3.3 Adding Symmetric Stabilizing Terms

When the residual-based stabilization methods of the previous section are applied to systems of convection-diffusion-reaction problems, this engenders couplings between the dependent variables but in general these couplings do not have any physical counterpart. In optimal control problems, residual-based stabilization methods lead to different discrete adjoint equations depending on whether the discretization of the problem or the construction of the adjoint is carried out first [BV07, BL08]. It has been observed that the asymmetry of the stabilizing term means that the computed control is significantly affected by the way in which the discrete optimality condition is defined. Moreover, in the case of transient problems, this asymmetric stabilization does not lead to diagonal matrices for the reaction term when a lumping technique (nodal quadrature) is applied; this is awkward for convection-dominated flows with zones of strong reaction. In the next two subsections we consider symmetric stabilization methods that avoid these failings.

3.3.1 Local Projection Stabilization

In residual-based stabilization methods with a given finite element space Y_h, several terms are added to the standard Galerkin method. For example, the streamline diffusion method adds

$$\sum_{T\in\mathcal{T}_h} \delta_T(-\varepsilon\Delta u + b\cdot\nabla u + cu - f,\, b\cdot\nabla v)_T,$$

but an inspection of how stabilization is achieved reveals that only the term

$$\sum_{T\in\mathcal{T}_h} \delta_T(b\cdot\nabla u, b\cdot\nabla v)_T \tag{3.84}$$

is responsible for the increased stability and consequent improved convergence properties. Thus it is natural to ask: in order to reduce the costs of assembling the discrete system, it is enough to add only a term like (3.84)? But with such a replacement, the consistency property of the method is lost. To retain the stability properties of the SDFEM, in the convection-dominated case choose $\delta_T = \mathcal{O}(h_T)$ in (3.84); then

$$\left|\sum_{T\in\mathcal{T}_h} \delta_T(b\cdot\nabla u, b\cdot\nabla v)_T\right| \le Ch^{1/2}|u|_1 \left(\sum_{T\in\mathcal{T}_h} \delta_T\|b\cdot\nabla v\|^2_{0,T}\right)^{1/2}$$

shows that the consistency error is $\mathcal{O}(\sqrt{h}\,)$ and the method will be suboptimal. The remedy presented here is to introduce a projection $\pi_h : L_2(\Omega) \to D_h$ into a second finite element space D_h, then to replace $b\cdot\nabla u$ by its fluctuations $\kappa_h(b\cdot\nabla u)$, where $\kappa_h := \mathrm{id} - \pi_h$ with $\mathrm{id} : L_2(\Omega) \to L_2(\Omega)$ the identity operator. The order of the consistency error can now be tuned by choosing an appropriate projection space D_h. Indeed, if π_h is the L_2 projection and D_h the space of discontinuous, piecewise polynomials of degree $k-1$ with $k \geq 1$, then

$$\|\kappa_h(b \cdot \nabla u)\|_{0,T} \leq C h_T^k \|u\|_{k+1,T},$$

and for $\delta_T = \mathcal{O}(h_T)$ it follows that

$$\Big| \sum_{T \in \mathcal{T}_h} \delta_T (\kappa_h(b \cdot \nabla u), \kappa_h(b \cdot \nabla v))_T \Big|$$
$$\leq C h^{k+1/2} \|u\|_{k+1} \left(\sum_{T \in \mathcal{T}_h} \delta_T \|\kappa_h(b \cdot \nabla v)\|_{0,T}^2 \right)^{1/2}.$$

Later we shall learn that the $\mathcal{O}(h^{k+1/2})$ estimation of the convection term for an approximation space Y_h with piecewise polynomials of degree k (which is already known for the SDFEM) can be preserved if there is an interpolant $j_h : H^2(\Omega) \to Y_h$ such that $w - j_h w$ is orthogonal to D_h.

This local projection stabilization (LPS) method is introduced for the Stokes problem in [BB01], extended to the transport equation in [BB04], and analysed for the lowest order ($r \leq 2$) discretizations of the Oseen equations in [BB06]. In all these papers a two-level approach is used where the projection space D_h lives on a mesh that is coarser than the mesh used by the approximation space Y_h. This has the disadvantage that the LPS scheme produces a stencil that is less compact than for the SDFEM stabilization. To overcome this difficulty, an alternative technique based on enrichment of the approximation space Y_h is proposed in [MST07]. We shall explain both approaches in a unified framework.

In the following the notation $\alpha \sim \beta$ means that there exist positive constants C_1 and C_2, which are independent of the meshsize h and of ε, such that

$$C_1 \alpha \leq \beta \leq C_2 \alpha.$$

Let $\mathcal{M}_h$ be a shape-regular decomposition of Ω into d-dimensional simplices, quadrilaterals or hexahedra. Each cell $M \in \mathcal{M}_h$ is called a macro-element and its diameter is denoted by h_M. Each macro-element M will be decomposed into one or more cells $T \in \mathcal{T}_h$, such that $\mathcal{T}_h$ also is shape-regular – one could for example generate $\mathcal{T}_h$ from $\mathcal{M}_h$ by some refinement rule. Then the projection space D_h will be a discontinuous finite element space defined on the macro-decomposition $\mathcal{M}_h$ while the approximation space $Y_h \subset H^1(\Omega)$ comprises continuous piecewise polynomial functions defined on $\mathcal{T}_h$. The case $\mathcal{T}_h = \mathcal{M}_h$ is permitted. We assume that the partitions $\mathcal{T}_h$ and $\mathcal{M}_h$ satisfy

$$h_T \sim h_M \qquad \forall T \subset M, \quad \forall M \in \mathcal{M}_h.$$

Let $D_h(M) := \{q_h|_M \ : \ q_h \in D_h\}$ be the local projection space. Define the global projection $\pi_h : L_2(\Omega) \to D_h$ by $(\pi_h w)|_M := \pi_M(w|_M)$, where

$\pi_M : L_2(M) \to D_h(M)$ is a local projection. Associate with the projection π_h the fluctuation operator $\kappa_h : L_2(\Omega) \to L_2(\Omega)$ defined by $\kappa_h := \mathrm{id} - \pi_h$, where $\mathrm{id} : L_2(\Omega) \to L_2(\Omega)$ is the identity.

Now we are ready to formulate the local projection stabilization (LPS) method for the convection-diffusion-reaction problem

$$-\varepsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{in } \Omega \subset \mathbb{R}^d, \quad u = 0 \quad \text{on } \Gamma, \tag{3.85}$$

where $\Gamma = \partial\Omega$, $d \geq 2$, the data b, c, f are sufficiently smooth, and $0 < \varepsilon \ll 1$ is a given small positive parameter. Assume that

$$c - \frac{1}{2}\,\mathrm{div}\, b \geq \omega > 0$$

which guarantees the unique solvability of the problem. Let $V_h = Y_h \cap H_0^1(\Omega)$ be the finite element space for approximating the weak solution $u \in H_0^1(\Omega)$ of (3.85). The corresponding stabilized discrete problem is:

Find $u_h \in V_h$ such that for all $v_h \in V_h$ one has

$$\varepsilon(\nabla u_h, \nabla v_h) + (b\cdot\nabla u_h + cu_h, v_h) + S_h(u_h, v_h) = (f, v_h), \tag{3.86a}$$

where the stabilizing term S_h is given by

$$S_h(u_h, v_h) := \sum_{M\in\mathcal{M}_h} \tau_M\Big(\kappa_h(b\cdot\nabla)u_h, \kappa_h(b\cdot\nabla)v_h\Big)_M \tag{3.86b}$$

with user-chosen constants τ_M. Define the mesh-dependent norm

$$|||v|||_{LPS} := \left(\varepsilon|v|_1^2 + \omega\|v\|_0^2 + \sum_{M\in\mathcal{M}_h} \tau_M\|\kappa_h(b\cdot\nabla)v\|_{0,M}^2\right)^{1/2} \tag{3.87}$$

associated with the discrete bilinear form implicitly defined by the left-hand side of (3.86a).

Remark 3.68. There is a close relation to stabilization by subgrid modelling [EG04, Gue99a], as we shall see in Section IV.4.5, but in subgrid modelling the stabilizing term uses gradients of fluctuations instead of fluctuations of gradients. ♣

The stability and convergence properties of the LPS method (3.86) will now be studied under the following assumptions.

Assumption A1: The approximation space Y_h is of order $r \in \mathbb{N}$. That is, there exists an interpolation operator $i_h : H^2(\Omega) \to Y_h$ with the properties that $i_h : H_0^1(\Omega) \cap H^2(\Omega) \to V_h$ and

$$\|w - i_h w\|_{0,T} + h_T|w - i_h w|_{1,T} \leq C\, h_T^l \|w\|_{l,T} \tag{3.88}$$

for all $w \in H^l(T)$, all $T \in \mathcal{T}_h$, and $2 \le l \le r+1$.

Assumption A2: The fluctuation operator κ_h has the approximation property

$$\|\kappa_h q\|_{0,M} \le C\, h_M^l |q|_{l,M} \qquad \forall q \in H^l(M),\ \forall M \in \mathcal{M}_h,\ 0 \le l \le r. \tag{3.89}$$

Remark 3.69. Let π_h be the L_2 projection in D_h and let the space $D_h(M)$ contain the space $P_{r-1}(M)$ of polynomials of degree at most $r-1$, where $r \ge 1$. Since D_h is allowed to be discontinuous across macro-element faces, the projection $\pi_M : L_2(M) \to D_h(M)$ is defined locally by

$$(\pi_M w - w, w_h)_M = 0 \qquad \forall w_h \in D_h(M),\ w \in L_2(M).$$

Then the L_2 projection $\pi_M : L_2(M) \to D_h(M)$ reduces to the identity mapping on the subspace $P_{r-1}(M) \subset H^l(M)$, and the Bramble-Hilbert lemma gives the approximation property of Assumption A2. ♣

Let $Y_h(M) := \{w_h|_M \ :\ w_h \in Y_h\} \cap H_0^1(M)$.

Assumption A3: There exists a constant $\beta_1 > 0$ such that for all $h > 0$ and all $M \in \mathcal{M}_h$ one has

$$\inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M}\, \|q_h\|_{0,M}} \ge \beta_1 > 0. \tag{3.90}$$

Remark 3.70. To satisfy Assumption A3, clearly $Y_h(M)$ has to be sufficiently rich compared with $D_h(M)$. In particular, it is necessary that

$$\dim Y_h(M) \ge \dim D_h(M). \tag{3.91}$$

On the other hand one cannot choose $D_h(M)$ too small to satisfy Assumption A3 since Assumption A2 should also be met. Later we try to fulfill both requirements for a given approximation space Y_h on $\mathcal{T}_h$ by choosing the projection space D_h as a discontinuous finite element space on the coarser mesh $\mathcal{M}_h$, where the dimension of $D_h(M)$ is small enough to satisfy Assumption A3 yet big enough to fulfil Assumption A2. A different strategy is used in the one-level approach where both spaces are defined on the same mesh: $D_h(M)$ is chosen such that Assumption A2 holds, then $Y_h(M)$ is enriched by additional functions in order to verify Assumption A3. ♣

Theorem 3.71. *Let Assumptions A1 and A3 be satisfied. Then there is an interpolation operator $j_h : H^2(\Omega) \to Y_h$, with $j_h : H_0^1(\Omega) \cap H^2(\Omega) \to V_h$, that has the following orthogonality and approximation properties:*

$$(w - j_h w, q_h) = 0 \tag{3.92a}$$

for all $q_h \in D_h$ and all $w \in H^2(\Omega)$, and

$$\|w - j_h w\|_{0,M} + h_M |w - j_h w|_{1,M} \le C\, h_M^l \|w\|_{l,M} \tag{3.92b}$$

for all $w \in H^l(\Omega)$ with $2 \le l \le r+1$, and all $M \in \mathcal{M}_h$.

Proof. Let $D_h(M)'$ denote the dual space of $D_h(M)$. Define the continuous linear operator $B_h : Y_h(M) \to D_h(M)'$ by

$$\langle B_h v_h, q_h \rangle_{D_h(M)} := (v_h, q_h)_M \quad \forall v_h \in Y_h(M),\ q_h \in D_h(M).$$

Set

$$W_h(M) := \{v_h \in Y_h(M) \ : \ (v_h, q_h) = 0 \ \ \forall q_h \in D_h(M)\},$$

and let $W_h(M)^\perp$ be the L_2-orthogonal complement of $W_h(M)$ in $Y_h(M)$. By [GR86, Lemma I.4.1], B_h is an isomorphism from $W_h(M)^\perp$ onto $D_h(M)'$ with

$$\beta_1 \|v_h\|_{0,M} \le \|B_h v_h\|_{D_h(M)'} \qquad \forall v_h \in W_h(M)^\perp$$

if and only if Assumption A3 holds true. Now, given $w \in H^2(\Omega)$, the mapping

$$q_h \mapsto (w - i_h w, q_h)_M$$

is linear and continuous on $D_h(M)$; hence for each $w \in H^2(\Omega)$ there is a unique $z_h(w) \in W_h(M)^\perp$ such that

$$\begin{aligned} \langle B_h z_h(w), q_h \rangle_{D_h(M)} &= (w - i_h w, q_h)_M \quad \forall q_h \in D_h(M), \\ \|z_h(w)\|_{0,M} &\le \frac{1}{\beta_1} \sup_{q_h \in D_h(M)} \frac{\langle B_h(z_h(w)), q_h \rangle_{D_h(M)}}{\|q_h\|_{0,M}} . \end{aligned}$$

The definition of $B_h : Y_h(M) \to D_h(M)'$ yields

$$(z_h(w), q_h)_M = (w - i_h w, q_h)_M \qquad \forall w \in H^2(\Omega), \forall q_h \in D_h(M), \tag{3.93a}$$

$$\|z_h(w)\|_{0,M} \le \frac{1}{\beta_1} \|w - i_h w\|_{0,M} \qquad \forall w \in H^2(\Omega). \tag{3.93b}$$

Set $j_h w\big|_M := i_h w\big|_M + z_h(w)$ for all $M \in \mathcal{M}_h$. Since $\bigoplus_{M \in \mathcal{M}_h} Y_h(M) \subset Y_h$, we then have a global interpolation operator $j_h : H^2(\Omega) \to Y_h$ such that

$$\|w - j_h w\|_{0,M} \le \left(1 + \frac{1}{\beta_1}\right) \|w - i_h w\|_{0,M} \le C\, h_M^l \|w\|_{l,M}$$

for all $M \in \mathcal{M}_h$, for all $w \in H^l(\Omega)$, $2 \le l \le r + 1$. That is, the L_2 approximation property of (3.92b) is verified.

The orthogonality property (3.92a) follows from (3.93a) and the definition of j_h. It remains to show the approximation property for the H^1 seminorm. To this end, apply an inverse inequality and (3.93b) to get

$$|z_h(w)|_{1,M} \le C h_M^{-1} \|z_h(w)\|_{0,M} \le C h_M^{-1} \|w - i_h w\|_{0,M} .$$

This inequality and the approximation property (3.88) then give

$$|w - j_h w|_{1,M} \le |w - i_h w|_{1,M} + |z_h(w)|_{1,M} \le C\, h_M^{l-1} \|w\|_{l,M} .$$

□

Remark 3.72. Following the analysis of [Ste99] and assuming a family of macro-elements that are equivalent to a reference macro-element, Assumption A3 reduces to showing that

$$N_M := \{q_h \in D_h(M) \ : \ (q_h, v_h)_M = 0 \quad \forall v_h \in V_h(M)\} = \{0\}.$$

♣

Example 3.73. Consider the case $\mathcal{T}_h = \mathcal{M}_h$. Let the approximation space Y_h comprise continuous piecewise linear functions enriched element by element with the bubble function b_T that is the product of the barycentric coordinates. Let the projection space D_h be discontinuous piecewise constant functions on $\mathcal{T}_h$. The usual piecewise linear nodal interpolation i_h satisfies the approximation property of Assumption A1 with $r = 1$, but it fails to satisfy (3.92a). Since $D_h(T) = \text{span}\,(1)$ and $Y_h(T) = \text{span}\,(b_T)$, Assumption A3 can be established by transforming the integrals in (3.90) to a reference cell. Thus there does exist an interpolation operator $j_h : H^2(\Omega) \to Y_h$ with the properties (3.92). It is given explicitly by a local definition on each cell T:

$$(j_h w)|_T(p_i) = w(p_i) \text{ for all vertices } p_i \in T, \quad (j_h w, 1)_T = (w, 1)_T \quad \forall T \in \mathcal{T}_h.$$

♣

Theorem 3.74. *Let the data of the problem be sufficiently smooth. Let Assumptions A1–A3 be fulfilled. If $\tau_M \sim h_M$ for all $M \in \mathcal{M}_h$, then there is a positive constant C, which is independent of ε and the mesh, such that*

$$|||u - u_h|||_{LPS} \leq C\,(\varepsilon^{1/2} + h^{1/2})h^r \|u\|_{r+1}\,.$$

Proof. The argument is standard: one demonstrates coercivity of the underlying discrete bilinear form

$$a_h(w, v) := \varepsilon(\nabla w, \nabla v) + (b \cdot \nabla w + cw, v) + S_h(w, v)$$

then estimates the approximation and consistency errors. Coercivity with respect to the $|||\cdot|||_{LPS}$ norm, i.e.,

$$a_h(v_h, v_h) \geq |||v_h|||^2_{LPS} \qquad \forall v_h \in V_h,$$

follows by integration by parts for all nonnegative τ_M. (This differs from the streamline diffusion method where an upper bound for δ_T is needed; compare the proof of Lemma 3.25.) Then for the interpolant $j_h u$ of the weak solution u of (3.85) and the solution u_h of the discrete problem (3.86) we have

$$|||j_h u - u_h|||^2_{LPS} \leq a_h(j_h u - u, j_h u - u_h) + a_h(u - u_h, j_h u - u_h)$$

whence

$$|||j_h u - u_h|||_{LPS} \le \sup_{w_h\in V_h} \frac{a_h(j_h u - u, w_h)}{|||w_h|||_{LPS}} + \sup_{w_h\in V_h} \frac{a_h(u-u_h, w_h)}{|||w_h|||_{LPS}}.$$

The first term here is the approximation error, the second term the consistency error. (The consistency error of a consistent method is zero.)

The tricky part in the estimation of the approximation error is the convection term which is split into two terms:

$$(b\cdot\nabla(j_h u - u), w_h) = -(j_h u - u, b\cdot\nabla w_h) - (\text{div } b\,(j_h u - u), w_h)$$

using integration by parts. For the first term, use the orthogonality and approximation properties of the special interpolant and $\tau_M \sim h_M$ to get

$$\begin{aligned}
\big|(j_h u - u, b\cdot\nabla w_h)\big| &= \big|(j_h u - u, \kappa_h(b\cdot\nabla)w_h)\big| \\
&\le C\left(\sum_{M\in\mathcal{M}_h} \tau_M^{-1} h_M^{2r+2} |u|^2_{r+1,M}\right)^{1/2} \left(\sum_{M\in\mathcal{M}_h} \tau_M \|\kappa_h(b\cdot\nabla)w_h\|^2_{0,M}\right)^{1/2} \\
&\le C\, h^{r+1/2}|u|_{r+1}\, |||w_h|||_{LPS}.
\end{aligned}$$

The estimation of the second term uses the approximation properties and the definition of the $|||\cdot|||_{LPS}$ norm:

$$\big|(\text{div } b\,(j_h u - u), w_h)\big| \le Ch^{r+1}|u|_{r+1}\|w_h\|_0 \le Ch^{r+1}|u|_{r+1}\,|||w_h|||_{LPS}.$$

Using (3.89), $\tau_M \sim h_M$, and $a_h(u-u_h, w_h) = S_h(u, w_h)$, the consistency error bound follows from

$$\begin{aligned}
|S_h(u,w_h)| &\le \sum_{M\in\mathcal{M}_h} \tau_M \|\kappa_h(b\cdot\nabla u)\|_{0,M}\, \|\kappa_h(b\cdot\nabla w_h)\|_{0,M} \\
&\le C\sum_{M\in\mathcal{M}_h} \tau_M h_M^r |b\cdot\nabla u|_{r,M}\, \|\kappa_h(b\cdot\nabla w_h)\|_{0,M} \\
&\le Ch^{r+1/2}\|u\|_{r+1}|||w_h|||_{LPS}.
\end{aligned}$$

It is now straightforward to finish the proof. ♣

Remark 3.75. An analogous theorem can be proved when the stabilizing term (3.86b) and the norm (3.87) are replaced by

$$S_h(u_h, v_h) := \sum_{M\in\mathcal{M}_h} \tau_M \Big(\kappa_h(\nabla u_h), \kappa_h(\nabla v_h)\Big)_M$$

and

$$|||v|||_{LPS} := \left(\varepsilon|v|_1^2 + \omega\|v\|_0^2 + \sum_{M\in\mathcal{M}_h} \tau_M \|\kappa_h(\nabla v)\|^2_{0,M}\right)^{1/2}$$

respectively. ♣

Fulfillment of Assumptions A1–A3 depends on the selections of the approximation space Y_h and the projection space D_h. Assumption A1 is satisfied for common finite element spaces that contain continuous piecewise polynomials of degree r. Assumption A2 can be easily satisfied by choosing the projection space D_h sufficiently large but Assumption A3 restricts the size of D_h for a given approximation space Y_h. Below we discuss examples of pairs of finite element spaces (Y_h, D_h) that satisfy Assumptions A1–A3 of Theorem 3.74 while referring the reader to [MST07] for the proofs.

Local Projection as a Two-level Approach

Consider the case where the partition $\mathcal{T}_h$ is formed by a suitable refinement of a macro-mesh $\mathcal{M}_h$. This is indicated by the notation $\mathcal{M}_h = \mathcal{T}_{2h}$. First we discuss simplicial elements in $\mathbb{R}^d$. A macro-element $M \in \mathcal{T}_{2h}$ is refined into $d+1$ elements $T \in \mathcal{T}_h$ by connecting the $d+1$ vertices of M with its barycentre; see Figure 3.8 for the cases $d = 2$ and $d = 3$. For the approximation

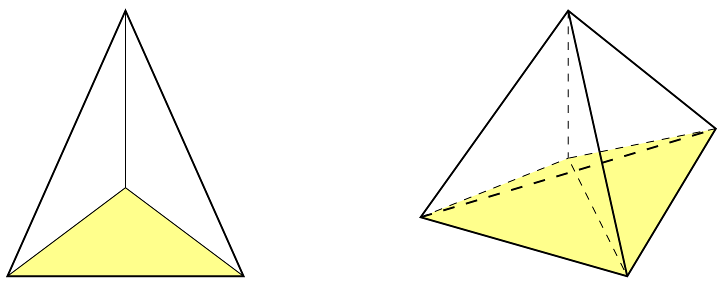

Fig. 3.8. Refinement of a macro-simplex $M \in \mathcal{T}_{2h}$ into cells $T \in \mathcal{T}_h$

space Y_h we choose a finite element space of continuous piecewise polynomials of degree $r \geq 1$. Let the projection space D_h comprise discontinuous piecewise polynomials of degree $r-1$ on $\mathcal{T}_{2h}$. This is summarized by writing $(Y_h, D_h) = (P_{r,h}, P^{\text{disc}}_{r-1,2h})$. Here and in what follows the superscript 'disc' indicates that the finite element space contains discontinuous functions. Then on shape-regular meshes Assumptions A1–A3 are satisfied [MST07].

Consider now hexahedral elements such as bricks. Let $\widehat{M} = (-1,1)^d$ denote the reference hyper-cube with 2^d vertices. This is refined into 2^d congruent cubes $\widehat{T}_i$, where $i = 1, \ldots, 2^d$. The multilinear mapping $F_M : \widehat{M} \to M$ maps $\widehat{M}$ onto a macro-cell $M \in \mathcal{T}_{2h}$ and induces a refinement of M into 2^d cells $T_i = F_M(\widehat{T}_i)$; see Figure 3.9 for the two-dimensional case. Furthermore, there is a bijective linear mapping $G_i : \widehat{T} \to \widehat{T}_i$ of the reference cell $\widehat{T} = (0,1)^d$ onto $\widehat{T}_i$ for $i = 1, \ldots, 2^d$. Now for each $T \in \mathcal{T}_h$ there are a unique $M \in \mathcal{M}_h$ and a unique $i \in \{1, \ldots, 4\}$ such that $T = T_i \subset M$ and $T = (F_M \circ G_i)(\widehat{T})$.

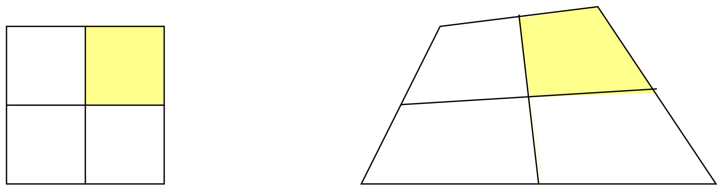

Fig. 3.9. Refinement of a macro-cell $M \in \mathcal{T}_{2h}$ (right) induced by a congruent refinement of the reference hyper-cube $\widehat{M}$ (left)

We write the bijective multilinear mapping $F_M \circ G_i$ as F_T for brevity. For the approximation space Y_h choose the standard space of mapped continuous piecewise polynomials of degree at most r in each variable, i.e., $Y_h = Q_{r,h}$. The projection space D_h lives on the coarser mesh $\mathcal{T}_{2h}$ and can be defined in two different ways, namely as an image of a space living on the reference macro-cell $\widehat{M}$ or directly on the macro-cell M. In general, this leads to two different finite element spaces. The mapped version of D_h has the advantage that the projection space defined locally on the reference macro-cell is always the same when moving from one element to another, but the approximation property of Assumption A2 is not satisfied on arbitrary families of shape-regular meshes [ABF02, Mat01]. This is apparently a great disadvantage but in practice the family of macro-element meshes is often generated by successively refining a given initial mesh, and for such a (restricted) mesh family Assumption A2 does hold true [Mat01]. The unmapped version of D_h satisfies Assumption A2 for any family of shape-regular meshes but the associated finite element spaces on the reference macro-cell differ from element to element. To distinguish between these two spaces we shall use the superscript 'unm' for the unmapped version of the finite element space D_h, with the understanding that all spaces lacking this superscript are mapped spaces.

The finite element pair $(Y_h, D_h) = (Q_{r,h}, Q^{\mathrm{disc}}_{r-1,2h})$ is our first example on hexahedral meshes; here

$$\begin{aligned} Q_{r,h} &:= \{ v \in H^1(\Omega) : v|_T \circ F_T \in Q_r(\widehat{T}) \quad \forall T \in \mathcal{T}_h \}, \\ Q^{\mathrm{disc}}_{r-1,2h} &:= \{ v \in L_2(\Omega) : v|_M \circ F_M \in Q_{r-1}(\widehat{M}) \quad \forall M \in \mathcal{T}_{2h} \}. \end{aligned}$$

Assumption A1 is clearly satisfied [Ape99, Clé75, SZ90]. Furthermore, since $P_{r-1}(M) \subset Q^{\mathrm{disc}}_{r-1}(M)$, one can verify Assumption A2 on arbitrary shape-regular families of meshes. For the proof of Assumption A3 see [MST07].

Alternatively, one can choose a smaller projection space by taking D_h to be

$$P^{\mathrm{disc}}_{r-1,2h} := \{ v \in L_2(\Omega) : v|_M \circ F_M \in P_{r-1}(\widehat{M}) \quad \forall M \in \mathcal{T}_{2h} \}.$$

This produces more stabilization in the sense that the stabilizing term vanishes on the smaller subset $P^{\mathrm{disc}}_{r-1,2h} \subset Q^{\mathrm{disc}}_{r-1,2h}$. Assumptions A1 and A3 are still valid but Assumption A2 can be guaranteed only on a restricted family of shape-regular meshes, e.g., on uniformly-refined families of meshes; see [ABF02] for quadrilateral meshes and [Mat01] for hexahedral meshes.

One could also investigate a choice of projection space D_h that is larger than $Q^{\text{disc}}_{r-1,2h}$ in order to minimize the stabilizing effect. Indeed, a dimensional analysis indicates that the inequality (3.91) is still satisfied for larger spaces D_h. For the choice $(Y_h, D_h) = (Q_{r,h}, Q^{\text{disc}}_{r-1,2h})$ one has

$$\dim Y_h(M) = (2r-1)^d \geq r^d = \dim D_h(M)$$

and only for $r = 1$ do the dimensions of both spaces coincide. In the case $r \geq 2$ a possible choice might be $D_h = Q^{\text{disc}}_{r,2h}$ since

$$\dim Y_h(M) = (2r-1)^d \geq (r+1)^d = \dim D_h(M), \qquad r \geq 2.$$

Now Assumption A1 still holds true without any change and Assumption A2 would be satisfied with a higher order of approximation than necessary. It is unclear however whether the inf-sup condition of Assumption A3 is valid.

Unmapped finite element spaces satisfy Assumption A2 on arbitrary shape-regular meshes. For example, take the approximation space to be again the space $Y_h = Q_{r,h}$ but for the projection space D_h select the space of discontinuous, piecewise polynomials of degree $r-1$ posed directly on the macro-cells $M \in \mathcal{M}_h$. That is, we choose

$$(Y_h, D_h) = (Q_{r,h}, P^{\text{disc,unm}}_{r-1,2h})$$

where

$$\begin{aligned}
Q_{r,h} &:= \{v \in H^1(\Omega) \,:\, v|_T \circ F_T \in Q_r(\widehat{T}) \quad \forall T \in \mathcal{T}_h\}\,, \\
P^{\text{disc,unm}}_{r-1,2h} &:= \{v \in L_2(\Omega) \,:\, v|_M \in P_{r-1}(M) \quad \forall M \in \mathcal{T}_{2h}\}\,.
\end{aligned}$$

Then Assumptions A1–A3 are satisfied on families of shape-regular meshes [MST07].

Local Projection by Enrichment of Approximation Spaces

One disadvantage of the local projection onto coarser meshes is that the support of the projected gradient $\kappa_h(b \cdot \nabla\varphi)$ of a basis function φ is in general larger than the support of $\nabla\varphi$, which leads to an increase in the stencil size that might not suit the data structure of an existing computer code. Bearing in mind that the key ingredient of the local projection method is the existence of an interpolation with the additional orthogonality property (3.92a), one can try to define the approximation and projection space on the same mesh $\mathcal{M}_h = \mathcal{T}_h$ and to satisfy Assumption A3 by an enrichment of the approximation space Y_h. This approach has been developed successfully in [MST07], as we now describe.

Use simplicial elements and set

$$\hat{b}(\hat{x}) := (d+1)^{d+1} \prod_{i=1}^{d+1} \hat{\lambda}_i(\hat{x}),$$

where $\hat{\lambda}_i$, $i = 1, \dots, d+1$, are barycentric coordinates on $\widehat{T}$. This bubble function $\hat{b}$ takes the value 1 at the barycentre of the reference simplex $\widehat{T}$. Then define the enriched space of continuous piecewise polynomials of degree r by

$$P_r^{\text{bubble}}(\widehat{T}) := P_r(\widehat{T}) + \hat{b} \cdot P_{r-1}(\widehat{T}) \,.$$

We choose the approximation and projection spaces

$$(Y_h, D_h) := (P_{r,h}^{\text{bubble}}, P_{r-1,h}^{\text{disc}})$$

to be the pair of finite element spaces defined via reference mappings by

$$\begin{aligned} P_{r,h}^{\text{bubble}} &:= \{v \in H^1(\Omega) \,:\, v|_T \circ F_T \in P_r^{\text{bubble}}(\widehat{T}) \quad \forall T \in \mathcal{T}_h\} \,, \\ P_{r-1,h}^{\text{disc}} &:= \{v \in L_2(\Omega) \,:\, v|_T \circ F_T \in P_{r-1}(\widehat{T}) \quad \forall T \in \mathcal{T}_h\} \,. \end{aligned}$$

Clearly Assumptions A1 and A2 are fulfilled. At first sight the enriched space seems large, but in fact

$$P_r(\widehat{T}) + \hat{b} \cdot P_{r-1}(\widehat{T}) = P_r(\widehat{T}) \oplus \left(\hat{b} \cdot \sum_{i=1}^{d} \tilde{P}_{r-i}(\widehat{T}) \right)$$

where

$$\tilde{P}_r(\widehat{T}) = \operatorname{span} \left\{ \prod_{i=1}^{d} \hat{x}_i^{\alpha_i} \,, \quad \sum_{i=1}^{d} \alpha_i = r \,, \quad (\hat{x}_1, \dots, \hat{x}_d) \in \widehat{K} \right\}.$$

The enrichment is minimal with respect to the required inequality (3.91). For, since the bubble part of the space $P_r(\widehat{T})$ is $\hat{b} \cdot P_{r-(d+1)}(\widehat{T})$, we have

$$\begin{aligned} \dim \widehat{Y}(\widehat{T}) &= \binom{r-(d+1)+d}{d} + \sum_{i=1}^{d} \left[\binom{r-i+d}{d} - \binom{r-i+d-1}{d} \right] \\ &= \binom{r-1}{d} + \binom{r-1+d}{d} - \binom{r-1}{d} \\ &= \dim D_h(\widehat{T}). \end{aligned}$$

When $(Y_h, D_h) := (P_{r,h}^{\text{bubble}}, P_{r-1,h}^{\text{disc}})$, Assumption A3 is satisfied [MST07].

If the mesh is quadrilateral or hexahedral, then the reference mapping $F_T : \widehat{T} \to T$ is in general no longer affine. Thus one has two different options for the projection space, the mapped version

$$P_{r-1,h}^{\text{disc}} := \{v \in L_2(\Omega) \,:\, v|_T \circ F_T \in P_{r-1}(\widehat{T}) \quad \forall T \in \mathcal{T}_h\}$$

and the unmapped version

$$P_{r-1,h}^{\text{disc,unm}} := \{v \in L_2(\Omega) \,:\, v|_T \in P_{r-1}(T) \quad \forall T \in \mathcal{T}_h\}.$$

To ensure the approximation property of Assumption A2 for the mapped version of the projection space, only families of uniformly-refined meshes will be considered [ABF02, Mat01]. For the unmapped version, Assumption A2 holds true on general shape-regular meshes. Choosing as approximation space $Y_h = Q_{r,h}$, i.e., the usual space of continuous piecewise mapped polynomials of degree at most r in each variable, one obtains the approximation property Assumption A1 but not the local inf-sup condition of Assumption A3. Therefore we search for suitable enrichments of the approximation space Y_h. Let

$$\hat{b}(\hat{x}) = \prod_{i=1}^{d}(1 - \hat{x}_i^2) \in Q_2(\widehat{T}) \,, \quad \hat{x} = (\hat{x}_1, \ldots, \hat{x}_d) \in \widehat{T} \,, \quad d = 2, 3 \,,$$

be a bubble function associated with the reference cell $\widehat{T} := (-1, 1)^d$. Our first enriched finite element space is

$$Q_r^{\text{bubble},1}(\widehat{T}) := Q_r(\widehat{T}) \oplus \operatorname{span}\left\{\hat{b}\, \hat{x}_i^{r-1} : i = 1, \ldots, d\right\}.$$

Select the finite element spaces

$$(Y_h, D_h) : = (Q_{r,h}^{\text{bubble},1}, P_{r-1,h}^{\text{disc}})$$

where

$$Q_{r,h}^{\text{bubble},1} \; : = \{v \in H^1(\Omega) \,:\, v|_T \circ F_T \in Q_r^{\text{bubble},1}(\widehat{T}) \quad \forall T \in \mathcal{T}_h\}.$$

Note that in general $Q_{r,h}^{\text{bubble},1}$ and $P_{r-1,h}^{\text{disc}}$ are not polynomial spaces. Since $Q_r(\widehat{T}) \subset Q_r^{\text{bubble},1}(\widehat{T})$, Assumption A1 is clearly satisfied. Assumption A2 holds on uniformly refined meshes – see [ABF02, Mat01]. For the proof of Assumption A3 we refer to [MST07].

Comparing the dimensions of the spaces $Y_h(T)$ and $D_h(T)$, one has

$$\dim \widehat{Y}(\widehat{T}) = (r-1)^d + d \geq \binom{r-1+d}{d} = \dim P_{r-1}(\widehat{T}) \quad \text{for all } r, d \in \mathbb{N}.$$

In particular the enrichment is minimal with respect to (3.91) for biquadratic and bicubic elements on quadrilaterals and for triquadratic elements on hexahedra.

Remark 3.76. It is remarkable that the space $Q_r^{\text{bubble},1}(\widehat{T})$ has, for all $r \geq 2$, precisely d basis functions more than $Q_r(\widehat{T})$. That is, the amount of enrichment is independent of the polynomial degree r. ♣

To satisfy Assumption A2 on arbitrary families of shape-regular (non-simplicial) meshes, we propose a second version of the enriched finite element space: set

$$Q_r^{\mathrm{bubble},2}(\widehat{T}) := Q_r(\widehat{T}) + \hat{b} \cdot Q_{r-1}(\widehat{T})$$

with the bubble function $\hat{b} \in Q_2(\widehat{T})$ and use the mapped enriched space

$$Q_{r,h}^{\mathrm{bubble},2} := \{v \in H^1(\Omega) \,:\, v|_T \circ F_T \in Q_r^{\mathrm{bubble},2}(\widehat{T}) \quad \forall T \in \mathcal{T}_h\}\,.$$

Thus

$$(Y_h, D_h) := (Q_{r,h}^{\mathrm{bubble},2}, P_{r-1,h}^{\mathrm{disc,unm}}).$$

and Assumptions A1–A3 are fulfilled [MST07].

Remark 3.77. The space $Q_{r,h}^{\mathrm{bubble},2}$ is more enriched than the space $Q_{r,h}^{\mathrm{bubble},1}$. Comparing the dimensions of the spaces $Y_h(T)$ and $D_h(T)$, one can surmise that the enriched space could be made smaller, but the validity of the local inf-sup condition Assumption A3 is then unresolved. ♣

Relationship to the Streamline Diffusion Method (SDFEM)

In Section 3.2.3 we started from the standard Galerkin finite element method with piecewise linears enriched by bubble functions on simplices and showed that elimination of the bubble part yields the streamline diffusion finite element method [BBF93, BR94]. Moreover, the shape of the bubble defined the SD parameter uniquely, but the symmetric version of the bubble

$$b_T := \prod_{i=1}^{d+1} \lambda_i^T, \qquad \lambda_i^T \text{ barycentric coordinates of } T,$$

as we saw in Remark 3.65, generated the SD parameter for the diffusion-dominated instead of the convection-dominated case. Several ideas have been developed to overcome this problem, ranging from the pseudo-residual-free bubble to the residual-free bubble method, where the bubbles are local solutions of the problem under consideration.

Here we shall examine the idea of eliminating the bubble part from the local projection method (3.86) for enriched approximation spaces. In problem (3.85) assume that one has piecewise constant functions b and f, and $c \equiv 0$. As in Section 3.2.3 suppose that V_h consists of piecewise linear functions and enrich this space by a bubble space B_h defined by

$$B_h := \mathrm{span}\,\{b_T \,:\, T \in \mathcal{T}_h\,\}.$$

Consider the local projection method on the enriched space $V_h \oplus B_h$ where the projection space D_h is the space of discontinuous piecewise constant functions on a triangulation $\mathcal{T}_h$:

Find $u_h \in V_h \oplus B_h$ such that for all $v_h \in V_h \oplus B_h$ one has

$$\varepsilon(\nabla u_h, \nabla v_h) + (b \cdot \nabla u_h, v_h) + S_h(u_h, v_h) = (f, v_h). \tag{3.94}$$

Here the stabilizing term S_h is given by (3.86b) with $\mathcal{M}_h = \mathcal{T}_h$. The dimension of the corresponding algebraic system of equations can be reduced by static condensation of the bubble part of the solution. To do this, write the solution as $u_h = u_L + u_B$, with $u_L \in V_h$ and $u_B \in B_h$, and use the test functions $v_h = v_L \in V_h$ and $v_h = v_B \in B_h$. As ∇v_L is piecewise constant, we get $\kappa_h(b \cdot \nabla)v_L = 0$ for all $v_L \in V_h$. Moreover, element-by-element integration by parts shows that $(\nabla v_L, \nabla v_B) = 0$ for all $v_L \in V_h$, $v_B \in V_B$. Hence (3.94) can be reformulated as:

Find $u_L \in V_h$ and $u_B \in B_h$ such that for all $v_L \in V_h$ and all $v_B \in B_h$,

$$\varepsilon(\nabla u_L, \nabla v_L) + (b \cdot \nabla(u_L + u_B), v_L) = (f, v_L), \tag{3.95a}$$

$$\varepsilon(\nabla u_B, \nabla v_B) + (b \cdot \nabla(u_L + u_B), v_B) + S_h(u_B, v_B) = (f, v_B). \tag{3.95b}$$

Now from the representation $u_B = \sum_{T \in \mathcal{T}_h} d_T b_T$, where the d_T, $T \in \mathcal{T}_h$, are unknown constants, (3.95b) becomes:

Given $u_L \in V_h$, find $\{d_T \in \mathbb{R} : T \in \mathcal{T}_h\}$ such that for each $T \in \mathcal{T}_h$,

$$\begin{aligned}\varepsilon(\nabla d_T b_T, \nabla b_T)_T + (b \cdot \nabla(u_L + d_T b_T), b_T)_T \\ + S_h(d_T b_T, b_T) = (f, b_T)_T.\end{aligned} \tag{3.96}$$

An integration by parts gives, using $\langle \cdot, \cdot \rangle$ to denote the $L_2(\Gamma)$ inner product,

$$d_T(b \cdot \nabla b_T, b_T)_T = \frac{d_T}{2} \left\langle b \cdot n, b_T^2 \right\rangle_{\partial T} = 0,$$

$$\pi_T(b \cdot \nabla) b_T = \frac{1}{|T|}\, b \cdot \int_T \nabla b_T \, dx = \frac{1}{|T|}\, b \cdot \int_{\partial T} b_T \, n \, d\gamma = 0$$

and (3.96) reduces to:

Given $u_L \in V_h$, find $\{d_T \in \mathbb{R} : T \in \mathcal{T}_h\}$ such that for each T,

$$d_T \left(\varepsilon |b_T|_{1,T}^2 + \tau_T \|b \cdot \nabla b_T\|_{0,T}^2 \right) = (f - b \cdot \nabla u_L, b_T)_T.$$

This has the solution

$$d_T = \frac{(1, b_T)_T}{\varepsilon |b_T|_{1,T}^2 + \tau_T \|b \cdot \nabla b_T\|_{0,T}^2} \, (f - b \cdot \nabla u_L)\big|_T. \tag{3.97}$$

Then (3.95a) can be rewritten as

$$\varepsilon(\nabla u_L, \nabla v_L) + (b \cdot \nabla u_L, v_L) + \sum_{T \in \mathcal{T}_h} d_T (b \cdot \nabla b_T, v_L)_T = (f, v_L).$$

The term $\sum_{T \in \mathcal{T}_h} \cdots$ does not appear in the standard Galerkin finite element method applied on the space V_h. One can rearrange it as

$$\sum_{T\in\mathcal{T}_h} d_T(b\cdot\nabla b_T, v_L)_T = -\sum_{T\in\mathcal{T}_h} d_T(b\cdot\nabla v_L, b_T)_T$$
$$= \sum_{T\in\mathcal{T}_h} \gamma_T(b\cdot\nabla u_L - f, b\cdot\nabla v_L)_T,$$

where, using (3.97), one sees that

$$\gamma_T = \frac{1}{|T|}\,\frac{|(1,b_T)_T|^2}{\varepsilon|b_T|^2_{1,T} + \tau_T\|b\cdot\nabla b_T\|^2_{0,T}}\,. \tag{3.98}$$

We have now eliminated the bubble component from (3.94), arriving at

$$\varepsilon(\nabla u_L,\nabla v_L) + (b\cdot\nabla u_L, v_L) + \sum_{T\in\mathcal{T}_h}\gamma_T(b\cdot\nabla u_L, b\cdot\nabla v_L)_T$$
$$= (f,v_L) + \sum_{T\in\mathcal{T}_h}\gamma_T(f, b\cdot\nabla v_L)_T \quad \text{for all } v_L\in V_h.$$

This is the streamline diffusion method (3.36) with the SD parameter $\delta_T \equiv \gamma_T$ given by (3.98). A scaling argument shows that $(1,b_T) \sim |T|$, $|b_T|^2_{1,T} \sim |T|/h_T^2$, and $\|b\cdot\nabla b_T\|^2_{0,T} \sim |T|\,\|b\|^2/h_T^2$, so $\gamma_T \sim h_T^2/(\varepsilon+\tau_T\|b\|^2)$. For $\tau_T = 0$ one has $\gamma_T \sim h_T^2/\varepsilon$ which corresponds to the diffusion-dominated case. Clearly γ_T is decreasing for increasing τ_T. The choice $\gamma_T \sim h_T/\|b\|$ in the convection-dominated case $\|b\|\,h_T/\varepsilon \gg 1$ corresponds to $\tau_T \sim h_T/\|b\|$. Letting $\tau_T\to\infty$, we obtain the standard Galerkin method that corresponds to $\gamma_T = 0$.

Comparing the residual-free bubble method with the local projection methods applied to the model problem (piecewise constant b and f, $c\equiv 0$), we see that via static condensation both methods recover the streamline diffusion method. But to generate the correct SD parameter, the RFB method needs to solve (at least approximately) local subproblems to find the correct bubble functions whereas for LPS the use of the simple bubble function $b_T = \prod_{i=1}^{d+1}\lambda_i^T$ suffices.

3.3.2 Continuous Interior Penalty Stabilization

Now we move on to the continuous interior penalty (CIP) stabilization method for the convection-diffusion problem

$$-\varepsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma, \tag{3.99}$$

where $\Gamma = \partial\Omega$, $\Omega\subset\mathbb{R}^d$ with $d = 2$ or 3, the data b, c, f are sufficiently smooth, and $0<\varepsilon\ll 1$ is a given small positive parameter. Assume as usual that

$$c - \frac{1}{2}\operatorname{div} b \ge \omega > 0,$$

which guarantees existence and uniqueness of a solution to (3.99). Let $\mathcal{T}_h$ be a shape-regular triangulation of the domain Ω into cells $T\in\mathcal{T}_h$ with $\mathcal{E}_h$ the set of all inner edges (faces in the three-dimensional case). Let $Y_h \subset H^1(\Omega)$ be a finite element space of piecewise polynomials of degree $r \geq 1$.

In the continuous interior penalty stabilization method, a symmetric term will be added to the Galerkin finite element discretization. Unlike other stabilization methods the Dirichlet boundary conditions are not incorporated into the finite element space Y_h but are imposed weakly on the discrete problem. We first discuss how Dirichlet-type boundary condition are implemented in a weak sense and address the CIP stabilization later.

Multiplying the differential equation $-\varepsilon\Delta u + b\cdot\nabla u + cu = f$ by a test function v, integrating over Ω and integrating by parts, we get

$$\varepsilon(\nabla u, \nabla v) + (b\cdot\nabla u + cu, v) - \varepsilon\left\langle \frac{\partial u}{\partial n}, v\right\rangle_\Gamma = (f, v)$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the inner product in $L_2(\Gamma)$. To obtain a lower bound like

$$(b\cdot\nabla v + cv, v) \geq \omega\|v\|_0^2 \qquad \forall v\in H_0^1(\Omega)$$

on the larger space $H^1(\Omega)$, subtract the term $\langle b\cdot n\, u, v\rangle_{\Gamma_-}$, which vanishes for $u\in H_0^1(\Omega)$ but not for $u\in H^1(\Omega)$. Here $\Gamma_- = \{x\in\Gamma : (b\cdot n)(x) < 0\}$ is the inflow part of the boundary. Then

$$\begin{aligned}(b\cdot\nabla v + cv, v)\; &- \langle b\cdot n\, v, v\rangle_{\Gamma_-}\\ &= \left(c - \frac{1}{2}\,\mathrm{div}\; b, v^2\right) + \frac{1}{2}\langle b\cdot n, v^2\rangle_\Gamma - \langle b\cdot n, v^2\rangle_{\Gamma_-}\\ &\geq \omega\|v\|_0^2 + \frac{1}{2}\|\,|b\cdot n|^{1/2}\, v\|_{0,\Gamma}^2.\end{aligned}$$

Furthermore, we add the term $\varepsilon\langle u, \frac{\partial v}{\partial n}\rangle_\Gamma$ to preserve the symmetry on $H^1(\Omega)$ of the diffusion term contribution and also add a penalty term to ensure coercivity. Then the statement of the *standard Galerkin method with weakly imposed boundary conditions* is:

Find $u_h\in Y_h$ such that for all $v_h\in Y_h$ one has

$$a_h(u_h, v_h) = (f, v_h)$$

where

$$\begin{aligned}a_h(u,v) = \varepsilon(\nabla u,\nabla v) + (b\cdot\nabla u + cu, v) - \varepsilon\left\langle\frac{\partial u}{\partial n}, v\right\rangle_\Gamma&\\ - \varepsilon\left\langle u, \frac{\partial v}{\partial n}\right\rangle_\Gamma - \langle b\cdot nu, v\rangle_{\Gamma_-} + \sum_{E\subset\Gamma}\frac{\varepsilon\gamma}{h_E}\langle u, v\rangle_E\,.&\end{aligned} \tag{3.100}$$

Lemma 3.78. *For all $v_h\in Y_h$, the bilinear form a_h given in* (3.100) *satisfies*

$$a_h(v_h,v_h) \geq \frac{1}{2}\left(\varepsilon|v_h|_1^2 + \omega\|v_h\|_0^2 + \|\,|b\cdot n|^{1/2}\, v\|_{0,\Gamma}^2 + \sum_{E\subset\Gamma}\frac{\varepsilon}{h_E}\|v_h\|_{0,E}^2\right)$$

provided that $\gamma\geq\gamma_0$ where γ_0 is sufficiently large (independently of ε and h).

Proof. It is already clear that

$$a_h(v_h, v_h) \geq \varepsilon |v_h|_1^2 + \omega \|v_h\|_0^2 - 2\varepsilon \left\langle \frac{\partial v_h}{\partial n}, v_h \right\rangle_\Gamma + \frac{1}{2} \| \, |b \cdot n|^{1/2} \, v\|_{0,\Gamma}^2 + \varepsilon\gamma \sum_{E \subset \Gamma} \frac{1}{h_E} \|v_h\|_{0,E}^2.$$

For $E \subset \partial T$, the Cauchy-Schwarz inequality and a trace inequality yield

$$2\varepsilon \left| \left\langle \frac{\partial v_h}{\partial n}, v_h \right\rangle_E \right| \leq 2\varepsilon \, C h_E^{-1/2} |v_h|_{1,T} \|v_h\|_{0,E} \leq \frac{\varepsilon}{2} |v_h|_{1,T}^2 + \frac{2\varepsilon C^2}{h_E} \|v_h\|_{0,E}^2.$$

Summing over all edges (faces) $E \subset \Gamma$ and taking $\gamma \geq 1/2 + 2C^2$ gives the desired result. $\square$

The above derivation of the bilinear form a_h shows that the standard Galerkin method with weakly imposed boundary condition is consistent, i.e., for a solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ of (3.99) one has

$$a_h(u, v_h) = (f, v_h) \qquad \forall v_h \in Y_h.$$

The CIP stabilized discrete problem is now defined to be:

Find $u_h \in Y_h$ such that for all $v_h \in Y_h$ one has

$$a_h(u_h, v_h) + J_h(u_h, v_h) = (f, v_h), \tag{3.101a}$$

where the stabilizing term J_h has the form

$$J_h(u, v) := \sum_{E \in \mathcal{E}_h} \tau_E \, \langle b_h \cdot [\nabla u]_E, b_h \cdot [\nabla v]_E \rangle_E \, . \tag{3.101b}$$

Here for each $E \in \mathcal{E}_h$ the τ_E are user-chosen parameters, $[w]_E$ is the jump of w across $E \in \mathcal{E}_h$ in a fixed direction n_E, i.e.,

$$([w]_E)(x) = \lim_{t \to +0} \{w(x + tn_E) - w(x - tn_E)\} \quad \text{for } x \in E,$$

and b_h is a continuous piecewise linear approximation of b that satisfies

$$\|b - b_h\|_{0,\infty,T} \leq C h_T \|b\|_{1,\infty,T}.$$

The form of the stabilizing term means that CIP stabilization is also called *edge stabilization*. For $u \in H^2(\Omega)$ one has $[u]_E = 0$ for all $E \in \mathcal{E}_h$ so CIP stabilization is consistent and enjoys the Galerkin orthogonality property.

Remark 3.79. Modifications of the stabilizing term are possible; see [BH04, Bur05, BFH06] ♣

A discrete bilinear form is associated with the left-hand side of (3.101a) in the usual way. To analyse this bilinear form we introduce the mesh-dependent norm

$$|||v|||_{CIP} := \left(\varepsilon|v|_1^2 + \omega\|v\|_0^2 + J_h(v,v) + \||b\cdot n|^{1/2} v\|_{0,\Gamma}^2 + \sum_{E\subset\Gamma} \frac{\varepsilon}{h_E}\|v\|_{0,E}^2 \right)^{1/2}.$$

Let

$$H^2(\mathcal{T}_h) := \{v : \Omega \to \mathbb{R} \,:\, v|_T \in H^2(T) \quad \forall T \in \mathcal{T}_h\}$$

be the space of piecewise H^2 functions. Then the key step in analysing the CIP stabilization is the following lemma.

Lemma 3.80. *There exists an interpolation operator $\pi_h^* : H^2(\mathcal{T}_h) \to Y_h$ and a positive constant C (independent of the mesh size) such that for all $v_h \in Y_h$ and all $T \in \mathcal{T}_h$ one has*

$$h_T \,\|b_h \cdot \nabla v_h - \pi_h^*(b_h \cdot \nabla v_h)\|_{0,T}^2 \le C \sum_{E\in\mathcal{E}_h(T)} \int_E h_E^2 |b_h \cdot [\nabla v_h]_E|^2 \, d\gamma, \quad (3.102)$$

where $\mathcal{E}_h(T) := \{E \in \mathcal{E}_h \,:\, E \cap T \neq \emptyset\}$.

Proof. Let $\mathcal{N}$ be the set of all nodes, i.e., those points p_i that are associated with the degrees of freedom $v_h(p_i)$ of Y_h. Thus each $v_h \in Y_h$ is uniquely defined by prescribing its values $v_h(p_i)$ for all $p_i \in \mathcal{N}$. For each node $p_i \in \mathcal{N}$, let m_i be the number of cells that contain p_i as a node. If $m_i = 1$ we call p_i an *inner node* – so a point $p_i \in \Gamma \cap \mathcal{N}$ that does not lie on an intersection of mesh lines is an 'inner' node. As in [Osw91, Sch00, Bur05, BFH06] introduce the quasi-interpolant $\pi_h^* v \in Y_h$ defined by

$$(\pi_h^* v)(p_i) := \frac{1}{m_i} \sum_{\{T: p_i \in T\}} v|_T(p_i) \qquad v \in H^2(\mathcal{T}_h).$$

Choose a discontinuous piecewise polynomial function Φ by setting

$$\Phi|_T := \Phi_T = (b_h \cdot \nabla v_h - \pi_h^*(b_h \cdot \nabla v_h))\Big|_T \in P_r(T).$$

Then $\Phi_T(p_j) = 0$ at all inner nodes p_j of T, owing to the definition of π_h^*. Hence, applying the norm equivalence of finite-dimensional spaces on the reference cell and using the scaling property, for shape-regular meshes one gets

$$\|\Phi_T\|_{0,T} \le C\, h_T^{1/2} \|\Phi_T\|_{0,\partial T} \qquad \forall T \in \mathcal{T}_h.$$

Next, define the (scaled) ℓ_1 norm of each $q_h \in P_r(E)$ by

$$\|q_h\|_{\ell_1,E} := |E|^{1/2} \sum_{\{j: p_j \in E\}} |q_h(p_j)|.$$

Appealing to norm equivalence on a reference edge (face), we find that there are positive constants C_1 and C_2 such that

$$C_1\|q_h\|_{0,E} \le \|q_h\|_{\ell_1,E} \le C_2\|q_h\|_{0,E} \qquad \forall q_h \in P_r(E),\ \forall E \in \mathcal{E}_h.$$

The continuity of b_h and the definition of the quasi-interpolant π_h^* imply that for all nodes $p_j \in E \subset \partial T$ we have

$$\Phi_T(p_j) \;= \frac{1}{m_j} \sum_{\{T':p_j\in T'\}} b_h(p_j)\cdot\left(\nabla v_h\big|_T(p_j) - \nabla v_h\big|_{T'}(p_j)\right),$$

so

$$|\Phi_T(p_j)| \le \frac{1}{m_j} \sum_{\{T':p_j\in T'\}} \sum_{E'\in P(T,T')} |b_h(p_j)\cdot[\nabla v_h]_{E'}(p_j)|,$$

where $P(T,T')$ denotes the set of all edges (faces) between T and T' (the shortest path); see Figure 3.10. If there are two paths with the same number of edges, choose one of them to make the definition of $P(T,T')$ unique. On

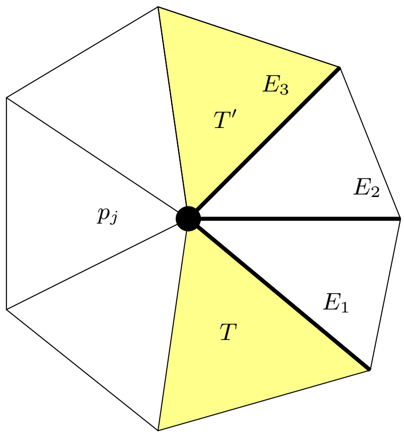

Fig. 3.10. Set of all edges E belonging to the shortest path $P(T,T') = \{E_1,E_2,E_3\}$

the skeleton $\mathcal{E}_h$ define the piecewise polynomial function

$$\Psi_E := b_h\cdot[\nabla v_h]_E \in P_r(E) \qquad \forall E \in \mathcal{E}_h$$

and denote the subset of edges containing the node p_j by

$$\mathcal{E}_{h,j} := \{E \in \mathcal{E}_h : p_j \in E\}.$$

The assumption that the family of meshes is shape-regular ensures that the sets $\mathcal{E}_{h,j}$ and $\mathcal{E}_h(T)$ each contain a bounded number of edges (faces). Moreover, $h_E \sim h_T$ for all $E \in \mathcal{E}_h(T)$ and $|E'| \sim |E|$ for all $E', E \in \mathcal{E}_h(T)$. Since

$$|\Phi_T(p_j)| \leq C \sum_{E' \in \mathcal{E}_{h,j}} |\Psi_{E'}(p_j)|$$

one obtains the estimate

$$\|\Phi_T\|_{\ell_1,E} \leq C \sum_{E' \in \mathcal{E}_h(T)} \|\Psi_{E'}\|_{\ell_1,E'} \qquad \forall E \subset \partial T.$$

Collecting the various inequalities, for each $T \in \mathcal{T}_h$ we deduce that

$$\begin{aligned}
h_T\|\Phi_T\|_{0,T}^2 &\leq Ch_T^2 \sum_{E \subset \partial T} \|\Phi_T\|_{0,E}^2 \;\leq\; Ch_T^2 \sum_{E \subset \partial T} \|\Phi_T\|_{\ell_1,E}^2 \\
&\leq Ch_T^2 \Bigg(\sum_{E' \in \mathcal{E}_h(T)} \|\Psi_{E'}\|_{\ell_1,E'} \Bigg)^2 \\
&\leq C \sum_{E' \in \mathcal{E}_h(T)} h_{E'}^2 \|\Psi_{E'}\|_{\ell_1,E'}^2 \;\leq\; C \sum_{E' \in \mathcal{E}_h(T)} h_{E'}^2 \|\Psi_{E'}\|_{0,E'}^2
\end{aligned}$$

where the inequality $(\sum a_i)^2 \leq C \sum a_i^2$ – valid for a bounded number of summands – was used. Recalling the definitions of Φ_T and $\Psi_{E'}$, the proof is complete. $\square$

Remark 3.81. It can be shown (see for example [BFH06]) that a positive constant C^* exists such that the lower bound

$$C^* \sum_{E \in \mathcal{E}_h(T)} \int_E h_E^2 |b_h \cdot [\nabla v_h]_E|^2 \, d\gamma \leq h_T \, \|b_h \nabla \cdot v_h - \pi_h^*(b_h \nabla \cdot v_h)\|_{0,T}^2$$

also holds true. Summing this inequality and (3.102) over T we get

$$C_1 J_h(v_h, v_h) \leq \sum_{T \in \mathcal{T}_h} h_T \, \|b_h \nabla \cdot v_h - \pi_h^*(b_h \nabla \cdot v_h)\|_{0,T}^2 \leq C_2 J_h(v_h, v_h)$$

when the parameter in (3.101b) is chosen so that $\tau_E \sim h_E^2$. ♣

Remark 3.82. In local projection stabilization we added a stabilizing term of the form

$$S_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \tau_T (\kappa_h(b_h \cdot \nabla u_h), \kappa_h(b_h \cdot \nabla v_h))_T$$

where $\kappa_h = \mathrm{id} - \pi_h$ is the fluctuation operator and π_h a local projection onto the (discontinuous) projection space D_h. If π_h is replaced by the quasi-interpolant $\pi_h^* : H^2(\mathcal{T}_h) \to Y_h$, then Lemma 3.80 enables us to replace the stabilizing term $S_h(\cdot,\cdot)$ on the discrete space Y_h by the stabilizing term

$$J_h(u_h, v_h) = \sum_{E \in \mathcal{E}_h} \tau_E \left\langle b_h \cdot [\nabla u_h]_E, b_h \cdot [\nabla v_h]_E \right\rangle_E .$$

The advantage of this replacement is the consistency of the CIP stabilization method; the LPS method is not consistent. ♣

Before investigating the convergence properties of the CIP stabilization method we describe the approximation properties of the global L_2 projection $i_h : L_2(\Omega) \to Y_h$.

Lemma 3.83. *The L_2 projection $i_h : L_2(\Omega) \to Y_h$ satisfies the global approximation properties*

$$\sum_{T \in \mathcal{T}_h} h_T^{2m} |u - i_h u|_{m,T}^2 \le C \sum_{T \in \mathcal{T}_h} h_T^{2r+2} |u|_{r+1,T}^2 \qquad \forall u \in H^{r+1}(\Omega),$$
$$\sum_{E \subset \Gamma} h_E |u - i_h u|_{0,E}^2 \le C \sum_{T \in \mathcal{T}_h} h_T^{2r+2} |u|_{r+1,T}^2 \qquad \forall u \in H^{r+1}(\Omega),$$

on shape-regular meshes $\mathcal{T}_h$ where $0 \le m \le r+1$ with $r \ge 1$.

Proof. Let $u^I \in Y_h$, $u \in H^2(\Omega)$, be the usual nodal interpolant that satisfies

$$h_T^m |u - u^I|_{m,T} \le C h_T^{r+1} |u|_{r+1,T} \qquad \forall u \in H^{r+1}(T)$$

where $0 \le m \le r+1$ and $r \ge 1$. Applying the Cauchy-Schwarz inequality to $(u - i_h u, u - i_h u) = (u - i_h u, u - u^I)$ yields the $L_2(\Omega)$ estimate

$$\|u - i_h u\|_0 \le \|u - u^I\|_0 \le C \sum_{T \in \mathcal{T}_h} h_T^{2r+2} |u|_{r+1,T}^2.$$

Estimates for the derivatives can then be deduced via a triangle inequality and an inverse estimate:

$$\begin{aligned} h_T^m |u - i_h u|_{m,T} &\le h_T^m |u - u^I|_{m,T} + h_T^m |u^I - i_h u|_{m,T} \\ &\le C h_T^{r+1} |u|_{r+1,T} + C\|u^I - u\|_{0,T} + C\|u - i_h u\|_{0,T} \\ &\le C h_T^{r+1} |u|_{r+1,T} + C\|u - i_h u\|_{0,T}. \end{aligned}$$

Squaring, summing and applying the above L_2 bound, we get the first of the desired estimates. For the second, a scaled version of a trace theorem gives

$$h_E^{1/2} \|v\|_{0,E} \le C(\|v\|_{0,T} + h_T |v|_{1,T}) \qquad \text{for all } v \in H^1(T). \tag{3.103}$$

Again square, sum, and apply the global L_2 and H^1 bounds. □

Remark 3.84. Lemma 3.83 does *not* imply that

$$\|u - i_h u\|_m^2 \le C \sum_{T\in\mathcal{T}_h} h_T^{2(r-m+1)} |u|_{r+1,T}^2 \qquad \forall u \in H^{r+1}(\Omega) \tag{3.104a}$$

$$\|u - i_h u\|_{0,\Gamma}^2 \le C \sum_{T\in\mathcal{T}_h} h_T^{2r+1} |u|_{r+1,T}^2 \qquad \forall u \in H^{r+1}(\Omega) \tag{3.104b}$$

for $m = 1, \ldots, r+1$, but on quasi-uniform meshes where $ch \le h_T \le h$ these inequalities are valid. ♣

Remark 3.85. Assume that the L_2 projection i_h is H^m stable, i.e.,

$$\|i_h u\|_m \le C_S \|u\|_m \qquad \forall u \in H^m(\Omega).$$

Then for the nodal interpolant u^I one has

$$\|u - i_h u\|_m \le \|u - u^I\|_m + \|i_h(u^I) - i_h u\|_m \le (1 + C_S)\|u - u^I\|_m$$

and (3.104a) follows. The L_2 projection is H^1 stable on quasi-uniform meshes and in [BPS01] this stability has been proved for the more general case of shape-regular meshes that satisfy a certain mesh condition. ♣

Theorem 3.86. *Let the data of the problem be sufficiently smooth, let γ be sufficiently large and assume that $\tau_E \sim h_E^2$. Then there is a positive constant C, which is independent of ε and the mesh, such that on quasi-uniform meshes one has*

$$|||u - u_h|||_{CIP} \le C\,(\varepsilon^{1/2} + h^{1/2})h^r \|u\|_{r+1}\,.$$

Proof. The proof follows a familiar pattern: demonstrate the coercivity of the underlying discrete bilinear form on Y_h with respect to the norm $|||\cdot|||_{CIP}$ then estimate the approximation error. By Lemma 3.78 one has

$$a_h(v_h, v_h) + J_h(v_h, v_h) \ge \frac{1}{2}|||v_h|||_{CIP}^2 \qquad \forall v_h \in Y_h,$$

for all nonnegative τ_E and γ large enough. Then, for any interpolant $i_h u \in Y_h$ of the weak solution u, with u_h the solution of the discrete problem, we get

$$\frac{1}{2}|||u_h - i_h u|||_{CIP}^2 \le a_h(u - i_h u, u_h - i_h u) + J_h(u - i_h u, u_h - i_h u)$$

whence

$$|||u_h - i_h u|||_{CIP} \le 2 \sup_{w_h\in Y_h} \frac{a_h(u - i_h u, w_h)}{|||w_h|||_{CIP}} + 2 \sup_{w_h\in Y_h} \frac{J_h(u - i_h u, w_h)}{|||w_h|||_{CIP}}\,.$$

Consider the individual terms in $a_h(u - i_h u, w_h)$ for all $w_h \in Y_h$. Integration by parts of the convection term gives

$$
\begin{aligned}
a_h(u - i_h u, w_h) = \varepsilon\,(\nabla(u - i_h u), \nabla w_h) &+ ((c - \ \mathrm{div}\ b)(u - i_h u), w_h) \\
&+ \langle b \cdot n\,(u - i_h u), w_h \rangle_{\Gamma_+} - (u - i_h u, b \cdot \nabla w_h) \\
&- \varepsilon \left\langle \frac{\partial (u - i_h u)}{\partial n}, w_h \right\rangle_{\Gamma} - \varepsilon \left\langle u - i_h u, \frac{\partial w_h}{\partial n} \right\rangle_{\Gamma} \\
&+ \sum_{E \subset \Gamma} \frac{\varepsilon \gamma}{h_E} \langle u - i_h u, w_h \rangle_E. \qquad (3.105)
\end{aligned}
$$

Here the fourth term is the most troublesome and we estimate it first. Adding and subtracting $b_h \cdot \nabla w_h$ gives

$$
(u - i_h u, b \cdot \nabla w_h) = (u - i_h u, (b - b_h) \cdot \nabla w_h) + (u - i_h u, b_h \cdot \nabla w_h);
$$

for the first term here an inverse inequality gives

$$
\begin{aligned}
|(u - i_h u, (b - b_h) \cdot \nabla w_h)| &\le C \sum_{T \in \mathcal{T}_h} \|u - i_h u\|_{0,T}\, h_T\, |w_h|_{1,T} \\
&\le C h^{r+1} \|u\|_{r+1}\, |||w_h|||_{CIP},
\end{aligned}
$$

while for the second term choose i_h to be the global L_2 projection in Y_h, and then the orthogonality of $u - i_h u$ with respect to Y_h and Lemma 3.80 imply that

$$
\begin{aligned}
|(u - i_h u, b_h \cdot \nabla w_h)| &= |\,(u - i_h u, b_h \cdot \nabla w_h - \pi_h^*(b_h \cdot \nabla w_h))\,| \\
&\le C \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|u - i_h u\|_{0,T}^2 \right)^{1/2} |||w_h|||_{CIP} \\
&\le C h^{r+1/2} \|u\|_{r+1}\, |||w_h|||_{CIP}.
\end{aligned}
$$

The other terms in (3.105) are bounded by means of standard arguments:

$$
\begin{aligned}
\varepsilon |(\nabla(u - i_h u), \nabla w_h)| &\le C \varepsilon^{1/2} h^r \|u\|_{r+1}\, |||w_h|||_{CIP}, \\
|((c - \nabla \cdot b)(u - i_h u), w_h)| &\le C h^{r+1} \|u\|_{r+1}\, |||w_h|||_{CIP}, \\
|\langle |b \cdot n| (u - i_h u), w_h \rangle_{\Gamma_+}| &\le C h^{r+1/2} \|u\|_{r+1}\, |||w_h|||_{CIP},
\end{aligned}
$$

where the scaled trace inequality (3.103) was used in deriving the last estimate. The Cauchy-Schwarz inequality shows that

$$
\left| \varepsilon \left\langle \frac{\partial (u - i_h u)}{\partial n}, w_h \right\rangle_{\Gamma} \right| \le C \varepsilon^{1/2} \left(\sum_{E \subset \Gamma} h_E \left\| \frac{\partial (u - i_h u)}{\partial n} \right\|_{0,E}^2 \right)^{1/2} |||w_h|||_{CIP}.
$$

An invocation of the scaled trace inequality (3.103) gives

$$
h_E^{1/2} \left\| \frac{\partial (u - i_h u)}{\partial n} \right\|_{0,E} \le C\,(|u - i_h u|_{1,T} + h_T |u - i_h u|_{2,T}) \qquad \forall E \subset \partial T;
$$

squaring then summing, we get

$$\left|\varepsilon\left\langle\frac{\partial(u-i_hu)}{\partial n},w_h\right\rangle_\Gamma\right|\le C\varepsilon^{1/2}h^r\|u\|_{r+1}\,|||w_h|||_{CIP}.$$

For the penultimate term in (3.105) one proceeds similarly, using an inverse inequality:

$$h_E^{1/2}\left\|\frac{\partial w_h}{\partial n}\right\|_{0,E}\le C\left(|w_h|_{1,T}+h_T|w_h|_{2,T}\right)\le C|w_h|_{1,T}\qquad\forall E\subset\partial T.$$

It follows that

$$\begin{aligned}\left|\varepsilon\left\langle(u-i_hu),\frac{\partial w_h}{\partial n}\right\rangle_\Gamma\right|&\le\sum_{E\subset\Gamma}\left(\frac{\varepsilon}{h_E}\right)^{1/2}\|u-i_hu\|_{0,E}\,(\varepsilon h_E)^{1/2}\left\|\frac{\partial w_h}{\partial n}\right\|_{0,E}\\&\le\left(\sum_{E\subset\Gamma}\frac{\varepsilon}{h_E}\|u-i_hu\|_{0,E}^2\right)^{1/2}\left(\varepsilon\sum_{T\in\mathcal{T}_h}|w_h|_{1,T}^2\right)^{1/2}\\&\le C\varepsilon^{1/2}h^r\|u\|_{r+1}\,|||w_h|||_{CIP}.\end{aligned}$$

The final term in (3.105) is handled by a Cauchy-Schwarz inequality, obtaining

$$\begin{aligned}\left|\sum_{E\subset\Gamma}\frac{\varepsilon\gamma}{h_E}\langle(u-i_hu,w_h\rangle_E\right|&\le\gamma\left(\sum_{E\subset\Gamma}\frac{\varepsilon}{h_E}\|u-i_hu\|_{0,E}^2\right)^{1/2}|||w_h|||_{CIP}\\&\le C\varepsilon^{1/2}h^r\|u\|_{r+1}\,|||w_h|||_{CIP}.\end{aligned}$$

Finally, using similar arguments to estimate the stabilizing term from the start of the proof, we get

$$\begin{aligned}|J_h(u-i_hu,w_h)|&\le Ch^{r+1/2}\|u\|_{r+1}\,|J_h(w_h,w_h)|^{1/2}\\&\le Ch^{r+1/2}\|u\|_{r+1}\,|||w_h|||_{CIP}.\end{aligned}$$

Combining the above estimates produces the desired error estimate. □

Remark 3.87. The proof of Theorem 3.86 assumed that the meshes were quasi-uniform. This assumption can be relaxed slightly [BFH06]. An alternative way of avoiding the assumption of quasi-uniformity is to replace the L_2 projection i_h by the standard nodal interpolation u^I. Although one cannot then appeal to an orthogonality property when estimating the convection term, nevertheless an $\mathcal{O}(h^r)$ error estimate (instead of the above $\mathcal{O}(h^{r+1/2})$) can be established; see [Sch07]. ♣

Remark 3.88. The continuous interior penalty approach is generalized to the *hp* version of the finite element method in [BE07]. In [BH04] the question of a discrete maximum principle is discussed. Local error estimates similar to those stated for the streamline diffusion method in Theorem 3.41 have been established in [BGL07]. ♣

Finally, we wish to point out the close relationship between the LPS and CIP analyses. The essential point in the error estimation of both methods is a special treatment of the convection term.

For the LPS method, after integrating by parts, the orthogonality property of a special interpolant j_h with respect to the projection space D_h is used:

$$(u - j_h u, b \cdot \nabla w_h) = (u - j_h u, b \cdot \nabla w_h - \pi_h(b \cdot \nabla w_h))$$

where $\pi_h : L_2(\Omega) \to D_h$ is a local projection into the discontinuous projection space D_h. Control over $\kappa_h(b \cdot \nabla w_h) = b \cdot \nabla w_h - \pi_h(b \cdot \nabla w_h)$ is achieved by adding a stabilizing term like (3.86b) which causes a consistency error, but this is sufficiently small provided that the projection space D_h is sufficiently large.

In the CIP stabilization method, the special interpolant j_h is replaced by the standard (global) L_2 projection $i_h : L_2(\Omega) \to V_h$ into the continuous finite element space Y_h and the L_2 projection π_h of LPS is replaced by the quasi-interpolant π_h^* into Y_h. The special construction of the quasi-interpolant π_h^* permits an L_2 control of $b_h \cdot \nabla w_h - \pi_h^*(b_h \cdot \nabla w_h)$ by (appropriately scaled) jumps in the gradient of w_h – see Lemma 3.80 above.

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