

Introduction

Number theory, one of the most beautiful and fascinating areas of mathematics, has made major progress over the last decades, and is still developing rapidly. In the beginning of the foreword to his book *Algebraic Number Theory*, J. Neukirch wrote

„Die Zahlentheorie nimmt unter den mathematischen Disziplinen eine ähnlich idealisierte Stellung ein wie die Mathematik selbst unter den anderen Wissenschaften.“ *)

Although the joint authors of the present book wish to reiterate this statement, we wish to stress also that number theory owes much of its current strong development to its interaction with almost all other mathematical fields. In particular, the geometric (and consequent functorial) point of view of arithmetic geometry uses techniques from, and is inspired by, analysis, geometry, group theory and algebraic topology. This interaction had already started in the 1950s with the introduction of group cohomology to local and global class field theory, which led to a substantial simplification and unification of this area.

The aim of the present volume is to provide a textbook for students, as well as a reference book for the working mathematician on cohomological topics in number theory. Its main subject is Galois modules over local and global fields, objects which are typically associated to arithmetic schemes. In view of the enormous quantity of material, we were forced to restrict the subject matter in some way. In order to keep the book at a reasonable length, we have therefore decided to restrict attention to the case of dimension less than or equal to one, i.e. to the global fields themselves, and the various subrings contained in them. Central and frequently used theorems such as the global duality theorem of *G. POITOU* and *J. TATE*, as well as results such as the theorem of *I. R. ŠAFAREVIČ* on the realization of solvable groups as Galois groups over global fields, had been part of algebraic number theory for a long time. But the proofs of statements like these were spread over many original articles, some of which contained serious mistakes, and some even remained unpublished. It was the initial motivation of the authors to fill these gaps and we hope that the result of our efforts will be useful for the reader.

In the course of the years since the 1950s, the point of view of class field theory has slightly changed. The classical approach describes the Galois groups

*) “Number theory, among the mathematical disciplines, occupies a similar idealized position to that held by mathematics itself among the sciences.”

of *finite* extensions using arithmetic invariants of the local or global ground field. An essential feature of the modern point of view is to consider infinite Galois groups instead, i.e. one investigates the set of all finite extensions of the field k at once, via the *absolute* Galois group G_k . These groups intrinsically come equipped with a topology, the Krull topology, under which they are Hausdorff, compact and totally disconnected topological groups. It proves to be useful to ignore, for the moment, their number theoretical motivation and to investigate topological groups of this type, the profinite groups, as objects of interest in their own right. For this reason, an extensive “algebra of profinite groups” has been developed by number theorists, not as an end in itself, but always with concrete number theoretical applications in mind. Nevertheless, many results can be formulated solely in terms of profinite groups and their modules, without reference to the number theoretical background.

The first part of this book deals with this “profinite algebra”, while the arithmetic applications are contained in the second part. This division should not be seen as strict; sometimes, however, it is useful to get an idea of how much algebra and how much number theory is contained in a given result.

A significant feature of the arithmetic applications is that classical reciprocity laws are reflected in duality properties of the associated infinite Galois groups. For example, the reciprocity law for local fields corresponds to Tate’s duality theorem for local cohomology. This duality property is in fact so strong that it becomes possible to describe, for an arbitrary prime p , the Galois groups of the maximal p -extensions of local fields. These are either free groups or groups with a very special structure, which are now known as Demuškin groups. This result then became the basis for the description of the full absolute Galois group of a p -adic local field by U. JANNSEN and the third author.

The global case is rather different. As was already noticed by J. TATE, the absolute Galois group of a global field is *not* a duality group. It is the geometric point of view, which offers an explanation of this phenomenon: the duality comes from the *curve* rather than from its generic point. It is therefore natural to consider the étale fundamental groups $\pi_1^{et}(\text{Spec}(\mathcal{O}_{k,S}))$, where S is a finite set of places of k . Translated to the language of Galois groups, the fundamental group of $\text{Spec}(\mathcal{O}_{k,S})$ is a quotient of the full group G_k , namely, the Galois group $G_{k,S}$ of the maximal extension of k which is unramified outside S . If S contains all places that divide the order of the torsion of a module M , the central *Poitou-Tate duality theorem* provides a duality between the localization kernels in dimensions one and two. In conjunction with Tate local duality, this can also be expressed in the form of a long 9-term sequence. The duality theorem of Poitou-Tate remains true for infinite sets of places S and, using topologically restricted products of local cohomology groups, the long exact sequence can be generalized to this case. The question of whether

the group $G_{k,S}$ is a duality group when S is finite was positively answered by the second author.

As might already be clear from the above considerations, the basic technique used in this book is Galois cohomology, which is essential for class field theory. For a more geometric point of view, it would have been desirable to have also formulated the results throughout in the language of étale cohomology. However, we decided to leave this to the reader. Firstly, the technique of sheaf cohomology associated to a Grothendieck topos is sufficiently covered in the literature (see [5], [139], [228]) and, in any case, it is an easy exercise (at least in dimension ≤ 1) to translate between the Galois and the étale languages. A further reason is that results which involve infinite sets of places (necessary when using Dirichlet density arguments) or infinite extension fields, can be much better expressed in terms of Galois cohomology than of étale cohomology of pro-schemes. When the geometric point of view seemed to bring a better insight or intuition, however, we have added corresponding remarks or footnotes. A more serious gap, due to the absence of Grothendieck topologies, is that we cannot use *flat* cohomology and the global flat duality theorem of Artin-Mazur. In chapter VIII, we therefore use an ad hoc construction, the group \mathbb{B}_S , which measures the size of the localization kernel for the first flat cohomology group with the roots of unity as coefficients.

Let us now examine the contents of the individual chapters more closely. The first part covers the algebraic background for the number theoretical applications. Chapter I contains well-known basic definitions and results, which may be found in several monographs. This is only partly true for chapter II: the explicit description of the edge morphisms of the Hochschild-Serre spectral sequence in §2 is certainly well-known to specialists, but is not to be found in the literature. In addition, the material of §3 is well-known, but contained only in original articles.

Chapter III considers abstract duality properties of profinite groups. Among the existing monographs which also cover large parts of the material, we should mention the famous *Cohomologie Galoisienne* by J.-P. SERRE and H. KOCH's book *Galoissche Theorie der p-Erweiterungen*. Many details, however, have been available until now only in the original articles.

In chapter IV, free products of profinite groups are considered. These are important for a possible non-abelian decomposition of global Galois groups into local ones. This happens only in rather rare, degenerate situations for Galois groups of global fields, but it is quite a frequent phenomenon for subgroups of infinite index. In order to formulate such statements (like the arithmetic form of Riemann's existence theorem in chapter X), we develop the concept of the free product of a *bundle* of profinite groups in §3.

Chapter V deals with the algebraic foundations of Iwasawa theory. We will not prove the structure theorem for Iwasawa modules in the usual way by using matrix calculations (even though it may be more acceptable to some mathematicians, as it is more concrete), but we will follow mostly the presentation found in Bourbaki, *Commutative Algebra*, with a view to more general situations. Moreover, we present results concerning the structure of these modules up to isomorphism, which are obtained using the homotopy theory of modules over group rings, as presented by U. JANNSEN.

The central technical result of the arithmetic part is the famous global duality theorem of Poitou-Tate. We start, in chapter VI, with general facts about Galois cohomology. Chapter VII deals with local fields. Its first three sections largely follow the presentation of J.-P. SERRE in *Cohomologie Galoisienne*. The next two sections are devoted to the explicit determination of the structure of local Galois groups. In chapter VIII, the central chapter of this book, we give a complete proof of the Poitou-Tate theorem, including its generalization to finitely generated modules. We begin by collecting basic results on the topological structure, universal norms and the cohomology of the S -idèle class group, before moving on to the proof itself, given in sections 4 and 6. In the proof, we apply the group theoretical theorems of Nakayama-Tate and of Poitou, proven already in chapter III.

In chapter IX, we reap the rewards of our efforts in the previous chapters. We prove several classical number theoretical results, such as the Hasse principle and the Grunwald-Wang theorem. In §4, we consider embedding problems and we present the theorem of K. IWASAWA to the effect that the maximal prosolvable factor of the absolute Galois group of \mathbb{Q}^{ab} is free. In §5, we give a complete proof of Šafarevič's theorem on the realization of finite solvable groups as Galois groups over global fields.

The main concern of chapter X is to consider restricted ramification. Geometrically speaking, we are considering the *curves* $\mathrm{Spec}(\mathcal{O}_{k,S})$, in contrast to chapter IX, where our main interest was in the *point* $\mathrm{Spec}(k)$. Needless to say, things now become much harder. Invariants like the S -ideal class group or the p -adic regulator enter the game and establish new arithmetic obstructions. Our investigations are guided by the analogy between number fields and function fields. We know a lot about the latter from algebraic geometry, and we try to establish analogous results for number fields. For example, using the group theoretical techniques of chapter IV, we can prove the number theoretical analogue of Riemann's existence theorem. The fundamental group of $\mathrm{Spec}(\mathcal{O}_k)$, i.e. the Galois group of the maximal unramified extension of the number field k , was the subject of the long-standing *class field tower* problem in number theory, which was finally answered negatively by E. S. GOLOD and

I. R. ŠAFAREVIČ. We present their proof, which demonstrates the power of the group theoretical and cohomological methods, in §8.

Chapter XI deals with Iwasawa theory, which is the consequent conceptual continuation of the analogy between number fields and function fields. We concentrate on the algebraic aspects of Iwasawa theory of p -adic local fields and of number fields, first presenting the classical statements which one can usually find in the standard literature. Then we prove more far-reaching results on the structure of certain Iwasawa modules attached to p -adic local fields and to number fields, using the homotopy theory of Iwasawa modules. The analytic aspects of Iwasawa theory will merely be described, since this topic is covered by several monographs, for example, the book [246] of *L. WASHINGTON*. Finally, the Main Conjecture of Iwasawa theory will be formulated and discussed; for a proof, we refer the reader to the original work of *B. MAZUR* and *A. WILES* ([134], [249]).

In the last chapter, we give a survey of so-called *anabelian geometry*, a program initiated by *A. GROTHENDIECK*. Perhaps the first result of this theory, obtained even before this program existed, is a theorem of *J. NEUKIRCH* and *K. UCHIDA* which asserts that the absolute Galois group of a global field, as a profinite group, characterizes the field up to isomorphism. We give a proof of this theorem for number fields in the first two sections. The final section gives an overview of the conjectures and their current status.

The reader will recognize very quickly that this book is not a basic textbook in the sense that it is completely self-contained. We use freely basic algebraic, topological and arithmetic facts which are commonly known and contained in the standard textbooks. In particular, the reader should be familiar with basic number theory. While assuming a certain minimal level of knowledge, we have tried to be as complete and as self-contained as possible at the next stage. We give full proofs of almost all of the main results, and we have tried not to use references which are only available in original papers. This makes it possible for the interested student to use this book as a textbook and to find large parts of the theory coherently ordered and gently accessible in one place. On the other hand, this book is intended for the working mathematician as a reference on cohomology of local and global fields.

Finally, a remark on the exercises at the end of the sections. A few of them are not so much exercises as additional remarks which did not fit well into the main text. Most of them, however, are intended to be solved by the interested reader. However, there might be occasional mistakes in the way they are posed. If such a case arises, it is an additional task for the reader to give the correct formulation.

We would like to thank many friends and colleagues for their mathematical examination of parts of this book, and particularly, *ANTON DEITMAR*, *TORSTEN FIMMEL*, *DAN HARAN*, *UWE JANNSEN*, *HIROAKI NAKAMURA* and *OTMAR VENJAKOB*. We are indebted to Mrs. *INGE MEIER* who \TeX ed a large part of the manuscript, and *EVA-MARIA STROBEL* receives our special gratitude for her careful proofreading. Hearty thanks go to *FRAZER JARVIS* for going through the entire manuscript, correcting our English.

Heidelberg, September 1999

Alexander Schmidt
Kay Wingberg

Preface to the Second Edition

The present second edition is a corrected and extended version of the first. We have tried to improve the exposition and reorganize the content to some extent; furthermore, we have included some new material. As an unfortunate result, the numbering of the first edition is not compatible with the second.

In the algebraic part you will find new sections on filtered cochain complexes, on the degeneration of spectral sequences and on Tate cohomology of profinite groups. Amongst other topics, the arithmetic part contains a new section on duality theorems for unramified and tamely ramified extensions, a careful analysis of 2-extensions of real number fields and a complete proof of Neukirch's theorem on solvable Galois groups with given local conditions.

Since the publication of the first edition, many people have sent us lists of corrections and suggestions or have contributed in other ways to this edition. We would like to thank them all. In particular, we would like to thank *JAKOB STIX* and *DENIS VOGEL* for their comments on the new parts of this second edition and *FRAZER JARVIS*, who again did a great job correcting our English.

Regensburg and Heidelberg, November 2007

Alexander Schmidt
Kay Wingberg



<http://www.springer.com/978-3-540-37888-4>

Cohomology of Number Fields

Neukirch, J.; Schmidt, A.; Wingberg, K.

2008, XV, 826 p., Hardcover

ISBN: 978-3-540-37888-4