$$-\frac{\partial^2 v_0}{\partial \xi^2} + \frac{\partial v_0}{\partial y} = 0, \quad v_0(0,y) = g(y), \quad v_0(\xi,0) = 0$$

has, on taking $g(y) := y^i$ for $i = 0, 1, \ldots$, infinitely many linearly independent solutions v_0. Each of these solutions generates its own necessary condition, but it should not be possible to satisfy infinitely many conditions with the finite number of parameters available in a scheme with a fixed number of nodes (see [AD01, Shi97b] for further details).

2.2 Layer-Adapted Meshes

2.2.1 Exponential Boundary Layers

This subsection examines a model problem with exponential layers at $x = 0$ and $y = 0$, namely

$$Lu := -\varepsilon \Delta u - b \cdot \nabla u + cu = f \quad \text{on } \Omega := (0,1) \times (0,1), \tag{2.9a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.9b}$$

where $b = (b_1(x,y), b_2(x,y)) > (\beta_1, \beta_2) > (0,0)$ on $\bar{\Omega}$ and b, c, f are smooth. If the differential equation were $-\varepsilon\Delta u + b \cdot \nabla u + cu = f$, whose solution has layers at $x = 1$ and $y = 1$, then the change of variables $x \mapsto 1-x$ and $y \mapsto 1-y$ converts this problem to (2.9).

First consider the simple upwind scheme

$$L^{up} u_h = f_h,$$

which, written out in full, is

$$\begin{aligned}
&- \frac{2\varepsilon}{h_i + h_{i+1}} \left(\frac{u_{i+1,j} - u_{ij}}{h_{i+1}} - \frac{u_{ij} - u_{i-1,j}}{h_i} \right) \\
&- \frac{2\varepsilon}{k_j + k_{j+1}} \left(\frac{u_{i,j+1} - u_{ij}}{k_{j+1}} - \frac{u_{ij} - u_{i,j-1}}{k_j} \right) \\
&- (b_1)_{ij} \frac{u_{i+1,j} - u_{i,j}}{h_i} - (b_2)_{i,j} \frac{u_{i,j+1} - u_{i,j}}{k_j} + c_{ij} = f_{ij}
\end{aligned} \tag{2.10a}$$

for $i = 1, \ldots, M-1$ and $j = 1, \ldots, N-1$, with

$$u_{ij} = 0 \quad \text{when } (x_i, y_j) \in \partial\Omega. \tag{2.10b}$$

We shall study this scheme on a tensor-product mesh $\omega_x \times \omega_y$, where ω_x and ω_y are one-dimensional Shishkin-type meshes (see Section I.2.4) having the same number of mesh points. Thus ω_x is obtained from the continuous mesh-generating function λ, where

$$\lambda(\xi) = \frac{\sigma\varepsilon}{\beta_1}\tilde{\lambda}(\xi) \quad \text{for } \xi \in [0, 1/2].$$

The function $\tilde{\lambda}$ is monotone with $\tilde{\lambda}(0) = 0$ and $\tilde{\lambda}(1/2) = \ln N$; on $[1/2, 1]$ the function λ is linear with $\lambda(1) = 1$. Recall that the mesh-characterizing function ψ is defined by $\psi = \exp(-\lambda)$. Figure 2.1 shows the typical structure of a tensor-product Shishkin-mesh for a problem with two exponential layers at $x = 1$ and $y = 1$; in this diagram (see Section I.2.4) $\lambda_x = C_x\varepsilon \ln N$ and $\lambda_y = C_y\varepsilon \ln N$, where N mesh intervals are used in each coordinate direction.

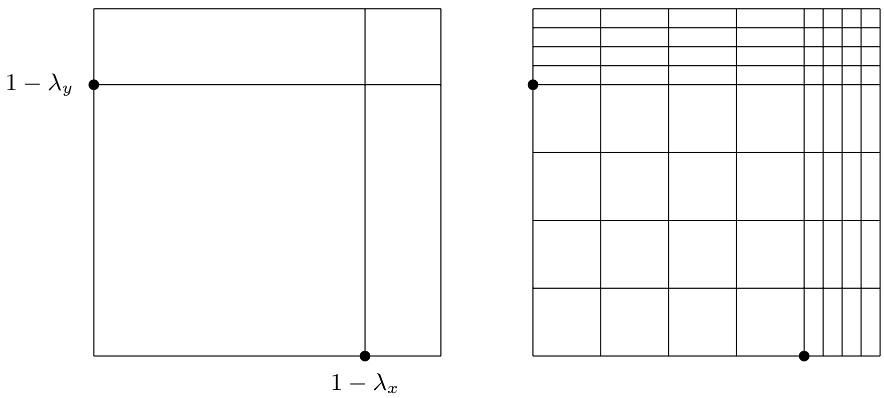

Fig. 2.1. Shishkin mesh for convection-diffusion with two outflow exponential layers

Theorem 2.7. *Assume that the solution of* (2.9) *can be decomposed similarly to Theorem 1.26 with* $\alpha = 1$ *and* $n = 3$. *Let the mesh-generating function be piecewise differentiable and satisfy*

$$\max \tilde{\lambda}'(\xi) \leq CN, \qquad \int_0^{1/2} \tilde{\lambda}'(\xi)^2 d\xi \leq CN.$$

Then for $\sigma \geq 2$, *the error of the simple upwind scheme satisfies*

$$|u(x_i, y_j) - u_{ij}| \leq \begin{cases} C\,N^{-1} & \text{for } i, j = N/2, \ldots, N, \\ C\,N^{-1} \max|\psi'| & \text{otherwise.} \end{cases}$$

For a piecewise-equidistant Shishkin mesh the mesh-characterizing function ψ introduces a factor $\ln N$ into the error estimate. On the other hand, the Bakhvalov-Shishkin mesh, for which $\psi(\xi) = 1-(1-1/N)2\xi$, yields the optimal error estimate CN^{-1} because $|\psi'|$ is uniformly bounded.

Proof. Recalling the decomposition of Theorem 1.26, split the numerical solution in a similar manner: define the mesh function S^N as the solution of

$$\left[L^{up} S^N\right]_{ij} = [LS]_{ij} \text{ for all } i \text{ and } j, \quad S^N = S \quad \text{on } \partial\Omega,$$

and define E_1^N, E_2^N and E_{12}^N analogously. For the smooth component S, standard classical arguments give

$$|S(x_i, y_j) - S_{ij}^N| \le C\, N^{-1} \text{ for all } i \text{ and } j.$$

For the layer term at $x = 0$ we have, like the proof of Lemma I.2.94,

$$0 \le E_{1,ij}^N \le W_{1,i} := C \prod_{\nu=1}^{i} \left(1 + \frac{\beta_1 h_\nu}{2\varepsilon}\right)^{-1} \quad \text{for } i, j = 0, \dots, N.$$

The smallness of E_1 on the coarse mesh leads to

$$\left|E_1(x_i, y_j) - E_{1,ij}^N\right| \le C\, N^{-1} \quad \text{for } i = N/2, \dots, N,\ j = 0, \dots, N.$$

A Taylor expansion gives

$$\left|L^{up}(E_1 - E_1^N)\right| \le C(N^{-1} + \varepsilon^{-1} W_{1,i} \max|\psi'|).$$

Appealing to a discrete comparison principle and using the barrier function $C(N^{-1} + W_{1,i} N^{-1} \max|\psi'|)$ yields

$$\left|E_1(x_i, y_j) - E_{1,ij}^N\right| \le C\, N^{-1} \max|\psi'| \quad \text{for } i = 0, \dots, N/2 - 1,\ j = 0, \dots, N.$$

Similar arguments are used for the terms E_2 and E_{12} corresponding to the boundary layer at $y = 0$ and the corner layer. $\Box$

When the solution is less regular, one can nevertheless prove some positive rate of convergence; see [Shi00].

In [LS99] a modified hybrid scheme on a tensor-product Shishkin mesh is considered. It is based on simple upwinding, but employs central differencing whenever the mesh allows one to do this without losing inverse-monotonicity. For this scheme the above proof avoids the factor $\ln N$ and gives the optimal error bound

$$\|u - u_h\|_{\infty,d} \le C N^{-1}.$$

Liseikin [Lis83] uses a tensor product of one-dimensional Bakhvalov-type meshes (see Section I.2.4). He assumes the validity of the estimates

$$\left|\frac{\partial^k u}{\partial x^k}(x, y)\right| \le C\left[1 + \varepsilon^{-k} e^{-\beta_1(1-x)/\varepsilon}\right]$$

and

$$\left|\frac{\partial^k u}{\partial y^k}(x, y)\right| \le C\left[1 + \varepsilon^{-k} e^{-\beta_2(1-y)/\varepsilon}\right]$$

on Ω for $0 \le k \le 3$. Such an assumption implies, as was seen in Chapter 1, that the data of the problem are smooth and satisfy strong compatibility conditions at the corners of $\bar{\Omega}$. The logarithmically graded mesh then controls the local truncation error of the simple upwind scheme. The computed solution satisfies

$$\|u - u_h\|_{\infty,d} \le CN^{-1}.$$

The proof of Theorem 2.7 used the discrete comparison principle and carefully chosen barrier functions. Alternatively, as we saw in Section I.2.4.2, one can use an improved stability result for the upwind scheme. Thus [And01] one has the following discrete stability analogue of the continuous stability bound of Theorem 1.22. Set $\bar{h}_i = (h_i + h_{i+1})/2$ and $\bar{k}_i = (k_i + k_{i+1})/2$ for each i. Define the discrete Green's function G_d by

$$L^{up} G_d(x_i, y_j; \xi_m, \eta_n) = \delta^d(x_i, \xi_m)\, \delta^d(y_j, \eta_n), \qquad G_d = 0 \text{ on } \partial\Omega,$$

with

$$\delta^d(x_i, \xi_m) = \begin{cases} (\bar{h}_i)^{-1} & \text{for } i = m, \\ 0 & \text{otherwise;} \end{cases}$$

the mesh function $\delta^d(y_j, \eta_n)$ is defined analogously.

Lemma 2.8. *The discrete Green's function G_d is nonnegative. One has the estimates*

$$\max_{x_i, y_j, \eta_n} \|G_d(x_i, y_j; \cdot, \eta_n)\|_{L_1,d} \le \frac{1}{\beta_2}$$

and

$$\max_{x_i, y_j, \xi_m} \|G_d(x_i, y_j; \xi_m, \cdot)\|_{L_1,d} \le \frac{1}{\beta_1},$$

where $\|\cdot\|_{L_1,d}$ is the one-dimensional discrete L_1 norm.

Proof. It suffices to prove the statement for $c = 0$ because its Green's function $\tilde{G}_d$ satisfies $0 \le G_d \le \tilde{G}_d$ owing to the inverse-monotonicity of the discrete problem. Thus assume that $c = 0$ in (2.9). Fix (x_i, y_j). Define the function of one variable $G^{\Sigma}(x_i, y_j; \cdot)$ by

$$G^{\Sigma}(x_i, y_j; \xi_m) := \sum_n G_d(x_i, y_j; \xi_m, \eta_n) \bar{k}_n.$$

As $G_d \ge 0$, this sum is simply the discrete L_1 norm of $G_d(x_i, y_j; \xi_m, \cdot)$. Analogously to the continuous Green's function, the function G_d satisfies the adjoint problem associated with (ξ_m, η_n) – for the simple upwind scheme, the term $-b_i D^+ u_i$ has adjoint $D^-(b_i u_i)$. Multiplying the adjoint equation by $\bar{k}_n$ then summing over n, we obtain a difference equation for G^{Σ} in its third argument:

$$-\varepsilon \delta^2 (G^{\Sigma})_m + D^-(b_1^* G^{\Sigma})_m = \delta^d(x_i, \xi_m) - F(x_i, y_j, \xi_m). \tag{2.11}$$

Here $b_1^* \ge \beta_1$ is derived from b_1 via a mean value theorem while

$$F(x_i, y_j, \xi_m) := b_{2,m,N-1} G_d(x_i, y_j; \xi_m, \eta_{N-1}) + \frac{\varepsilon}{\bar{k}_{N-1}} G_d(x_i, y_j; \xi_m, \eta_{N-1}) + \frac{\varepsilon}{\bar{k}_1} G_d(x_i, y_j; \xi_m, \eta_1).$$

Denoting by G_d^* the Green's function associated with the one-dimensional operator on the left-hand side of (2.11), one then has

$$0 \le G_d^* \le \frac{1}{\beta_1}\,.$$

Now the solution representation

$$G^{\Sigma}(x_i, y_j, \xi_m) = G_d^*(x_i, y_j, \xi_m, \xi_m) - \sum_n G_d^*(x_i, y_j, \xi_m, z_n) F(x_i, y_j, z_n) \bar{h}_n$$

gives us immediately the second estimate of the lemma because $F \ge 0$. The other inequality is proved similarly. □

Consider now the discrete boundary value problem

$$L^{up} u_h = f_h \text{ in } \Omega, \qquad u_h = 0 \text{ on } \partial\Omega.$$

The solution representation

$$u_h(x_i, y_j) = \sum_{m,n} \bar{h}_m \bar{k}_n G_d(x_i, y_j, \xi_m, \eta_n) f_h(\xi_m, \eta_n)$$

yields

Theorem 2.9. *Assume that $b_1 > \beta_1 > 0$. Then the simple upwind operator enjoys the anisotropic stability estimate*

$$\|v_h\|_{\infty,d} \le C \|L^{up} v_h\|_{1\otimes\underline{\infty},d}.$$

The notation here is analogous to Theorem 1.22: first apply the maximum norm in the y-direction and then the discrete L_1 norm with respect to x.

When $b_1 > 0$ and $b_2 > 0$ on $\bar{\Omega}$, Theorem 2.9 gives an alternative proof of

$$\|u - u_h\|_{\infty,d} \le C \begin{cases} N^{-1} & \text{for a Bakhvalov mesh,} \\ N^{-1} \ln N & \text{for a Shishkin mesh.} \end{cases}$$

Remark 2.10. (Reaction-diffusion problem) For the reaction-diffusion problem discussed in Remark 1.27 with $\Omega = (0,1)^2$ one expects uniform convergence on a Shishkin mesh for the standard finite difference method obtained by setting $b_1 \equiv b_2 \equiv 0$ in (2.10). With sufficient compatibility to ensure that $u \in C^{4,\alpha}(\bar{\Omega})$ and that one has a suitable decomposition of u, it is straightforward to prove

$$\|u - u_h\|_{\infty,d} \le C(N^{-1} \ln N)^2. \tag{2.12}$$

This was shown in [CGO05] using a barrier function technique; alternatively, one could use an improved stability estimate based on the Green's function as in the one-dimensional problem considered by Savin [Sav95].

Remarkably, Andreev [And06] was able to avoid the use of compatibility conditions (see the discussion on solution decomposition in Remark 1.27) when proving

$$\|u - u_h\| \leq C\, N^{-2} (\ln N)^4$$

for this scheme. Subsequently Andreev [And] and Andreev and Kopteva [AK08] extended these results to problems with stronger corner singularities. In the first of these papers, Dirichlet and Neumann boundary conditions meet at a corner of the unit square, while in the second an L-shaped domain with Dirichlet boundary conditions is treated. In both cases the solution lies only in $C^{0,\alpha}(\bar{\Omega})$ with $0 < \alpha < 1$. The analysis combines layer-adapted meshes with geometrically graded meshes near the corner singularity; for related work, see [Mel02].

Kopteva [Kop07a, Kop07b] studies semilinear problems of the type

$$\begin{aligned} Lu := -\varepsilon \triangle u + b(x,u) = 0 \quad & \text{in } \Omega, \\ u = g \quad & \text{on } \Gamma, \end{aligned}$$

in a domain with a smooth boundary while assuming some standard stability property of the reduced solution. The discretization combines features of finite differences and finite elements. In a strip "parallel" to the boundary, whose thickness corresponds to the construction of a Bakhvalov or a Shishkin mesh, a finite difference method is used, and in the interior of Ω the difference scheme is generated via linear finite elements on a Delaunay triangulation. Optimal error bounds in the maximum norm are derived.

Systems of reaction-diffusion problems in two-dimensional domains are solved in [KLS, KMS08, Shi07b] using standard schemes on Bakhvalov and Shishkin meshes and error bounds like (2.12) are proved. ♣

Are there second-order schemes for convection-diffusion problems? In [Kop03] Kopteva derives an error expansion for the simple upwind scheme on a piecewise equidistant mesh. This expansion is used to show that Richardson extrapolation generates a robust almost (i.e., up to a logarithmic factor) second-order method.

Comparing numerical results for simple upwinding, a hybrid scheme, central differencing and defect correction on a Shishkin mesh, it is concluded in [LS01b] that defect correction is the most efficient of these because it combines the accuracy of central differencing with the good stability properties of upwinding. But up to now, no complete analysis of defect correction for two-dimensional convection-diffusion problems has been given.

We do not know of any theoretical result for central differencing on layer-adapted meshes for (2.9), but there are some results for related schemes generated by finite element methods; see Chapter 3.

2.2.2 Parabolic Layers

Compared with exponential layers, satisfactory convergence results for difference schemes for parabolic boundary layers are thin on the ground. Let us consider the model problem

$$Lu := -\varepsilon \Delta u + u_x + cu = f \quad \text{in } \Omega := (0,1)\times(0,1), \tag{2.13a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.13b}$$

The problem has an exponential layer of width $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ at the outflow boundary $x = 1$ and parabolic layers of width $\mathcal{O}(\sqrt{\varepsilon}\,|\ln\varepsilon|)$ at the characteristic boundaries $y = 0$ and $y = 1$.

We know already that on standard meshes it is impossible to construct difference schemes that are uniformly convergent pointwise for problems with parabolic boundary layers. Thus for (2.13) one could combine fitted schemes in the x-direction with a layer-adapted scheme in the y-direction, or use layer-adapted meshes in both directions to try to achieve uniform convergence. In a finite difference framework, we know of only one paper that avails itself of the former strategy: Shishkin [Shi86] uses the one-dimensional Il'in-Allen-Southwell scheme to approximate $-\varepsilon u_{xx} - u_x$ and central differencing to approximate $-\varepsilon u_{yy}$. He proves the following result, where the mesh uses N points in each coordinate direction – equidistant in x but layer-adapted in y.

Theorem 2.11. *(Il'in-Allen-Southwell scheme and a Bahkvalov mesh) Assume that $c, f \in C^3(\bar{\Omega})$, that $u \in C^4(\bar{\Omega})$, and $f(1,0) = f(1,1) = 0$. Then*

$$\|u - u_h\|_{\infty,d} \le CN^{-1/4}.$$

If more smoothness and compatibility of the data are assumed, then the conclusion of the Theorem becomes $\|u - u_h\|_{\infty,d} \le CN^{-1/2}$.

As tensor products of layer-adapted meshes were quite successful for exponential layers, we now introduce a mesh of this type for (2.13), using the mesh transition parameters (see Section I.2.4)

$$\lambda_x = \min\left\{1/2,\ \sigma_x \varepsilon \ln N\right\}, \qquad \lambda_y = \min\left\{1/4,\ \sigma_y \sqrt{\varepsilon}\,\ln N\right\}$$

where N mesh intervals are used in each coordinate direction and the mesh is fine at $x = 0$ and at $y = 0$, $y = 1$. Note that the ε of the exponential layer transition point becomes $\sqrt{\varepsilon}$ for the parabolic layer; this is due to the different asymptotic structure of these layers. Figure 2.2 shows the typical structure of such a mesh for (2.13); in each coordinate direction, half the mesh intervals are in the coarse mesh and half in the fine mesh. For simplicity, only the standard Shishkin mesh is discussed here but Shishkin-type and Bakhvalov-type meshes are also possible.

Numerical results in several papers [CMOS01, FHS96a, FHS96b, FHS96c, HMOS95] and the monograph [FHM+00] demonstrate numerically the almost

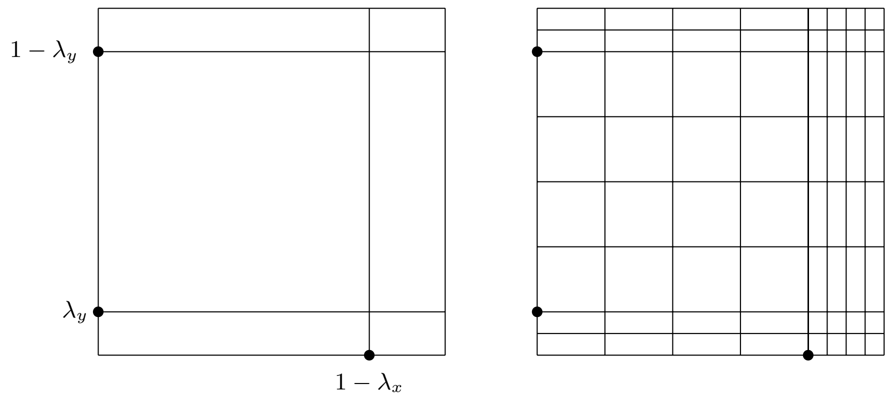

Fig. 2.2. Shishkin mesh for one exponential and two parabolic layers

first-order uniform convergence of the simple upwind scheme, but a rigorous proof of this convergence under minimal regularity assumptions is not easy. Using the decomposition (1.29) and the estimates (1.30), Shishkin [Shi90b] proved that

$$\|u - u_h\|_{\infty,d} \le C(N^{-1}\ln N)^p$$

with $p = 1/18$ or $p = 1/14$ depending on the precise assumptions on the problem data. If we assume the validity of the decomposition

$$u = S + E_1 + E_2 + E_{12}$$

and of bounds like (1.32) for the third-order derivatives needed in the analysis of the simple upwind scheme, then it should be possible to prove that

$$\|u - u_h\|_{\infty,d} \le CN^{-1}\ln N.$$

In fact [OS07a] derives the error estimate $\|u - u_h\|_{\infty,d} \le CN^{-1}(\ln N)^2$.

Remark 2.12. (The A-mesh) In [Wes96] Wesseling implicitly assumes the existence of a decomposition of the solution (by ignoring higher-order terms in an asymptotic expansion) for a problem with parabolic boundary layers and a weak exponential layer. He proves first-order uniform convergence for an upwind scheme with a refined piecewise equidistant mesh near the characteristic boundaries but his choice of transition point is

$$\lambda_y = \min\{1/4, \sigma_y\sqrt{\varepsilon}\,|\ln\varepsilon|\},$$

i.e., the factor $\ln N$ of the Shishkin mesh is replaced by $|\ln\varepsilon|$. This mesh is sometimes called the A-mesh. Numerical experiments in [HMOS97] demonstrate however that this choice of transition point for a piecewise equidistant mesh is not as good as Shishkin's if one wants also to approximate scaled derivatives of the solution. ♣

Remark 2.13. The authors of [CGLS02] study a problem with Robin boundary conditions on the characteristic boundary, so the parabolic boundary layer is weak. Assuming the existence of a decomposition of the type (1.31) with estimates similar to (1.32) for the fourth-order derivatives, it is shown that

$$\|u - u_h\|_{\infty,d} \le C\left[(N^{-1}\ln N)^2 + \sqrt{\varepsilon}N^{-1}\ln N\right]$$

for a scheme which in the x-direction is related to the midpoint upwind scheme. ♣

Remark 2.14. (Interior parabolic layers) In [HS94], Hemker and Shishkin study a singularly perturbed parabolic equation with a discontinuous initial condition that generates an *interior parabolic layer* and construct a uniformly convergent (fitted) scheme on an equidistant mesh. Unlike the situation with parabolic boundary layers, the equation determining the layer correction now has only one solution (the classical error function).

One would expect a similar result for an elliptic problem of type (2.13) with constant coefficients, if a discontinuous boundary condition generates an interior parabolic layer at the subcharacteristic through the point of discontinuity. For nonconstant coefficients (curved subcharacteristics) the situation is more complicated and is unclear. ♣

Remark 2.15. (Hemker's problem) In [Hem97] Hemker proposes the following benchmark problem: solve

$$-\varepsilon\triangle u + u_x = 0$$

in the plane region exterior to the unit circle with the boundary conditions

$$u(x,y) = 1 \text{ for } x^2+y^2 = 1, \qquad u(x,y) \to 0 \text{ as } x^2+y^2 \to \infty.$$

This is a complicated problem: the solution has an exponential layer and two interior parabolic layers – in particular the asymptotic situation is quite complicated at the points $(0,\pm1)$ where the parabolic layers are "born" from the exponential layer. (The unboundedness of the domain is unimportant.)

Numerical results for this problem can be found in [HHH00] (here the so-called over-set grid technique is used) and [NH00], where an adaptive sparse-grid technique is developed. ♣

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