2

Failure Rate and Mean Remaining Lifetime

Reliability engineering, survival analysis and other disciplines mostly deal with positive random variables, which are often called *lifetimes*. As a random variable, a lifetime is completely characterized by its distribution function. A realization of a lifetime is usually manifested by a failure, death or some other 'end event'. Therefore, for example, information on the probability of failure of an operating item in the next (usually sufficiently small) interval of time is really important in reliability analysis. The failure (hazard) rate function $\lambda(t)$ defines this probability of interest. If this function is increasing, then our object is usually degrading in some suitable probabilistic sense, as the conditional probability of failure in the corresponding infinitesimal interval of time increases with time. For example, it is well known that the failure (mortality) rate of adult humans increases exponentially with time; the failure rate of many mechanically wearing devices is also increasing. Thus, understanding and analysing the shape of the failure rate is an essential part of reliability and lifetime data analysis. Similar to the distribution function $F(t)$, the failure rate also completely characterizes the corresponding random variable. It is well known that there exists a simple, meaningful exponential representation for the absolutely continuous distribution function in terms of the corresponding failure rate (Section 2.1).

The study of the failure rate function, the main topic of this book, is impossible without considering other reliability measures. The mean remaining (residual) lifetime function is probably first among these; it also plays a crucial role in the aforementioned disciplines. These functions complement each other nicely: the failure rate gives a description of the random variable in an infinitesimal interval of time, whereas the mean remaining lifetime describes it in the whole remaining interval of time. Moreover, these two functions are connected *via* the corresponding differential equation and asymptotically, as time approaches infinity, one tends to the reciprocal of the other (Section 2.4.3).

In this introductory chapter, we consider only some basic facts, definitions and properties. We will use well-known results and approaches to the extent sufficient for the presentation of other chapters. The topic of reversed failure rate, which has attracted considerable interest recently, and the rather specific Section 2.4.3 on the limiting behaviour of the mean remaining life function can be skipped at first reading.

This chapter is, in fact, a mathematically oriented introduction to some of the main reliability notions and approaches. Recent books by Lai and Xie (2006), Marshall and Olkin (2007), a classic monograph by Barlow and Proschan (1975) and a useful textbook by Rausand and Hoyland (2004) can be used for further reading and as sources of numerous reliability-related results and facts.

2.1 Failure Rate Basics

Let $T \geq 0$ be a continuous lifetime random variable with a cumulative distribution function (Cdf)

$$F(t) = \begin{cases} \Pr[T \leq t], & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Unless stated specifically, we will implicitly assume that this distribution is 'proper', *i.e.*, $F^{-1}(1) = \infty$, and that $F(0) = 0$. The support of $F(t)$ will usually be $[0, \infty)$, although other intervals of $\Re_{+}$ = $[0, \infty)$ will also be used. We can view T as some time to failure (death) of a technical device (organism), but other interpretations and parameterizations are possible as well. Inter-arrival times in a sequence of ordered events or the amount of monotonically accumulated damage on the failure of a mechanical item are also relevant examples of lifetimes.

Denote the expectation of the lifetime variable $E[T]$ by m and assume that it is finite, *i.e.*, $m < \infty$. Assume also that $F(t)$ is absolutely continuous, and therefore the probability density function (pdf) $f(t) = F'(t)$ exists (almost everywhere). Recall that a function $g(t)$ is absolutely continuous in some interval $[a, b]$, $0 \leq a < b \leq \infty$, if for every positive number ε, no matter how small, there is a positive number δ such that whenever a sequence of disjoint subintervals $[x_k, y_k]$, $k = 1,2,...,n$ satisfies

$$\sum_{1}^{n} | y_k - x_k | < \delta,$$

the following sum is bounded by ε:

$$\sum_{1}^{n} | g(y_k) - g(x_k) | < \varepsilon.$$

Owing to this definition, the uniform continuity in $[a, b]$, and therefore the 'ordinary' continuity of the function $g(t)$ in this interval, immediately follows.

In accordance with the definition of $E[T]$ and integrating by parts:

$$\begin{aligned} m &= \lim_{t \to \infty} \int_0^t x f(x) dx \\ &= \lim_{t \to \infty} \left[tF(t) - \int_0^t F(x) dx \right] \end{aligned}$$

$$= \lim_{t\to\infty}\left[-t\bar{F}(t)+\int_0^t \bar{F}(x)dx\right],$$

where

$$\bar{F}(t)=1-F(t)=\Pr[T>t]$$

denotes the corresponding survival (reliability) function. As $0<m<\infty$, it is easy to conclude that

$$m=\int_0^\infty \bar{F}(x)dx\,, \tag{2.1}$$

which is a well-known fact for lifetime distributions. Thus, the area under the survival curve defines the mean of T.

Let an item with a lifetime T and a Cdf $F(t)$ start operating at $t=0$ and let it be operable (alive) at time $t=x.$ The remaining (residual) lifetime is of significant interest in reliability and survival analysis. Denote the corresponding random variable by T_x. The Cdf $F_x(t)$ is obtained using the law of conditional probability (on the condition that an item is operable at $t=x$), *i.e.*,

$$\begin{aligned} F_x(t)=\Pr[T_x\le t]&=\frac{\Pr[x<T\le x+t]}{\Pr[T>x]}\\ &=\frac{F(x+t)-F(x)}{\bar{F}(x)}. \end{aligned} \tag{2.2}$$

The corresponding conditional survival probability is given by

$$\bar{F}_x(t)=\Pr[T_x>t]=\frac{\bar{F}(x+t)}{\bar{F}(x)}\,. \tag{2.3}$$

Although the main focus of this book is on failure rate modelling, analysis of the remaining lifetime, and especially of the mean remaining lifetime (MRL), is often almost as important. We will use Equations (2.2) and (2.3) for definitions of the next section.

Now we are able to define the notion of failure rate, which is crucial for reliability analysis and other disciplines. Consider an interval of time $(t,t+\Delta t]$. We are interested in the probability of failure in this interval given that it did not occur before in $[0,t]$. This probability can be interpreted as the risk of failure (or of some other harmful event) in $(t,t+\Delta t]$ given the stated condition. Using a relationship similar to (2.2), *i.e.*,

$$\begin{aligned} \Pr[t<T\le t+\Delta t\mid T>t]&=\frac{\Pr[t<T\le t+\Delta t]}{\Pr[T>t]}\\ &=\frac{F(t+\Delta t)-F(t)}{\bar{F}(t)}. \end{aligned}$$

Consider the following quotient:

$$\lambda_{\Delta t}(t) = \frac{F(t+\Delta t) - F(t)}{\bar{F}(t)\Delta t}$$

and define the failure rate $\lambda(t)$ as its limit when $\Delta t \to 0$. As the pdf $f(t)$ exists,

$$\begin{aligned}\lambda(t) &= \lim_{\Delta t \to 0} \frac{\Pr[t < T \le t + \Delta t \mid T > t]}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\bar{F}(t)\Delta t} = \frac{f(t)}{\bar{F}(t)} \ .\end{aligned} \tag{2.4}$$

Therefore, when $\Delta(t)$ is sufficiently small,

$$\Pr[t < T \le t + \Delta t \mid T > t] \approx \lambda(t)\Delta t \ ,$$

which gives a very popular and important interpretation of $\lambda(t)\Delta t$ as an approximate conditional probability of a failure in $(t, t+\Delta t]$. Note that $f(t)\Delta t$ defines the corresponding approximate unconditional probability of a failure in $(t, t+\Delta t]$. It is very likely that, owing to this interpretation, failure rate plays a pivotal role in reliability analysis, survival analysis and other fields. In actuarial and demographic disciplines, it is usually called the *force of mortality* or the *mortality rate*. To be precise, the force of mortality in demographic literature is usually the infinitesimal version ($\Delta t \to 0$), whereas the term *mortality rate* more often describes the discrete version when Δt is set equal to a calendar year. For convenience, we will always use the term *mortality rate* as an equivalent of *failure rate* when discussing demographic applications. Chapter 10 will be devoted entirely to some aspects of mortality rate modelling. Note that, when considering real populations, the mortality rate becomes a function of two variables: age t and calendar time x. This creates many interesting problems in the corresponding stochastic analysis. We will briefly discuss some of them in this chapter. For a general introduction to mathematical demography, where the mortality rate also plays a pivotal role, the interested reader is referred to Keyfitz and Casewell (2005).

Definition 2.1. The failure rate $\lambda(t)$, which corresponds to the absolutely continuous Cdf $F(t)$, is defined by Equation (2.4) and is approximately equal to the probability of a failure in a small unit interval of time $(t, t+\Delta t]$ given that no failure has occurred in $[0, t]$.

The following theorem shows that the failure rate uniquely defines the absolutely continuous lifetime Cdf:

Theorem 2.1. Exponential Representation of $F(t)$ by Means of the Failure Rate
Let T be a lifetime random variable with the Cdf $F(t)$ and the pdf $f(t)$.

Then

$$F(t) = 1 - \exp\left(-\int_0^t \lambda(u)du \right). \tag{2.5}$$

Proof. As $f(t) = F'(t)$, we can view Equation (2.4) as an elementary first-order differential equation with the initial condition $F(0) = 0$. Integration of this equation results in the main *exponential formula of reliability and survival analysis* (2.5). ■

The importance of this formula is hard to overestimate as it presents a simple characterization of $F(t)$ *via* the failure rate. Therefore, along with the Cdf $F(t)$ and the pdf $f(t)$, the failure rate $\lambda(t)$ uniquely describes a lifetime T. At many instances, however, this characterization is more convenient, which is often due to the meaningful probabilistic interpretation of $\lambda(t)\Delta t$ and the simplicity of Equation (2.5).

Equation (2.5) has been derived for an absolutely continuous Cdf. Does the probability of failure in a small unit interval of time (which always exists) define the corresponding distribution function of a random variable under weaker assumptions? This question will be addressed in the next chapter.

Remark 2.1 Equation (2.4) can be used for defining the simplest empirical estimator for the failure rate. Assume that there are $N >> 1$ independent, statistically identical items (*i.e.*, having the same Cdf) that started operating in a common environment at $t = 0$. A population of this kind in the life sciences is often called a *cohort*. Failure times of items are recorded, and therefore the number of operating items $N(t), N(0) = N$ at each instant of time $t \geq 0$ is known. Thus, for $N \to \infty$, Equation (2.4) is equivalent to

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{N(t)\Delta t}, \tag{2.6}$$

which can be used as an estimate for the failure rate for finite N and Δt, whereas $(N(t + \Delta t) - N(t)) / N(t)$ is an estimate for the probability of failure in $(t, t + \Delta t]$.

2.2 Mean Remaining Lifetime Basics

How much longer will an item of age x live? This question is vital for reliability analysis, survival analysis, actuarial applications and other disciplines. For example, how much time does an average person aged 65 (which is the typical retirement age in most countries) have left to live? The distribution of this remaining lifetime T_x, $T_0 \equiv T$ is given by Equation (2.2). Note that this equation defines a conditional probability, *i.e.*, the probability on condition that the item is operating at time $t = x$.

Assume, as previously, that $E[T] \equiv m < \infty$. Denote $E[T_t] \equiv m(t)$, $m(0) = m$, where, for the sake of notation, the variable x in Equation (2.2) has been interchanged with the variable t. The function $m(t)$ is called the mean remaining (residual) life (MRL) function. It defines the mean lifetime left for an item of age t.

Along with the failure rate, it plays a crucial role in reliability analysis, survival analysis, demography and other disciplines. In demography, for example, this important population characteristic is called the "life expectancy at time t" and in risk analysis the term "mean excess time" is often used.

Whereas the failure rate function at t provides information on a random variable T about a small interval after t, the MRL function at t considers information about the whole remaining interval (t,∞) (Guess and Proschan, 1988). Therefore, these two characteristics complement each other, and reliability analysis of, *e.g.*, engineering systems is often carried out with respect to both of them. It will be shown in this section that, similar to the failure rate, the MRL function also uniquely defines the Cdf of T and that the corresponding exponential representation is also valid.

In accordance with Equations (2.1) and (2.3),

$$\begin{aligned} m(t) &= E[T_t] = E[T - t \mid T > t] \\ &= \int_0^\infty \bar{F}_t(u)du \\ &= \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)}. \end{aligned} \tag{2.7}$$

Assuming that the failure rate exists and using Equation (2.5), Equation (2.7) can be transformed into

$$m(t) = \int_0^\infty \exp\left\{-\int_t^{t+u} \lambda(x)dx\right\}du .$$

It easily follows from these equations that the MRL function, which corresponds to the constant failure rate λ, is also constant and is equal to $1/\lambda$.

Definition 2.2. The MRL function $m(t) = E[T_t]$, $m(0) \equiv m < \infty$, is defined by Equation (2.7), obtained by integrating the survival function of the remaining lifetime T_t.

Alternatively, integrating by parts, similar to (2.1),

$$\int_t^\infty uf(u)du = \int_t^\infty \bar{F}(u)du + t\bar{F}(t) .$$

Therefore, the last integral in (2.7) can be obtained from this equation, which results in the equivalent expression

$$m(t) = \frac{\int_t^\infty uf(u)du}{\bar{F}(t)} - t . \tag{2.8}$$

Equation (2.8) can be sometimes helpful in reliability analysis.

Assume that $m(t)$ is differentiable. Differentiation in (2.7) yields

$$m'(t) = \frac{\lambda(t)\int_t^\infty \bar{F}(u)du - \bar{F}(t)}{\bar{F}(t)}$$
$$= \lambda(t)m(t) - 1 \,. \tag{2.9}$$

From Equation (2.9) the following relationship between the failure rate and the MRL function is obtained:

$$\lambda(t) = \frac{m'(t)+1}{m(t)} \,. \tag{2.10}$$

This simple but meaningful equation plays an important role in analysing the shapes of the MRL and failure rate functions.

Consider now the following lifetime distribution function:

$$F_e(t) = \frac{\int_0^t \bar{F}(u)du}{m} \,, \tag{2.11}$$

where, as usual, $m(0) \equiv m$. The right-hand side of Equation (2.11) defines an equilibrium distribution, which plays an important role in renewal theory (Ross, 1996). This distribution will help us to prove the following simple but meaningful theorem. An elegant idea of the proof belongs to Meilijson (1972).

Theorem 2.2. Exponential Representation of $F(t)$ by Means of the MRL Function Let T be a lifetime random variable with the Cdf $F(t)$, the pdf $f(t)$ and with finite first moment: $m = m(0) < \infty$.

Then

$$\bar{F}(t) = \frac{m}{m(t)} \exp\left\{ -\int_0^t \frac{1}{m(u)} du \right\} . \tag{2.12}$$

Proof. It follows from Equation (2.11) that

$$\bar{F}_e(t) = 1 - \frac{\int_0^t \bar{F}(u)du}{\int_0^\infty \bar{F}(u)du} = \frac{\int_t^\infty \bar{F}(u)du}{m}$$

and that $f_e(t) = \overline{F}(t)/m$. Therefore, the failure rate, which corresponds to the equilibrium distribution $F_e(t)$, is

$$\lambda_e(t) = \frac{f_e(t)}{\overline{F}_e(t)} = \frac{1}{m(t)}. \tag{2.13}$$

Applying Theorem 2.1 to $\overline{F}_e(t)$ results in

$$\overline{F}_e(t) = \exp\left(-\int_0^t \frac{1}{m(u)}du\right). \tag{2.14}$$

Therefore, the corresponding pdf is

$$f_e(t) = \frac{1}{m(t)}\exp\left(-\int_0^t \frac{1}{m(u)}du\right).$$

Finally, substitution of this density into the equation $\overline{F}(t) = mf_e(t)$ results in Equation (2.12). ■

On differentiating Equation (2.12), we obtain the pdf $f(t)$ that is also expressed in terms of the MRL function $m(t)$ (Lai and Xie, 2006), *i.e.*,

$$f(t) = \frac{m(m'(t)+1)}{m^2(t)}\exp\left(-\int_0^t \frac{1}{m(u)}du\right).$$

Theorem 2.2 has meaningful implications. Firstly, it defines another useful exponential representation of the absolutely continuous distribution $F(t)$. Whereas (2.5) is obtained in terms of the failure rate $\lambda(t)$, Equation (2.12) is expressed in terms of the MRL function $m(t)$. Secondly, it shows that, under certain assumptions, $\lambda(t)$ and $1/m(t)$ could be close, at least in some sense to be properly defined. This topic will be discussed in the next section, where the shapes of the failure rate and the MRL functions will be studied.

Equation (2.12) can be used for 'constructing' distribution functions when $m(t)$ is specified. Zahedi (1991) shows that in this case, differentiable functions $m(t)$ should satisfy the following conditions:

- $m(t) > 0,\ t \in [0,\infty)$;
- $m(0) < \infty$;
- $m'(t) > -1, t \in (0,\infty)$;
- $\int_0^\infty \frac{1}{m(u)}du = \infty$;

The first two conditions are obvious. The third condition is obtained from Equation (2.10) and states that $\lambda(t)m(t)$ is strictly positive for $t > 0$. Note that, $m(0)\lambda(0) = 0$ when $\lambda(0) = 0$. The last condition states that the cumulative failure rate

$$\int_0^t \lambda_e(u)du = \int_0^\infty \frac{1}{m(u)} du$$

of equilibrium distribution (2.11) should tend to infinity as $t \to \infty$. This condition ensures a proper Cdf, as $\lim_{t\to\infty} \bar{F}_e(t) = 0$ in this case.

In accordance with Equation (2.3) and exponential representation (2.5), the survival function for T_t can be written as

$$\bar{F}_t(x) = \Pr[T_t > x] = \exp\left\{-\int_t^{t+x} \lambda(u)du\right\}. \tag{2.15}$$

This equation means that the failure rate, which corresponds to the remaining lifetime T_t, is a shift of the baseline failure rate, namely

$$\lambda_t(x) = \lambda(t+x). \tag{2.16}$$

Assume that $\lambda(t)$ is an increasing (decreasing) function. Note that, in this book, as usual, by *increasing* (*decreasing*) we actually mean *non-decreasing* (*non-increasing*). The first simple observation based on Equation (2.15) tells us that in this case, for each fixed $x > 0$, the function $\bar{F}_t(x)$ is decreasing (increasing), and therefore, in accordance with (2.7), the MRL function $m(t)$ is decreasing (increasing). The inverse is generally not true, *i.e.*, a decreasing $m(t)$ does not necessarily lead to an increasing $\lambda(t)$. This topic will be addressed in Section 2.4.

The operation of conditioning in the definition of the MRL function is performed with respect to the event that states that an item is operating at time t. In this approach, an item is considered as a 'black box' without any additional information on its state. Alternatively, we can define the information-based MRL function, which makes sense in many situations when this information is available. The following example (Finkelstein, 2001) illustrates this approach.

Example 2.1 Information-based MRL
Consider a parallel system of two components with independent, identically distributed (i.i.d.) exponential lifetimes defined by the failure rate λ. The survival function of this structure is

$$\bar{F}(t) = 2\exp\{-\lambda t\} - \exp\{-2\lambda t\},$$

and therefore, the corresponding failure rate is defined by

$$\lambda(t) = \frac{2\lambda\exp\{-\lambda t\} - 2\lambda\exp\{-2\lambda t\}}{2\exp\{-\lambda t\} - \exp\{-2\lambda t\}}.$$

It can easily be seen that $\lambda(t)$ monotonically increases from $\lambda(0)=0$ to λ as $t \to \infty$. The corresponding MRL function, in accordance with (2.7), is

$$m(t) = \frac{1}{\lambda}\frac{(4-\exp\{-\lambda t\})}{(4-2\exp\{-\lambda t\})}.$$

This function decreases from $3/2\lambda$ to $1/\lambda$ as $t \to \infty$. Therefore, the following bounds are obvious for $t \in (0,\infty)$:

$$\frac{1}{\lambda} < m(t) < \frac{3}{2\lambda} = m(0). \tag{2.17}$$

These inequalities can be interpreted in the following way. The left-hand side defines the information-based MRL when observation of the system confirms that only one component is operating at $t \in (0,\infty)$, whereas the right-hand side is the information-based MRL when observation confirms that both components are operating. Thus the values of the information-based MRL are the bounds for $m(t)$ in this simple case.

For the case of independent components with different failure rates λ_1, λ_2 ($\lambda_1 < \lambda_2$), the result of the comparison appears to be dependent on the time of observation. The corresponding survival function is defined as

$$\bar{F}(t) = \exp\{-\lambda_1 t\} + \exp\{-\lambda_2 t\} - \exp\{-(\lambda_1+\lambda_2)t\},$$

and the system's failure rate is

$$\lambda(t) = \frac{\lambda_1\exp\{-\lambda_1 t\} + \lambda_2\exp\{-\lambda_2 t\} - (\lambda_1+\lambda_2)\exp\{-(\lambda_1+\lambda_2)t\}}{\exp\{-\lambda_1 t\} + \exp\{-\lambda_2 t\} - \exp\{-(\lambda_1+\lambda_2)t\}}.$$

It can be shown that the function $\lambda(t)$ ($\lambda(0)=0$) is monotonically increasing in $[0, t_{\max}]$ and monotonically decreasing in $(t_{\max}, \infty)$, asymptotically approaching λ_1 from above as $t \to \infty$, as stated in Barlow and Proschan (1975). It crosses the line $y = \lambda_1$ at $t = t_c < t_{\max}$. The value of $t_{\max}$ is uniquely obtained from the equation

$$\lambda_2^2\exp\{-\lambda_1 t\} + \lambda_1^2\exp\{-\lambda_2 t\} = (\lambda_1-\lambda_2)^2; \quad \lambda_1 \neq \lambda_2.$$

As in the previous case, the MRL function can be explicitly obtained, but we are more interested in discussing the information-based bounds. When both components are operating at $t > 0$, then, similar to the right-hand inequality in (2.17), the MRL function $m(t)$ is bounded from above by $m(0)$:

$$m(t) < \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}.$$

Now, let only the second component be operating at the time of observation. As this component is the worst one $(\lambda_2 < \lambda_1)$, the system's MRL should be better: $m(t) > 1/\lambda_2$. On the other hand, if only the first component is operable at time t, then

$$m(t) \le \frac{1}{\lambda_1}, \quad t \in [t_c, \infty). \tag{2.18}$$

This inequality immediately follows by combining the shape of the failure rate (*i.e.*, $\lambda(t)$ is larger than λ_1 for $t > t_c$), Equation (2.15) and the definition of the MRL function in (2.7). It is also clear that $m(t) > 1/\lambda_1$ for sufficiently small values of t, as two components are 'better' than one component in this case. This fact suggests that there should be some equilibrium point $\tilde{t}$ in $(0, t_c)$, where $m(\tilde{t}) = 1/\lambda_1$.

2.3 Lifetime Distributions and Their Failure Rates

There are many lifetime distributions used in reliability theory and in practice. In this section, we briefly discuss the important properties of several important lifetime distributions that we will use in this book. Complete information on the subject can be found in Johnson *et al.* (1994, 1995). A recent book by Marshall and Olkin (2007) also presents a thorough analysis of statistical distributions with an emphasis on reliability theory.

2.3.1 Exponential Distribution

The exponential distribution (or negative exponential), owing to its simplicity and relevance in many applications, is still probably the most popular distribution in practical reliability analysis. Many engineering devices (especially electronic) have a constant failure rate $\lambda > 0$ during the usage period. The Cdf and the pdf of the exponential distribution are given by

$$F(t) = \Pr[T \le t] = 1 - \exp\{-\lambda t\} \tag{2.19}$$

and

$$f(t) = \lambda \exp\{-\lambda t\},$$

respectively.

The expected value and variance are respectively given by

$$E[T] = \frac{1}{\lambda}, \quad \mathrm{var}(T) = \frac{1}{\lambda^2}.$$

The MRL function is also a constant, *i.e.*,

$$m(t) \equiv m = E[T].$$

The exponential distribution is the only distribution that possesses the memoryless property:

$$F(t \mid x) = F(t), \forall x, t \geq 0 ,$$

and therefore, it is the only non-trivial solution of the functional equation

$$\bar{F}(t+x) = \bar{F}(t)\bar{F}(x) .$$

As the failure rate λ is constant, the items described by the exponential distribution do not age in the sense to be defined in Section 2.4.1. The exponential distribution has many characterizations (Marshall and Olkin, 2007). The simplest is *via* the constant failure rate. Another natural characterization is as follows: a distribution is exponential if and only if its mean remaining lifetime is a constant. The memoryless property can also be used as a characterization for this distribution.

2.3.2 Gamma Distribution

Consider the sum of n i.i.d. exponential random variables:

$$T = X_1 + X_2 + \ldots + X_n .$$

The corresponding $(n-1)$-fold convolution of Cdf (2.19) with itself results in the following Cdf for this sum:

$$F(t) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} \exp\{-\lambda t) , \qquad (2.20)$$

whereas the pdf is

$$f(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} \exp\{-\lambda t\} .$$

For $n = 1$, this distribution reduces to the exponential one. Therefore, (2.20) can be considered a generalization of the exponential distribution. The mean and variance are respectively

$$E[T] = \frac{n}{\lambda}, \quad \mathrm{var}(T) = \frac{n}{\lambda^2} ,$$

and the failure rate is given by the following equation:

$$\lambda(t) = \frac{\lambda^n t^{n-1}}{(n-1)! \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}} . \qquad (2.21)$$

It can easily be seen from this formula that $\lambda(t)$ ($\lambda(0) = 0$) is an increasing function asymptotically approaching λ from below, *i.e.*,

$$\lim_{t\to\infty} \lambda(t) = \lambda .$$

This distribution, which is a special case of the gamma distribution for integer n, is often called the *Erlangian* distribution. It plays an important role in reliability engineering. For example, the distribution function of the time to failure of a 'cold' standby system, where the lifetimes of components are exponentially distributed, follows this rule. As $\lambda(t)$ increases, this system ages.

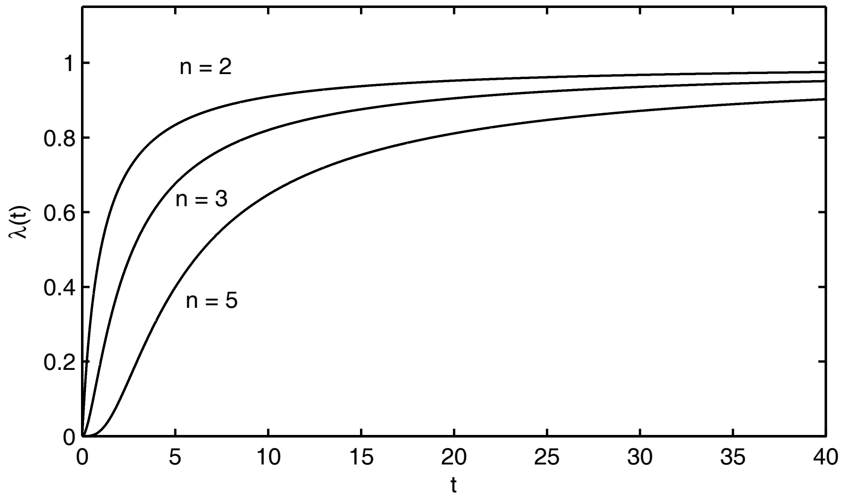

Figure 2.1. The failure rate of the Erlangian distribution ($\lambda = 1$)

We will use this graph for deterioration curve modelling in Chapter 5.

The probability density function for a non-integer n, which for the sake of notation is denoted by α, is

$$f(t) = \frac{\lambda^{\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \exp\{-\lambda t\}, \tag{2.22}$$

where the gamma function is defined in the usual way as

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} \exp\{-u\} du$$

and the scale parameter λ and the shape parameter α are positive. For non-integer α, the corresponding Cdf does not have a 'closed form' as in the integer case (2.20). Equation (2.22) defines a standard two-parameter gamma distribution that is very popular in various applications. The gamma distribution naturally appears in statistical analyses as the distribution of the sum of squares of independent normal variables.

It can be shown (Lai and Xie, 2006) that the failure rate of the gamma distribution can be represented in the following way:

$$\frac{1}{\lambda(t)} = \int_0^\infty \left(1+\frac{u}{t}\right)^{\alpha-1} \exp\{-\lambda u\} du \,.$$

It follows from this equation that $\lambda(t)$ is an increasing function for $\alpha \geq 1$ and is decreasing for $0 < \alpha \leq 1$. When $\alpha = 1$, we arrive at the exponential distribution, which has a failure rate 'that is increasing and decreasing at the same time'.

As we stated in the previous section, it follows from Equations (2.15) and (2.7) that for increasing (decreasing) $\lambda(t)$, the MRL function $m(t)$ is decreasing (increasing). This is a general fact, which means in the case of the gamma distribution that $m(t)$ is a decreasing function for $\alpha \geq 1$ and is increasing for $0 < \alpha \leq 1$. Govil and Agraval (1983) have shown that

$$m(t) = \frac{\lambda^{\alpha-1} t^{\alpha} \exp\{-\lambda t\}}{\Gamma(\alpha)\bar{F}(t)} + \frac{\alpha}{\lambda} - t \,,$$

where $\bar{F}(t)$ is the survival function for the gamma distribution. It can be verified by direct differentiation that the monotonicity properties of $m(t)$ defined by this equation comply with those obtained from general considerations. As the corresponding integrals can usually be calculated explicitly, the gamma distribution is often used in stochastic and statistical modelling. For example, it is a prime candidate for a mixing distribution in mixture models (Chapters 6 and 7).

2.3.3 Exponential Distribution with a Resilience Parameter

The two-parameter distribution obtained from the exponential distribution by introducing a resilience parameter r has not received much attention in the literature (Marshall and Olkin, 2007). However, when r is an integer, similar to the Erlangian distribution, it plays an important role in reliability, as it defines the time-to-failure distribution of a parallel system of r exponentially distributed components. Therefore, the Cdf and the pdf are defined respectively as

$$F(t) = (1-\exp\{-\lambda t\})^r \,,\ \lambda, r > 0 \,,$$

$$f(t) = \lambda r \exp\{-\lambda t\}(1-\exp\{-\lambda t\})^{r-1} ,\ \lambda, r > 0 \,.$$

The failure rate is

$$\lambda(t) = \frac{\lambda r \exp\{-\lambda t\}(1-\exp\{-\lambda t\})^{r-1}}{1-(1-\exp\{-\lambda t\})^r} \,. \tag{2.23}$$

It is easy to show by direct computation that $\lambda(t)$ is increasing for $r > 1$. Therefore, the described parallel system is ageing. Using L'Hospital's rule, it can also be shown that for $r > 0$,

$$\lim_{t\to\infty} \lambda(t) = \lambda,$$

which, similar to the case of the Erlangian distribution, also follows from the definition of the failure rate as a conditional characteristic. Also: $\lambda(0) = 0$ for $r > 1$ and $\lambda(t) \to \infty$ as $t \to 0$ for $0 < r < 1$.

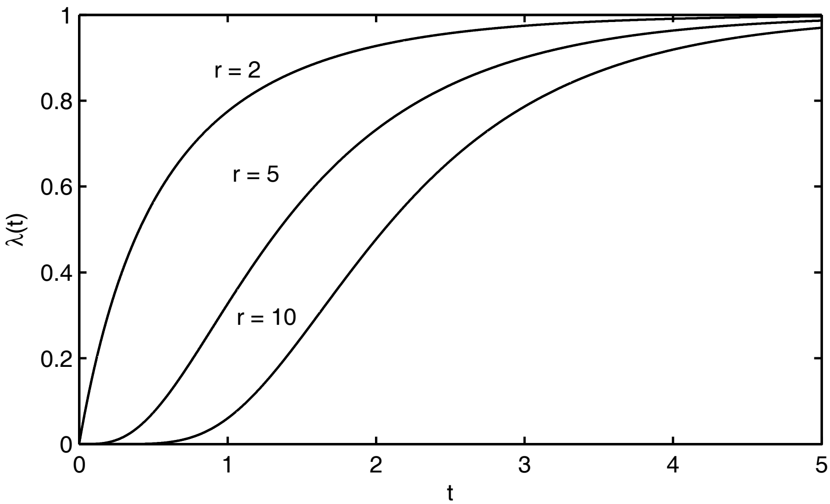

Figure 2.2. The failure rate of the exponential distribution ($\lambda = 1$) with a resilience parameter

2.3.4 Weibull Distribution

The Weibull distribution is one of the most popular distributions for modelling stochastic deterioration. It has been widely used in reliability analysis of ball bearings, engines, semiconductors, various mechanical devices and in modelling human mortality as well. It also appears as a limiting distribution for the smallest of a large number of the i.i.d. positive random variables. If, for example, a series system of n i.i.d. components is considered, then the time to failure of this system is asymptotically distributed ($n \to \infty$) as the Weibull distribution. The monograph by Murthy *et al.* (2003) covers practically all topics on the theory and practical usage of this distribution.

The standard two-parameter Weibull distribution is defined by the following survival function:

$$\bar{F}(t) = \exp\{-(\lambda t)^{\alpha}\}, \quad \lambda, \alpha > 0. \tag{2.24}$$

The failure rate is

$$\lambda(t) = \alpha\lambda(\lambda t)^{\alpha-1} . \qquad (2.25)$$

For $\alpha \geq 1$, it is an increasing function and therefore is suitable for deterioration modelling. When $0 < \alpha \leq 1$, this function is decreasing and can be used, *e.g.*, for infant-mortality modelling. The corresponding expectation is given by

$$m(0) = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right).$$

In general, $m(t)$ has a rather complex form, but for some specific cases (Lai and Xie, 2006) it can be reasonably simple. On the other hand, as $\lambda(t)$ is monotone, $m(t)$ is also monotone: it is increasing for $0 < \alpha \leq 1$ and is decreasing for $\alpha \geq 1$.

2.3.5 Pareto Distribution

The Pareto distribution can be viewed as another interesting generalization of the exponential distribution. We will derive it using mixtures of distributions, which is a topic of Chapters 6 and 7 of this book. Therefore, the following can be considered as a meaningful example illustrating the operation of mixing.

Assume that the failure rate in (2.19) is random, *i.e.*,

$$\lambda = Z ,$$

where Z is a gamma-distributed random variable with parameters α (shape) and β (scale). When considering mixing distributions, we will usually use the notation β for the scale parameter and not λ as in (2.23). Thus, if $Z = z$, the pdf of the random variable T is given by

$$f(t \mid Z = z) \equiv f(t,z) = z\exp\{-zt\} .$$

Denote the pdf of Z by $\pi(z)$. The marginal (or observed) pdf of T is

$$f(t) = \int_0^\infty f(t,z)\pi(z)dz = \frac{\alpha\beta^\alpha}{(\beta+t)^{\alpha+1}}$$

and the corresponding survival function is given by

$$\bar{F}(t) = \left(1+\frac{t}{\beta}\right)^{-\alpha}, \alpha,\beta > 0 . \qquad (2.26)$$

Equation (2.26) defines the Pareto distribution of the second kind (the Lomax distribution) for $t \geq 0$. Note that the survival function of the Pareto distribution of the first kind is usually given by $\bar{F}(t) = t^{-c}$, where $c > 0$ is the corresponding shape parameter. Therefore, this distribution has a support in $[1,\infty)$, whereas (2.26) is defined in $[0,\infty)$, which is usually more convenient in applications.

The failure rate is given by a very simple relationship:

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\alpha}{(\beta + t)}, \tag{2.27}$$

which is a decreasing function. Therefore, the MRL function $m(t)$ is increasing. Oakes and Dasu (1990) show that it can be a linear function for some specific values of parameters α and β. The expectation is

$$m(0) = \frac{\beta}{\alpha - 1}, \quad \alpha > 1.$$

Unlike exponentially decreasing functions, survival function (2.26) is a 'slowly decreasing' function. This property makes the Pareto distribution useful for modelling of extreme events.

2.3.6 Lognormal Distribution

The most popular statistical distribution is the normal distribution. However, it is not a lifetime distribution, as its support is $(-\infty,+\infty)$. Therefore, usually two 'modifications' of the normal distribution are considered in practice for positive random variables: the lognormal distribution and the truncated normal distribution.

A random variable $T \geq 0$ follows the lognormal distribution if $Y = \ln T$ is normally distributed. Therefore, we assume that Y is $N(\alpha,\sigma^2)$, where α and σ^2 are the mean and the variance of Y, respectively. The Cdf in this case is given by

$$F(t) = \Phi\left\{\frac{\ln t - \alpha}{\sigma}\right\}, t \geq 0, \tag{2.28}$$

where, as usual, $\Phi(\cdot)$ denotes the standard normal distribution function. The pdf is given by

$$f(t) = \frac{\exp\left\{-\frac{(\ln t - \alpha)^2}{2\sigma^2}\right\}}{(t\sqrt{2\pi}\sigma)},$$

and it can be shown (Lai and Xie, 2006) that the failure rate is

$$\lambda(t) = \frac{1}{t\sqrt{2\pi}\sigma} \frac{\exp\left\{-\frac{(\ln at)^2}{2\sigma^2}\right\}}{1 - \Phi\left\{\frac{\ln at}{\sigma}\right\}}, \quad a \equiv \exp\{-\alpha\}. \tag{2.29}$$

The expected value of T is

$$m(0) = \exp\left\{\alpha + \frac{\sigma^2}{2}\right\}.$$

The MRL function for this distribution will be discussed in the next section. The shape of the failure rate for $\alpha = 0$ is illustrated by Figure 2.3. Sweet (1990) showed that the failure rate has the upside-down bathtub shape (see the next section) and that $\lim_{t\to\infty} \lambda(t) = 0$, $\lim_{t\to 0} \lambda(t) = 0$.

It is worth noting that, along with the Weibull distribution, the lognormal distribution is often used for fatigue analysis, although it models different dynamics of deterioration than the dynamics described by the Weibull law. It is also considered as a good candidate for modelling the repair time in engineering systems.

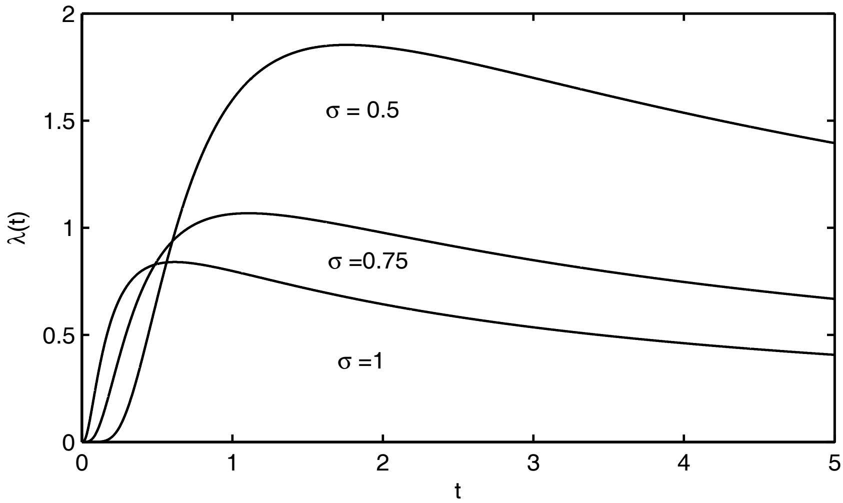

Figure 2.3. The failure rate of the lognormal distribution

2.3.7 Truncated Normal Distribution

The density of the truncated normal distribution is given by

$$f(t) = c\exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\}, \sigma > 0, -\infty < \mu < \infty,\ t \ge 0\,,$$

where

$$c = \frac{1}{\sqrt{2\pi\sigma^2}}\frac{1}{\Phi(\mu/\sigma)}.$$

The corresponding failure rate then follows as

$$\lambda(t)=\frac{1}{\sqrt{2\pi\sigma^2}}\left(1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right)^{-1}\exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\}.$$

It can be shown that this failure rate is increasing and asymptotically approaches the straight line, as defined by (Navarro and Hernandez, 2004):

$$\lim_{t\to\infty}\lambda(t)=\sigma^{-2}.$$

If $\mu+3\sigma>>0$, then the truncated normal distribution practically coincides for $t\geq 0$ with the corresponding standard normal distribution, which is known to have an increasing failure rate.

2.3.8 Inverse Gaussian Distribution

This distribution is popular in reliability, as it defines the first passage time probability for the Wiener process with drift. Although realizations of this process are not monotone, it is widely used for modelling deterioration. The distribution function of the inverse Gaussian distribution is defined by the following equation:

$$F(t)=\Phi\left\{\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}-1\right)\right\}+\exp\left\{\frac{2\lambda}{\mu}\right\}\Phi\left\{-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}+1\right)\right\},\ t\geq 0, \tag{2.30}$$

where λ and μ are parameters. The pdf of the inverse Gaussian distribution is

$$f(t)=\sqrt{\frac{\lambda}{2\pi t^3}}\exp\left\{-\frac{\lambda}{2\mu^2 t}(t-\mu)^2\right\}.$$

The mean and the variance are respectively

$$E[T]=\mu,\ \mathrm{var}(T)=\frac{\mu^3}{\lambda}.$$

We will show in Section 2.4 that its failure rate has an upside-down bathtub shape. The MRL function will also be analysed.

2.3.9 Gompertz and Makeham–Gompertz Distributions

These distributions have their origin in demography and describe the mortality of human populations.

Gompertz (1825) was the first to suggest the following exponential form for the mortality (failure) rate of humans (see Chapter 10 for more details):

$$\lambda(t)=a\exp\{bt\},\ a,b>0. \tag{2.31}$$

The data on human mortality in various populations are in good agreement with this curve. In Section 10.1, we will present a simple original 'justification' of this model, but in fact, there is no suitable biological explanation of exponentiality in (2.31) so far. Therefore, this distribution should only be considered as an empirical law. Note that this is the first distribution in this section that is defined directly *via* the failure (mortality) rate. The corresponding survival function is

$$\bar{F}(t) = \exp\left\{-\int_0^t \lambda(u)du\right\} = \exp\left\{-\frac{a}{b}(\exp\{bt\}-1)\right\}. \tag{2.32}$$

The mortality rate (2.31) is increasing, therefore the corresponding MRL function is decreasing.

The Makeham–Gompertz distribution is a slight generalization of (2.32). It takes into account the initial period, where the mortality is approximately constant and is mostly due to external causes (accidents, suicides, *etc.*). This distribution was also defined in Makeham (1867) directly *via* the mortality rate, although the equation-based explanation was also provided by this author (Chapter 10):

$$\lambda(t) = A + a\exp\{bt\},\ A, a, b > 0\,.$$

The corresponding survival function in this case is

$$\bar{F}(t) = \exp\left\{-At + \frac{a}{b}(\exp\{bt\}-1\right\}. \tag{2.33}$$

Both of these distributions are still widely used in demography. Numerous generalizations and alterations have been suggested in the literature and applied in practice.

2.4 Shape of the Failure Rate and the MRL Function

2.4.1 Some Definitions and Notation

Understanding the shape of the failure rate is important in reliability, risk analysis and other disciplines. The conditional probability of failure in $(t, t+dt]$ describes the ageing properties of the corresponding distributions, which are crucial for modelling in many applications. A qualitative description of the monotonicity properties of the failure rate can be very helpful in the stochastic analysis of failures, deaths, disasters, *etc*. As the failure rate of the exponential distribution is constant (as is the corresponding MRL function), this distribution describes stochastically non-ageing lifetimes.

Survival and failure data are frequently modelled by monotone failure rates. This may be inappropriate when, *e.g.*, the course of a disease is such that the mortality reaches a peak after some finite interval of time and then declines (Gupta, 2001). In such a case, the failure rate has an *upside-down bathtub shape* and the data should be analysed with the help of, *e.g.*, lognormal or inverse Gaussian distributions. On the other hand, many engineering devices possess a period of 'infant mortality' when the failure rate declines in an initial time interval, reaches a minimum and then increases. In such a case, the failure rate has a *bathtub shape* and can be modelled, *e.g.*, by mixtures of distributions. Navarro and Hernandez (2004) show how to obtain the bathtub-shaped failure rates from the mixtures of truncated normal distributions. Many other relevant examples can be found in Section 2.8 of Lai and Xie (2006) and in references therein. We will consider in this section only some basic facts, which will be helpful for obtaining and discussing the results in the rest of this book.

Most often, the Cdf and the failure rate of a lifetime are modelled or estimated only on the basis of the corresponding failures (deaths). However, one can also use information (if available) on the process of a 'failure development'. If, *e.g.*, a failure occurs when the accumulated random damage or wear exceeds a predetermined level, then the failure rate can be derived analytically for some simple stochastic processes of wear. The shape of the failure rate in this case can also be analysed using properties of underlying stochastic processes (Aalen and Gjeissing, 2001). These underlying processes are largely unknown. However, this does not imply that they should be ignored. Some simple models of this kind will be discussed in Chapter 10.

As we saw in the previous section, many popular parametric lifetime models are described by monotone failure rates. If $\lambda(t)$ increases (decreases) in time, then we say that the corresponding distribution belongs to the increasing (decreasing) failure rate (IFR (DFR)) class. These are the simplest nonparametric classes of ageing distributions. A natural generalization on the non-monotone failure rates is when

$$\frac{\int_0^t \lambda(u)du}{t} \tag{2.34}$$

is increasing (decreasing) in t. These classes are called IFRA (DFRA), where "A" stands for "average".

We say that the Cdf $F(x)$ belongs to the decreasing (increasing) mean remaining lifetime (DMRL (IMRL)) class if the corresponding MRL function $m(t)$ is decreasing (increasing). These classes are in some way dual to IFR (DFR) classes. See Section 3.3.2 for formal definitions of IFR (DFR) and DMRL (IMRL) classes.

The Cdf $F(x)$ is said to be new better (worse) than used (NBU (NWU)) if

$$\bar{F}(x \mid t) \le (\ge) \bar{F}(x), \forall x, t \ge 0\,. \tag{2.35}$$

This definition means that an item of age t has a stochastically smaller (larger) remaining lifetime (Definition 3.4) than a new item at age $t = 0$.

The described classes will usually be sufficient for presentation in this book. Each of them has a clear, simple 'physical' meaning describing some kind of deterioration. A variety of other ageing classes of distributions can be found in the literature (Barlow and Proschan, 1975; Rausand and Hoyland, 2004; Lai and Xie, 2006; Marshall and Olkin, 2007, to name a few). Many of them do not have this clear interpretation and are of mathematical interest only.

Note that IFR (DFR) and DMRL (IMRL) classes are defined directly by the shape of the failure rate and the MRL function, respectively. If $\lambda(t)$ is monotonically (strictly) increasing (decreasing) in time, we say that it is I (D) shaped and for brevity write $\lambda(t) \in$ I (D). A similar notation will be used for the DMRL (IMRL) classes, *i.e.*, $m(t) \in$ D (I).

Figure 1.1 of Chapter 1 gives an illustration of the bathtub shape of a failure rate with a useful period, where it is approximately constant. This can be the case in practical life-cycle applications, but formally we will define the bathtub shape without a useful period plateau of this kind.

Definition 2.3. The differentiable failure rate $\lambda(t)$ has a bathtub shape if

$$\lambda'(t) < 0 \text{ for } t \in [0, t_0), \lambda'(t_0) = 0, \; \lambda'(t) > 0 \text{ for } t \in (t_0, \infty),$$

and it has an upside-down bathtub shape if

$$\lambda'(t) > 0 \text{ for } t \in [0, t_0), \lambda'(t_0) = 0, \; \lambda'(t) < 0 \text{ for } t \in (t_0, \infty).$$

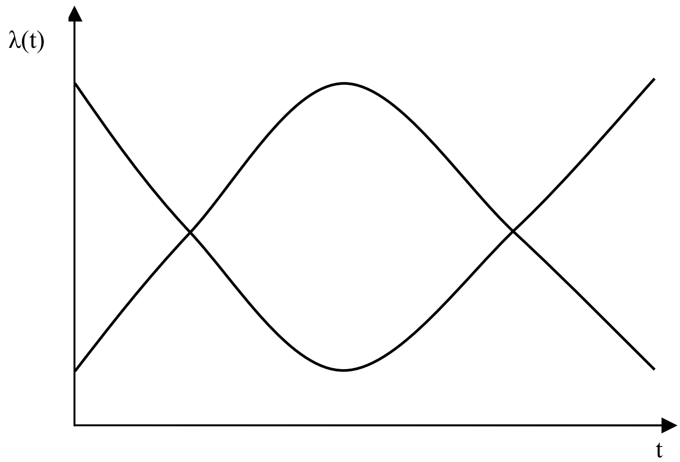

Figure 2.4. The BT and the UBT shapes of the failure rate

We will use the notation $\lambda(t) \in$ BT and $\lambda(t) \in$ UBT, respectively. There can be modifications and generalizations of these shapes (*e.g.*, when there is more than one minimum or maximum for the function $\lambda(t)$), but for simplicity, only BT and UBT shapes will be considered.

2.4.2 Glaser's Approach

As we have already stated, the lognormal and the inverse Gaussian distributions have a UBT failure rate. We will see in Chapter 6 that many mixing models with an increasing baseline failure rate result in a UBT shape of the mixture (observed) failure rate. For example, mixing in a family of increasing (as a power function) failure rates (the Weibull law) 'produces' the UBT shape of the observed failure rate. From this point of view, the BT shape is 'less natural' and often results as a combination of different standard distributions defined for different time intervals. For example, infant mortality in $[0,t_0]$ is usually described by some DFR distribution in this interval, whereas the wear-out in (t_0,∞) is modelled by an IFR distribution. However, mixing of specific distributions can also result in the BT shape of the failure rate as, *e.g.*, in Navarro and Fernandez (2004). Note that the infant mortality curve can also be explained *via* the concept of mixing, as, *e.g.*, mixtures of exponential distributions are always DFR (Chapter 6).

The function

$$\eta(t) = -\frac{f'(t)}{f(t)} \tag{2.36}$$

appears to be extremely helpful in the study of the shape of the failure rate $\lambda(t) = f(t)/\bar{F}(t)$. This function contains useful information about $\lambda(t)$ and is simpler because it does not involve $\bar{F}(t)$. In particular, the shape of $\eta(t)$ often defines the shape of $\lambda(t)$ (Gupta, 2001).

Assume that the pdf $f(t)$ is a twice differentiable, positive function in $(0,\infty)$. Define a function $g(t)$ as the reciprocal of the failure rate, *i.e.*,

$$g(t) = \frac{1}{\lambda(t)} = \frac{\bar{F}(t)}{f(t)}. \tag{2.37}$$

Then

$$g'(t) = g(t)\eta(t) - 1, \tag{2.38}$$

which means that the turning point of $\lambda(t)$ is the solution of the equation $\lambda(t) = \eta(t)$ (compare with Equation (2.9)). It can also be verified that (Gupta, 2001)

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \eta(t).$$

Using Equations (2.37) and (2.38):

$$\begin{aligned} g'(t) &= \int_t^\infty \left[\frac{f(y)}{f(t)}\right]\eta(t)dy - 1 \\ &= \int_t^\infty \left[\frac{f(y)}{f(t)}\right][\eta(t) - \eta(y)]dy + \int_t^\infty \left[\frac{f(y)}{f(t)}\right]\eta(y)dy - 1. \end{aligned}$$

Taking into account that

$$\int_t^\infty \left[\frac{f(y)}{f(t)}\right]\eta(y)dy = -\frac{1}{f(t)}\int_t^\infty f'(y)dy = 1,$$

we arrive eventually at

$$g'(t) = \int_t^\infty \left[\frac{f(y)}{f(t)}\right][\eta(t) - \eta(y)]dy. \tag{2.39}$$

Using (2.39) as a supplementary result, we are now able to prove Glaser's theorem, which is crucial for the analysis of the shape of the failure rate function (Glaser, 1980).

Theorem 2.3.

- If $\eta(t) \in \mathrm{I}$, then also $\lambda(t) \in \mathrm{I}$;
- If $\eta(t) \in \mathrm{D}$, then also $\lambda(t) \in \mathrm{D}$;
- If $\eta(t) \in \mathrm{BT}$ and there exists y_0 such that $g'(y_0) = 0$, then $\lambda(t) \in \mathrm{BT}$, otherwise $\lambda(t) \in \mathrm{I}$;
- If $\eta(t) \in \mathrm{UBT}$ and there exists y_0 such that $g'(y_0) = 0$, then $\lambda(t) \in \mathrm{UBT}$, otherwise $\lambda(t) \in \mathrm{D}$.

Proof. If $\eta(t) \in \mathrm{I}$, then $g'(t)$, as follows from Equation (2.39), is negative for all $t > 0$. Therefore, $g(t) \in \mathrm{D}$ and $\lambda(t) \in \mathrm{I}$. The proof of the second statement is similar.

Let us prove the first part of the third statement. This proof follows the original proof in Glaser (1980). Another proof, which is obtained using more general considerations, can be found in Marshall and Olkin (2007). It follows from the definition of the BT shape that $\eta(t) \in \mathrm{BT}$ if

$$\eta'(t) < 0 \text{ for } t \in [0, t_0), \eta'(t_0) = 0, \ \eta'(t) > 0 \text{ for } t \in (t_0, \infty). \tag{2.40}$$

Assume that $g''(y_0) < 0$. Since $g'(y_0) = 0$ in accordance with the conditions of the theorem, it follows from the differentiation of (2.38) that

$$g''(y_0) = g(y_0)\eta'(y_0).$$

Therefore,

$$g''(y_0) < 0 \Leftrightarrow \eta'(y_0) < 0 \Leftrightarrow y_0 < t_0.$$

Thus, if our assumption is true, then $y_0 < t_0$. Suppose the opposite: $y_0 \ge t_0$. From Equations (2.39) and (2.40) it follows that $g'(t) < 0$ for $t \ge t_0$. Therefore, $g'(y_0) < 0$, which contradicts the condition of the theorem stating that $g'(y_0) = 0$. Hence $y_0 < t_0$ and $g''(y_0) < 0$. On the other hand, it is clear that $y = y_0$ is the only root of equation $g'(y) = 0$ and that $g(t)$ attains its maximum at this point.

The proof of the second part is simpler: indeed, either $g'(t) > 0$ for all $t > 0$ or $g'(t) < 0$. It follows from Equation (2.39) that $g'(t) < 0$ for all $t \ge t_0$. Therefore, $g'(t) < 0$ for all $t > 0$ and $\lambda(t) \in \mathrm{I}$.

The proof of the last statement is similar. ■

This important theorem states that the monotonicity properties of $\lambda(t)$ are defined by those of $\eta(t)$, and because $\eta(t)$ is often much simpler than $\lambda(t)$, its analysis is more convenient. The simplest meaningful example is the standard normal distribution. Although it is not a lifetime distribution, the application of Glaser's theorem is very impressive in this case. Indeed, the failure rate of the normal distribution does not have an explicit expression, whereas the function $\eta(t)$, as can be easily verified, is very simple:

$$\eta(t) = (t - \mu) / \sigma^2 .$$

Therefore, as $\eta(t) \in$ I, the failure rate is also increasing, which is a well-known fact for the normal distribution.

Note that Gupta and Warren (2001) generalized Glaser's theorem to the case where $\lambda(t)$ has two or more turning points.

Example 2.2 Failure Rate Shape of the Truncated Normal Distribution
The function $\eta(t)$ in this case is the same as for the normal distribution, and therefore the failure rate is also increasing. Navarro and Hernandez (2004) also show that

$$\lambda(t) > (t - \mu) / \sigma^2, t \geq 0 .$$

Example 2.3 Failure Rate Shapes of Lognormal and Inverse Gaussian Distributions
The function $\eta(t)$ for the lognormal distribution is

$$\eta(t) = -\frac{f'(t)}{f(t)} = \frac{1}{\sigma^2 t}(\sigma^2 + \ln t - \alpha) . \tag{2.41}$$

It can be shown that $n(t) \in$ UBT (Lai and Xie, 2006) and that the second condition in the last statement of Theorem 2.3 is also satisfied, since, in accordance with Equation (2.29),

$$\lim_{t \to 0} \lambda(t) = 0 , \lim_{t \to \infty} \lambda(t) = 0 .$$

Therefore, $\lambda(t) \in$ UBT, and this is illustrated by Figure 2.2.

The $\eta(t)$ function for the inverse Gaussian distribution (2.30) is

$$\eta(t) = \frac{3\mu^2 t + \lambda(t^2 - \mu^2)}{2\mu^2 t^2} . \tag{2.42}$$

Using arguments similar to those used in the case of the lognormal distribution, it can be shown (Lai and Xie, 2006) that $\lambda(t) \in$ UBT. The exact MRL function for this distribution (Gupta, 2001) is very cumbersome to derive.

Glaser's approach was generalized by Block *et al.* (2002) by considering the ratio of two functions

$$G(t) = \frac{N(t)}{D(t)}, \tag{2.43}$$

where the functions on the right-hand side are continuously differentiable and $D(t)$ is positive and strictly monotone. As with (2.36), where the numerator is the derivative of $f(t)$ and the denominator is the derivative of $\overline{F}(t)$, we define the function $\eta(t)$ as

$$\eta(t) = \frac{N'(t)}{D'(t)}. \tag{2.44}$$

These authors show that the monotonicity properties of $G(t)$ are 'close' to those of $\eta(t)$, as in the case where $\eta(t)$ is defined by (2.36). Consider, for example, the MRL function

$$m(t) = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)}.$$

We can use it as $G(t)$. It is remarkable that $\eta(t)$ in this case is simply the reciprocal of the failure rate, *i.e.*,

$$\eta(t) = \frac{\overline{F}(t)}{f(t)} = \frac{1}{\lambda(t)}.$$

Therefore, the functions $m(t)$ and $1/\lambda(t)$ can be close in some suitable sense; this will be discussed in Section 2.4.3.

Glaser's theorem defines sufficient conditions for monotonic or BT (UBT) shapes of the failure rate. The next three theorems establish relationships between the shapes of $\lambda(t)$ and $m(t)$. The first one is obvious and in fact has already been used several times.

Theorem 2.4. If $\lambda(t) \in$ I (or $(\lambda(t))^{-1} \in D$), then $m(t) \in D$.

Proof. The result follows immediately from Equations (2.7) and (2.15). The symmetrical result is also evident: if $\lambda(t) \in$ D, then $m(t) \in$ I. ■

Thus, a monotone failure rate always corresponds to a monotone MRL function. The inverse is true only under additional conditions.

Theorem 2.5. Let the MRL function $m(t)$ be twice differentiable and the failure rate $\lambda(t)$ be differentiable in $(0,\infty)$. If $m(t) \in$ D (I) and is a convex (concave) function, then $\lambda(t) \in$ I (D).

Proof. Differentiation of both sides of Equation (2.9) gives

$$m''(t) = m'(t)\lambda(t) + m(t)\lambda'(t) .$$

If $m(t)$ is strictly decreasing, then its derivative is negative for all $t \in (0,\infty)$. Owing to convexity defined by $m''(t) \geq 0$ and taking into account that the functions $\lambda(t)$ and $m(t)$ are positive in $(0,\infty)$, $\lambda'(t)$ should be positive as well. This means that $\lambda(t) \in$ I. The 'symmetrical' result is proved in a similar way. ■

Gupta and Kirmani (2000) state that if $\lambda(t)$ is concave, then $m(t)$ is a convex function. Theorem 2.5 gives the sufficient conditions for the monotonicity of the failure rate in terms of the monotonicity of $m(t)$. The following theorem generalizes the foregoing results to a non-monotone case (Gupta and Akman, 1995; Mi, 1995; Finkelstein, 2002a). It states that the BT (UBT) failure rate under certain assumptions can correspond to a monotone MRL function (compare with Theorem 2.4, which gives a simpler correspondence rule).

Theorem 2.6. Let $\lambda(t)$ be a differentiable BT failure rate in $[0,\infty)$.

- If
$$m'(0) = \lambda(0)m(0) - 1 \leq 0 , \tag{2.45}$$
then $m(t) \in$ D;
- If $m'(0) > 0$, then $m(t) \in$ UBT.

Let $\lambda(t)$ be a differentiable UBT failure rate in $[0,\infty)$.

- If $m'(0) \geq 0$, then $m(t) \in$ I;
- If $m'(0) < 0$, then $m(t) \in$ BT.

Proof. We will prove only the first statement. Other results follow in the same manner. Denote the numerator in (2.9) by $d(t)$, *i.e.*,

$$d(t) = \lambda(t)\int_t^{\infty} \bar{F}(u)du - \bar{F}(t) . \tag{2.46}$$

The sign of $d(t)$ in (2.9) defines the sign of $m'(t)$. On the other hand,

$$d'(t) = \lambda'(t)\int_t^{\infty} \bar{F}(u)du , \tag{2.47}$$

and the monotonicity properties of $\lambda(t)$ are the same as for $d(t)$. Recall that t_0 is the change (turning) point for the BT failure rate. Therefore,

$$\lambda'(t_0) = d'(t_0) = 0 \text{ ; } \lambda(t) > \lambda(t_0) \text{ for } t > t_0$$

and

$$d(t_0) = \lambda(t_0)\int_{t_b}^{\infty}\overline{F}(u)du - \overline{F}(t_0)$$

$$< \int_{t_b}^{\infty}\lambda(u)\overline{F}(u)du - \overline{F}(t_0) = 0. \qquad (2.48)$$

Owing to the assumption $m'(0) \le 0$ and to Equation (2.9), the function $d(t)$ is negative at $t = 0$. It then follows from (2.47) that $d(t)$ decreases to $d(t_0)$ and then increases in (t_0, ∞), being negative. The latter can be seen from Inequality (2.48), where t_0 can be substituted by any $t > t_0$. Therefore, in accordance with (2.9), $m'(t) < 0$ in $(0, \infty)$, which completes the proof. ■

Corollary 2.1. Let $\lambda(0) = 0$. If $\lambda(t)$ is a differentiable UBT failure rate, then $m(t)$ has a bathtub shape.

Proof. This statement immediately follows from Theorem 2.6, as Equation (2.45) reads $m'(0) = \lambda(0)m(0) - 1 = -1 \le 0$ in this case. ■

Example 2.4 (Gupta and Akman, 1995)
Consider a lifetime distribution with $\lambda(t) \in \mathrm{BT}$, $t \in [0, \infty)$ of the following specific form:

$$\lambda(t) = \frac{(1 + 2.3t^2) - 4.6t}{1 + 2.3t^2}.$$

It can easily be obtained using Equation (2.22) that the corresponding MRL is

$$m(t) = \frac{1}{1 + 2.3t^2},$$

which is a decreasing function. Obviously, the condition $\lambda(0) \le 1/m(0)$ is satisfied.

2.4.3 Limiting Behaviour of the Failure Rate and the MRL Function

In this section, we will discuss and compare the simplest asymptotic (as $t \to \infty$) properties of $\lambda(t)$ and $1/m(t)$. When a lifetime T has an exponential distribution, these functions are equal to the same constant. It has already been mentioned that Block *et al.* (2001) stated that the monotonicity properties of the function $G(t)$ defined by Equation (2.43) are 'close' to those of the function $\eta(t)$ defined by Equation (2.44). When we choose $G(t) = m(t)$, the function $\eta(t)$ is equal to $1/\lambda(t)$, and therefore the monotonicity properties of these functions are similar. Moreover, we will show now that they are asymptotically equivalent.

Denote $r(t) \equiv 1/m(t)$ and, as in Finkelstein (2002a), rewrite Equation (2.10) in form that connects the failure rate and the reciprocal of the MRL function

$$\lambda(t) = -\frac{r'(t)}{r(t)} + r(t). \qquad (2.49)$$

The following obvious result is a direct consequence of Equation (2.49).

Theorem 2.7. Let $\lim_{t\to\infty} r(t) = c,\ 0 < c \le \infty$.
Then $r(t)$ is asymptotically equivalent to $\lambda(t)$ in the following sense:

$$\lim_{t\to\infty} \left|\lambda(t) - r(t)\right| = 0, \tag{2.50}$$

if and only if

$$\left|\frac{r'(t)}{r(t)}\right| = \left|\frac{m'(t)}{m(t)}\right| \to 0 \text{ as } t \to \infty. \tag{2.51}$$

Let, *e.g.*, $r(t) = t^{\beta}$; $\beta > 0$. Then Theorem 2.7 holds and the reciprocal of the MRL function for the Weibull distribution with an increasing failure rate can be approximated as $t \to \infty$ by this failure rate. The exact formula for the MRL function in this case is rather cumbersome, and thus this result can be helpful for asymptotic analysis. Note that Relationship (2.51) does not hold for sharply increasing functions $r(t)$, such as, *e.g.*, $r(t) = \exp\{t\}$ or $r(t) = \exp\{t^2\}$.

Remark 2.2 Applying L'Hopital's rule to the right-hand side of (2.7), the following asymptotic relation can be obtained (Calabra and Pulchini, 1987; Bradley and Gupta, 2003):

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} \frac{1}{\lambda(t)},$$

provided the latter limit exists and is finite. It is clear that this statement differs from the stronger one (2.50) only when $\lim_{t\to\infty} \lambda(t) = \infty$.

The asymptotic equivalence in (2.50) is a very strong one, especially when $\lim_{t\to\infty} r(t) = \infty$ and $\lim_{t\to\infty} \lambda(t) = \infty$. Therefore, it is reasonable to consider the following relative distance between $\lambda(t)$ and $r(t)$:

$$\frac{|\lambda(t) - r(t)|}{r(t)} = m'(t).$$

This distance tends to zero when

$$\lim_{t\to\infty} |m'(t)| = \lim_{t\to\infty} \left|\frac{r'(t)}{r^2(t)}\right| = 0, \tag{2.52}$$

which, in fact, is equivalent to the following asymptotic relationship:

$$\lambda(t) = r(t)(1 + o(1)) \text{ as } t \to \infty, \tag{2.53}$$

where, as usual, the notation $o(1)$ means $\lim_{t\to\infty} o(1) = 0$. Asymptotic relationships of this kind are also often written as $\lambda(t) \sim r(t)$, meaning that

$$\lim_{t\to\infty}\frac{r(t)}{\lambda(t)}=1. \tag{2.54}$$

We will use both types of asymptotic notation.

It can easily be verified that $|m'(t)|\to 0$, *e.g.*, for functions $r(t)=\exp\{t\}$ or $r(t)=\exp\{t^2\}$, for which (2.51) does not hold.

When $\lim_{t\to\infty} r(t)=0$ $(\lim_{t\to\infty} m(t)=\infty)$, which corresponds to $\lambda(t)\to 0$ as $t\to\infty$, the reasoning should be slightly different. Relationships (2.50) and (2.52) do not make much sense in this case. Therefore, the corresponding reciprocal values should be considered. From Equation (2.10):

$$\frac{1}{\lambda(t)}=\frac{m(t)}{m'(t)+1}$$

and

$$\frac{1}{\lambda(t)}-m(t)=-\frac{m'(t)m(t)}{m'(t)+1}.$$

The relative distance in this case is

$$\frac{1}{\lambda(t)m(t)}-1=-\frac{m'(t)}{m'(t)+1}.$$

Therefore, Relationship (2.52) is also valid if

$$\lim_{t\to\infty}|m'(t)|=0.$$

Example 2.5 (Bradley and Gupta, 2003)
Consider the linear MRL function

$$m(t)=a+bt,\ a,b>0.$$

The corresponding failure rate is

$$\lambda(t)=\frac{1+b}{a+bt}.$$

Thus, Condition (2.52) is not satisfied, and therefore (2.53) does not hold.

Remark 2.3 Assume that $r(t)$ is ultimately (*i.e.*, for large t) increasing. It is easy to see from (2.49) that $\lambda(t)$ is also ultimately increasing if $r'(t)/r(t)$ is ultimately decreasing, which holds, *e.g.*, for the power law.

Many of the standard distributions have failure rates that are polynomials or ratios of polynomials. The same is true for the MRL function. Theorem 2.7 can be generalized to these rather general classes of functions by assuming that $r(t)$ is a regularly varying function (Bingham *et al.*, 1987). A regularly varying function is defined as a function with the following structure:

$$r(t) = t^{\beta} l(t)(1 + o(1)) \, , \; t \to \infty \, ; -\infty < \beta < \infty \, , \; \beta \neq 0 \, ,$$

where $l(t)$ is a slowly varying function: $l(kt)/l(t) \to 1$ for all $k > 0$. Therefore, as $t \to \infty$, it is asymptotically equivalent to the product of a power function and a function, which, *e.g.*, increases slower than any increasing power function (for example, $\ln t$).

Theorem 2.8. Let the function $r(t)$ in Theorem 2.7 be a regularly varying function with $\beta > 0$. Assume that $r'(t)$ is ultimately monotone.

Then Relationship (2.51) holds, and therefore (2.50) is also true.

Proof (Finkelstein, 2002a). In accordance with the Monotone Density Theorem (Bingham *et al.*, 1987), the ultimately monotone $r'(t)$ can be written in the following way:

$$r'(t) = t^{\beta - 1} \tilde{l}(t)(1 + o(1)) \text{ as } t \to \infty \, ,$$

where $\tilde{l}(t)$ is a slowly varying function. Using expressions for regularly varying $r(t)$ and $r'(t)$:

$$\left| \frac{r'(t)}{r(t)} \right| = t^{-1} \hat{l}(t)(1 + o(1)) \text{ as } t \to \infty \, ,$$

where $\hat{l}(t)$ is another slowly varying function. Owing to the definition of the slowly varying function, $t^{-1} \hat{l}(t) \to 0$ as $t \to \infty$, and therefore Relationship (2.51) holds.

2.5 Reversed Failure Rate

2.5.1 Definitions

As stated earlier, the failure rate plays a crucial role in reliability and survival analysis. The interpretation of $\lambda(t)dt$ as the conditional probability of failure of an item in $(t, t + dt]$ given that it did not fail before in $[0, t]$ is meaningful. It describes the chances of failure of an operable object in the next infinitesimal interval of time.

The reversed failure (hazard) rate (RFR) function was introduced by von Mises in 1936 (von Mises, 1964). It has been largely ignored in the literature primarily because it was believed that this function did not have the strong intuitive probabilistic content of the failure rate (Marshall and Olkin, 2007). In the next section, we will show that it still has an interesting probabilistic meaning, although not similar to that of the 'ordinary' failure rate. Most likely owing to this meaning, the properties of the reversed failure rate have attracted considerable interest among researchers (Block *et al.*, 1998; Chandra, and Roy, 2001; Gupta and Nanda, 2001; Finkelstein, 2002, to name a few). Here we will only consider definitions and some of the simplest general properties. For more details, the reader is referred to the above-mentioned papers and references therein.

Definition 2.4. The RFR $\rho(t)$ is defined by the following equation:

$$\rho(t) = \frac{f(t)}{F(t)}. \tag{2.55}$$

Thus, $\rho(t)dt$ can be interpreted as an approximate probability of a failure in $(t-dt,t]$ given that the failure had occurred in $[0,t]$.

Similar to exponential representation (2.5), it can be easily shown solving, for instance, the elementary differential equation $F'(t) = \rho(t)F(t)$ with the initial condition $F(0) = 0$ that the following analogue of (2.5) holds:

$$F(t) = \exp\left\{-\int_t^\infty \rho(u)du\right\} \tag{2.56}$$

and that the corresponding pdf is given by

$$f(t) = \rho(t)\exp\left\{-\int_t^\infty \rho(u)du\right\}.$$

Therefore, $\rho(t)$ defines another characterization for the absolutely continuous Cdf $F(t)$. Note that for proper lifetime distributions,

$$\int_0^\infty \rho(u)du = \infty, \quad \int_t^\infty \rho(u)du \neq \infty, \forall t > 0, \tag{2.57}$$

which means that

$$\lim_{t\to 0} \rho(t) = \infty,$$

and $F(0) = 0$ should also be understood as the corresponding limit.

Unlike $\lambda(t)$, the RFR $\rho(t)$ cannot be a constant or an increasing function in $(a,\infty), a \geq 0$. It is easy to verify that (2.57) holds, *e.g.*, for the power function $\rho(t) = t^{-\alpha}, \alpha > 1$.

After a simple transformation, the following relationship between $\rho(t)$ and $\lambda(t)$ can be obtained:

$$\rho(t) = \frac{\lambda(t)\overline{F}(t)}{1-\overline{F}(t)} = \lambda(t)\frac{1}{(\overline{F}(t))^{-1}-1} = \frac{\lambda(t)}{\exp\left\{\int_0^t \lambda(u)du\right\}-1}. \tag{2.58}$$

Let, *e.g.*, $\lambda(t)$ be a constant: $\lambda(t) = \lambda$. In accordance with Equation (2.58),

$$\rho(t) = \frac{\lambda}{\exp\{\lambda t\} - 1},$$

and therefore, $\rho(t)$ decreases exponentially as $t \to \infty$, whereas its behaviour for $t \to 0$ is defined by the function t^{-1}.

It follows from Equation (2.58) that if $\lambda(t)$ is decreasing, then $\rho(t)$ is also decreasing. For $t \to \infty$, Equation (2.55) can be written asymptotically as

$$\rho(t) = f(t)(1 + o(1)).$$

Thus $\rho(t)$ and $f(t)$ are asymptotically equivalent, which means that the study of the RFR function is relevant only for finite time.

Example 2.6 Consider a series system of two independent components with survival functions $\bar{F}_1(t), \bar{F}_2(t)$, failure rates $\lambda_1(t), \lambda_2(t)$ and RFRs $\rho_1(t), \rho_2(t)$, respectively. As the survival function of the system in this case is the product of the components' survival functions $\bar{F}_s(t) = \bar{F}_1(t)\bar{F}_2(t)$, it follows from (2.5) that $\lambda_s(t) = \lambda_1(t) + \lambda_2(t)$, where $\lambda_s(t)$ denotes the failure rate of the system. On the other hand, $F_s(t)$ can be written in terms of the RFRs as

$$\begin{aligned} F_s(t) &= 1 - \bar{F}_1(t)\bar{F}_2(t) \\ &= 1 - \left(1 - \exp\left\{-\int_t^\infty \rho_1(u)du\right\}\right)\left(1 - \exp\left\{-\int_t^\infty \rho_2(u)du\right\}\right), \end{aligned} \tag{2.59}$$

and the system's RFR can be obtained using Definition 2.4. This will be a much more cumbersome expression than the self-explanatory $\lambda_1(t) + \lambda_2(t)$.

Using the same notation, consider now a parallel system of two independent components. The failure rate of this system is defined by the distribution $F_i(t)F_2(t)$ which, similar to (2.59), does not give a 'nice' expression for $\lambda_s(t)$. The RFR for this system, however, is simply the sum of individual reversed failure rates, *i.e.*,

$$\rho_s(t) = \rho_1(t) + \rho_2(t),$$

which can be seen by substituting (2.56) into the product $F_1(t)F_2(t)$. A similar result is obviously valid for more than two independent components in parallel.

Remark 2.4 It is well known that the probability that the i th component is the cause of the failure of the series system described in Example 2.6 (given that this failure had occurred in $(t, t + dt]$) is $\lambda_i(t) / \lambda_s(t),\ i = 1,2$. It can easily be seen, however (Cha and Mi, 2008), that a similar relationship holds for the probability that the i th component is the last to fail in the described parallel system (given that the failure of a system had occurred in $(t, t + dt]$) and that probability is $\rho_i(t) / \rho_s(t), i = 1,2$.

The foregoing reasoning indicates that some characteristics of parallel systems can be better described *via* the RFR than *via* the 'ordinary' failure rate.

2.5.2 Waiting Time

It turns out that the RFR is closely related to another important lifetime characteristic: the waiting time since failure. Indeed, as the condition of a failure in $[0,t]$ is already imposed in the definition of the RFR, it is of interest in different applications (reliability, actuarial science, survival analysis) to describe the time that has elapsed since the failure time T to the current time t. Denote this random variable by $T_{w,t}$. Similar to (2.3), the corresponding survival function with support in $[0,t]$ (Finkelstein, 2002b) is

$$\begin{aligned}\bar{F}_{w,t}(x) &= P\{t-T > x \mid T \le t\} \\ &= \frac{F(t-x)}{F(t)},\ x \in [0,t],\end{aligned} \tag{2.60}$$

and the corresponding pdf is

$$f_{w,t}(x) = \frac{f(t-x)}{F(t)},\ x \in [0,t],$$

which, taking into account (2.55), leads to an intuitively evident relationship

$$\rho(t) = f_{w,t}(0).$$

Similar to Equation (2.7):

Definition 2.5. The mean waiting time (MWT) function $m_w(t)$ for an item that had failed in the interval $[0,t]$ is

$$\begin{aligned}m_w(t) \equiv E[T_{w,t}] &= \int_0^t \bar{F}_{w,t}(u)du \\ &= \frac{\int_0^t F(u)du}{F(t)}.\end{aligned} \tag{2.61}$$

Assume that $m_w(t)$ is differentiable. Differentiating (2.61) and similar to (2.9), the following equation is obtained:

$$m'_w(t) = 1 - \rho(t) m_w(t). \tag{2.62}$$

Equivalently,

$$\rho(t) = \frac{1 - m'_w(t)}{m_w(t)}. \tag{2.63}$$

Substituting the RFR defined by Equation (2.63) into the right-hand side of Equation (2.56), we arrive at the exponential representation for the Cdf $F(t)$, which can also be considered as another characterization of the absolutely continuous distribution function *via* the MWT function $m_w(t)$:

$$F(t) = \exp\left\{ -\int_t^{\infty} \frac{1 - m_w'(u)}{m_w(u)} du \right\}. \tag{2.64}$$

Remark 2.5 Sufficient conditions for the function $m_w(t)$ to be a MWT function for some proper lifetime distribution are similar to the corresponding conditions for the MRL function in Section 2.2.

Note that the properties of $m_w(x)$ and $m(x)$ differ significantly, which can be illustrated by the following example.

Example 2.7 Let $\lambda(t) = \lambda$. Then $m(t) = \lambda^{-1}$, whereas

$$m_w(t) = \frac{\int_0^t F(u)du}{F(t)} = \frac{t + \lambda^{-1}(\exp\{-\lambda t\} - 1)}{1 - \exp\{-\lambda t\}}.$$

It can be shown that

$$sign(m_w'(t)) = sign(\exp\{-\lambda t\} - 1 - \lambda t) > 0,$$

and therefore $m_w(t)$ is increasing in $t \in [0, \infty)$.

Transform (2.61) in the following way:

$$m_w(t) = \frac{\int_0^t F(u)du}{F(t)} = \frac{t - \int_0^t \overline{F}(u)du}{1 - \overline{F}(t)}, \tag{2.65}$$

and, as usual, assume that $E[T] = m(0) < \infty$. Then (2.65) results in the following asymptotic relationship:

$$m_w(t) = (t - m(0))(1 + o(1)), \quad t \to \infty.$$

As $m(0) = m$ is the mean time to failure, this relationship means that for t sufficiently large, $m_w(t)$ is approximately equal to the corresponding unconditional mean waiting time, when the condition that the failure had occurred in $[0, t]$ is not imposed. This result is intuitively evident.

2.6 Chapter Summary

In this chapter, we have discussed the definitions and basic properties of the failure rate, the mean remaining lifetime function and of the reversed failure rate. These facts are essential for our presentation in the following chapters. Exponential representation (2.5) for an absolutely continuous Cdf *via* the corresponding failure rate plays an important role in understanding, interpreting and applying reliability concepts.

We have considered a number of lifetime distributions which are most popular in applications. Complete information on the subject can be found in Johnson *et al.* (1994, 1995).

The classical Glaser result (Theorem 2.3) helps to analyse the shape of the failure rate, which is important for understanding the ageing properties of distributions. Various generalizations and extensions can be found, *e.g.*, in Lai and Xie (2006). The shape of the failure rate can also be analysed using properties of underlying stochastic processes (Aalen and Gjeissing, 2001). Some examples of this approach are considered in Chapter 10.

In Section 2.4.1, several of the simplest, most popular classes of ageing distributions were defined. It is clear that the IFR ($\lambda(t) \in \mathrm{I}$) property is the simplest and the most natural one for describing deterioration. On the other hand, the decreasing in time mean remaining lifetime also shows a monotone deterioration of an item. Note that Theorem 2.5 states that the decreasing MRL defines a more general type of ageing than the increasing failure rate.

The properties of the reversed failure (hazard) rate have recently attracted considerable interest. Although the corresponding definition seems to be rather artificial, the concept of the waiting time described in Section 2.5.2 makes it relevant for reliability applications. Another possible advantage of the reversed failure rate is that the analysis of parallel systems can usually be simpler using this characteristic than using the 'ordinary' failure rate.

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