2

Plasma Modelling for Magnetic Control

The aim of this chapter is to derive a linearized mathematical model describing the interaction between the plasma ring and the voltages applied to the poloidal field coils. This model will be of fundamental importance in the design of the plasma magnetic control system. To start with, the equations of the ideal magnetohydrodynamics theory will be presented; these equations will be used to characterize the equilibrium configurations of a plasma in a tokamak machine, and to derive a nonlinear dynamical model. Then, it is shown how to obtain a finite dimensional linear time-invariant model. This model will be completed in the next chapter including the output equations describing the parameters used to characterize the plasma shape and position.

2.1 The Ideal Magnetohydrodynamics Theory

Magnetohydrodynamics (MHD) [10, 41, 42] describes the basic behaviour of a magnetically confined plasma. In this theory the plasma is considered as a single fluid, that is, no distinction is made among the various particles constituting the plasma. The plasma is completely described once the local mass density ρ and the fluid velocity $\boldsymbol{v}$ vector are assigned.

The fundamental laws that link these quantities are the mass conservation law

$$\frac{\partial}{\partial t}\,\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0\,, \tag{2.1}$$

and Newton's law applied to an infinitesimal plasma element

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\,\boldsymbol{v} = \boldsymbol{J}\times\boldsymbol{B} - \boldsymbol{\nabla}\,p\,, \tag{2.2}$$

where $\boldsymbol{J}$ is the current density field, $\boldsymbol{B}$ is the magnetic induction field, and p is the kinetic pressure inside the plasma. The coupling between the plasma and the electromagnetic field is given by the Lorentz force term $\boldsymbol{J}\times\boldsymbol{B}$ in Equation 2.2.

Moreover, the electromagnetic fields have to satisfy Maxwell's equations

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.3a}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \tag{2.3b}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}\,. \tag{2.3c}$$

Equation 2.3a is Gauss's law for the magnetic induction field, Equation 2.3b is Ampere's law, which gives the relationship between the current density and the magnetic field intensity $\boldsymbol{H}$, and Equation 2.3c is Faraday's law; in Faraday's law $\boldsymbol{E}$ represents the electrical field. In Ampere's law the time derivative of the displacement electric field (usually denoted by $\boldsymbol{D}$) is neglected; this corresponds to neglecting parasitic capacitive effects. This assumption is consistent with the time scale of the phenomena involved.

Finally the constitutive relations

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} \tag{2.4a}$$

$$\eta \boldsymbol{J} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \tag{2.4b}$$

complete the set of ideal magnetohydrodynamic equations. These equations are summarized in Table 2.1.

Table 2.1. The ideal magnetohydrodynamical equations

$\nabla \cdot \boldsymbol{B} = 0$	$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$
$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$	$\rho \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = \boldsymbol{J} \times \boldsymbol{B} - \nabla p$
$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$	$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J}$

2.2 Magnetohydrodynamics in Axisymmetric Toroidal Geometry: the Poloidal Flux Function

Since a tokamak is an axisymmetric toroidal machine, it is convenient to write the magnetohydrodynamics equations in a three-dimensional cylindrical coordinates system (r, φ, z), where the axis $r = 0$ is the rotational axis of the tokamak.

In what follows:

- $\boldsymbol{r}$ will denote a generic point with cylindrical coordinate (r, φ, z), where r is the radial coordinate, φ is the toroidal angle, and z is the height; $\boldsymbol{i}_r$, $\boldsymbol{i}_\varphi$ and $\boldsymbol{i}_z$ will denote the axis unit vectors;

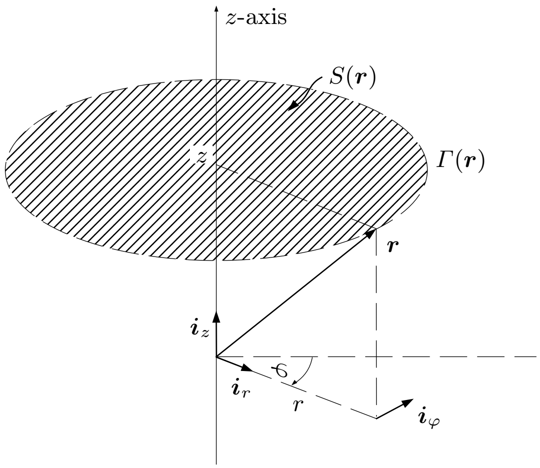

Figure 2.1. The cylindrical coordinate system

- $\Gamma(\boldsymbol{r})$ will denote the circumference given by the rotation of the point $\boldsymbol{r}$ around the $r = 0$ axis;
- $S(\boldsymbol{r})$ will denote a surface having $\Gamma(\boldsymbol{r})$ as edge (see also Figure 2.1).

Given a generic vector $\boldsymbol{A}$, its components along the unit vectors will be denoted by A_r, A_φ, and A_z, respectively, so as to have

$$\boldsymbol{A} = A_r \boldsymbol{i}_r + A_\varphi \boldsymbol{i}_{\boldsymbol{\varphi}} + A_z \boldsymbol{i}_z \,.$$

In each point the direction parallel to the unit vector $\boldsymbol{i}_\varphi$ is called toroidal, while the plane perpendicular to this direction is called poloidal; this plane is characterized by a constant toroidal angle φ.

Moreover, due to the toroidal axisymmetric geometry of a tokamak machine, it is possible to assume that all the quantities involved do not depend on the toroidal angle; as a consequence, again with reference to a generic vector $\boldsymbol{A}$, it is possible to assume

$$\frac{\partial}{\partial \varphi} \boldsymbol{A} = 0 \,.$$

Making use of the axisymmetric assumption, Gauss's law (Equation 2.3a) in cylindrical coordinates can be written as

$$\frac{1}{r} \frac{\partial}{\partial r} r B_r + \frac{\partial}{\partial z} B_z = 0 \,. \tag{2.5}$$

Now it is convenient to introduce the poloidal flux function

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi} \int_{S(\boldsymbol{r})} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \,. \tag{2.6}$$

Since the surface integral in (2.6) does not depend on the particular surface $S(\boldsymbol{r})$, but only on its edge $\Gamma(\boldsymbol{r})$, choosing $S(\boldsymbol{r})$ perpendicular at each point to $\boldsymbol{i}_z$ (as in Figure 2.1), one obtains

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi}\int_0^r\int_0^{2\pi} B_z(\rho,z)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi = \int_0^r \rho B_z(\rho,z)\,\mathrm{d}\rho\,. \tag{2.7}$$

Differentiation of Equation 2.7 with respect to r gives

$$\frac{\partial}{\partial r}\psi = rB_z\,,$$

while differentiating the same equation with respect to z, and taking into account Equation 2.5, results in

$$\frac{\partial}{\partial z}\psi = -rB_r\,.$$

Hence the poloidal flux function and the magnetic inductance field are linked by the following equations

$$B_r = \frac{1}{r}\frac{\partial}{\partial z}\psi \tag{2.8a}$$

$$B_z = -\frac{1}{r}\frac{\partial}{\partial r}\psi\,. \tag{2.8b}$$

Equations 2.8, taking into account that $\boldsymbol{\nabla}\varphi = r^{-1}\boldsymbol{i}_\varphi$, can be written in vectorial notation

$$\boldsymbol{B}_p = B_r\boldsymbol{i}_r + B_z\boldsymbol{i}_z = \boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\varphi\,, \tag{2.9}$$

where $\boldsymbol{B}_p$ represents the projection of the magnetic induction field on the poloidal plane.

Note that the existence of a scalar function ψ satisfying Equations 2.8 is a consequence only of the divergenceless of the magnetic induction field. Now, applying the divergence operator to Ampere's law (2.3b), it is simple (taking into account that the divergence of a rotor is zero) to show that also the current density vector $\boldsymbol{J}$ is divergenceless. Hence there will exist a scalar function f satisfying the relations

$$J_r = -\frac{1}{r}\frac{\partial}{\partial z}f \tag{2.10a}$$

$$J_z = \frac{1}{r}\frac{\partial}{\partial r}f\,. \tag{2.10b}$$

Ampere's law (2.3b), combined with the constitutive relation (2.4a) and the axisymmetric assumption, is written in cylindrical coordinates as

$$-\frac{\partial}{\partial z}B_\varphi = \mu_0 J_r \tag{2.11a}$$

$$\frac{\partial}{\partial z}B_r - \frac{\partial}{\partial r}B_z = \mu_0 J_\varphi \tag{2.11b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}rB_\varphi = \mu_0 J_z\,. \tag{2.11c}$$

Combining Equations 2.10 with Equations 2.11a and 2.11c gives

$$B_\varphi = \mu_0 \frac{f}{r} \,.$$

Letting $F(r,z) = \mu_0 f(r,z)$, the toroidal component of the magnetic induction field can be written as

$$B_\varphi = F \boldsymbol{\nabla}\, \varphi \cdot \boldsymbol{i}_\varphi \,. \tag{2.12}$$

Finally the magnetic induction field can be expressed through the two scalar functions ψ and F as

$$\boldsymbol{B} = \boldsymbol{\nabla}\, \psi \times \boldsymbol{\nabla}\, \varphi + F \boldsymbol{\nabla}\, \varphi \,. \tag{2.13}$$

As already said, the first term on the right-hand side of Equation 2.13 gives the projection of the magnetic induction field on the poloidal plane (poloidal magnetic induction field), while the second term gives the toroidal component (toroidal magnetic inductance field). Substituting Equation 2.13 in Ampere's law (2.3b) gives

$$\begin{aligned} \boldsymbol{J} &= \mu_0^{-1} \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla}\, \psi \times \boldsymbol{\nabla}\, \varphi + F \boldsymbol{\nabla}\, \varphi \right) \\ &= -\mu_0^{-1} \Delta^*\, \psi \boldsymbol{\nabla}\, \varphi + \mu_0^{-1} \boldsymbol{\nabla}\, F \times \boldsymbol{\nabla}\, \varphi \,, \end{aligned} \tag{2.14}$$

where Δ^* is the differential elliptic operator defined by the relation

$$\Delta^*\, \chi = r^2 \boldsymbol{\nabla} \cdot \left(r^{-2} \boldsymbol{\nabla}\, \chi \right) = r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}\, \chi \right) + \frac{\partial^2}{\partial z^2} \chi \,.$$

Projection of Equation 2.14 along the toroidal direction gives

$$\Delta^*\, \psi = -\mu_0 r J_\varphi \,, \tag{2.15}$$

where J_φ is the toroidal current density.

Another useful relation can be found between the toroidal component of the electrical field and the time derivative of the poloidal flux function. Starting from Faraday's law (2.3c) and applying the Kelvin–Stokes theorem, one obtains

$$\begin{aligned} \oint_{\Gamma(\boldsymbol{r})} \boldsymbol{E} \cdot \mathbf{d}\boldsymbol{l} &= -\frac{\partial}{\partial t} \int_{S(\boldsymbol{r})} \boldsymbol{B} \cdot \mathbf{d}\boldsymbol{S} \\ &= -2\pi \frac{\partial}{\partial t}\, \psi \,, \end{aligned}$$

from which it can be easily obtained that

$$E_\varphi = -\frac{1}{r} \frac{\partial}{\partial t}\, \psi \,. \tag{2.16}$$

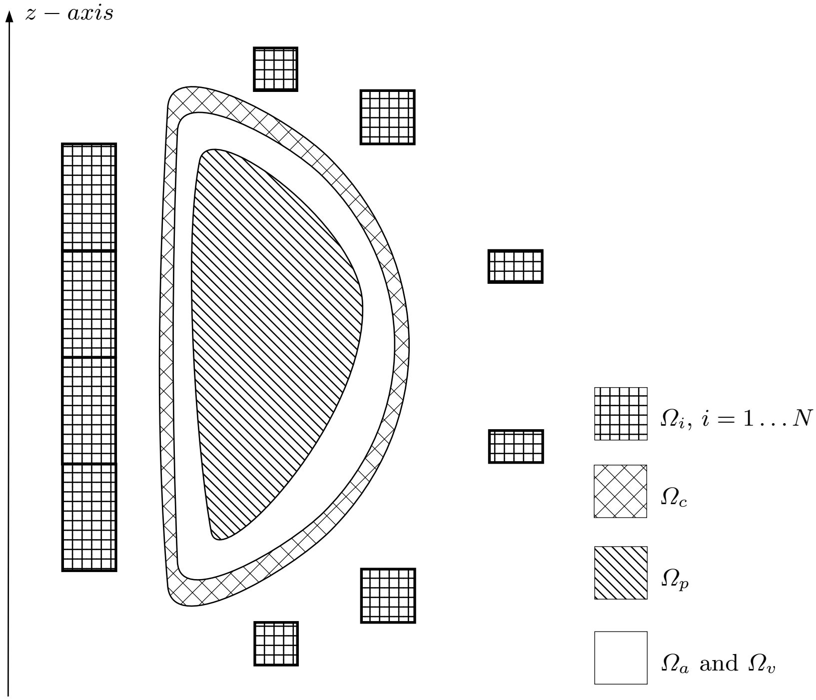

Figure 2.2. The poloidal cross-section of a tokamak machine can be partitioned into regions occupied by the plasma (Ω_p), by the conducting structure (Ω_c), by the poloidal field coils (Ω_i, $i = 1 \ldots N$), by the air (Ω_a) and by the vacuum (Ω_v).

2.3 A Plasmaless Model

In this section an electromagnetic model of a tokamak in the absence of the plasma will be derived. This model will enable one to evaluate the poloidal flux function at each point in space, given the voltages applied to the poloidal field coils. With reference to Figure 2.2 the poloidal plane can be divided into two types of region: the air and vacuum region ($\Omega_a \cup \Omega_v$), and the region $\Omega_m = \Omega_c \cup \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_N$ occupied by conducting materials. The set $\mathcal{L} = \Omega_p \cup \Omega_v$ is the vacuum vessel region, that is the space inside the tokamak that the plasma can occupy.

In Ω_m Ohm's law takes the form

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{E_m}), \tag{2.17}$$

where σ is the conductivity (the inverse of the resistivity η) of the materials involved, and $\boldsymbol{E_m}$ is the electromotive field supplying the electromotive force to the poloidal field coils ($\boldsymbol{E_m} = 0$ in Ω_c).

Integrating Equation 2.17 along the circuit individuated by the closed line $\Gamma(\boldsymbol{r})$ gives

$$\oint_{\Gamma(\boldsymbol{r})} \boldsymbol{J} \cdot \mathbf{d}\boldsymbol{l} = \sigma \left(\oint_{\Gamma(\boldsymbol{r})} \boldsymbol{E} \cdot \mathbf{d}\boldsymbol{l} + \oint_{\Gamma(\boldsymbol{r})} \boldsymbol{E_m} \cdot \mathbf{d}\boldsymbol{l} \right) ,$$

from which, invoking again the axisymmetric assumption, and taking into account Equation 2.16, can be obtained

$$2\pi r J_\varphi = -2\pi\sigma \frac{\partial}{\partial t} \psi + \sigma V , \tag{2.18}$$

where

$$V = \oint_{\Gamma(\boldsymbol{r})} \boldsymbol{E_m} \cdot \mathbf{d}\boldsymbol{l} ,$$

represents the electromotive force on the $\Gamma(\boldsymbol{r})$ circuit. It is evident that the electromotive force is different from zero only in the regions where the active poloidal field coils are located; moreover, in each of these regions, V can be assumed to be constant, and so it is possible to write

$$V(\boldsymbol{r}, t) = \sum_{i=1}^{N} V_i(t) g_i(\boldsymbol{r}) , \tag{2.19}$$

where g_i is the characteristic function of the Ω_i set, that is

$$g_i(\boldsymbol{r}) = \begin{cases} 1 , & \text{if } \boldsymbol{r} \in \Omega_i \\ 0 , & \text{if } \boldsymbol{r} \notin \Omega_i . \end{cases}$$

Combining Equations 2.15 and 2.18, and considering that $J_\varphi = 0$ in $\mathcal{L}$ and Ω_a, one obtains that the poloidal flux function in the absence of the plasma must satisfy the following partial differential equation

$$\Delta^* \psi = \begin{cases} 0 , & \text{if } \boldsymbol{r} \in \Omega_v \cup \Omega_a \\ \mu_0 \sigma \frac{\partial}{\partial t} \psi , & \text{if } \boldsymbol{r} \in \Omega_c \\ \mu_0 \sigma \frac{\partial}{\partial t} \psi - \frac{\mu_0 \sigma}{2\pi} V_i , & \text{if } \boldsymbol{r} \in \Omega_i , \quad i = 1 \ldots N . \end{cases} \tag{2.20}$$

To find a unique solution to this equation, initial and boundary conditions must be provided

$$\psi(r, z, t)|_{t=0} = \psi_0(r, z) \tag{2.21a}$$
$$\psi(r, z, t)|_{r=0} = 0 \tag{2.21b}$$
$$\lim_{\boldsymbol{r} \to \infty} \psi(r, z, t) = 0 . \tag{2.21c}$$

The initial condition (2.21a) provides the flux distribution at the starting time; if it is assumed that at this time there is no current distribution in the conducting region, then this initial distribution can be assumed to be zero everywhere. The boundary condition (2.21b) is a consequence of the poloidal flux definition, whereas (2.21c) is a regularity assumption of the magnetic induction field as $\boldsymbol{r} \to \infty$.

Once the time behaviour of the voltages V_i applied to the poloidal field coils is assigned, it is possible in principle to integrate the partial differential Equation 2.20 with the conditions (2.21) to evaluate the poloidal flux function at each point of the poloidal plane. The difficulties in finding an analytical solution to this problem justify the use of a numerical approach based on finite element and Galerkin methods. Following [43], let

$$J_\varphi(\boldsymbol{r}, t) = \sum_{h=1}^{n_c} I_h(t) q_h(\boldsymbol{r}) , \tag{2.22}$$

in such a way as to approximate the toroidal current density with the sum of n_c base functions q_h, weighted by unknown coefficients I_h. Each base function q_h has a compact support D_h (*i.e.*, it is zero outside D_h), and satisfies the conditions

$$\boldsymbol{\nabla} \cdot (q_h \boldsymbol{i}_\varphi) = 0 \tag{2.23a}$$

$$\frac{\partial}{\partial \varphi} q_h = 0 \tag{2.23b}$$

$$\int_{D_h} q_h \mathrm{d}S = 1 . \tag{2.23c}$$

In this way the region Ω_m is discretized in a finite number of circuits, having D_h as cross-section on the poloidal plane. The subsets D_h satisfy the properties

$$\bigcup_{h=1}^{n_c} D_h = \Omega_m \tag{2.24}$$

$$D_h \cap D_k = \emptyset \text{ when } h \neq k \tag{2.25}$$

$$\exists k \in \{c, 1, \ldots, N\} : D_h \cap \Omega_k \neq \emptyset \Rightarrow D_h \subseteq \Omega_k . \tag{2.26}$$

Therefore the subsets D_h completely cover the Ω_m region, the intersection between two of these subsets is empty, and finally each domain D_h can have a no empty intersection with at most one of the regions Ω_c, $\Omega_1, \ldots, \Omega_N$.

In Section A.1 it is shown that a general solution of Equation 2.15 can be written as

$$\psi(\boldsymbol{r}, t) = \int_{\mathbb{R}^2} J_\varphi(\boldsymbol{r}', t) G_0(\boldsymbol{r}, \boldsymbol{r}') \mathrm{d}S' , \tag{2.27}$$

where $G_0(\boldsymbol{r}, \boldsymbol{r}')$ is the free space Green's function defined in (A.4). Equation 2.27 enables one to write

$$\psi(\boldsymbol{r},t) = \sum_{h=1}^{n_c} I_h(t)\tilde{q}_h(\boldsymbol{r})\,, \tag{2.28}$$

where

$$\tilde{q}_h(\boldsymbol{r}) = \int_{\Omega_m} q_h(\boldsymbol{r}')G_0(\boldsymbol{r},\boldsymbol{r}')\,\mathrm{d}S'\,.$$

Equation 2.18 can be written as

$$\frac{1}{\sigma}J_\varphi = -\frac{1}{r}\frac{\partial}{\partial t}\psi + \frac{1}{2\pi r}V\,. \tag{2.29}$$

Multiplying both sides of (2.29) by q_h, and integrating over the volume V_m obtained by rotating the domain Ω_m around the z-axis, the following equality is obtained

$$\int_{V_m} \frac{1}{\sigma}J_\varphi q_h\,\mathrm{d}\tau = -\int_{V_m}\frac{1}{r}q_h\frac{\partial}{\partial t}\psi\,\mathrm{d}\tau + \frac{1}{2\pi}\int_{V_m}\frac{1}{r}Vq_h\,\mathrm{d}\tau\,. \tag{2.30}$$

Now

$$\begin{aligned}
\int_{V_m}\frac{1}{\sigma}J_\varphi q_h \mathrm{d}\tau &= \sum_{k=1}^{n_c} I_k \int_{V_m}\frac{q_k q_h}{\sigma}\mathrm{d}\tau \\
&= \sum_{k=1}^{n_c} I_k \int_0^{2\pi}\int_{\Omega_m}\frac{q_k q_h}{\sigma} r\,\mathrm{d}\varphi\,\mathrm{d}S \\
&= \sum_{k=1}^{n_c} I_k 2\pi \int_{\Omega_m} r\frac{q_k q_h}{\sigma}\,\mathrm{d}S \\
&= \sum_{k=1}^{n_c} R_{hk} I_k\,;
\end{aligned}$$

similarly

$$\begin{aligned}
\int_{V_m}\frac{1}{r}q_h\frac{\partial}{\partial t}\psi\,\mathrm{d}\tau &= \sum_{k=1}^{n_c}\dot{I}_k\int_{V_m}\frac{\tilde{q}_k q_h}{r}\,\mathrm{d}\tau \\
&= \sum_{k=1}^{n_c}\dot{I}_k 2\pi\int_{\Omega_m}\tilde{q}_k q_h\,\mathrm{d}S \\
&= \sum_{k=1}^{n_c} L_{hk}\dot{I}_k\,,
\end{aligned}$$

and finally

$$\begin{aligned}\frac{1}{2\pi}\int_{V_m}\frac{1}{r}Vq_h\,\mathrm{d}\tau &= \frac{1}{2\pi}\sum_{l=1}^{N}V_l\int_{V_m}\frac{g_l q_h}{r}\,\mathrm{d}\tau\\ &= \sum_{l=1}^{N}V_l\int_{\Omega_m}g_l q_h\,\mathrm{d}S\\ &= \sum_{l=1}^{N}B_{hl}V_l\,,\end{aligned}$$

where the dot over a time-varying function denotes its time derivative, and

$$R_{hk} = 2\pi\int_{\Omega_m} r\frac{q_k q_h}{\sigma}\,\mathrm{d}S \tag{2.31a}$$

$$L_{hk} = 2\pi\int_{\Omega_m}\tilde{q}_k q_h\,\mathrm{d}S \tag{2.31b}$$

$$B_{hl} = \int_{\Omega_m} g_l q_h\,\mathrm{d}S\,, \tag{2.31c}$$

with $h = 1,\ldots,n_c$, $k = 1,\ldots,n_c$, and $l = 1,\ldots,N$. Equation 2.30 can be written in the form

$$\sum_{k=1}^{n_c}L_{hk}\dot{I}_k + \sum_{k=1}^{n_c}R_{hk}I_k = \sum_{l=1}^{N}B_{hl}V_l\,. \tag{2.32}$$

Note that

$$B_{hl} = \begin{cases}1\,, & \text{if } D_h \subseteq \Omega_l\\ 0\,, & \text{if } D_h \not\subseteq \Omega_l\end{cases} \tag{2.33}$$

$$R_{hk} = 0 \text{ if } h \neq k\,. \tag{2.34}$$

Now defining the matrices $L_c \in \mathbb{R}^{n_c\times n_c}$, $R_c \in \mathbb{R}^{n_c\times n_c}$, and $B_c \in \mathbb{R}^{n_c\times N}$, whose elements are the scalars L_{hk}, R_{hk}, and B_{hl}, respectively, and the vectors $\tilde{x} = \left(I_1\ I_2\ \ldots\ I_{n_c}\right)^T \in \mathbb{R}^{n_c}$, $u = \left(V_1\ V_2\ \ldots V_N\right)^T \in \mathbb{R}^N$, the n_c equations (2.32) can be written in matrix form as

$$L_c\,\dot{\tilde{x}} + R_c\,\tilde{x} = B_c\,u\,. \tag{2.35}$$

As can be noted, Equation 2.35 is in the same form as a system consisting of n_c circuits with inductors, resistors, and voltage sources. The generic element L_{hk} of the L_c matrix corresponds to the mutual inductance between the circuit h and the circuit k, while the diagonal element L_{hh} corresponds to the self-inductance of the circuit h; it is a basic property of circuit theory that the inductance matrix L is symmetric, diagonal dominant and invertible. Similarly R_{hh} corresponds to the resistance of the circuit h; therefore the diagonal matrix R_c is called the resistance matrix. A voltage source is present only on the circuits contained in one of the Ω_k regions. To simplify the notation, it can be assumed that $D_h = \Omega_h$ for $h = 1, 2, \dots N$; in such a way the first N subsets D_h are equal to the N regions Ω_k containing the poloidal field coils, and these regions are discretized in only one circuit. If this is the case, the L_c, R_c and B_c matrices, and the vector $\tilde{x}$ can be decomposed as

$$L_c = \begin{pmatrix} L_a & L_{ab} \\ L_{ab}^T & L_b \end{pmatrix} \tag{2.36a}$$

$$R_c = \begin{pmatrix} R_a & 0 \\ 0 & R_b \end{pmatrix} \tag{2.36b}$$

$$B_c = \begin{pmatrix} I \\ 0 \end{pmatrix} \tag{2.36c}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_e \end{pmatrix}, \tag{2.36d}$$

where the matrices L_{aa} and R_{aa} are of dimension n_c, and I denotes the identity matrix. In this way (2.35) can be divided into two equations: one related to the currents flowing in the poloidal field coils (the active circuits), and one related to the currents flowing in the other conducting structures (the passive circuits)

$$L_a\, \dot{x}_a + L_{ab}\, \dot{x}_e + R_a\, x_a = u \tag{2.37a}$$

$$L_{ab}^T\, \dot{x}_a + L_e\, \dot{x}_e + R_e\, x_e = 0\,. \tag{2.37b}$$

Equations 2.37 show that the active circuits currents can be controlled using the input voltages vector u; the time variations of these currents are opposed by the *eddy currents* induced in the conducting structures. Note that once the vector $\tilde{x}$ is assigned, which is equivalent to assigning the distribution of the toroidal current density, using Equation 2.28 it is possible to evaluate the poloidal flux function at each point of the poloidal plane.

2.4 The Plasma Equilibrium

In the previous section a model describing the electromagnetic behaviour of a tokamak machine in the absence of the plasma, has been derived. As Equation 2.35 shows, this model is linear; in the next sections of this chapter it will be shown that the presence of the plasma makes the model nonlinear, and that the added complexity can be overcome by resorting to a linearized model valid in the neighbourhood of an equilibrium point. The first step, therefore, aims at characterizing the plasma equilibrium configurations.

The balance between the plasma pressure and the magnetic confinement forces can be studied with the aid of the equations written in Sections 2.1 and 2.2. The basic condition for equilibrium is that the overall force acting on an infinitesimal plasma volume is zero; this is expressed by the equation

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B}\,, \tag{2.38}$$

from which it can be obtained that

$$\boldsymbol{B} \cdot \nabla p = 0 \tag{2.39a}$$
$$\boldsymbol{J} \cdot \nabla p = 0\,. \tag{2.39b}$$

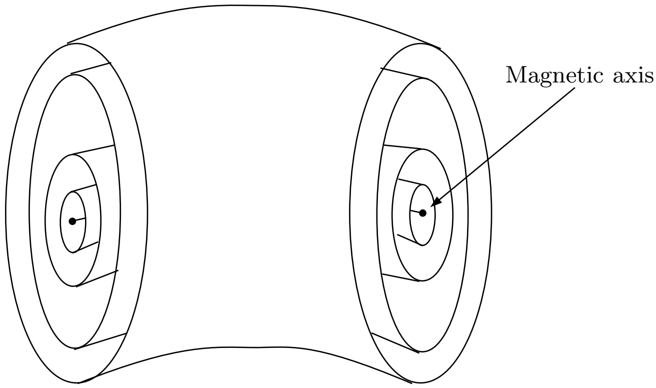

Figure 2.3. Isobaric surfaces in a plasma equilibrium configuration

Equations 2.39 show that the field lines of the magnetic induction and of the current density lie on isobaric surfaces (surfaces where the pressure is constant). For most plasma equilibria the pressure is maximum near the centre of the poloidal cross-section of the plasma, and the isobaric surfaces are toroidally nested as shown in Figure 2.3 (see [10] for a detailed explanation). As a consequence of the fact that the magnetic field lines lie on the isobaric surfaces, these surfaces are also called magnetic surfaces. The limiting magnetic surface, which approaches a single magnetic line where the pressure is maximum, is called the magnetic axis.

Now it follows from Equation 2.8 that

$$\boldsymbol{B} \cdot \nabla \psi = 0\,,$$

so the magnetic (or isobaric) surfaces also coincide with the constant poloidal flux surfaces. Hence on the poloidal plane the current density, the magnetic induction and the pressure are constant on each line level of the ψ function. As a consequence, it is possible to consider these quantities (and the others related to them) as dependent only on the poloidal flux

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}(\psi) \\ \boldsymbol{J} &= \boldsymbol{J}(\psi) \\ p &= p(\psi)\,. \end{aligned}$$

Starting from the force equilibrium Equation 2.38, using the equalities (2.13) and (2.14), and taking into account that p and F are functions only of ψ, it is possible to obtain that

$$\boldsymbol{\nabla}\, p = -\frac{1}{r} J_\varphi \boldsymbol{\nabla}\, \psi - \frac{1}{\mu_0 r^2} F \boldsymbol{\nabla}\, F\,. \tag{2.40}$$

Considering that

$$\boldsymbol{\nabla}\, p = \frac{\mathrm{d}}{\mathrm{d}\psi}\, p \boldsymbol{\nabla}\, \psi \tag{2.41}$$

$$\boldsymbol{\nabla}\, F = \frac{\mathrm{d}}{\mathrm{d}\psi}\, F \boldsymbol{\nabla}\, \psi\,, \tag{2.42}$$

then

$$J_\varphi = -r \frac{\mathrm{d}}{\mathrm{d}\psi}\, p - \frac{1}{\mu_0 r} F \frac{\mathrm{d}}{\mathrm{d}\psi}\, F\,. \tag{2.43}$$

Putting together Equations 2.15 and 2.43 the celebrated *Grad–Shafranov* equation is obtained

$$\Delta^*\, \psi = -\mu_0 r^2 \frac{\mathrm{d}}{\mathrm{d}\psi}\, p - F \frac{\mathrm{d}}{\mathrm{d}\psi}\, F\,. \tag{2.44}$$

The plasma equilibrium is then completely characterized by the following nonlinear partial differential problem

$$\Delta^*\, \psi = \begin{cases} 0\,, & \text{if } \boldsymbol{r} \in \Omega_v \cup \Omega_a \\ \mu_0 r J_\varphi\,, & \text{if } \boldsymbol{r} \in \Omega_m \\ -\mu_0 r^2 \dfrac{\mathrm{d}}{\mathrm{d}\psi}\, p - F \dfrac{\mathrm{d}}{\mathrm{d}\psi}\, F\,, & \text{if } \boldsymbol{r} \in \Omega_p\,. \end{cases} \tag{2.45}$$

$$\psi(r, z)|_{r=0} = 0 \tag{2.46a}$$

$$\lim_{\boldsymbol{r}\to\infty} \psi(r, z) = 0\,. \tag{2.46b}$$

This problem can be solved when the current density external to the plasma region, and the functions $p(\psi)$ and $F(\psi)$ have been assigned. Note that this is a free boundary problem, the boundary $\partial\Omega_p$ of Ω_p being one of the unknowns to be determined.

Several numerical codes [44, 45, 46, 47] have been developed to solve this problem. The choice of the functions $p(\psi)$ and $F(\psi)$ determine the toroidal

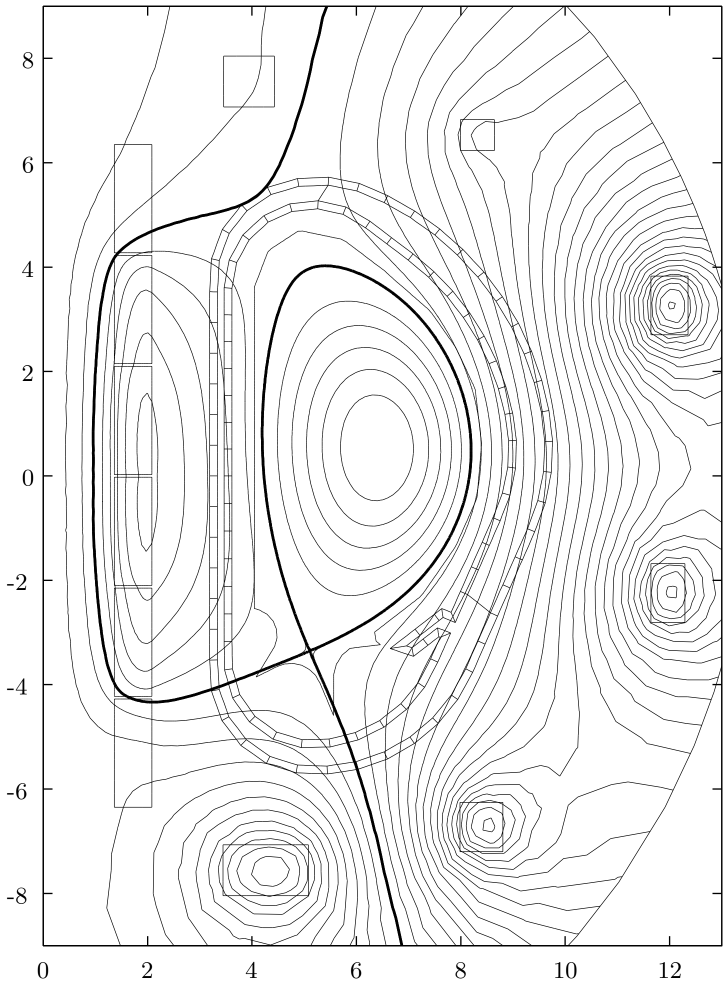

Figure 2.4. Constant level curves of the poloidal flux function for a plasma equilibrium as determined by the CREATE-L numerical code [48]. The thicker line corresponds to the value of the poloidal flux at the plasma boundary.

current density inside the plasma (see Equation 2.43). Although the problem of determining this current density could be, in principle, included in Equations 2.45 and 2.46 adding a certain number of equations related to the diffusion and to the transport of the plasma particles, it is simpler to adopt here an approach based on experimental evidence [49] and assign J_φ inside the plasma as a parameterized function. The parameters used to characterize the toroidal current density are the total plasma current I_p, the poloidal beta β_p, and the internal inductance l_i; these quantities are defined as

$$I_p = \int_{\Omega_p} J_\varphi \, \mathrm{d}S \tag{2.47a}$$

$$\beta_p = \frac{4}{\mu_0 r_c I_p^2} \int_{V_p} p \, \mathrm{d}\tau \tag{2.47b}$$

$$l_i = \frac{4}{\mu_0 r_c I_p^2} \int_{V_p} \frac{\|\boldsymbol{B_p}\|^2}{2\mu_0} \, \mathrm{d}\tau \,, \tag{2.47c}$$

where r_c is the horizontal coordinate of the plasma current centroid (r_c, z_c) defined as

$$r_c = \left(\frac{1}{I_p} \int_{\Omega_p} r^2 J_\varphi \, \mathrm{d}S \right)^{\frac{1}{2}} \tag{2.48a}$$

$$z_c = \frac{1}{I_p} \int_{\Omega_p} z J_\varphi \, \mathrm{d}S \,. \tag{2.48b}$$

The total plasma current is the current flowing through the poloidal plane in the plasma region. The poloidal beta is a measure of the efficiency of the plasma confinement: indeed, it is a measure of the ratio between the pressure energy and the magnetic energy in the plasma. The internal inductance is a dimensionless quantity and it is linked to the magnetic energy in the plasma region. The plasma current centroid is a sort of geometrical centre for the plasma region.

Coming back to the problem of characterizing the toroidal current density inside the plasma region, in [49] the following expression is proposed

$$J_\varphi = \lambda \left[\beta_0 \frac{r}{r_0} + (1 - \beta_0) \frac{r_0}{r} \right] (1 - \tilde{\psi}^m)^n \,, \tag{2.49}$$

where

$$\tilde{\psi} = \frac{\psi - \psi_a}{\psi_b - \psi_a}$$

is the so-called normalized flux, ψ_b and ψ_a being the flux values at the plasma boundary and at the magnetic axis, respectively, r_0 the horizontal coordinate of a characteristic point inside the vacuum vessel (typically the centre of the chamber), and λ, β_0, m and n parameters which are related to β_p, l_i and I_p. Once these parameters are assigned, it is possible to solve the problem given by Equations 2.45 and 2.46, evaluate the poloidal flux function, and hence characterize the plasma equilibrium.

As seen in Section 2.3 the toroidal current density J_φ outside the plasma region is completely defined by the vector $\tilde{x}$, whose components represent the current flowing in each region D_h; while inside the plasma the toroidal current density is completely defined by the two-dimensional vector $w = \left(\beta_p \; l_i\right)^T$ and by I_p. Therefore at each point $\boldsymbol{r}$ of the poloidal plane it is possible to write

$$\psi(\boldsymbol{r}) = \gamma_1(\boldsymbol{r}, \tilde{x}, w, I_p) \,, \tag{2.50}$$

where the function γ_1 is not given analytically, but can be computed numerically by a solver of the problem (2.45) and (2.46). A typical result obtained using the equilibrium solver of the CREATE-L code [48] is shown in Figure 2.4.

2.5 A Linearized Model for Plasma Behaviour

In this section a linearized model of the plasma will be derived. This model describes the plasma behaviour, in a neighbourhood of an equilibrium configuration, from an electromagnetic point of view. This model will be used for the plasma shape and position control system design in later chapters. The fundamental assumption made to derive this model is that the mass density of the plasma can be considered very small, so that the inertial term in Equation 2.2 becomes negligible. This assumption is certainly satisfied on the typical time scale considered in the shape and position control design problem. If this is the case, the plasma equilibrium Equation 2.38 is satisfied at each time instant. In other words the plasma evolves through a sequence of static equilibria. The only dynamic behaviour is in the time evolution of the currents flowing in the conducting structures, for which it is possible to obtain a finite-dimensional model using the approach of Section 2.3.

Since Equation 2.15 continues to hold, it is possible to use again the free space Green's function defined in (A.4) to express the poloidal flux function

$$\begin{aligned}\psi(\boldsymbol{r},t) &= \int_{\mathbb{R}^2} J_\varphi(\boldsymbol{r}',t)G_0(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}S' \\ &= \int_{\Omega_m} J_\varphi(\boldsymbol{r}',t)G_0(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}S' + \int_{\Omega_p} J_\varphi(\boldsymbol{r}',t)G_0(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}S' \\ &= \psi_m(\boldsymbol{r},t) + \psi_p(\boldsymbol{r},t)\,. \end{aligned} \tag{2.51}$$

The first term in (2.51) gives the flux produced by the current flowing in the conducting structures, while the second term gives the flux produced by the current distribution in the plasma. Now

$$E_\varphi = -\frac{1}{r}\frac{\partial}{\partial t}\psi = -\frac{1}{r}\frac{\partial}{\partial t}\psi_m - \frac{1}{r}\frac{\partial}{\partial t}\psi_p\,, \tag{2.52}$$

therefore, starting again from Equation 2.30 and repeating the same mathematical derivations of Section 2.3, it is possible to arrive at the equation

$$\sum_{k=1}^{n_c} L_{hk}\dot{I}_k + \sum_{k=1}^{n_c} R_{hk}I_k + \dot{\Psi}_{p_h} = \sum_{l=1}^{N} B_{hl}V_l\,, \tag{2.53}$$

where L_{hk}, R_{hk}, B_{hl} have been introduced in Equations 2.31, and

$$\Psi_{p_h} = 2\pi\int_{\Omega_m} \psi_p q_h\,\mathrm{d}S\,. \tag{2.54}$$

The term $\dot{\Psi}_{p_h}$ in Equation 2.53 represents the electromotive force which appears on the circuit h as a consequence of the time variations of the current density inside the plasma. These variations are due to changes in the plasma current internal profile, or also to the movements of the plasma ring. Defining the vector $\Psi_p = \left(\Psi_{p_1}\ \Psi_{p_2}\ \dots\ \Psi_{p_{n_c}}\right)^T \in \mathbb{R}^{n_c}$, and considering Equation 2.53 for each circuit, the following matrix equation is obtained

$$L_c\,\dot{\tilde{x}} + R_c\,\tilde{x} + \dot{\Psi}_p = B_c\,u\,. \tag{2.55}$$

Equation 2.55 defines the dynamic behaviour of the currents flowing in the conducting structures in the presence of the plasma. The Ψ_p vector depends only on the flux produced by the plasma on these structures; it can be calculated solving an equilibrium problem when the vectors $\tilde{x}$ and w, and the plasma current I_p have been assigned. In other words it is possible to write

$$\Psi_p = \gamma_2(\tilde{x}, w, I_p)\,, \tag{2.56}$$

where the vectorial function γ_2 is computed using one of the numerical codes cited in Section 2.4.

Putting together Equations 2.55 and 2.56, the following finite-dimensional nonlinear differential equation is obtained

$$L_c\,\dot{\tilde{x}} + R_c\,\tilde{x} + \frac{\mathrm{d}}{\mathrm{d}t}\,\gamma_2(\tilde{x}, w, I_p) = B_c\,u\,. \tag{2.57}$$

Equation 2.57 can be linearized in the neighbourhood of an equilibrium point $(\tilde{x}_0, w_0, I_{p_0})$; indeed letting

$$\begin{aligned}
\tilde{x} &= \tilde{x}_0 + \delta\tilde{x} \\
w &= w_0 + \delta w \\
I_p &= I_{p_0} + \delta I_p \\
u &= u_0 + \delta u = R_c\tilde{x}_0 + \delta u\,,
\end{aligned}$$

and using the standard linearization procedure, it is possible to write

$$\left(L_c + \left[\frac{\partial}{\partial\tilde{x}}\,\gamma_2\right]_0\right)\delta\dot{\tilde{x}} + R_c\delta\tilde{x} + \left[\frac{\partial}{\partial w}\,\gamma_2\right]_0\delta\dot{w} + \left[\frac{\partial}{\partial I_p}\,\gamma_2\right]_0\delta\dot{I}_p = B\delta u\,, \tag{2.58}$$

where the subscript 0 denotes that the Jacobian matrices have to be evaluated at the considered equilibrium point. In Equation 2.58 the plasma current variation δI_p appears as an input parameter; in other words it cannot be determined by the equation itself, but it has to be assigned. Since the plasma current is one of the parameters that are controlled, this problem needs to be solved, since it is preferable to be able to express δI_p as an output of the model. Several methods can be used to overcome this problem; most of them are based on neglecting the plasma resistivity and assuming the conservation of some physical quantity: the plasma current itself, the poloidal flux averaged on the plasma region, the poloidal flux at the magnetic axis, *etc.* These approaches give for the plasma current variation an equation of the type

$$L_p \delta \dot{I}_p + M_{pc} \delta \dot{\tilde{x}} + M_{pw} \delta \dot{w} = 0\,. \tag{2.59}$$

Interestingly, Equation 2.59 shows that, from the point of view of the total current, the plasma can be seen as just another circuit which is added to the ones used to model the conducting structures of the machine; the absence in Equation 2.59 of a resistive term is a consequence of the assumption on plasma resistivity.

Now letting $x = \begin{pmatrix} \tilde{x} & I_p \end{pmatrix}^T$, and defining

$$L^* = \begin{pmatrix} L_c + \left[\frac{\partial}{\partial \tilde{x}} \gamma_2 \right]_0 & \left[\frac{\partial}{\partial I_p} \gamma_2 \right]_0 \\ M_{pc} & L_p \end{pmatrix} \tag{2.60}$$

$$R = \begin{pmatrix} R_c & 0 \\ 0 & 0 \end{pmatrix} \tag{2.61}$$

$$E = \begin{pmatrix} \left[\frac{\partial}{\partial w} \gamma_2 \right]_0 \\ 0 \end{pmatrix} \tag{2.62}$$

$$B = \begin{pmatrix} B_c \\ 0 \end{pmatrix}, \tag{2.63}$$

one obtains the final linearized model

$$L^* \, \delta \dot{x} + R \, \delta x + E \, \delta \dot{w} = B \, \delta u\,, \tag{2.64}$$

where the L^* matrix is often called the *modified inductance matrix*. This model gives only the evolution of the currents in the conducting structures and of the total plasma current. These variables play the role of state variables of the plant to be controlled; Equation 2.64 has to be completed with the static equation relating the inputs (the u vector) and the state variables to the output variables to be controlled. This equation will be derived in Chapter 3.

The matrices in Equation 2.64 are calculated by using numerical codes, for instance those described in [48, 26]. The dimension of the state space vector x depends on the number of finite elements used to discretize the tokamak structure; typical values range from about 100 to 200 depending on the size of the machine.

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