Roth [28] for a very good description of the solution approach and a more detailed explanation of the axioms. The remarkable result due to Nash is that there is a bargaining solution that satisfies the above axioms and it is unique.

**Theorem 4 [27].** *There is a unique solution that satisfies all the “axioms”. This solution, for every two-person bargaining game  $\langle F, d \rangle$  is obtained by solving:*

$$\arg \max_{x=(x_1, x_2) \in F, x \geq d} (x_1 - d_1)(x_2 - d_2).$$

The axiomatic approach, though simple, can be used as a building block for much more complex bargaining problems. Even though the axiomatic approach is prescriptive, descriptive non-cooperative models of negotiation such as the Nash demand game [29] and the alternating offer game [30], reach similar conclusions as Nash bargaining. This somehow justifies the Nash bargaining approach to model negotiations. In our discussion, we have only provided a description of the bargaining problem and its solution between two players. However, this result can easily be generalised to any number of players simultaneously negotiating for allocations in a feasible set.

## 2.13 References

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