Chapter 2
Introductory Mathematical Concepts for Mining Equipment Reliability, Maintainability, and Safety Analysis

2.1 Introduction

As in other areas of engineering analysis, various mathematical concepts play a pivotal role in mining equipment reliability, maintainability, and safety analysis. Although the history of our current number symbols can be traced back to the stone columns erected by the Scythian emperor Asoka of India in 250 B.C., the application of mathematical concepts in engineering in general is relatively new [1].

In particular, probability plays a central role in the analysis of mining equipment reliability, maintainability, and safety problems; its history may only be traced back to the 16th-century writings of Girolamo Cardano (1501–1576) [1, 2]. In these writings, Cardano considered some interesting questions on probability. In the 17th century, the problem of dividing the winnings in a game of chance was solved independently and correctly by Blaise Pascal (1623–1662) and Pierre Fermat (1601–1665). In the 18th century, probability concepts were further developed and successfully applied to areas other than games of chance by Pierre Laplace (1749–1827) and Karl Gauss (1777–1855) [2, 3].

A detailed history of mathematics including probability is available in Refs. [1, 2]. This chapter presents various introductory mathematical concepts considered useful for performing mining equipment reliability, maintainability, and safety analysis [4, 5].

2.2 Range, Arithmetic Mean, Mean Deviation, and Standard Deviation

Many statistical measures are used to analyze reliability-, maintainability-, and safety-related data. This section presents a number of such measures considered useful for application in the area of mining equipment reliability, maintainability, and safety.

2.2.1 Range

This is a measure of dispersion or variation. More specifically, the range of a data set is the difference between the largest and the smallest values in the set.

Example 2.1

A mining facility reported the following monthly equipment failures over a period of 12 months:

$$40,\ 5,\ 10,\ 15,\ 20,\ 46,\ 50,\ 19,\ 25,\ 17,\ 35,\ \text{and}\ 16\ .$$

Find the range of the above data set values.

By examining the given data values, we conclude that the largest and the smallest values are 50 and 5, respectively. Thus, the range, R, of the given data set is expressed by

$$R = \text{Largest value} - \text{Smallest value} = 50 - 5 = 45\ .$$

Thus, the range of the given data set is 45.

2.2.2 Arithmetic Mean

The arithmetic mean is defined by

$$m = \frac{\sum_{j=1}^{n} m_j}{n}\ , \tag{2.1}$$

where

m is the mean value,
m_j is the data value j; for $j = 1, 2, 3, \ldots, n$,
n is the total number of data values.

Example 2.2

A mining equipment manufacturing organization inspected ten identical mining systems and found 5, 10, 3, 2, 7, 15, 20, 1, 9, and 8 defects in each system. Calculate the average number of defects per mining system (*i.e.*, arithmetic mean of the data set).

Inserting the specified data values into Eq. (2.1) we obtain

$$m = \frac{5+10+3+2+7+15+20+1+9+8}{10} = 8\,.$$

Thus, the average number of defects per mining system (*i.e.*, arithmetic mean of the data set) is 8.

2.2.3 Mean Deviation

This is a widely used measure of dispersion that indicates the degree to which given data tend to spread about a mean value. The mean deviation is defined by

$$m_{\mathrm{d}} = \frac{\sum_{j=1}^{n} \left| m_j - m \right|}{n}\,, \tag{2.2}$$

where

n	is the total number of data values,
m_j	is the data value j; for $j = 1, 2, 3, \ldots, n$,
m_d	is the mean deviation,
m	is the mean value,
$\left\| m_j - m \right\|$	is the absolute value of the deviation of m_j from m.

Example 2.3

Calculate the mean deviation of the data values given in Example 2.2. Using the given data and calculated values of Example 2.2 in Eq. (2.2) yields

$$\begin{aligned} m_{\mathrm{d}} &= \frac{|5-8|+|10-8|+|3-8|+|2-8|+|7-8|+|15-8|}{10} \\ &\quad + \frac{|20-8|+|1-8|+|9-8|+|8-8|}{10} \\ &= \frac{3+2+5+6+1+7+12+7+1+0}{10} \\ &= 4.4\,. \end{aligned}$$

Thus, the mean deviation of the given data values is 4.4.

2.2.4 Standard Deviation

This is another measure of dispersion of data in a data set about the mean value. The standard deviation is defined by [3]

$$\sigma = \left[\frac{\sum_{j=1}^{n}(m_j - m)^2}{n}\right]^{1/2}, \tag{2.3}$$

where

σ is the standard deviation.

The following three standard deviation properties are associated with the normal distribution presented later in the chapter:

- 99.73% of the all data values are included between $m-3\sigma$ and $m+3\sigma$.
- 95.45% of the all data values are included between $m-2\sigma$ and $m+2\sigma$.
- 68.27% of the all data values are included between $m-\sigma$ and $m+\sigma$.

Example 2.4

Calculate the standard deviation of the data values given in Example 2.2.

Using the given data and calculated value of Example 2.2 in Eq. (2.3) we get

$$\begin{aligned}\sigma &= \left[\frac{(5-8)^2+(10-8)^2+(3-8)^2+(2-8)^2+(7-8)^2+(15-8)^2}{10}\right.\\ &\qquad \left.+\frac{(20-8)^2+(1-8)^2+(9-8)^2+(8-8)^2}{10}\right]^{1/2}\\ &= \left[\frac{9+4+25+36+1+49+144+49+1+0}{10}\right]^{1/2} = 5.64\,.\end{aligned}$$

Thus, the standard deviation of the data values given in Example 2.2 is 5.64.

2.3 Boolean Algebra Laws and Probability Definition and Properties

Boolean algebra plays an important role in probability theory and is named after mathematician George Boole (1813–1864). Some of its laws are as follows [6, 7]:

$$C + D = D + C\,, \tag{2.4}$$

where

C	is a set or an event,
D	is a set or an event,
$+$	denotes the union of events or sets

$$C \cdot D = D \cdot C\,, \tag{2.5}$$

where

dot between C and D or D and C denotes the intersection of events or sets. Sometimes the intersection of events is written without the dot (*e.g.*, CD), but it still conveys exactly the same meaning.

$$DD = D\,, \tag{2.6}$$

$$C + C = C\,, \tag{2.7}$$

$$C(C + D) = C\,, \tag{2.8}$$

$$D + DC = D\,, \tag{2.9}$$

$$C(D + E) = CD + CE\,, \tag{2.10}$$

where

E is a set or an event.

$$C + 0 = C\,, \tag{2.11}$$

$$(C + D)(C + E) = C + DE\,. \tag{2.12}$$

Probability may be defined as the likelihood of occurrence of a given event. Mathematically, it is expressed as follows [8]:

$$P(Y) = \lim_{n \to \infty} \left[\frac{M}{n} \right]\,, \tag{2.13}$$

where

$P(Y)$	is the probability of occurrence of event Y,
M	is the number of times event Y occurs in the n repeated experiments.

Some probability properties are as follows [8]:

- The probability of occurrence of event, say X, is

$$0 \leq P(X) \leq 1\,. \tag{2.14}$$

- The probability of occurrence and nonoccurrence of an event, say X, is always

$$P(X) + P(\bar{X}) = 1\,, \tag{2.15}$$

where

$P(X)$ is the probability of occurrence of event X,
$P(\bar{X})$ is the probability of nonoccurrence of event X.

- The probability of an intersection of K independent events is given by

$$P(X_1 X_2 X_3 \ldots X_K) = P(X_1)P(X_2)P(X_3) \cdot \ldots \cdot P(X_K) \,, \tag{2.16}$$

where

$P(X_i)$ is the probability of occurrence of event X_i, for $i = 1, 2, 3, \ldots, K$.

- The probability of the union of K independent events is expressed by

$$P(X_1 + X_2 + \ldots + X_K) = 1 - \prod_{i=1}^{K} (1 - P(X_i)) \ . \tag{2.17}$$

- The probability of the union of K mutually exclusive events is

$$P(X_1 + X_2 + \ldots + X_K) = \sum_{i=1}^{K} P(X_i) \ . \tag{2.18}$$

Example 2.5

Assume that in Eqs. (2.17) and (2.18) we have $K = 2$, $P(X_1) = 0.05$, and $P(X_2) = 0.12$. Calculate the probability of the union of events X_1 and X_2 using Eqs. (2.17) and (2.18) and comment on the resulting probability values.

Inserting the given data into Eq. (2.17) we get

$$\begin{aligned} P(X_1 + X_2) &= P(X_1) + P(X_2) - P(X_1)P(X_2) \\ &= 0.05 + 0.12 - (0.05)(0.12) \\ &= 0.164 \,. \end{aligned}$$

Using the specified data values in Eq. (2.18) we get

$$\begin{aligned} P(X_1 + X_2) &= P(X_1) + P(X_2) \\ &= 0.05 + 0.12 \\ &= 0.17 \,. \end{aligned}$$

This means that the probability of the union of mutually exclusive events X_1 and X_2 is higher than the probability of the union of independent events X_1 and X_2.

2.4 Useful Mathematical Definitions

This section presents a number of mathematical definitions considered useful to perform reliability, maintainability, and safety studies in the mining industry [3, 8].

2.4.1 Cumulative Distribution Function

For continuous random variables, the cumulative distribution function is defined by

$$F(t) = \int_{-\infty}^{t} f(x)\,\mathrm{d}x\,, \tag{2.19}$$

where

$F(t)$	is the cumulative distribution function,
$f(x)$	is the probability density function of continuous random variable x,
t	is time.

For $t = \infty$, Eq. (2.19) yields

$$F(\infty) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1\,. \tag{2.20}$$

This simply means that the total area under the probability density curve is equal to unity.

Usually, in reliability work Eq. (2.19) is simply expressed as

$$F(t) = \int_{0}^{t} f(x)\,\mathrm{d}x\,. \tag{2.21}$$

2.4.2 Probability Density Function

For continuous random variables, the probability density function is defined by

$$f(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t}\,, \tag{2.22}$$

where

$f(t)$	is the probability density function (in reliability work, it is often called failure density function).

2.4.3 Reliability Function

The reliability function is defined by

$$R(t) = 1 - F(t) = 1 - \int_0^t f(x)\,\mathrm{d}x\,, \tag{2.23}$$

where

$R(t)$ is the reliability function or simply reliability at time t.

2.4.4 Expected Value

For continuous random variables, this is defined by

$$E(t) = m = \int_{-\infty}^{\infty} t f(t)\,\mathrm{d}t\,, \tag{2.24}$$

where

$E(t)$ is the expected value of the continuous random variable t.
m is the mean of the continuous random variable t. In reliability work, it is referred as mean time to failure.

2.4.5 Variance

The variance of a random variable t is defined by

$$\sigma^2(t) = E(t^2) - [E(t)]^2 \tag{2.25}$$

or

$$\sigma^2(t) = \int_0^{\infty} t^2 f(t)\,\mathrm{d}t - m^2\,, \tag{2.26}$$

where

$\sigma^2(t)$ is the variance of random variable t.

2.4.6 Laplace Transform

This is defined by

$$f(s) = \int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t\,, \tag{2.27}$$

where

s	is the Laplace transform variable,
t	is the time variable,
$f(s)$	is the Laplace transform of the function $f(t)$.

Laplace transforms of some commonly occurring functions in mining equipment reliability, maintainability, and safety studies are presented in Table 2.1 [9, 10].

2.4.7 Laplace Transform: Final Value Theorem

If the following limits exist, then the final-value theorem may be stated as

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[sf(s)\right]\,. \tag{2.28}$$

Table 2.1 Laplace transforms of some commonly occurring functions in mining equipment reliability, maintainability, and safety studies

$f(t)$	$f(s)$
$\mathrm{e}^{-\lambda t}$	$\dfrac{1}{(s+\lambda)}$
$t\,\mathrm{e}^{-\lambda t}$	$\dfrac{1}{(s+\lambda)^2}$
$t f(t)$	$-\dfrac{\mathrm{d}f(s)}{\mathrm{d}s}$
c (a constant)	$\dfrac{\mathrm{c}}{s}$
$\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}$	$sf(s)-f(0)$
t^m, for $m=1,2,3,\ldots$	$\dfrac{m!}{s^{m+1}}$
$\int_0^t f(t)\,\mathrm{d}t$	$\dfrac{f(s)}{s}$

2.5 Probability Distributions

Over the years, a large number of probability distributions have been developed to perform various types of statistical analysis [11]. This section presents some of these probability distributions considered useful for application in the area of mining equipment reliability, maintainability, and safety.

2.5.1 Binomial Distribution

This is a discrete random variable distribution and it was developed by Jakob Bernoulli (1654–1705) [1]. Thus, it is also called a Bernoulli distribution. The distribution is used in situations where one is concerned with the probabilities of outcome such as the total number of occurrences (*e.g.*, failures) in a sequence of given number of trials. More specifically, each of these trials has two possible outcomes (*e.g.*, success or failure), but the probability of each trial remains constant or unchanged.

The binomial probability density function, $f(x)$, is defined by

$$f(x) = \frac{m!}{x!(m-x)!} p^x q^{m-x} , \quad \text{for } x = 0, 1, 2, 3, \ldots, m , \tag{2.29}$$

where

- p is the single trial probability of occurrence (*e.g.*, success),
- q is the single trial probability of nonoccurrence (*e.g.*, failure),
- x is the number of nonoccurrences (*e.g.*, failures) in m trials.

The cumulative distribution function is given by [8, 11]

$$F(x) = \sum_{i=0}^{x} \frac{m!}{i!(m-i)!} p^i q^{m-i} , \tag{2.30}$$

where

- $F(x)$ is the probability of x or less nonoccurrences in m trials.

2.5.2 Exponential Distribution

This is a continuous random variable distribution and is widely used in reliability, maintainability, and safety work. Two principal reasons for its widespread use are as follows:

- Easy to handle in performing various types of reliability, maintainability, and safety analyses.

- Constant failure rates of many engineering items during their useful life periods, particularly electronic ones [12].

The distribution probability density function is expressed by

$$f(t) = \lambda\,\mathrm{e}^{-\lambda t}\,, \quad \text{for } \lambda > 0,\ t \geq 0\,, \tag{2.31}$$

where

$f(t)$	is the probability density function,
t	is time,
λ	is the distribution parameter. In reliability work, it is known as the constant failure rate.

Substituting Eq. (2.31) into Eq. (2.21) we get the following expression for the cumulative distribution function:

$$F(t) = \int_0^t \lambda\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x = 1 - \mathrm{e}^{-\lambda t}\,. \tag{2.32}$$

2.5.3 Rayleigh Distribution

This continuous random variable distribution is named after its originator, John Rayleigh (1842–1919) [1]. The distribution probability density function is defined by

$$f(t) = \frac{2}{\alpha^2} t\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^2}\,, \quad \text{for } \alpha > 0,\ t \geq 0\,, \tag{2.33}$$

where

α is the distribution parameter.

Using Eq. (2.33) in Eq. (2.21) yields the following cumulative distribution function:

$$F(t) = \int_0^t \frac{2}{\alpha^2} x\,\mathrm{e}^{-\left(\frac{x}{\alpha}\right)^2}\,\mathrm{d}x = 1 - \mathrm{e}^{-\left(\frac{t}{\alpha}\right)^2}\,. \tag{2.34}$$

2.5.4 Weibull Distribution

This continuous random variable distribution is named after W. Weibull, a Swedish mechanical engineering professor, and it can be used to represent many different physical phenomena [13]. The distribution probability density function is expressed by

$$f(t) = \frac{\theta}{\alpha^\theta} t^{\theta-1} \mathrm{e}^{-\left(\frac{t}{\alpha}\right)^\theta}, \quad \text{for } t \geq 0,\ \alpha > 0,\ \theta > 0\,, \tag{2.35}$$

where

θ and α are the distribution shape and scale parameters, respectively.

By substituting Eq. (2.35) into Eq. (2.21), we get the following equation for the cumulative distribution function:

$$F(t) = \int_0^t \frac{\theta}{\alpha^\theta} x^{\theta-1} \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^\theta} \mathrm{d}x = 1 - \mathrm{e}^{-\left(\frac{t}{\alpha}\right)^\theta} \tag{2.36}$$

It is to be noted that both exponential and Rayleigh distributions are the special cases of Weibull distribution for $\theta = 1$ and $\theta = 2$, respectively.

2.5.5 Normal Distribution

This is one of the most widely used continuous random variable distributions and is also known as the Gaussian distribution after Carl Friedrich Gauss (1777 – 1855). However, the distribution was actually discovered by De Moivre in 1733 [11].

The probability density function of the distribution is defined by

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right], \quad \text{for } -\infty < t < +\infty\,, \tag{2.37}$$

where

μ is the distribution mean,
σ is the distribution standard deviation.

Using Eq. (2.37) in Eq. (2.21) yields the following equation for the cumulative distribution function:

$$F(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^t \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}x\,. \tag{2.38}$$

2.5.6 Lognormal Distribution

This is another continuous random variable distribution and is often used to represent failed equipment repair times. The distribution probability density function is expressed by

$$f(t) = \frac{1}{t\theta\sqrt{2\pi}} \exp\left[-\frac{(lnt-m)^2}{2\theta^2}\right], \quad \text{for } t \geq 0\,, \tag{2.39}$$

where

m and θ are the distribution parameters.

Using Eq. (2.39) in Eq. (2.21) yields the following cumulative distribution function:

$$F(t) = \frac{1}{\theta\sqrt{2\pi}} \int_{-\infty}^{t} \frac{1}{x} \exp\left[-\frac{(lnx-m)^2}{2\theta^2}\right] \mathrm{d}x\,. \tag{2.40}$$

2.6 Solving Differential Equations Using Laplace Transforms

Sometimes mining equipment reliability, maintainability, and safety studies may require finding solutions to a system of linear first-order differential equations. Under such circumstances, the application of Laplace transforms has proven to be a very effective approach. The following example demonstrates the application of Laplace transforms to finding solutions to a set of linear first-order differential equations describing a mining system:

Example 2.6

Assume that an engineering system used in mines can be, at any time t, in either of the three distinct states: working normally, failed in open mode, or failed in short mode. The following three linear first-order differential equations describe the mining system:

$$\frac{\mathrm{d}P_w(t)}{\mathrm{d}t} + (\lambda_{om} + \lambda_{sm}) P_w(t) = 0\,, \tag{2.41}$$

$$\frac{\mathrm{d}P_{om}(t)}{\mathrm{d}t} - \lambda_{om} P_\omega(t) = 0\,, \tag{2.42}$$

$$\frac{\mathrm{d}P_{sm}(t)}{\mathrm{d}t} - \lambda_{sm} P_\omega(t) = 0\,, \tag{2.43}$$

where

$P_j(t)$ is the probability that the mining system is in state j at time t, $j = \mathrm{w}$ (working normally), $j = \mathrm{om}$ (failed in open mode), and $j = \mathrm{sm}$ (failed in short mode),
λ_{sm} is the mining system constant short mode failure rate,
λ_{om} is the mining system constant open mode failure rate.

At time $t = 0$, $P_w(0) = 1$, $P_{om}(0) = 0$, and $P_{sm}(0) = 0$.

Find solutions to differential Eqs. (2.41)–(2.43) using Laplace transforms.
Using Table 2.1, Eqs. (2.41)–(2.43) and the given initial conditions we get

$$sP_{\mathrm{w}}(s) - 1 + (\lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}})\, P_{\mathrm{w}}(s) = 0\,, \tag{2.44}$$

$$sP_{\mathrm{om}}(s) - \lambda_{\mathrm{om}} P_{\mathrm{w}}(s) = 0\,, \tag{2.45}$$

$$sP_{\mathrm{sm}}(s) - \lambda_{\mathrm{sm}} P_{\mathrm{w}}(s) = 0\,. \tag{2.46}$$

Solving Eqs. (2.44)–(2.46) we obtain

$$P_{\mathrm{w}}(s) = \frac{1}{s + \lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}}}\,, \tag{2.47}$$

$$P_{\mathrm{om}}(s) = \frac{\lambda_{\mathrm{om}}}{s\,(s + \lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}})}\,, \tag{2.48}$$

$$P_{\mathrm{sm}}(s) = \frac{\lambda_{\mathrm{sm}}}{s\,(s + \lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}})}\,. \tag{2.49}$$

Taking the inverse Laplace transforms of Eqs. (2.47)–(2.49) we get

$$P_{\mathrm{w}}(t) = \mathrm{e}^{-(\lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}})t}\,, \tag{2.50}$$

$$P_{\mathrm{om}}(t) = \frac{\lambda_{\mathrm{om}}}{\lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}}} \left[1 - \mathrm{e}^{-(\lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}})t}\right], \tag{2.51}$$

and

$$P_{\mathrm{sm}}(t) = \frac{\lambda_{\mathrm{sm}}}{\lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}}} \left[1 - \mathrm{e}^{-(\lambda_{\mathrm{om}} + \lambda_{\mathrm{sm}})t}\right]. \tag{2.52}$$

Thus, Eqs. (2.50)–(2.52) are the solutions to differential Eqs. (2.41)–(2.43).

2.7 Problems

1. A mining equipment manufacturing company inspected eight identical mining systems and found 10, 11, 2, 20, 6, 9, 4, and 5 defects in each system. Calculate the average number of defects per mining system (*i.e.*, arithmetic mean of the data set).
2. Calculate the mean deviation of the data values given in the above problem (*i.e.*, Problem 1).
3. Discuss the history of probability.
4. Define standard deviation.
5. Prove Eq. (2.12).
6. Discuss five important properties of probability.
7. Mathematically, define probability.
8. Determine expected value of Eq. (2.31).

9. Write down probability density functions for the following statistical distributions:
   - Normal distribution
   - Weibull distribution
10. Obtain the Laplace transform for the following function:

$$f(t) = t\,e^{-\lambda t}\,, \tag{2.53}$$

where

t	is the time variable,
λ	is a constant.

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