2

Redundant Models

It would be necessary to incorporate redundancy into the design of systems to meet the demand for high reliability. We discuss analytically the number of units of redundant systems and their maintenance times mainly based our original work. As some examples of applications, we present typical redundant models in various fields and analyze them from reliability viewpoints. From such results, we can learn practically how to make the design of redundancy and when to do some maintenance. It would be useful for us to acquire redundant techniques in practical situations in other fields.

High system reliability can be achieved by redundancy and maintenance. The most typical model is a standard parallel system that consists of n units in parallel. It was shown by graph that the system can operate for a specified mean time by either changing the replacement time or increasing the number of units [2]. The reliabilities of many redundant systems were computed and summarized [5]. A variety of redundant systems with multiple failure modes and their optimization problems were discussed in detail [27]. Reliabilities of parallel and parallel-series systems with dependent failures of components were derived [28].

First, we summarize our research results for a parallel system with n units in Sect. 2.1 [29, 30]: The optimum number of units and times of two replacement policies are derived analytically. These results are easily extended to a k-out-of-n system [31]. Furthermore, we consider two replacement models where the system is replaced at periodic times if the total number of failed units exceeds a threshold level [29].

Next, in Sect. 2.2, we take up series-parallel and parallel-series systems and analyze theoretically the stochastic behavior of two systems with the same number of series and parallel units. An optimum number of units for a series-parallel system with complexity ([32], see Chap. 9) is also derived.

As one example of analyzing redundant systems, we consider three redundant systems in Sect. 2.3; (1) a one-unit system with n-hold mean time, (2) an n-unit parallel system, and (3) an n-unit standby system. Various kinds of reliability measures of the three systems are computed and compared.

The notion and techniques of redundancy are indispensable in a communication system [12]. Some data transmission models and their optimum schemes were formulated and discussed analytically and numerically [33]. Section 2.4 adopts three schemes of ARQ (Automatic Repeat Request) as the data transmission and discusses which model is the best among the three schemes.

Many stochastic redundant models exist in the general public. Finally, as practical applications, we give three redundant models in Sect. 2.5 [30]; (1) transmission with redundant bits, (2) redundant networks, and (3) redundant copies. The optimum designs of redundancy for the three models are discussed analytically. Redundant techniques of computer systems for improving reliability and achieving fault tolerance have been classified in Sect. 1.2.

2.1 Parallel Systems

System reliabilities can be improved by redundant units. This section summarizes the known results for parallel redundant systems [29,30,34]: First, we derive an optimum number of units for a parallel system with n units. It is shown that similar discussions can be had about a k-out-of-n system. Next, we discuss two replacement policies where the system is replaced at time T. Furthermore, we take up two replacement policies where the system is replaced at periodic times if the total number of failed units exceeds a threshold level.

2.1.1 Number of Units and Replacement Time

(1) Number of Units

Consider a parallel redundant system that consists of n identical units and fails when all units have failed, *i.e.*, when at least one of n units is operating, the system is also operating. Each unit has an independent and identical failure distribution $F(t)$ with a finite mean μ_1. It is assumed that the failure rate is $h(t) \equiv f(t)/\overline{F}(t)$, where $\overline{F}(t) \equiv 1 - F(t)$ and $f(t)$ is a density function of $F(t)$ *i.e.*, $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$. Because the system with n units has a failure distribution $F(t)^n$, its mean time to failure is

$$\mu_n = \int_0^\infty [1 - F(t)^n]\,\mathrm{d}t \qquad (n = 1, 2, \dots), \tag{2.1}$$

that increases strictly with n from μ_1 to ∞. Therefore, the expected cost rate is [34,35]

$$C(n) = \frac{nc_1 + c_R}{\mu_n} \qquad (n = 1, 2, \dots), \tag{2.2}$$

where c_1 = acquisition cost for one unit and c_R = replacement cost for a failed system.

We find an optimum number n^* that minimizes $C(n)$. Forming the inequality $C(n+1) - C(n) \geq 0$,

$$\frac{\mu_n}{\mu_{n+1}-\mu_n} - n \geq \frac{c_R}{c_1} \qquad (n = 1, 2, \dots), \tag{2.3}$$

whose left-hand side increases strictly to ∞ because

$$\frac{\mu_n}{\mu_{n+1}-\mu_n} - n \geq \frac{\mu_1}{\mu_{n+1}-\mu_n} - 1 \qquad \text{for } n \geq 1.$$

Thus, there exists a finite and unique minimum n^* $(1 \leq n^* < \infty)$ that satisfies (2.3) because $\mu_{n+1} - \mu_n$ goes to zero as $n \to \infty$.

In the particular case of $F(t) = 1 - \mathrm{e}^{-\lambda t}$,

$$\mu_n = \int_0^\infty [1 - (1 - \mathrm{e}^{-\lambda t})^n]\,\mathrm{d}t = \frac{1}{\lambda}\sum_{j=1}^{n}\frac{1}{j}, \tag{2.4}$$

that is given approximately by

$$\mu_n \approx \frac{1}{\lambda}(C + \log n) \qquad \text{for large } n,$$

where C is Euler's constant and $C = 0.577215\ldots$. It was also shown [2, p. 65] that when $F(t)$ is IFR (Increasing Failure Rate), *i.e.*, $h(t)$ increases, a parallel system has a IFR property and

$$\mu_1 \leq \mu_n \leq \mu_1 \sum_{j=1}^{n}\frac{1}{j}.$$

In addition, (2.3) becomes

$$(n+1)\sum_{j=1}^{n}\frac{1}{j} - n = \sum_{j=1}^{n}\frac{n+1}{j+1} \geq \frac{c_R}{c_1} \qquad (n = 1, 2, \dots), \tag{2.5}$$

whose left-hand side increases strictly from 1 to ∞. Note that an optimum number n^* does not depend on the mean failure time μ_1 of each unit. Because

$$\sum_{j=1}^{n}\frac{n+1}{j+1} - n = \sum_{j=1}^{n}\frac{n-j}{j+1} \geq 0,$$

if $n - 1 < c_R/c_1 \leq n$, then $n^* \leq n$. Conversely, because

$$\sum_{j=1}^{n}\frac{n+1}{j+1} \leq \sum_{j=1}^{n} j = \frac{n(n+1)}{2},$$

if $\sum_{j=1}^{n} j < c_R/c_1 \leq \sum_{j=1}^{n+1} j$, then $n^* \geq n$.

(2) Replacement Times

Suppose that a parallel system is replaced at time T $(0 < T \leq \infty)$ or at failure, whichever occurs first. Then, the mean time to replacement is

$$\mu_n(T) \equiv \int_0^T [1 - F(t)^n] \, \mathrm{d}t, \tag{2.6}$$

where note that $\mu_n(\infty) = \mu_n$ in (2.1). Thus, the expected cost rate is [29]

$$C_1(T) = \frac{nc_1 + c_R F(T)^n}{\mu_n(T)}. \tag{2.7}$$

When $n = 1$, $C_1(T)$ agrees with the expected cost rate for the standard age replacement [1, p. 72].

We find an optimum replacement time T_1^* that minimizes $C_1(T)$ for a given n $(n \geq 2)$. It is assumed that the failure rate $h(t)$ increases. Then, differentiating $C_1(T)$ with respect to T and setting it equal to zero,

$$H(T)\mu_n(T) - F(T)^n = \frac{nc_1}{c_R}, \tag{2.8}$$

where

$$H(t) \equiv \frac{nh(t)[F(t)^{n-1} - F(t)^n]}{1 - F(t)^n}.$$

It is easily proved that

$$\frac{1 - F(t)^{n-1}}{1 - F(t)^n} = \frac{\sum_{j=0}^{n-2} F(t)^j}{\sum_{j=0}^{n-1} F(t)^j}$$

decreases strictly with t from 1 to $(n-1)/n$ for $n \geq 2$, and

$$\lim_{t \to \infty} \frac{n[F(t)^{n-1} - F(t)^n]}{1 - F(t)^n} = 1.$$

Thus, $H(t)$ increases strictly with t to $h(\infty)$ for $n \geq 2$. Therefore, denoting the left-hand side of (2.8) by $Q_1(T)$, it follows that $\lim_{T \to 0} Q_1(T) = 0$,

$$\frac{\mathrm{d}Q_1(T)}{\mathrm{d}T} = H'(T)\mu_n(T) > 0, \qquad \lim_{T \to \infty} Q_1(T) = \mu_n h(\infty) - 1,$$

where μ_n is given in (2.1).

Therefore, we have the following optimum policy:

(i) If $\mu_n h(\infty) > (nc_1 + c_R)/c_R$, then there exists a finite and unique T_1^* $(0 < T_1^* < \infty)$ that satisfies (2.8), and the resulting cost rate is

$$C_1(T_1^*) = c_R H(T_1^*). \tag{2.9}$$

(ii) If $\mu_n h(\infty) \leq (nc_1 + c_R)/c_R$, then $T_1^* = \infty$, *i.e.*, the system is replaced only at failure, and the expected cost rate $C_1(\infty)$ is given in (2.2).

Next, suppose that a parallel system is replaced only at time T, *i.e.*, the system remains in a failed state for the time interval from a system failure to its detection at time T. Then, the expected cost rate is [29]

$$C_2(T) = \frac{nc_1 + c_D \int_0^T F(t)^n \, \mathrm{d}t}{T}, \tag{2.10}$$

where c_D = downtime cost per unit of time from system failure to replacement. When $n = 1$, $C_2(T)$ agrees with the expected cost rate for the model with no replacement at failure [1, p. 120]. Differentiating $C_2(T)$ with respect to T and setting it equal to zero,

$$\int_0^T [F(T)^n - F(t)^n] \, \mathrm{d}t = \frac{nc_1}{c_D}. \tag{2.11}$$

The left-hand side of (2.11) increases strictly from 0 to μ_n.

Therefore, the optimum policy is as follows:

(iii) If $\mu_n > nc_1/c_D$, then there exists a finite and unique T_2^* $(0 < T_2^* < \infty)$ that satisfies (2.11), and the resulting cost rate is

$$C_2(T_2^*) = c_D F(T_2^*)^n. \tag{2.12}$$

(iv) If $\mu_n \leq nc_1/c_D$, then $T_2^* = \infty$.

Example 2.1. Suppose that the failure time of each unit is exponential, *i.e.*, $F(t) = 1 - \mathrm{e}^{-\lambda t}$. Then, we have the respective optimum replacement times T_1^* and T_2^* that minimize $C_1(T)$ in (2.7) and $C_2(T)$ in (2.10) as follows: From the optimum policies (i) and (ii), if $\sum_{j=2}^n 1/j > nc_1/c_R$ for $n \geq 2$, then T_1^* is given by a unique solution of the equation

$$\frac{n\mathrm{e}^{-\lambda T}(1 - \mathrm{e}^{-\lambda T})^{n-1}}{1 - (1 - \mathrm{e}^{-\lambda T})^n} \sum_{j=1}^n \frac{1}{j}(1 - \mathrm{e}^{-\lambda T})^j - (1 - \mathrm{e}^{-\lambda T})^n = \frac{nc_1}{c_R}, \tag{2.13}$$

and the resulting cost rate is

$$C_1(T_1^*) = \frac{c_R n\lambda \mathrm{e}^{-\lambda T_1^*}(1 - \mathrm{e}^{-\lambda T_1^*})^{n-1}}{1 - (1 - \mathrm{e}^{-\lambda T_1^*})^n}. \tag{2.14}$$

From (iii) and (iv), if $\sum_{j=1}^n 1/j > n\lambda c_1/c_D$, then T_2^* is a unique solution of the equation

$$\frac{1}{\lambda}\sum_{j=1}^n \frac{1}{j}(1 - \mathrm{e}^{-\lambda T})^j - T[1 - (1 - \mathrm{e}^{-\lambda T})^n] = \frac{nc_1}{c_D}, \tag{2.15}$$

and the resulting cost rate is

$$C_2(T_2^*) = c_D(1 - \mathrm{e}^{-\lambda T_2^*})^n. \tag{2.16}$$

Tables 2.1 and 2.2 present the optimum times T_1^* and T_2^* for c_R/c_1 and c_1/c_D when $1/\lambda = 50$ and $n = 2$, 3, 5, 15, and 20 (see Example 2.2). ■

(3) k-out-of-n System

Suppose that the system consists of a k-out-of-n system $(1 \leq k \leq n)$, *i.e.*, it is operating if and only if at least k units of n units are operating [2]. The reliability characteristics of such a system were investigated [36,37]. The number of units that should be on-line to assure that a minimum of k units will be available to complete an assignment for mass transit and computer systems was determined [38]. A k-out-of-n code is also used as a totally self-checking checker for error detecting codes [12]. A good survey of multi-state and consecutive k-out-of-n systems was done [31,39].

The mean time to system failure is [2]

$$\mu_{n,k} = \sum_{j=k}^{n} \binom{n}{j} \int_0^\infty [\overline{F}(t)]^j [F(t)]^{n-j}\,\mathrm{d}t, \tag{2.17}$$

and the expected cost rate is

$$C(n;k) = \frac{nc_1 + c_R}{\mu_{n,k}} \qquad (n = k, k+1, \dots). \tag{2.18}$$

When $F(t) = 1 - \mathrm{e}^{-\lambda t}$, the expected cost rate is simplified as

$$C(n;k) = \frac{nc_1 + c_R}{(1/\lambda)\sum_{j=k}^{n}(1/j)}, \tag{2.19}$$

and an optimum number that minimizes $C(n,k)$ is obtained by a finite and unique minimum n^* $(k \leq n^* < \infty)$ such that

$$(n+1)\sum_{j=k}^{n}\frac{1}{j} - n \geq \frac{c_R}{c_1} \qquad (n = k, k+1, \dots). \tag{2.20}$$

It is natural that n^* increases with k.

Similarly, the expected cost rates in (2.7) and (2.10) are easily written as, respectively,

$$C_1(T,k) = \frac{nc_1 + c_R\sum_{j=0}^{k-1}\binom{n}{j}[\overline{F}(T)]^j[F(T)]^{n-j}}{\sum_{j=k}^{n}\binom{n}{j}\int_0^T[\overline{F}(t)]^j[F(t)]^{n-j}\mathrm{d}t}, \tag{2.21}$$

$$C_2(T,k) = \frac{nc_1 + c_D\sum_{j=0}^{k-1}\binom{n}{j}\int_0^T[\overline{F}(t)]^j[F(t)^{n-j}]\mathrm{d}t}{T}, \tag{2.22}$$

where all results agree with those of **(1)** and **(2)** when $k = 1$.

In particular, when $k = n - 1$, an $(n-1)$-out-of-n $(n \geq 3)$ system can be identified with a fault tolerant system with a single bit error correction and referred to a fail-safe design in reliability theory [2, p. 216]. In addition, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, the expected cost rate in (2.21) is rewritten as

$$C_1(T) = \frac{nc_1 + c_R[1 - n\mathrm{e}^{-(n-1)\lambda T} + (n-1)\mathrm{e}^{-n\lambda T}]}{(1/\lambda)\left\{[n/(n-1)][1 - \mathrm{e}^{-(n-1)\lambda T}] - [(n-1)/n][1 - \mathrm{e}^{-n\lambda T}]\right\}}. \tag{2.23}$$

Differentiating $C_1(T)$ with respect to T and setting it equal to zero,

$$\frac{n(1 - \mathrm{e}^{-\lambda T}) - (1 - \mathrm{e}^{-n\lambda T})}{(n-1)(1 - \mathrm{e}^{-\lambda T}) + 1} = \frac{nc_1}{c_R}. \tag{2.24}$$

The left-hand side of (2.24) increases strictly from 0 to $(n-1)/n$. Thus, if $(n-1)/n > nc_1/c_R$, then there exists a finite and unique T_1^* $(0 < T_1^* < \infty)$ that satisfies (2.24), and the resulting cost rate is

$$C_1(T_1^*) = \frac{c_R \lambda n(n-1)(1 - \mathrm{e}^{-\lambda T_1^*})}{(n-1)(1 - \mathrm{e}^{-\lambda T_1^*}) + 1}. \tag{2.25}$$

Similarly, the expected cost rate in (2.22) is

$$C_2(T) = \frac{nc_1 - (c_D/\lambda)\left\{[n/(n-1)][1 - \mathrm{e}^{-(n-1)\lambda T}] - [(n-1)/n][1 - \mathrm{e}^{-n\lambda T}]\right\}}{T} + c_D. \tag{2.26}$$

Differentiating $C_2(T)$ with respect to T and setting it equal to zero,

$$\frac{n}{n-1}\left[1 - \mathrm{e}^{-(n-1)\lambda T}\right] - \frac{n-1}{n}\left[1 - \mathrm{e}^{-n\lambda T}\right] - \lambda T\left[n\mathrm{e}^{-(n-1)\lambda T} - (n-1)\mathrm{e}^{-n\lambda T}\right] = \frac{n\lambda c_1}{c_D}. \tag{2.27}$$

The left-hand side of (2.27) increases strictly from 0 to $1/n + 1/(n-1)$. Thus, if

$$\frac{1}{n} + \frac{1}{n-1} > \frac{n\lambda c_1}{c_D},$$

then there exists a finite and unique T_2^* that satisfies (2.27), and the resulting cost rate is

$$C_2(T_2^*) = c_D[1 - n\mathrm{e}^{-(n-1)\lambda T_2^*} + (n-1)\mathrm{e}^{-n\lambda T_2^*}]. \tag{2.28}$$

2.1.2 Replacement Number of Failed Units

Suppose that the replacement may be done at planned time jT $(j = 1, 2, \dots)$, where T means a day, a week, a month, and so on. Similar preventive maintenance models were considered [1, p. 54, 40]. If the total number of failed units in a parallel system with n units exceeds N $(1 \le N \le n-1)$ until time $(j+1)T$, then the system is replaced before failure at time $(j+1)T$ $(j = 0,1,2,\dots)$.

The system is replaced at failure or at time $(j+1)T$ when the total number of failed units has exceeded N, whichever occurs first. Then, the probability that the system is replaced at failure is [29]

$$\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i[F((j+1)T)-F(jT)]^{n-i}, \tag{2.29}$$

and the probability that it is replaced before failure, *i.e.*, when $N, N+1,\dots,n-1$ units have failed until $(j+1)T$, is

$$\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i\sum_{k=N-i}^{n-i-1}\binom{n-i}{k}[F((j+1)T)-F(jT)]^k[\overline{F}((j+1)T)]^{n-i-k}, \tag{2.30}$$

where note that (2.29) + (2.30) = 1. Thus, the mean time to replacement is

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i\int_{jT}^{(j+1)T} t\,\mathrm{d}\{[F(t)-F(jT)]^{n-i}\} \\
&\quad+\sum_{j=0}^{\infty}[(j+1)T]\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i \\
&\quad\times\sum_{k=N-i}^{n-i-1}\binom{n-i}{k}[F((j+1)T)-F(jT)]^k[\overline{F}((j+1)T)]^{n-i-k} \\
&=\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i\int_{jT}^{(j+1)T}\{[\overline{F}(jT)]^{n-i}-[F(t)-F(jT)]^{n-i}\}\,\mathrm{d}t.
\end{aligned} \tag{2.31}$$

Therefore, the expected cost rate is, from (2.7),

$$C_1(N)=\frac{nc_1+c_R\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i[F((j+1)T)-F(jT)]^{n-i}}{\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i\int_{jT}^{(j+1)T}\{[\overline{F}(jT)]^{n-i}-[F(t)-F(jT)]^{n-i}\}\mathrm{d}t} \quad (N=1,2,\dots,n). \tag{2.32}$$

When $N = n$, the system is replaced only at failure, and $C_1(n)$ agrees with $C(n)$ in (2.2).

Next, suppose that the system is replaced only at time $(j+1)T$ $(j = 0,1,2,\dots)$ when the total number of failed units has exceeded N until time $(j+1)T$. Then, the mean time from system failure to replacement is

$$\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i\int_{jT}^{(j+1)T}[(j+1)T-t]\,\mathrm{d}\big\{[F(t)-F(jT)]^{n-i}\big\}$$
$$=\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i\int_{jT}^{(j+1)T}[F(t)-F(jT)]^{n-i}\,\mathrm{d}t, \tag{2.33}$$

and the mean time to replacement is

$$\sum_{j=0}^{\infty}[(j+1)T]\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i$$
$$\times\sum_{k=N-i}^{n-i}\binom{n-i}{k}[F((j+1)T)-F(jT)]^k[\overline{F}((j+1)T)]^{n-i-k}$$
$$=T\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i[\overline{F}(jT)]^{n-i}, \tag{2.34}$$

where note that (2.31) + (2.33) = (2.34).

Therefore, the expected cost rate is, from (2.10),

$$C_2(N)=\frac{nc_1+c_D\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i\int_{jT}^{(j+1)T}[F(t)-F(jT)]^{n-i}\mathrm{d}t}{T\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}[F(jT)]^i[\overline{F}(jT)]^{n-i}}$$
$$(N=1,2,\ldots,n). \tag{2.35}$$

Example 2.2. We compute the respective optimum numbers N_1^* and N_2^* $(1\le N_i^*\le n)$ that minimize $C_1(N)$ and $C_2(N)$ for a fixed $T>0$ when $F(t)=1-\mathrm{e}^{-\lambda t}$. In this case, the expected cost rate in (2.32) is rewritten as

$$C_1(N)=\frac{nc_1+c_R\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i[\mathrm{e}^{-j\lambda T}-\mathrm{e}^{-(j+1)\lambda T}]^{n-i}}{(1/\lambda)\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i(\mathrm{e}^{-j\lambda T})^{n-i}\sum_{k=1}^{n-i}[(1-\mathrm{e}^{-\lambda T})^k/k]}$$
$$(N=1,2,\ldots,n). \tag{2.36}$$

Forming the inequality $C_1(N+1)-C_1(N)\ge 0$,

$$L_1(N)\ge\frac{nc_1}{c_R}\qquad(N=1,2,\ldots,n-1), \tag{2.37}$$

where

$$L_1(N)\equiv\frac{(1-\mathrm{e}^{-\lambda T})^{n-N}}{\sum_{k=1}^{n-N}[(1-\mathrm{e}^{-\lambda T})^k/k]}$$
$$\times\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i(\mathrm{e}^{-j\lambda T})^{n-i}\sum_{k=1}^{n-i}\frac{(1-\mathrm{e}^{-\lambda T})^k}{k}$$
$$-\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i[\mathrm{e}^{-j\lambda T}-\mathrm{e}^{-(j+1)\lambda T}]^{n-i}.$$

Because

$$L_1(N+1)-L_1(N)=\sum_{j=0}^{\infty}\sum_{i=0}^{N}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i(\mathrm{e}^{-j\lambda T})^{n-i}\sum_{k=1}^{n-i}\frac{(1-\mathrm{e}^{-\lambda T})^k}{k}$$
$$\times\left\{\frac{(1-\mathrm{e}^{-\lambda T})^{n-N-1}}{\sum_{k=1}^{n-N-1}[(1-\mathrm{e}^{-\lambda T})^k/k]}-\frac{(1-\mathrm{e}^{-\lambda T})^{n-N}}{\sum_{k=1}^{n-N}[(1-\mathrm{e}^{-\lambda T})^k/k]}\right\}$$
$$>0,$$

$L_1(N)$ increases strictly with N. Thus, if $L_1(n-1)\geq nc_1/c_R$, then there exists a unique minimum N_1^* $(1\leq N_1^*\leq n-1)$ that satisfies (2.37), and otherwise, $N_1^*=n$, *i.e.*, the system is replaced only at failure.

The expected cost rate $C_2(N)$ in (2.35) is rewritten as

$$C_2(N)=\frac{\begin{array}{c}nc_1+c_D\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i(\mathrm{e}^{-j\lambda T})^{n-i}\\ \times\left\{T-(1/\lambda)\sum_{k=1}^{n-i}(1-\mathrm{e}^{-\lambda T})^k/k]\right\}\end{array}}{T\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i(\mathrm{e}^{-j\lambda T})^{n-i}}$$
$$(N=1,2,\ldots,n).\qquad(2.38)$$

From the inequality $C_2(N+1)-C_2(N)\geq 0$,

$$L_2(N)\geq\frac{n\lambda c_1}{c_D}\qquad(N=1,2,\ldots,n-1),\qquad(2.39)$$

where

$$L_2(N)\equiv\sum_{j=0}^{\infty}\sum_{i=0}^{N-1}\binom{n}{i}(1-\mathrm{e}^{-j\lambda T})^i(\mathrm{e}^{-j\lambda T})^{n-i}\sum_{k=n-N+1}^{n-i}\frac{(1-\mathrm{e}^{-\lambda T})^k}{k}.$$

It can be easily seen that $L_2(N)$ increases strictly with N. Thus, if $L_2(n-1)\geq n\lambda c_1/c_D$, then there exists a unique minimum N_2^* $(1\leq N_2^*\leq n-1)$ that satisfies (2.39), and otherwise, $N_2^*=n$, *i.e.*, the system is replaced only after failure.

Tables 2.1 and 2.2 present the optimum numbers N_1^* and N_2^* for c_R/c_1 and c_1/c_D when $1/\lambda=50, T=4$, and $n=2$, 3, 5, 15, and 20. For example, when $n=5$ and $c_R/c_1=10$, the mean failure time of the system is $\mu_5=114.2$, and the optimum time is $T_1^*=75.0$, *i.e.*, the system should be replaced at $(75.0/114.2)\times 100=65.4\%$ of its mean time from Table 2.1. Such percentages increase with n and decrease with c_R/c_1. In the same case, $N_1^*=3$, *i.e.*, the system should be replaced when at least three of five units have failed at some jT. Such optimum numbers also increase with n and decrease with c_R/c_1. We can give a similar explanation in Table 2.2. It is of interest that the system should be replaced when $n-1$ or $n-2$ units have failed in both tables. ■

Table 2.1. Optimum time T_1^* and number N_1^* when $1/\lambda = 50$, $T = 4$

c_R/c_1	5		10		20		30		40		50	
n	T_1^*	N_1^*	T_1^*	N_1^*	T_1^*	N_1^*	T_1^*	N_1^*	T_1^*	N_1^*	T_1^*	N_1^*
2	99.6	1	40.9	1	23.2	1	16.2	1	14.4	1	12.5	1
3	99.6	1	51.5	1	33.4	1	26.9	1	23.3	1	21.1	1
5	135.3	3	75.0	3	53.0	3	44.9	3	40.4	2	37.4	2
15	∞	13	159.2	13	114.9	13	101.5	13	94.1	13	89.3	12
20	∞	18	197.2	18	135.7	18	119.7	18	111.3	18	105.8	18

Table 2.2. Optimum time T_2^* and number N_2^* when $1/\lambda = 50$, $T = 4$

c_1/c_D	0.05		0.10		0.5		1		5		10	
n	T_2^*	N_2^*	T_2^*	N_2^*	T_2^*	N_2^*	T_2^*	N_2^*	T_2^*	N_2^*	T_2^*	N_2^*
2	107.3	1	67.0	1	30.9	1	23.2	1	12.6	1	9.8	1
3	142.8	2	88.1	2	44.1	2	34.7	2	21.0	1	17.1	1
5	229.0	4	122.7	4	65.4	4	53.6	3	36.0	3	30.8	3
15	∞	15	287.1	14	127.1	14	109.0	14	82.9	13	75.1	13
20	∞	20	∞	19	146.6	19	126.2	19	97.6	18	89.2	18

2.2 Series and Parallel Systems

System reliabilities can be improved by redundant compositions of units. An optimum number of subsystems for a parallel-series system was obtained by considering two failures of open-circuits and short-circuits [2]. Reliability optimization of parallel-series and series-parallel systems was discussed [27]. It has been well-known that the reliability of series-parallel system with n subsystems in series, each subsystem having m units in parallel (Fig. 2.1), goes to 1 as $m \to \infty$ and to 0 as $n \to \infty$. Of interest is the question of what the stochastic behavior of such a system with the same number of series and parallel units is, *i.e.*, $n = m$, as $n \to \infty$. We answer this question mathematically and investigate several characteristics of series-parallel and parallel-series systems.

2.2.1 Series-parallel System

We consider a series-parallel system that consists of n $(n \geq 1)$ subsystems in series, each subsystem having identical m $(m \geq 1)$ units in parallel(Fig. 2.1). It is assumed that each unit has an identical and independent reliability $q \equiv 1-p$ $(0 < q \leq 1)$. Then, the system reliability is [2]

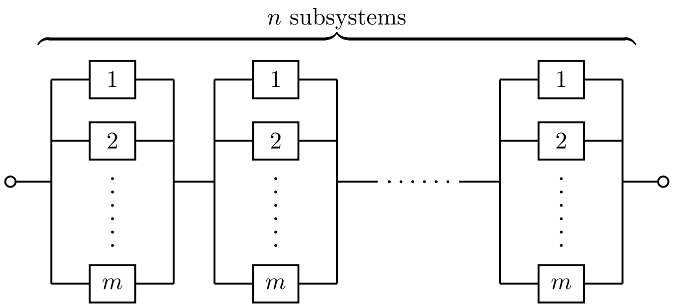

Fig. 2.1. Series-parallel system

$$R_{n,m}(q) = [1 - (1-q)^m]^n = (1-p^m)^n \qquad (n, m = 1, 2, \dots). \tag{2.40}$$

We investigate the characteristics of $R_{n,m}(q)$:

(1) $R_{n,m}(q)$ is an increasing function of q from 0 to 1 because

$$\lim_{q \to 0} R_{n,m}(q) = 0, \qquad \lim_{q \to 1} R_{n,m}(q) = 1.$$

(2) For a fixed p $(0 < p < 1)$, $m < \infty$, and $n < \infty$,

$$\lim_{n \to \infty} R_{n,m}(q) = 0, \quad \lim_{m \to \infty} R_{n,m}(q) = 1. \tag{2.41}$$

(3) Using the binomial expansion in (2.40) for $n \geq 2$ and a fixed p $(0 < p < 1)$,

$$1 - np^m < (1-p^m)^n < 1 - np^m + \frac{n(n-1)}{2} p^{2m}, \tag{2.42}$$

and hence,

$$0 < R_{n,m}(q) - (1 - np^m) < \frac{n(n-1)}{2} p^{2m}. \tag{2.43}$$

When $n = m$, we investigate the characteristics of reliability

$$R_n(p) \equiv (1-p^n)^n \qquad (n = 1, 2, \dots). \tag{2.44}$$

(4) If n increases to $n+1$, then the number of units increases in order of n^2. From (2.42), for a fixed p $(0 < p < 1)$,

$$\lim_{n \to \infty} R_n(p) \geq \lim_{n \to \infty} (1 - np^n) = 1.$$

Thus,

$$\lim_{n \to \infty} R_n(p) = 1. \tag{2.45}$$

Table 2.3. Values of $(1-p^n)^n$ and $1-np^n$ when $p=0.1$, and λMTTF when $p=1-\mathrm{e}^{-\lambda t}$

n	$(1-p^n)^n$	$1-np^n$	λMTTF
1	0.90000	0.90000	1.00000
2	0.98010	0.98000	0.91667
3	0.99700	0.99700	0.97897
4	0.99960	0.99960	1.05830
5	0.99995	0.99995	1.13653

Example 2.3. We compute the reliability $(1-p^n)^n$ and its approximation $1-np^n$ in Table 2.3 for $n=1$, 2, 3, 4, and 5 when $p=0.1$. The accuracy of approximation becomes better as p is smaller. In general, it would be sufficient to compute $1-np^m$ for a series-parallel system for a small p.

In particular, when $p=1-\mathrm{e}^{-\lambda t}$, the MTTF (Mean Time to Failure) of the system is

$$\int_0^\infty [1-(1-\mathrm{e}^{-\lambda t})^n]^n \,\mathrm{d}t = \frac{1}{\lambda}\int_0^1 \frac{(1-x^n)^n}{1-x}\,\mathrm{d}x. \tag{2.46}$$

In particular, when $n=2$, λMTTF is 11/12. Table 2.3 also presents λMTTF. It is of great interest that these values are minimum at $n=2$ and increase strictly with n $(n \geq 2)$. ■

(5) To investigate the monotonic property $R_n(p)$ $(0<p<1)$ in (2.44) for n, we obtain a solution p_n that satisfies

$$(1-p^{n+1})^{n+1} = (1-p^n)^n \qquad (n=1,2,\dots). \tag{2.47}$$

In the particular case of $n=1$, $p_1=(-1+\sqrt{5})/2 \approx 0.618$ that is equal to the golden ratio and plays a role in analyzing the system. We expect that p_n would increase with n. In the approximation $1-np^n$, a solution to satisfy

$$1-(n+1)p^{n+1} = 1-np^n \tag{2.48}$$

is given by $\widetilde{p}_n = n/(n+1)$, that increases from 1/2 to 1. Thus, $1-np^n$ increases with n from q to 1 for $0<p<1/2$.

Example 2.4. Table 2.4 presents the values p_n and $\widetilde{p}_n$ for $n=1$, 2, 3, 4, 5, and 10. This indicates that p_n increases with n and $p_n > \widetilde{p}_n$. Thus, if $0<p<p_1$, then $R_n(p)$ increases with n. In general, because p is a failure probability, its value would be lower than p_1. Therefore, it might be said in actual fields that $R_n(p)$ could be regarded as an increasing function of n.

It has been well-known that $R_n(p)$ has an S-shape [2, p. 198]. Figure 2.2 draws the reliability $R_n(p)$ for p when $n=1$, 2, 3, and 4. Because

Table 2.4. Values of p_n, $\widetilde{p}_n$, $\widehat{p}_n$, and $[1/(n+1)]^{1/n}$

n	p_n	$\widetilde{p}_n$	$\widehat{p}_n$	$[1/(n+1)]^{1/n}$
1	0.61803	0.50000	0.61803	0.50000
2	0.75488	0.66667	0.68233	0.57735
3	0.81917	0.75000	0.72449	0.62996
4	0.85668	0.80000	0.75809	0.66874
5	0.88127	0.83333	0.77809	0.69883
10	0.93607	0.90909	0.84440	0.78963

$$\frac{\mathrm{d}^2 R_n(p)}{\mathrm{d}p^2} = -(n-1)n^2(p - p^{n+1})^{n-2}[1-(n+1)p^n],$$

the inflection point is $[1/(n+1)]^{1/n}$ for $0 < p < 1$. These points increase with n $(1 \le n < \infty)$ from 0.5 to 1 because the function $[x/(1+x)]^x$ decreases from 1 to 0.5 for $0 < x \le 1$. This is also obtained by setting the approximation reliability is equal to that of one unit, *i.e.*,

$$1-(n+1)p^{n+1} = 1-p.$$

Table 2.4 also presents the value $\widehat{p}_n$ of a solution of the equation

$$(1-p^{n+1})^{n+1} = 1-p,$$

and the inflection points $[1/(n+1)]^{1/n}$ for $n = 1, 2, 3, 4, 5$, and 10. It is obvious that $p_n > \widehat{p}_n > [1/(n+1)]^{1/n}$ for $n \ge 2$. From Table 2.4 and Fig. 2.2, if $p > p_1$ that is the golden ratio, then we should not build up such a redundancy system. For example, when the failure time of each unit is exponential, *i.e.*, $p = 1 - \mathrm{e}^{-\lambda t}$, we should work this system in the interval less than $t = -[\log(1-p_1)]/\lambda \approx 0.9624/\lambda$, that is a little smaller than the mean time $1/\lambda$ of a unit. ■

2.2.2 Parallel-series System

We consider a parallel-series system that consists of m $(m \ge 1)$ subsystems in parallel, each subsystem having identical n $(n \ge 1)$ units in series (Fig. 2.3). When each unit has an identical reliability q $(0 < q \le 1)$, the system reliability is [2]

$$R_{m,n}(q) = 1-(1-q^n)^m \qquad (n, m = 1, 2, \dots). \tag{2.49}$$

When $q = \mathrm{e}^{-\lambda t}$, the MTTF is

$$\int_0^\infty [1-(1-\mathrm{e}^{-n\lambda t})^m]\mathrm{d}t = \frac{1}{n\lambda}\sum_{j=1}^m \frac{1}{j} \qquad (m = 1, 2, \dots). \tag{2.50}$$

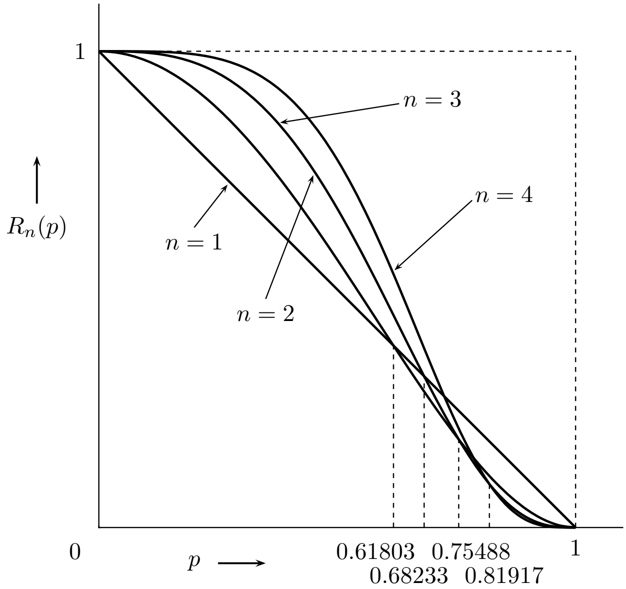

Fig. 2.2. Reliability of series-parallel systems

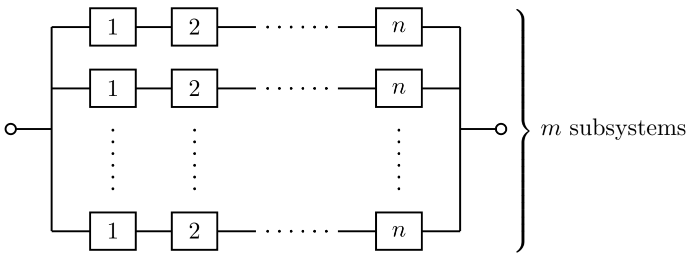

Fig. 2.3. Parallel-series system

When $n = m$, the MTTF decreases with n from $1/\lambda$ to 0.

We investigate the characteristics of $R_{m,n}(q)$:

(1) $R_{m,n}(q)$ increases with q from 0 to 1.

(2) For a fixed q $(0 < q \leq 1)$, $m < \infty$, and $n < \infty$,

$$\lim_{n\to\infty} R_{m,n}(q) = 0, \qquad \lim_{m\to\infty} R_{m,n}(q) = 1. \tag{2.51}$$

(3) For $m \geq 2$ and a fixed q $(0 < q < 1)$,

$$mq^n - \frac{m(m-1)}{2}q^{2n} < 1 - (1-q^n)^m < mq^n, \tag{2.52}$$

and hence,

$$0 < mq^n - R_{m,n}(q) < \frac{m(m-1)}{2}q^{2n}. \tag{2.53}$$

When $n = m$, we investigate the characteristics of the reliability

$$R_n(q) \equiv 1 - (1-q^n)^n \qquad (n = 1, 2, \dots). \tag{2.54}$$

(4) From (2.52), for a fixed q $(0 < q < 1)$,

$$\lim_{n\to\infty} R_n(q) \le \lim_{n\to\infty} nq^n = 0,$$

and hence

$$\lim_{n\to\infty} R_n(q) = 0. \tag{2.55}$$

(5) To investigate the monotonic property of $R_n(q)$ $(0 < q < 1)$ for n, we obtain a solution q_n that satisfies

$$(1-q^{n+1})^{n+1} = (1-q^n)^n \qquad (n = 1, 2, \dots). \tag{2.56}$$

It is clear that q_n is calculated by replacing p_n in Table 2.4 with q_n. Thus, if $p_1 < q < 1$, then $1 - (1-q^n)^n$ decreases with n. In general, because q is the reliability of a unit, its value would be larger than p_1. Therefore, $R_n(q)$ could be regarded as a decreasing function of n in actual fields.

(6) Comparing two reliabilities of series-parallel and parallel-series systems, for $0 < q < 1$,

$$[1-(1-q)^n]^n \ge 1 - (1-q^n)^n \qquad (n = 1, 2, \dots). \tag{2.57}$$

We set to prove the inequality (2.57) that

$$f_n(q) \equiv [1-(1-q)^n]^n - 1 + [(1-q^n)]^n.$$

It is easily seen that

$$\lim_{q\to 0} f_n(q) = \lim_{q\to 1} f_n(q) = 0,$$

$$f_1(q) = 0, \qquad f_2(q) = 2q^2(1-q)^2 > 0,$$

$$\begin{aligned}
\frac{\mathrm{d}f_n(q)}{\mathrm{d}q} &= n^2\left\{[1-(1-q)^n]^{n-1}(1-q)^{n-1} - (1-q^n)^{n-1}q^{n-1}\right\} \\
&= n^2[q(1-q)]^{n-1}\left\{\left[\frac{1-(1-q)^n}{q}\right]^{n-1} - \left[\frac{1-q^n}{1-q}\right]^{n-1}\right\} \\
&= n^2[q(1-q)]^{n-1}\left\{\left[\sum_{j=0}^{n-1}(1-q)^j\right]^{n-1} - \left[\sum_{j=0}^{n-1}q^j\right]^{n-1}\right\}.
\end{aligned}$$

Hence, $\mathrm{d}f_n(q)/\mathrm{d}q > 0$ for $0 < q < 1/2$, 0 for $q = 1/2$, and < 0 for $1/2 < q < 1$. Thus $f_n(q)$ is a concave function of q $(0 < q < 1)$ and takes 0 at $q = 0, 1$. This completes the proof of inequality (2.57). The inequality holds only when $n = 1$, and its difference is maximum at $q = 1/2$.

Table 2.5. Optimum number n^* of series-parallel system with complexity

α	q		
	$1-10^{-1}$	$1-10^{-2}$	$1-10^{-3}$
10^{-1}	1	1	1
10^{-2}	2	1	1
10^{-3}	2	2	1
10^{-4}	3	2	2
10^{-5}	4	2	2
10^{-6}	4	3	2
10^{-7}	5	3	2
10^{-8}	5	4	3

2.2.3 Complexity of Series-parallel System

We define the complexity of redundant systems as the number P_a of paths and its reliability as $\exp\{-\alpha[P_a - 1]\}$ that will be denoted in Chap. 9. Based on such definitions, the number of paths of a series-parallel system is $P_a = m^n$, and hence, its reliability is, from (9.6),

$$R_s(n,m) = \exp[-\alpha(m^n - 1)][1-(1-q)^m]^n \qquad (n,m = 1,2,\dots) \tag{2.58}$$

for $0 \leq \alpha < \infty$. More detailed studies on system complexity will be done in Chap. 9.

Example 2.5. We can obtain the optimum number n^* that maximizes $R_s(n,n)$. The reliability of a series-parallel system increases with n for large q, however, the reliability of the complexity decreases with n. Table 2.5 presents the optimum n^* for $\alpha = 10^{-1}$–10^{-8} and $q = 1-10^{-1}$, $1-10^{-2}$, and $1-10^{-3}$. This indicates naturally that n^* decreases with both α and q. ■

2.3 Three Redundant Systems

As one application of redundant techniques, this section considers the following three typical redundant systems and evaluates them to make the optimization design of system redundancies:

(1) System 1: One unit system with n-fold mean time.
(2) System 2: n-unit parallel redundant system.
(3) System 3: n-unit standby redundant system.

When $n = 1$, all systems are identical. When the failure time of each unit is exponential, we compute the reliability quantities of the three systems.

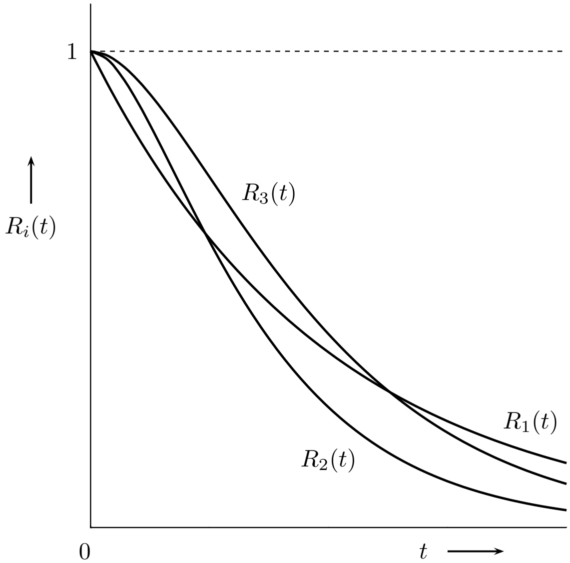

Fig. 2.4. Reliabilities of the three systems when $n = 2$

Furthermore, we obtain the expected costs for each system and compare them. The scheduling problem in which a job has a random working time and is achieved by a system will be discussed in Sect. 5.3. In this model, we define the reliability as the probability that the work of a job is accomplished by a system without failure, derive the reliabilities of the three systems, and compare them.

2.3.1 Reliability Quantities

When the failure time of each unit is exponential, *i.e.*, the failure distribution is $F(t) = 1 - \mathrm{e}^{-\lambda t}$, we calculate the following reliability quantities [2]:

(i) Reliability function $R(t)$

(1) $$R_1(t) = \mathrm{e}^{-\lambda t/n}, \tag{2.59}$$

(2) $$R_2(t) = 1 - (1 - \mathrm{e}^{-\lambda t})^n, \tag{2.60}$$

(3) $$R_3(t) = \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t}. \tag{2.61}$$

Figure 2.4 shows the reliabilities $R_i(t)$ $(i = 1, 2, 3)$ of the three systems when $n = 2$. We can prove that $R_3(t) > R_2(t)$ for $t > 0$ and $n \geq 2$, *i.e.*,

$$(1 - \mathrm{e}^{-\lambda t})^n > \sum_{j=n}^{\infty} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t} \qquad (n = 2, 3, \dots), \tag{2.62}$$

by mathematical induction: When $n = 2$, we denote $Q(t)$ by

$$Q(t) \equiv (1 - \mathrm{e}^{-\lambda t})^2 - [1 - (1 + \lambda t)\mathrm{e}^{-\lambda t}].$$

Then, it is clearly seen that $Q(0) = Q(\infty) = 0$, and

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \lambda \mathrm{e}^{-\lambda t}[2(1 - \mathrm{e}^{-\lambda t}) - \lambda t],$$

that implies $Q(t) > 0$ for $t > 0$ because $Q(t)$ is a concave function. Assuming that when $n = k$,

$$(1 - \mathrm{e}^{-\lambda t})^k > \sum_{j=k}^{\infty} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t},$$

we prove that

$$(1 - \mathrm{e}^{-\lambda t})^{k+1} > \sum_{j=k+1}^{\infty} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t}.$$

We easily have

$$\begin{aligned}
&(1 - \mathrm{e}^{-\lambda t})^{k+1} - \sum_{j=k+1}^{\infty} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t} \\
&> (1 - \mathrm{e}^{-\lambda t}) \sum_{j=k}^{\infty} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t} - \sum_{j=k+1}^{\infty} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t} \\
&= \mathrm{e}^{-\lambda t} \left[\frac{(\lambda t)^k}{k!} - \sum_{j=k}^{\infty} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t} \right] \\
&= \frac{(\lambda t)^k}{k!} \mathrm{e}^{-2\lambda t} \left[\sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} - \sum_{j=k}^{\infty} \frac{(\lambda t)^j}{j!} \frac{k!}{(\lambda t)^k} \right] \\
&= \frac{(\lambda t)^k}{k!} \mathrm{e}^{-2\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!(j+k)!} [(j+k)! - j!k!] > 0.
\end{aligned}$$

This concludes that $R_3(t) > R_2(t)$ for $n = 2, 3, \ldots$ and $t > 0$.

(ii) Mean time μ and standard deviation σ

(1) $\quad \mu_1 = \dfrac{n}{\lambda}, \qquad \sigma_1 = \dfrac{n}{\lambda},$ (2.63)

(2) $\quad \mu_2 = \dfrac{1}{\lambda} \displaystyle\sum_{j=1}^{n} \frac{1}{j}, \qquad \sigma_2 = \dfrac{1}{\lambda} \sqrt{\displaystyle\sum_{j=1}^{n} \frac{1}{j^2}},$ (2.64)

(3) $\quad \mu_3 = \dfrac{n}{\lambda}, \qquad \sigma_3 = \dfrac{\sqrt{n}}{\lambda}.$ (2.65)

Note that $\mu_1 = \mu_3 > \mu_2$ and $\sigma_1 > \sigma_3 > \sigma_2$ for $n = 2, 3, \ldots$.

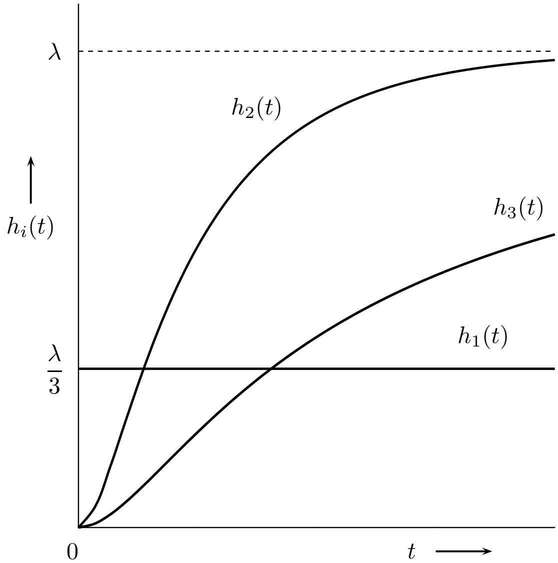

Fig. 2.5. Failure rates of the three systems when $n = 3$

(iii) Failure rate $h(t)$

$$(1) \quad h_1(t) = \frac{\lambda}{n}, \tag{2.66}$$

$$(2) \quad h_2(t) = \frac{n\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}}{1 - (1 - e^{-\lambda t})^n}, \tag{2.67}$$

$$(3) \quad h_3(t) = \frac{\lambda(\lambda t)^{n-1}/(n-1)!}{\sum_{j=0}^{n-1}[(\lambda t)^j/j!]}. \tag{2.68}$$

Both $h_2(t)$ and $h_3(t)$ increase strictly from 0 to λ for $n \geq 2$. It seems certain that $h_2(t) \geq h_3(t)$. Unfortunately, we cannot prove this inequality mathematically. Figure 2.5 shows the three failure rates $h_i(t)$ when $n = 3$.

(iv) Complexity

When a redundant system has the number n of paths, we define its complexity as $P_e = \log_2 n$ and its reliability as $R_e(n) = \exp(-\alpha \log_2 n)$ for parameter $\alpha > 0$ as shown in Sect. 9.2. Because we count that the numbers of paths are 1 for System 1 and n for Systems 2 and 3, the complexities for System i are $\log_2 1$, $\log_2 n$, and $\log_2 n$, respectively. Thus, the reliabilities of complexity are $\exp(-\alpha \log_2 1)$ for System 1 and $\exp(-\alpha \log_2 n)$ for Systems 2 and 3. If the reliabilities of a whole system with complexity is given by the product of the reliabilities of the system and complexity, then from (2.59)–(2.61),

Table 2.6. Reliabilities $R_i(2)$ (%) of the three systems when $\lambda t = 1$

α	$R_1(2)$	$R_2(2)$	$R_3(2)$
0.2	60.7	49.2	60.2
0.1	60.7	54.3	66.6
0.01	60.7	59.4	72.8
0.001	60.7	60.0	73.5
0.0001	60.7	60.0	73.6

$$(1)\quad R_1(n) = \mathrm{e}^{-\lambda t/n} \exp(-\alpha \log_2 1) = \mathrm{e}^{-\lambda t/n}, \tag{2.69}$$

$$(2)\quad R_2(n) = [1 - (1 - \mathrm{e}^{-\lambda t})^n] \exp(-\alpha \log_2 n), \tag{2.70}$$

$$(3)\quad R_3(n) = \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t} \exp(-\alpha \log_2 n). \tag{2.71}$$

Example 2.6. Table 2.6 presents reliabilities $R_i(n)$ of the three systems for $\alpha = 0.2$, 0.1, 0.01, 0.001, and 0.0001 when $n = 2$ and $\lambda t = 1$. This indicates that System 3 is better than System 2 for any $\alpha > 0$, as shown in (i). When α is larger, System 1 is better than System 3, and when α is smaller, System 3 is better than System 1. When $\alpha = 0.193$, the reliabilities of Systems 1 and 3 are equal to each other. ■

2.3.2 Expected Costs

We introduce the following costs for the three systems:

$$(1)\quad C_1(n) = c_1(n) + b + c, \tag{2.72}$$

$$(2)\quad C_2(n) = an + bn + c, \tag{2.73}$$

$$(3)\quad C_3(n) = an + b + cn, \tag{2.74}$$

where $c_1(n)$ and an are production costs for Systems 1 and 2, 3, where $c_1(1) = a$, b and bn are operating costs for System 1, 3 and 2, and c and cn are replacement costs for System 1, 2 and 3, respectively.

Comparing the three costs,

$$\text{(i)}\quad c_1(n) \geq an + b(n-1) \Longleftrightarrow C_1(n) \geq C_2(n). \tag{2.75}$$

$$\text{(ii)}\quad c_1(n) \geq an + c(n-1) \Longleftrightarrow C_1(n) \geq C_3(n). \tag{2.76}$$

$$\text{(iii)}\quad b \geq c \Longleftrightarrow C_2(n) \geq C_3(n). \tag{2.77}$$

Furthermore, we obtain the following expected cost rates by dividing $C_i(n)$ by the mean times μ_i $(i = 1, 2, 3)$:

$$(1) \quad \widetilde{C}_1(n) = \frac{c_1(n)+b+c}{n/\lambda}, \tag{2.78}$$

$$(2) \quad \widetilde{C}_2(n) = \frac{an+bn+c}{(1/\lambda)\sum_{j=1}^{n}(1/j)}, \tag{2.79}$$

$$(3) \quad \widetilde{C}_3(n) = \frac{an+b+cn}{n/\lambda}. \tag{2.80}$$

Comparing the above three costs,

$$\text{(iv)} \quad \frac{c_1(n)+b+c}{n}\sum_{j=1}^{n}\frac{1}{j} \geq an+bn+c \Longleftrightarrow \widetilde{C}_1(n) \geq \widetilde{C}_2(n). \tag{2.81}$$

$$\text{(v)} \quad c_1(n) \geq an+c(n-1) \Longleftrightarrow \widetilde{C}_1(n) \geq \widetilde{C}_3(n). \tag{2.82}$$

$$\text{(vi)} \quad an+bn+c \geq \frac{an+b+cn}{n}\sum_{j=1}^{n}\frac{1}{j} \Longleftrightarrow \widetilde{C}_2(n) \geq \widetilde{C}_3(n). \tag{2.83}$$

Note that the above results do not depend on the failure rate λ of a unit.

(vii) When $c_1(n) = an^2$, we find an optimum number n^* that minimizes

$$\frac{\widetilde{C}_1(n)}{\lambda} = an + \frac{b+c}{n} \qquad (n=1,2,\dots). \tag{2.84}$$

From the inequality $\widetilde{C}_1(n+1) - \widetilde{C}_1(n) \geq 0$,

$$\frac{n(n+1)}{2} \geq \frac{b+c}{2a}. \tag{2.85}$$

Thus, there exists a finite and unique minimum n^* that satisfies (2.85). Note that the left-hand side represents the summation of integers from 1 to n and will appear often in partition models of Sect. 3.1.

(viii) We find an optimum number n^* to minimize $\widetilde{C}_2(n)$ in (2.79). From the inequality $\widetilde{C}_2(n+1) - \widetilde{C}_2(n) \geq 0$,

$$(n+1)\sum_{j=1}^{n}\frac{1}{j+1} \geq \frac{c}{a+b}. \tag{2.86}$$

The left-hand side of (2.86) agrees with that of (2.5) and increases strictly to ∞. Thus, there exists a finite and unique minimum n^* that satisfies (2.86).

2.3.3 Reliabilities with Working Time

Suppose that a positive random variable S with distribution $W(t) = \Pr\{S \leq t\}$ is the working time of a job that has to be achieved by each system. Then, we define the reliabilities of each system by

$$R_i \equiv \int_0^\infty R_i(t)\,\mathrm{d}W(t) \qquad (i = 1, 2, 3), \tag{2.87}$$

that represent the probabilities that a job with working time S is accomplished by each system without failure. Several properties of these reliabilities and optimization problems are summarized in Sect. 5.3.

From this definition, the reliabilities of the three systems are

$$(1) \quad R_1 = \int_0^\infty \mathrm{e}^{-\lambda t/n}\,\mathrm{d}W(t), \tag{2.88}$$

$$(2) \quad R_2 = \int_0^\infty [1 - (1 - \mathrm{e}^{-\lambda t})^n]\,\mathrm{d}W(t), \tag{2.89}$$

$$(3) \quad R_3 = \sum_{j=0}^{n-1} \int_0^\infty \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t}\,\mathrm{d}W(t). \tag{2.90}$$

When $W(t) = 1 - \mathrm{e}^{-\omega t}$, the above reliabilities are rewritten as

$$(1) \quad R_1 = \frac{n\omega}{\lambda + n\omega}, \tag{2.91}$$

$$(2) \quad R_2 = 1 - \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{\omega}{\omega + j\lambda}, \tag{2.92}$$

$$(3) \quad R_3 = 1 - \left(\frac{\lambda}{\lambda + \omega}\right)^n. \tag{2.93}$$

We have the following results:

(i) When $\lambda = \omega$,

$$R_1 = R_2 = \frac{n}{n+1}, \quad R_3 = 1 - \frac{1}{2^n}, \tag{2.94}$$

and $R_3 > R_1 = R_2$ for $n = 2, 3, \ldots$.

(ii) We compare R_1 and R_3 for $n = 2, 3, \ldots$. Because

$$\begin{aligned}\frac{\lambda}{\lambda + n\omega} - \left(\frac{\lambda}{\lambda + \omega}\right)^n &= \frac{\lambda}{(\lambda + n\omega)(\lambda + \omega)^n}[(\lambda + \omega)^n - \lambda^{n-1}(\lambda + n\omega)] \\ &= \frac{\lambda}{(\lambda + n\omega)(\lambda + \omega)^n} \sum_{j=0}^{n-2} \binom{n}{j} \lambda^j \omega^{n-j} > 0,\end{aligned}$$

$R_3 > R_1$ for $n = 2, 3, \ldots$.

(iii) $R_3 > R_2$ for $n = 2, 3, \ldots$ from $R_3(t) > R_2(t)$.

(iv) When $n = 2, 3$, it is easily proved that

$$\omega > \lambda \Longleftrightarrow R_2 > R_1.$$

Furthermore, it seems that this result holds for $n = 4, 5, \ldots$. Unfortunately, we cannot prove it mathematically. Figure 2.6 shows R_1, R_2, and R_3 for $\lambda = a\omega$ $(0 \le a \le 1)$ when $n = 4$.

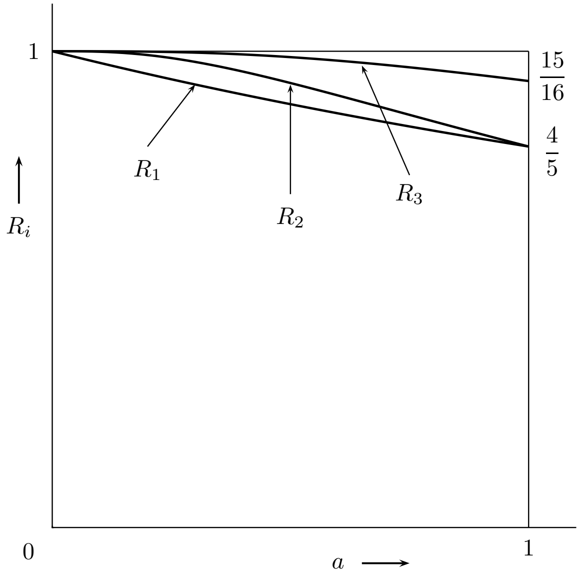

Fig. 2.6. Reliabilities of the three systems when $n = 4$

2.4 Redundant Data Transmissions

Data transmissions in a communication system fail due to some errors that have been generated by disconnection, cutting, warping, noise, or distortion in a communication line. To transmit accurate data, we have to prepare error control schemes that automatically detect and correct errors. The following three control schemes have been used mainly in communication systems [41–43]: (1) FEC (Forward Error Connection) scheme, (2) ARQ (Automatic Repeat Request) scheme, and (3) Hybrid ARQ scheme. A variety of such error-correcting strategies and a great many protocols of ARQ schemes were proposed and appeared in actual systems [33].

Scheme 2 has been widely used in data transmissions between two points because the error control is simple and easy. This section considers three simple models of Scheme 2 and obtains the expected costs until the success of data transmission [44]. We discuss analytically which model is the best among three models. The techniques used in this section would be useful for other schemes.

2.4.1 Three Models

We transmit an amount of data from a sender to a receiver that is called *unit data*. To detect and correct errors, we consider three redundant models, where we transmit two, two plus one, and three unit data simultaneously to a receiver.

Suppose that the transmission of unit data fails with probability p ($0 \leq p < 1$) due to errors that have occurred independently of each other. If there is no failure of the transmission, all transmitted data are the same ones at a receiver. Let c_n ($n = 1, 2, \ldots$) be the cost required for the transmissions of n unit data; this includes all costs of editing, transmission, and checking. It is assumed that $c_2 + c_1 > c_3 > c_2 > c_1$.

(1) Model 1

We transmit two unit data simultaneously to a receiver who checks two data:

(1) If the two data are not the same, then the receiver cancels such data and informs the sender. We call it a transmission failure.
(2) If the two data are the same, the receiver accepts the data and informs the sender. We call it a transmission success.
(3) When the transmission has failed, the sender transmits two units data again and continues the above transmission until its success.

The expected cost until transmission success is given by a renewal equation:

$$C_1 = (1-p)^2 c_2 + [1-(1-p)^2](c_2 + C_1). \tag{2.95}$$

Solving (2.95) for C_1,

$$C_1 = \frac{c_2}{(1-p)^2}. \tag{2.96}$$

(2) Model 2

(1) If the two data are not the same, the sender transmits one unit data again and the receiver checks it with two former data. If the retransmitted data are not the same as either of two data, we call it a transmission failure and transmit the two unit data from the beginning.
(2) If the two data are the same or if the retransmitted data are the same as either of two former data, we call it a transmission success.
(3) The sender continues the above transmission until its success.

The expected cost is

$$C_2 = (1-p)^2 c_2 + 2p(1-p)^2(c_2 + c_1) + [2p^2(1-p) + p^2](c_2 + c_1 + C_2),$$

i.e.,

$$C_2 = \frac{c_2 + [1-(1-p)^2]c_1}{(1-p)^2(1+2p)}. \tag{2.97}$$

(3) Model 3

We transmit three unit data simultaneously to a receiver who checks them:

(1) If none of the three data are the same, the receiver cancels such data and informs the sender. We call it a transmission failure.
(2) If at least two of the three data are the same, the receiver accepts the data. We call it a transmission success.
(3) The sender continues the above transmission until its success.

The probability that at least two of the three data are the same is

$$\sum_{j=0}^{1} \binom{3}{j} p^j (1-p)^{3-j} = (1-p)^2(1+2p),$$

that agrees with the denominator in (2.97). Thus, the expected cost is

$$C_3 = \frac{c_3}{(1-p)^2(1+2p)}. \tag{2.98}$$

Note that all expected costs increase with p from $C_1 = C_2 = c_2$ and $C_3 = c_3$ to ∞.

If we transmit n $(n \geq 3)$ unit data simultaneously and if at least two of n data are the same, we call it a transmission success. Then, the expected cost is similarly given by

$$C_3 = \frac{c_n}{1 - np^{n-1} + (n-1)p^n}. \tag{2.99}$$

2.4.2 Optimum Policies

We compare the three expected costs C_1, C_2, and C_3. From (2.96) and (2.97),

$$C_1 - C_2 = \frac{1}{(1-p)^2(1+2p)}[2p(c_2 - c_1) + p^2 c_1] > 0,$$

that implies $C_1 > C_2$. In addition, from (2.96) – (2.98),

$$C_3 - C_2 = \frac{1}{(1-p)^2(1+2p)}[(1-p)^2 c_1 - (c_2 + c_1 - c_3)],$$
$$C_1 - C_3 = \frac{1}{(1-p)^2(1+2p)}[2pc_2 - (c_3 - c_2)].$$

Therefore, we have the following optimum policy:

(i) If $(1-p)^2 \leq (c_2 + c_1 - c_3)/c_1$, then $C_1 > C_2 \geq C_3$.
(ii) If $(1-p)^2 > (c_2 + c_1 - c_3)/c_1$ and $2p > (c_3 - c_2)/c_2$, then $C_1 > C_3 \geq C_2$.
(iii) If $2p \leq (c_3 - c_2)/c_2$, then $C_3 \geq C_1 > C_2$.

Table 2.7. Expected costs C_i $(i = 1, 2, 3)$ when $c_n = 2 + n$

p	C_1	C_2	C_3
0.5	16.00	12.50	10.00
0.2	6.25	5.67	5.58
0.1	4.94	4.70	5.14
0.01	4.08	4.02	4.95
0.001	4.01	4.01	5.00
0.0001	4.00	4.00	5.00

Table 2.8. Data length L_0 when $C_2 = C_3$ and $c_n = c_0 + n$

p_1	c_0				
	1	2	3	4	5
10^{-3}	346.4	202.6	143.8	111.5	91.1
10^{-4}	3465.7	2027.3	1438.4	1115.7	911.6
10^{-5}	34657.4	20273.3	14384.1	111571.7	91160.7
10^{-6}	346573.6	202732.5	143841.0	1115717.0	911607.0

In general, the probability p of transmission failure of unit data is not constant and depends on its length L and bit error rate p_1. Suppose that $p \equiv 1 - (1 - p_1)^L$, *i.e.*, $L = \log(1-p)/\log(1-p_1)$. Then, the above policy is:

(i)′ If $L \geq \log[(c_2 + c_1 - c_3)/c_1]/[2\log(1-p_1)]$, then $C_1 > C_2 \geq C_3$.
(ii)′ If $\log[(3c_2 - c_3)/(2c_2)]\ /\log(1-p_1) < L < \log[(c_2 + c_1 - c_3)/c_1]/[2\log(1-p_1)]$, then $C_1 > C_3 \geq C_2$.
(iii)′ If $L \leq \log[(3c_2 - c_3)/(2c_2)]/\log(1-p_1)$, then $C_3 \geq C_1 > C_2$.

Example 2.7. When $c_n = 2 + n$ $(n = 1, 2, 3)$, Table 2.7 presents the expected costs C_i $(i = 1, 2, 3)$ for p. In this case, when $p = p_0 \equiv 1 - \sqrt{2/3} = 0.1835$, C_2 is equal to C_3. If $p > p_0$, then C_3 is smaller than C_2, and *vice versa.*

In addition, when $p = 1 - (1-p_1)^L$ and $c_n = c_0 + n$, Table 2.8 presents the data length L_0 at which C_2 is equal to C_3, *i.e.*,

$$L_0 = \frac{\log[c_0/(c_0+1)]}{2\log(1-p_1)}$$

for p_1 and $c_0 = 1$, 2, 3, 4, and 5. For example, when $p_1 = 10^{-4}$ and $c_0 = 2$, if $L > 2028$, then C_3 is smaller than C_2. It is of interest that $p_1 L_0$ is almost constant for a specified c_0, *i.e.*, $p_1 L_0 \approx (1/2)\log[(c_0+1)/c_0]$. ■

2.5 Other Redundant Models

Using the properties of redundant systems, we apply them to the following three redundant models:

(1) Redundant Bits

The BASIC mode data transmission control procedure is one typical method of data transmission on a public transmission line and is simply called basic procedure [45]. To reduce failures of transmissions, data in basic procedure are often divided into some small blocks, each of which has redundant bits such as heading, control characters, bit check character, and *etc.*

If we send one block to a receiver and he or she finds no error, we call it a block transmission success. If a receiver detects some errors by redundant bits, he or she informs it to us, and we send the same block again. The process is repeated until the success of all block transmissions, *i.e.*, transmission success. This is called ARQ scheme [41].

It is assumed that bit errors occur independently of each other, and its rate (BER) is constant p_1 $(0 < p_1 < 1)$ for any transmission. In addition, we divide unit data with length S into N blocks. Then, the error rate of one block is

$$p = 1 - (1-p_1)^{S/N}. \tag{2.100}$$

In addition, we attach n redundant bits to each block, so that the length of one block is $S/N + n$. We transmit each block successively to a receiver. If some errors of a block are detected, we retransmit it until block transmission success. Let $M(N)$ be the total expected number of blocks that have been transmitted until data transmission success. Because errors of each block occur with probability p in (2.100), we have a renewal equation:

$$M(N) = \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} [M(j) + N] \qquad (N = 1, 2, \dots),$$

where $M(0) \equiv 0$. Solving for $M(N)$,

$$M(N) = \frac{N}{1-p}. \tag{2.101}$$

Thus, the total average length $L(N)$ of transmission data until transmission success is

$$L(N) = \left(\frac{S}{N} + n\right) M(N) = \frac{S + nN}{(1-p_1)^{S/N}} \qquad (N = 1, 2, \dots). \tag{2.102}$$

Note that $L(N)$ increases with n. However, when n is small, we might not detect some occurrences of errors and cannot trust the accuracy of data transmission even if it succeeds.

Table 2.9. Optimum number N^* and average data length $L(N^*)$

S	$p_1 = 10^{-4}$				$p_1 = 10^{-5}$			
	$n = 64$		$n = 128$		$n = 64$		$n = 128$	
	N^*	$L(N^*)$	N^*	$L(N^*)$	N^*	$L(N^*)$	N^*	$L(N^*)$
1,024	1	1,205	1	1,276	1	1,099	1	1,164
2,048	3	2,398	2	2,552	1	2,159	1	2,221
4,096	5	4,793	4	5,105	2	4,311	1	4,401
8,192	11	9,584	8	10,210	3	8,616	2	8,801
16,384	21	19,167	15	20,417	7	17,230	5	17,591

Example 2.8. We can easily compute the optimum number N^* that minimizes $L(N)$ in (2.102) for specified S, n, and p_1. Table 2.9 presents the optimum N^* and the resulting length $L(N^*)$ for $p_1 = 10^{-4}$, 10^{-5} and $n = 64$, 128. This indicates that N^* increases with p_1 and S and decreases with n. For example, when $S = 2,048$, $n = 64$, and $p_1 = 10^{-4}$, the optimum number is $N^* = 3$, and the average data length is $L(N^*) = 2,398$, that is 17.1% longer than an original data length S. The rate $L(N^*)/S$ decreases slowly with S. ■

(2) Redundant Networks

Consider a network system with two terminals that consists of N ($N \geq 1$) networks (see Fig. 9.9 in Chap. 9): Customers arrive at the system according to an exponential distribution $(1 - \mathrm{e}^{-\lambda t})$, and their usage times also have an identical exponential distribution $(1 - \mathrm{e}^{-\mu t})$, *i.e.*, this process forms an M/M/N(∞) queueing one. Then, the probability in the steady-state that there are j customers in the system is [46]

$$p_j = \begin{cases} \dfrac{a^j}{j!} p_0 & (0 \leq j \leq N), \\ \dfrac{N^N \rho^j}{N!} p_0 & (j \geq N), \end{cases} \tag{2.103}$$

where $a \equiv \lambda/\mu$, $\rho \equiv a/N < 1$, and

$$p_0 = 1 \Big/ \left[\sum_{j=0}^{N-1} \frac{a^j}{j!} + \frac{a^N}{(N-1)!(N-a)} \right].$$

We define the probability that customers can use a network without waiting, *i.e.*, the availability of the system is

Table 2.10. Optimum number N^* and system efficiency $C(N^*)/c_1$ when $a = 0.5$

c_0/c_1	N^*	$C(N^*)/c_1$
0.1	1	2.20
0.2	1	2.40
0.5	2	2.78
1.0	2	3.33
2.0	2	4.44
5.0	2	7.78
10.0	3	13.20

$$A(N) \equiv \sum_{j=0}^{N-1} p_j. \tag{2.104}$$

Let $c_1 N + c_0$ be the construction cost for system with N networks. By arguments similar to those in **(1)** of Sect. 2.1.1, we give a system efficiency as

$$C(N) = \frac{c_1 N + c_0}{\sum_{j=0}^{N-1} p_j} \qquad (N = 1, 2, \dots). \tag{2.105}$$

From the inequality $C(N+1) - C(N) \geq 0$, an optimum network N^* to minimize $C(N)$ is given by a minimum that satisfies

$$\frac{1}{p_N} \sum_{j=0}^{N-1} p_j - N \geq \frac{c_0}{c_1} \qquad (N = 1, 2, \dots). \tag{2.106}$$

It can be easily seen that if p_N decreases strictly with N, then the left-hand side of (2.106) increases strictly with N and tends to ∞ as $N \to \infty$ when $p_N \to 0$ because

$$\frac{1}{p_N} \sum_{j=0}^{N-1} p_j - N > \frac{p_0}{p_N} - 1 \qquad \text{for } N \geq 2.$$

Example 2.9. Table 2.10 presents the optimum N^* and the resulting efficiency $C(N^*)/c_1$ for c_0/c_1 when $a = 0.5$, *i.e.*, $1/\lambda = 2/\mu$, that means that the mean arrival time of customers is two times their mean usage time. In this case, there exits a finite N^* always exists because $p_N \to 0$ as $N \to \infty$. This indicates that the optimum N^* increases slowly as c_0/c_1 increases. ■

(3) N Copies

One of the most important things in modern societies is the diversification of information and risks. We have to take some copies of important goods to prevent their loss and store them separately in other places. It is assumed that the probabilities of losing all N copies and of at least one of N copies being stolen are p^N and $1 - q^N$, respectively, where $1 \geq q > p > 0$. In addition, we introduce the following costs: A storage cost of N copies is $c_1 N + c_0$, c_2 is the cost of losing all copies and c_3 is the cost of at least one copy being stolen.

We give the total expected cost with N copies as

$$C(N) = c_1 N + c_0 + c_2 p^N + c_3(1 - q^N) \qquad (N = 1, 2, \ldots). \tag{2.107}$$

From the inequality $C(N+1) - C(N) \geq 0$, an optimum number N^* that minimizes $C(N)$ is a minimum such that

$$\frac{c_1 + c_3 q^N (1-q)}{p^N (1-p)} \geq c_2 \qquad (N = 1, 2, \ldots). \tag{2.108}$$

The left-hand side of (2.108) increases strictly with N to ∞. Thus, there exists a finite and unique minimum N^* $(1 \leq N^* < \infty)$ that satisfies (2.108). For example, when $p = 0.2$, $q = 0.99$, $c_2/c_1 = 500$, and $c_3/c_1 = 100$, the optimum number N^* that minimizes $C(N)$ is $N^* = 4$.

http://www.springer.com/978-1-84800-293-7

Advanced Reliability Models and Maintenance Policies
Nakagawa, T.
2008, IX, 246 p., Hardcover
ISBN: 978-1-84800-293-7