

Chapter II

Some Homological Algebra

§1. Spectral Sequences

If G is a profinite group and H a closed normal subgroup, then one may ask whether the cohomology groups $H^n(G, A)$ can be computed from the cohomology groups of the smaller groups H and G/H . We have already seen a relation of this type, namely the isomorphism

$$H^n(G/H, H^0(H, A)) \cong H^n(G, A)$$

if $H^n(H, A) = 0$ for all $n \geq 1$, which follows from (1.6.7). There is a quite general relation, which is denoted by

$$H^p(G/H, H^q(H, A)) \Rightarrow H^n(G, A)$$

and is called a “spectral sequence”. The situation is slightly involved. It roughly says that there is a canonical decreasing filtration of $H^n = H^n(G, A)$,

$$H^n = F^0 H^n \supseteq F^1 H^n \supseteq \dots \supseteq F^n H^n \supseteq F^{n+1} H^n = 0,$$

such that the quotient $F^p H^n / F^{p+1} H^n$ is isomorphic not directly to the group $H^p(G/H, H^{n-p}(H, A))$, but to a certain subquotient of it. The notion of **spectral sequence** is very general and of utmost importance in cohomology theory. The general set-up, which can be generalized in several directions if the underlying category has inductive limits, is the following.

Let \mathcal{A} be an abelian category. A (decreasing) **filtration** of an object A is a family $(F^p A)_{p \in \mathbb{Z}}$ of subobjects $F^p A$ of A such that $F^p A \supseteq F^{p+1} A$ for all p . Write

$$gr_p A = F^p A / F^{p+1} A.$$

By convention, we put $F^\infty A = 0$ and $F^{-\infty} A = A$. We say that the filtration is **finite** if there exist $n, m \in \mathbb{Z}$ with $F^m A = 0$ and $F^n A = A$. Given filtered objects A and B in \mathcal{A} , a morphism $f: A \rightarrow B$ is said to be compatible with the filtration if $f(F^p A) \subseteq F^p B$ for all $p \in \mathbb{Z}$.

Let m be a natural number. An **E_m -spectral sequence**^{*)} in \mathcal{A} is a system

$$E = (E_r^{pq}, E^n),$$

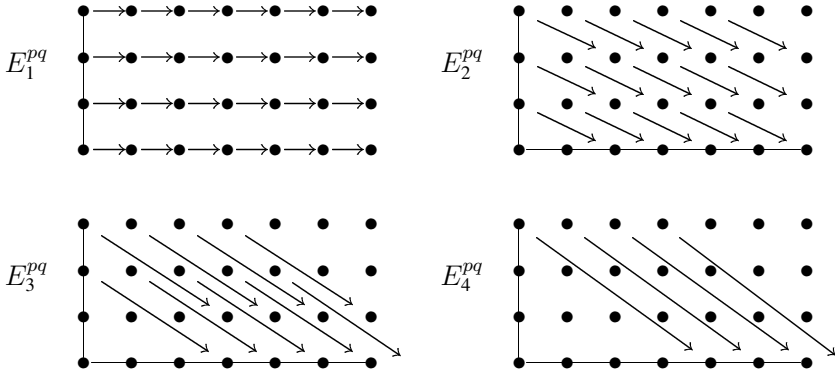
^{*)}In the applications usually only E_1 - and E_2 -spectral sequences occur.

consisting of

- a) objects $E_r^{pq} \in \mathcal{A}$ for all $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ and any integer $r \geq m$,
- b) morphisms $d = d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r, q-r+1}$ with $d \circ d = 0$ and such that for each fixed pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ the morphisms d_r^{pq} and $d_r^{p-r, q+r-1}$ vanish for sufficiently large r ,
- c) isomorphisms $\alpha_r^{pq} : \ker(d_r^{pq}) / \text{im}(d_r^{p-r, q+r-1}) \xrightarrow{\sim} E_{r+1}^{pq}$,
- d) finitely filtered objects $E^n \in \mathcal{A}$ for all $n \in \mathbb{Z}$,
- e) isomorphisms $\beta^{pq} : E_\infty^{pq} \xrightarrow{\sim} gr_p E^{p+q}$.

By b) and c), the objects E_r^{pq} are independent of r for r sufficiently large and are then denoted by E_∞^{pq} . These are the objects occurring in e).

In other words, for each $r \geq m$, the system E_r^{pq} is a system of complexes whose cohomology groups are the objects E_{r+1}^{pq} of the next system. A spectral sequence is like a book with (infinitely many) pages $E_m^{pq}, E_{m+1}^{pq}, E_{m+2}^{pq}, \dots$ and a limit page E^n at the end.



For an E_m -spectral sequence $E = (E_r^{pq}, E^n)$, one usually writes

$$E_m^{pq} \Rightarrow E^{p+q}$$

or $E_m^{pq} \Rightarrow E^n$. The E_m^{pq} are called the **initial terms**, the E^n the **limit terms** and the d_r^{pq} **differentials**. By forgetting the first $m' - m$ pages, an E_m -spectral sequence induces an $E_{m'}$ -spectral sequence for all $m' \geq m$ in a natural way.

A morphism of E_m -spectral sequences

$$\varphi : E = (E_r^{pq}, E^n) \longrightarrow E' = (E_r'^{pq}, E'^n)$$

in \mathcal{A} is a system of morphisms

$$\varphi_r^{pq} : E_r^{pq} \longrightarrow E_r'^{pq}, \quad \varphi^n : E^n \longrightarrow E'^n,$$

where the φ^n are compatible with the filtrations of E^n and E'^n and the $\varphi_r^{pq}, \varphi^n$ commute with d_r^{pq}, α_r^{pq} and β^{pq} .

If $E_r^{pq} = 0$ for $p < 0$ or $q < 0$, one speaks of a **first quadrant** spectral sequence. In this case we have

$$E_r^{pq} = E_\infty^{pq} \quad \text{for } r > \max(p, q + 1), r \geq m.$$

Once a first quadrant spectral sequence is given, we obtain a realm of homomorphic connections. We restrict to the most important case of an E_2 -spectral sequence. Of basic importance are the two homomorphisms

$$E_2^{n,0} \longrightarrow E^n \longrightarrow E_2^{0,n},$$

the so-called **edge morphisms**. The first one is the composite of the morphisms

$$E_2^{n,0} \longrightarrow E_3^{n,0} \longrightarrow \dots \longrightarrow E_\infty^{n,0} \longrightarrow E^n,$$

which are well-defined because $F^{n+1}E^n = 0$ and $E_r^{n+r, -r+1} = 0$ for $r \geq 2$ (so that $E_{r+1}^{n,0}$ is a quotient of $E_r^{n,0}$). The second one is the composite of the morphisms

$$E^n \longrightarrow E_\infty^{0,n} \longrightarrow \dots \longrightarrow E_3^{0,n} \longrightarrow E_2^{0,n},$$

which are well-defined because $F^0E^n = E^n$ and $E_r^{-r, n+r-1} = 0$ for $r \geq 2$ (so that $E_{r+1}^{0,n}$ is embedded in $E_r^{0,n}$). A direct consequence of the definition of the edge morphisms is the following

(2.1.1) Proposition. *For any first quadrant E_2 -spectral sequence the sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow E^1 \longrightarrow E_2^{0,1} \xrightarrow{d} E_2^{2,0} \longrightarrow E^2$$

*is exact. It is called the associated **five term exact sequence**.*

We get a generalization of this result under the assumption that $E_2^{pq} = 0$ for $0 < q < n$. Namely, in this case we have isomorphisms $E_{n+1}^{0,n} \simeq \dots \simeq E_2^{0,n}$ and $E_2^{n+1,0} \simeq \dots \simeq E_{n+1}^{n+1,0}$. Therefore the differential $E_{n+1}^{0,n} \rightarrow E_{n+1}^{n+1,0}$ induces a homomorphism

$$E_2^{0,n} \xrightarrow{d} E_2^{n+1,0}.$$

We obtain the following

(2.1.2) Lemma. *Assume that, in a first quadrant E_2 -spectral sequence, the terms E_2^{pq} vanish for $0 < q < n$ and all p . Then*

$$E_2^{m,0} \cong E^m$$

for $m < n$ and the sequence

$$0 \longrightarrow E_2^{n,0} \longrightarrow E^n \longrightarrow E_2^{0,n} \xrightarrow{d} E_2^{n+1,0} \longrightarrow E^{n+1}$$

is exact.

The proof of this lemma (and also of the results below) is elementary and we refer to [21], chap. XV, §5. The most frequent application of spectral sequences are in the following special cases.

(2.1.3) Lemma. Assume that a first quadrant E_2 -spectral sequence is given.

(i) If $E_2^{pq} = 0$ for all $q > 1$ and all p , then we have a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_2^{1,0} & \longrightarrow & E_d^1 & \longrightarrow & E_2^{0,1} \\ & & & & \searrow & & \searrow \\ & & & & E_2^{2,0} & \longrightarrow & E_d^2 \longrightarrow E_2^{1,1} \\ & & & & & & \searrow \\ & & & & E_2^{3,0} & \longrightarrow & E^3 \longrightarrow E_2^{2,1} \longrightarrow \dots \end{array}$$

(ii) If $E_2^{pq} = 0$ for all $p > 1$ and all q , then the sequences

$$0 \longrightarrow E_2^{1,n-1} \longrightarrow E^n \longrightarrow E_2^{0,n} \longrightarrow 0$$

are exact for all $n \geq 1$.

(2.1.4) Lemma. Assume that, in a first quadrant E_2 -spectral sequence, the term E_2^{pq} vanishes for all (p, q) with $(p - m) \cdot (q - n) < 0$. Then

$$E_2^{mn} \cong E^{m+n}.$$

In particular, if $E_2^{pq} = 0$ for all $q > 0$, then

$$E_2^{m,0} \cong E^m \quad \text{for all } m.$$

Proof. If $p > m$ and $q < n$ or if $p < m$ and $q > n$, then $E_r^{pq} = 0$ for all $r \geq 2$, since it is a subquotient of E_2^{pq} , and hence $E_\infty^{pq} = 0$. Therefore on the line $p + q = m + n$, all terms E_∞^{pq} are zero up to E_∞^{mn} and consequently $E_\infty^{mn} \cong E^{m+n}$. The maps

$$E_r^{m-r, n+r-1} \xrightarrow{d_r} E_r^{m, n} \xrightarrow{d_r} E_r^{m+r, n-r+1}$$

are zero for all $r \geq 2$, hence $E_2^{mn} = E_3^{mn} = \dots = E_r^{mn} = E_\infty^{mn}$. \square

In practice, the differentials of a spectral sequence are often difficult to calculate. We are in a rather comfortable situation if they vanish from a certain point on. In this case one says that the spectral sequence *degenerates*. The precise definition is the following.

(2.1.5) Definition. An E_m -spectral sequence **degenerates** at $E_{m'}$ for some $m' \geq m$ if the differentials

$$d_r^{pq}: E_r^{pq} \longrightarrow E_r^{p+r, q-r+1}$$

vanish for all $r \geq m'$ and all $(p, q) \in \mathbb{Z} \times \mathbb{Z}$. Hence, in this case

$$E_{m'}^{pq} = E_{m'+1}^{pq} = E_{m'+2}^{pq} = \dots = E_\infty^{pq}$$

for all $p, q \in \mathbb{Z} \times \mathbb{Z}$.

§2. Filtered Cochain Complexes

Many spectral sequences arise from filtered cochain complexes. Let \mathcal{A} be an abelian category. By a (cochain) **complex** A^\bullet in \mathcal{A} we understand a sequence $A^\bullet = (A^n, d^n)_{n \in \mathbb{Z}}$ of objects and homomorphisms

$$\dots \longrightarrow A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} A^{n+2} \longrightarrow \dots$$

with $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. The d 's are called **differentials**. We say that A^\bullet is **bounded below** (resp. **bounded above**, resp. **bounded**) if $A^n = 0$ for $n \ll 0$ (resp. for $n \gg 0$, resp. for $n \ll 0$ and $n \gg 0$). We set

$$\begin{aligned} Z^n(A^\bullet) &= \ker(A^n \xrightarrow{d^n} A^{n+1}), \\ B^n(A^\bullet) &= \operatorname{im}(A^{n-1} \xrightarrow{d^{n-1}} A^n). \end{aligned}$$

The elements of $Z^n(A^\bullet)$ and $B^n(A^\bullet)$ are called the **n -cocycles** and **n -coboundaries**, respectively. As $d \circ d = 0$, we have $B^n(A^\bullet) \subseteq Z^n(A^\bullet)$. The factor group

$$H^n(A^\bullet) = Z^n(A^\bullet) / B^n(A^\bullet)$$

is called the **n -dimensional cohomology group** of A^\bullet .

A homomorphism of complexes $f: A^\bullet \rightarrow B^\bullet$ is a sequence $f = (f^n)_{n \in \mathbb{Z}}$ of homomorphisms $f^n: A^n \rightarrow B^n$ with $f^{n+1} \circ d^n = d^n \circ f^n$ for all $n \in \mathbb{Z}$. A homomorphism $f: A^\bullet \rightarrow B^\bullet$ of complexes induces homomorphisms

$$H^n(f): H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

on the cohomology, and we call f a **quasi-isomorphism** if $H^n(f)$ is an isomorphism for all n .

A **filtration by subcomplexes** of A^\bullet is a filtration $F^\bullet A^n$ of A^n for all $n \in \mathbb{Z}$ such that for each n , $F^n A^\bullet$ is a subcomplex of A^\bullet . We say that the filtration $F^\bullet A^\bullet$ is **biregular**, if, for each $n \in \mathbb{Z}$, the filtration $F^\bullet A^n$ is finite.

Examples: For any complex A^\bullet we have the following filtrations

1. The **trivial filtration** $tr^\bullet A^\bullet$ defined by

$$tr^p A^\bullet = \begin{cases} A^\bullet, & \text{for } p \leq 0, \\ 0, & \text{for } p \geq 1. \end{cases}$$

We have

$$H^q(tr^n A^\bullet) = \begin{cases} H^q(A^\bullet), & \text{for } n \leq 0, \\ 0, & \text{for } n \geq 1. \end{cases}$$

2. Consider for $p \in \mathbb{Z}$ the subcomplex $\tau_{\leq p}(A^\bullet)$ of A^\bullet given by

$$\tau_{\leq p}(A^\bullet)^n = \begin{cases} A^n, & \text{for } n \leq p-1, \\ \ker(A^n \xrightarrow{d} A^{n+1}), & \text{for } n = p, \\ 0, & \text{for } n \geq p+1. \end{cases}$$

We have

$$H^q(\tau_{\leq p}(A^\bullet)) = \begin{cases} H^q(A^\bullet), & \text{for } q \leq p, \\ 0, & \text{for } q \geq p+1. \end{cases}$$

The **canonical filtration** $\tau^\bullet A^\bullet$ on A^\bullet is the decreasing filtration defined by $\tau^p A^n = \tau_{\leq -p}(A^\bullet)^n$.

A biregular filtration $F^\bullet A^\bullet$ induces an E_1 -spectral sequence in the following way.

(2.2.1) Proposition. *Let $F^\bullet A^\bullet$ be a biregularly filtered cochain complex. For $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ and $r \in \mathbb{Z} \cup \{\infty\}$, we put*

$$\begin{aligned} Z_r^{pq} &= \ker \left(F^p A^{p+q} \rightarrow A^{p+q+1} / F^{p+r} A^{p+q+1} \right), \\ B_r^{pq} &= d(F^{p-r} A^{p+q-1}) \cap F^p A^{p+q}, \\ E_r^{pq} &= Z_r^{pq} / (B_{r-1}^{pq} + Z_{r-1}^{p+1, q-1}), \\ F^p H^{p+q}(A^\bullet) &= \operatorname{im} (H^{p+q}(F^p A^\bullet) \rightarrow H^{p+q}(A^\bullet)). \end{aligned}$$

Then the differential d of the complex A^\bullet induces homomorphisms

$$d = d_r^{pq}: E_r^{pq} \longrightarrow E_r^{p+r, q-r+1}$$

for all $r \in \mathbb{Z}$ in a natural way. There are canonical isomorphisms

$$\alpha_r^{pq}: \ker(d_r^{pq}) / \operatorname{im}(d_r^{p-r, q+r-1}) \xrightarrow{\sim} E_{r+1}^{pq}$$

for all $r \in \mathbb{Z}$. For fixed $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, the morphisms d_r^{pq} and $d_r^{p-r, q+r-1}$ vanish for sufficiently large r and we have natural isomorphisms

$$E_r^{pq} \xrightarrow{\sim} E_\infty^{pq}, \quad r \gg 0.$$

Finally, there exist natural isomorphisms

$$\beta^{pq}: E_\infty^{pq} \xrightarrow{\sim} gr_p H^{p+q}(A^\bullet).$$

In particular, these data define a spectral sequence

$$E_1^{pq} \Rightarrow H^{p+q}(A^\bullet).$$

Remark: We have $E_r^{pq} = gr_p A^{p+q}$ for all $r \leq 0$. For $r \leq -1$ the differentials d_r^{pq} are zero and $\alpha_r^{pq} = id_{gr_p A^{p+q}}$. The sequence of isomorphisms α_r^{pq} for $r \geq 0$ starts with

$$\alpha_0^{pq}: H^q(E_0^{p\bullet}) = H^q(gr_p A^{p\bullet}) \xrightarrow{\sim} E_1^{pq}.$$

Proof of (2.2.1): The fact that d induces homomorphisms d_r^{pq} and the existence of natural isomorphism α_r^{pq} can be easily verified using the definition of the objects occurring (cf. [21], I.3.1.5). As the filtration is biregular, for each fixed pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ the morphisms d_r^{pq} and $d_r^{p-r, q+r-1}$ vanish for sufficiently large r . The natural isomorphisms $E_r^{pq} = E_\infty^{pq}$ for $r \gg 0$ and $E_\infty^{pq} \xrightarrow{\sim} gr_p H^{p+q}(A^\bullet)$ can be read off directly from the definition, using the biregularity of the filtration. \square

(2.2.2) Definition. We call the spectral sequence of (2.2.1) the spectral sequence associated to the biregularly filtered complex $F^\bullet A^\bullet$.

Examples: 1. For the trivial filtration $tr^\bullet A^\bullet$, the associated spectral sequence has the following shape:

$$E_1^{pq} = \begin{cases} H^q(A^\bullet), & \text{for } p = 0, \\ 0, & \text{for } p \neq 0. \end{cases}$$

The spectral sequence degenerates at E_1 , i.e. $E_1^{pq} = E_\infty^{pq}$ for all p, q .

2. For the canonical filtration $\tau^\bullet A^\bullet$, the associated spectral sequence has the following shape:

$$E_1^{pq} = \begin{cases} H^{-p}(A^\bullet), & \text{for } p + q = -p, \\ 0, & \text{for } p + q \neq -p. \end{cases}$$

The spectral sequence degenerates at E_1 , i.e. $E_1^{pq} = E_\infty^{pq}$ for all p, q .

In both examples, the filtration on the limit terms has only one nontrivial graded piece, which is not the case in general. Quite often, spectral sequences arise from double complexes.

(2.2.3) Definition. A double complex $A^{\bullet\bullet}$ is a collection of objects $A^{pq} \in \mathcal{A}$, $p, q \in \mathbb{Z}$, together with differentials $d^{pq}: A^{pq} \rightarrow A^{p+1, q}$ and $d''^{p, q}: A^{pq} \rightarrow A^{p, q+1}$ such that $d' \circ d' = 0 = d'' \circ d''$ and $d' \circ d'' + d'' \circ d' = 0$. The associated **total complex**

$$A^\bullet = \text{tot}(A^{\bullet\bullet})$$

is the single complex with $A^n = \bigoplus_{p+q=n} A^{pq}$ whose differential $d: A^n \rightarrow A^{n+1}$ is given by the sum of the maps

$$d = d' + d'': A^{pq} \longrightarrow A^{p+1, q} \oplus A^{p, q+1}, \quad p + q = n.$$

Example: Let (C^\bullet, d_C) and (D^\bullet, d_D) be two complexes of abelian groups. Their **tensor product** is the double complex

$$A^{\bullet\bullet} = C^\bullet \otimes D^\bullet$$

with $A^{pq} = C^p \otimes D^q$, $p, q \in \mathbb{Z}$, and with the following differentials:

$$d'^{pq} = d_C^p \otimes id_{D^q}, \quad d''^{pq} = (-1)^p id_{C^p} \otimes d_D^q.$$

As there is no danger of confusion, the associated total complex $\text{tot}(C^\bullet \otimes D^\bullet)$ is usually also called the tensor product of C^\bullet and D^\bullet .

Given a double complex $A^{\bullet\bullet}$, we have the **natural filtration** $F^p A^\bullet$ of the total complex defined by

$$F^p A^n = \bigoplus_{i \geq p} A^{i, n-i}.$$

Example: Let D^\bullet be any complex of abelian groups and let C^\bullet be the complex given by

$$C^i = \begin{cases} \mathbb{Z}, & \text{for } i = 0, \\ 0, & \text{for } i \neq 0. \end{cases}$$

Then $\text{tot}(C^\bullet \otimes D^\bullet) = D^\bullet$ and the induced filtration on D^\bullet is the trivial filtration.

Let $A^{\bullet\bullet}$ be a double complex. We will assume in the following that, for each n , there are only finitely many nonzero A^{pq} on the line $p+q = n$. Then the above filtration on $A^\bullet = \text{tot}(A^{\bullet\bullet})$ is biregular and induces a spectral sequence

$$E_1^{pq} \Rightarrow H^{p+q}(A^\bullet)$$

converging to the cohomology of A^\bullet . The initial terms E_1^{pq} are obtained by taking cohomology in direction q :

$$E_1^{pq} = H^q(A^{p\bullet}, d'').$$

The E_1 terms give a complex

$$H^q(A^{\bullet\bullet}) : \dots \xrightarrow{d'} H^q(A^{p-1, \bullet}) \xrightarrow{d'} H^q(A^{p, \bullet}) \xrightarrow{d'} H^q(A^{p+1, \bullet}) \xrightarrow{d'} \dots,$$

whose cohomology yields the E_2 -terms:

$$E_2^{pq} = H^p(H^q(A^{\bullet\bullet})).$$

For reasons that will become apparent later, one often forgets the E_1 -page and calls the spectral sequence

$$E_2^{pq} = H^p(H^q(A^{\bullet\bullet})) \Rightarrow H^{p+q}(\text{tot } A^{\bullet\bullet})$$

the **spectral sequence associated to the double complex** $A^{\bullet\bullet}$.

Remark: For a double complex $A^{\bullet\bullet}$ of abelian groups, the differentials $d_2^{pq} : E_2^{pq} \rightarrow E_2^{p+2, q-1}$ may be described as follows. For each class $c \in E_2^{pq}$, there are elements $x \in A^{pq}$ and $y \in A^{p+1, q-1}$ with the following properties:

- 1) $d''x = 0$, $d''y = -d'x$,
- 2) c is represented by x and $d_2^{pq}c$ by $d'y$.

(2.2.4) Lemma. *Let $A^{\bullet\bullet}$ be a first quadrant double complex, i.e. $A^{pq} = 0$ if $p < 0$ or $q < 0$. Assume that for each $q \geq 0$ the horizontal complex $A^{0q} \rightarrow A^{1q} \rightarrow A^{2q} \rightarrow \dots$ is exact (i.e. trivial cohomology in dimension ≥ 1). Then $E^n = H^n(B^\bullet)$, where B^\bullet is the complex $\ker(A^{0\bullet} \rightarrow A^{1\bullet})$.*

Proof: Setting $'A^{pq} = A^{qp}$, we obtain a double complex $'A^{\bullet\bullet}$ with the same total complex A^\bullet as $A^{\bullet\bullet}$, and hence a new spectral sequence $'E_2^{pq} \Rightarrow E^n$ with the same limit terms E^n (with different filtrations, however). Now the vertical sequences $'A^{p0} \rightarrow 'A^{p1} \rightarrow \dots$ are exact, so that $'E_2^{pq} = 0$, i.e. $'E_\infty^{pq} = 0$ for $q > 0$ and all $p \geq 0$. This means that $E^n = 'F^n E^n \cong 'E_\infty^{n,0} = 'E_2^{n,0} = H^q(H^p(A^{\bullet\bullet}))$ for $p = 0, q = n$. \square

Our first application of spectral sequences is the following

(2.2.5) Proposition. *Let R be a commutative ring with unit. Let $f: D^\bullet \rightarrow D'^\bullet$ be a quasi-isomorphism of complexes of R -modules and let C^\bullet be a complex consisting of flat R -modules. Assume that one of the following conditions is fulfilled:*

- (i) C^\bullet is bounded above,
- (ii) D^\bullet and D'^\bullet are bounded below,
- (iii) R is a Dedekind domain.

Then the induced homomorphism

$$id_{C^\bullet} \otimes f : \text{tot}(C^\bullet \otimes_R D^\bullet) \longrightarrow \text{tot}(C^\bullet \otimes_R D'^\bullet)$$

is a quasi-isomorphism.

Proof: We start with the special case $D'^\bullet = 0$, i.e. D^\bullet is exact, and we have to show that $\text{tot}(C^\bullet \otimes_R D^\bullet)$ is exact.

Let us first assume that D^\bullet is bounded above. Then, if (i) or (ii) holds, the natural filtration on $\text{tot}(C^\bullet \otimes_R D^\bullet)$ is biregular, and for the associated spectral sequence

we have

$$E_1^{pq} = H^q(C^p \otimes D^\bullet) = 0$$

as C^p is flat for all p . Hence all limit terms $H^{p+q}(\text{tot}(C^\bullet \otimes_R D^\bullet))$ vanish.

If D^\bullet is not bounded above, we write $D^\bullet = \varinjlim_n \tau_{\leq n}(D^\bullet)$ and obtain

$$H^k(\text{tot}(C^\bullet \otimes_R D^\bullet)) = \varinjlim_n H^k(\text{tot}(C^\bullet \otimes_R \tau_{\leq n}(D^\bullet))) = 0$$

for all k .

Keeping the assumption $D'^\bullet = 0$, we now assume that R is a Dedekind domain. Then each submodule of a flat R -module is again flat. Hence the complexes $\tau_{\leq n}(C^\bullet)$ consist of flat R -modules for all n , and we obtain

$$H^k(\text{tot}(C^\bullet \otimes_R D^\bullet)) = \varinjlim_n H^k(\text{tot}(\tau_{\leq n}(C^\bullet) \otimes_R D^\bullet)) = 0$$

for all k . This settles the case $D'^\bullet = 0$.

In the general case we consider the mapping cone $C(f)^\bullet$ of f . It is defined by

$$C(f)^n = D^{n+1} \oplus D^n$$

with differential $d((a, b)) = (-d(a), d'(b) + f(a))$ for $a \in D^{n+1}$, $b \in D^n$. The natural long exact sequence

$$\cdots \longrightarrow H^n(D^\bullet) \longrightarrow H^n(D'^\bullet) \longrightarrow H^n(C(f)^\bullet) \longrightarrow \cdots$$

shows that $C(f)^\bullet$ is exact, as f is a quasi-isomorphism. Moreover, the construction of the mapping cone commutes with tensor products, i.e.

$$C(id_{C^\bullet} \otimes f)^\bullet = \text{tot}(C^\bullet \otimes C(f)^\bullet).$$

By the first part of the proof, $\text{tot}(C^\bullet \otimes C(f)^\bullet)$ is exact if one of the conditions (i)–(iii) is satisfied. Hence $C(id_{C^\bullet} \otimes f)^\bullet$ is exact and therefore $id_{C^\bullet} \otimes f$ is a quasi-isomorphism. \square

Exercise 1. Calculate the spectral sequence associated to the *stupid* filtration

$$\sigma_{\geq p}(A^\bullet)^n = \begin{cases} A^n, & \text{for } n \geq p, \\ 0, & \text{for } n \leq p-1. \end{cases}$$

Does it degenerate?

Exercise 2. Let p be a prime number and $R = \mathbb{Z}/p^2\mathbb{Z}$. Consider the exact and flat complex of R -modules

$$C^\bullet = \cdots \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \cdots.$$

Show that $H^i(\text{tot}(C^\bullet \otimes_R C^\bullet)) \neq 0$ for all i .

§3. Degeneration of Spectral Sequences

In this section we investigate the degeneration of the spectral sequence attached to a biregularly filtered cochain complex. The first and easiest case is degeneration at E_1 .

(2.3.1) Theorem. *For the spectral sequence*

$$E_1^{pq} \Rightarrow H^{p+q}(A^\bullet)$$

associated to a biregularly filtered complex $F^\bullet A^\bullet$ the following assertions are equivalent:

- (i) *The spectral sequence degenerates at E_1 .*
- (ii) *For all n, p we have $F^p A^n \cap d(A^{n-1}) = d(F^p A^{n-1})$.*
- (iii) *For all n, p the natural map $H^n(F^p A^\bullet) \rightarrow H^n(A^\bullet)$ is injective.*

If, moreover, the maps in (iii) are split-injections, we obtain a (non-canonical) splitting

$$H^n(A^\bullet) \cong \bigoplus_{p+q=n} E_1^{pq}.$$

Proof: Without loss of generality, we may work in the category of modules over a ring. Assume that (i) holds. For sufficiently large p , we have $F^p A^{n-1} = 0$, hence assertion (ii) holds for $p \gg 0$. We fix $n \in \mathbb{Z}$ and proceed by decreasing induction on p . Let $x \in F^p A^n \cap d(A^{n-1})$. As the filtration is biregular, $x \in d(F^m A^{n-1})$ for some m , which we choose as large as possible ($m = \infty$ allowed). We assume that $m = p - r$ for some $r \geq 1$, and show that this yields a contradiction. Let $y \in F^{p-r} A^{n-1}$ be a pre-image of x . By construction, $y \in Z_r^{p-r, n-1-p+r}$. By assumption, the differential $d_r: E_r^{p-r, n-1-p+r} \rightarrow E_r^{p, n-p}$ is the zero map. Hence $x = dy \in B_{r-1}^{p, n-p} + Z_{r-1}^{p+1, n-p-1}$, i.e. we may write x in the form

$$x = dy' + x', \quad y' \in F^{p-r+1} A^{n-1}, \quad x' \in F^{p+1} A^n.$$

As x and dy' are coboundaries, the same holds for x' . By our inductive assumption, we have $x' \in d(F^{p+1} A^{n-1})$, which implies $x \in F^p A^n \cap d(F^{p-r+1} A^{n-1})$, contradicting the maximality of $m = p - r$. Hence $m \geq p$, showing that $x \in d(F^p A^{n-1})$. This proves (i) \Rightarrow (ii). The implication (ii) \Rightarrow (i), as well as the equivalence (ii) \Leftrightarrow (iii) are elementary.

Finally, assume that (i)–(iii) hold. By definition, $F^p H^n(A^\bullet)$ is the image of the natural map $H^n(F^p A^\bullet) \rightarrow H^n(A^\bullet)$, which is a split-injection. As the filtration on $H^n(A^\bullet)$ is finite, we recursively obtain a splitting

$$H^n(A^\bullet) \cong \bigoplus_p F^p H^n(A^\bullet) / F^{p+1} H^n(A^\bullet)$$

into a finite direct sum. By (i), we have

$$E_1^{p,n-p} = E_\infty^{p,n-p} = \text{gr}_p H^n(A^\bullet) = F^p H^n(A^\bullet) / F^{p+1} H^n(A^\bullet),$$

showing the last assertion of the proposition. \square

By a formal reindexing procedure, we can displace a spectral sequence in the following sense: Assume we are given an E_m -spectral sequence $E_m^{pq} \Rightarrow E^n$. Putting

$$\tilde{E}^n = E^n, \quad F^p \tilde{E}^n = F^{p+n} E^n \quad \text{and} \quad \tilde{E}_r^{pq} = E_{r+1}^{2p+q, -p} \quad \text{for } r \geq m-1,$$

we obtain an E_{m-1} -spectral sequence converging to the same limit terms, but with a shifted filtration, which we call the **displaced spectral sequence**. It is a remarkable fact that, if the spectral sequence E arises from a biregular filtered cochain complex, then the displaced spectral sequence \tilde{E} arises from another filtration on the same complex, the displaced filtration. This will be useful in showing that a spectral sequence degenerates at E_2 , just by showing that the displaced spectral sequence \tilde{E} satisfies the conditions of (2.3.1).

(2.3.2) Definition. Let $F^\bullet A^\bullet$ be a biregular filtered cochain complex. The filtration

$$\text{Dis}(F)^p A^n = Z_1^{p+n, -p} = \{a \in F^{p+n} A^n \mid da \in F^{p+n+1} A^{n+1}\}$$

is called the **displaced filtration**. We denote the complex A^\bullet , together with the filtration $\text{Dis}(F)$, by $\text{Dis}(A^\bullet)$.

One easily verifies that $\text{Dis}(A^\bullet)$ is a filtered complex:

$$d(Z_1^{p+n, -p}) \subseteq F^{p+1+n} A^{n+1} \cap \ker(d) \subseteq Z_\infty^{p+n+1, -p} \subseteq Z_1^{p+n+1, -p},$$

and the filtration $\text{Dis}(F)$ is obviously biregular.

Example: Displacing the trivial filtration tr on A^\bullet , we obtain the canonical filtration τ :

$$\begin{aligned} \text{Dis}(tr)^p A^n &= Z_1^{p+n, -p} = \{a \in tr^{p+n} A^n \mid da \in tr^{p+n+1} A^{n+1}\} \\ &= \begin{cases} A^n, & \text{for } n \leq -p-1, \\ \ker(A^n \rightarrow A^{n+1}), & \text{for } n = -p, \\ 0, & \text{for } n \geq -p+1, \end{cases} \\ &= \tau^p A^n. \end{aligned}$$

(2.3.3) Proposition. *For all $r \geq 1$, there are natural isomorphism*

$$E_r^{pq}(\text{Dis}(A^\bullet)) \xrightarrow{\sim} E_{r+1}^{2p+q, -p}(A^\bullet)$$

commuting with the corresponding differentials. The displaced spectral sequence is the spectral sequence associated to the displaced filtration.

Proof: For $r \geq 0$ we have

$$\begin{aligned} \text{Dis}_F Z_r^{pq} &= \{x \in \text{Dis}(F)^p A^{p+q} \mid dx \in \text{Dis}(F)^{p+r} A^{p+q+1}\} \\ &= \{x \in F^{2p+q} A^{p+q} \mid dx \in F^{2p+q+r+1} A^{p+q+1}\} \\ &= {}_F Z_{r+1}^{2p+q, -p}. \end{aligned}$$

Analogously,

$$\begin{aligned} \text{Dis}_F B_r^{pq} &= d(\text{Dis}(F)^{p-r} A^{p+q-1}) \cap \text{Dis}(F)^p A^{p+q} \\ &= d(F^{2p+q-r-1} A^{p+q-1}) \cap F^{2p+q} A^{p+q} \\ &= {}_F B_{r+1}^{2p+q, -p}. \end{aligned}$$

By definition, $E_r^{pq} = Z_r^{pq} / (B_{r-1}^{pq} + Z_{r-1}^{p+1, q-1})$. Hence the natural identifications above induce isomorphisms $E_r^{pq}(\text{Dis}(A^\bullet)) \xrightarrow{\sim} E_{r+1}^{2p+q, -p}(A^\bullet)$ for $r \geq 1$. Therefore, the E_r^{pq} -terms of the displaced spectral sequence and of the spectral sequence associated to the displaced filtration are canonically isomorphic. The same holds for the limit terms, which can easily be seen from their definitions. \square

As an application, we obtain the following degeneration result. It should not be mistaken for the well-known Künneth-formula, which arises from another filtration on the tensor product (see the exercise below).

(2.3.4) Theorem. *Let R be a Dedekind domain and let C^\bullet and D^\bullet be complexes of R -modules such that the natural filtration on their tensor product is biregular.*) If C^\bullet consists of flat (i.e. torsion-free) R -modules, then the spectral sequence of the double complex $A^{\bullet\bullet} = C^\bullet \otimes_R D^\bullet$ degenerates at E_2 . Furthermore, we have a non-canonical splitting*

$$H^n(\text{tot}(C^\bullet \otimes_R D^\bullet)) \cong \bigoplus_{p+q=n} E_2^{pq}.$$

*) e.g. both complexes are bounded above, or both complexes are bounded below, or one of the complexes is bounded.

Proof: Let $A^\bullet = \text{tot}(C^\bullet \otimes_R D^\bullet)$, and let $F^p A^n = \bigoplus_{i \geq p} C^i \otimes_R D^{n-i}$ be the natural filtration. We want to show that the spectral sequence of the double complex degenerates at E_2 . By (2.3.3), it suffices to show that the spectral sequence associated to the displaced filtration degenerates at E_1 . Thus, by (2.3.1), we have to show that the natural maps

$$H^n(\text{Dis}(F)^p A^\bullet) \longrightarrow H^n(A^\bullet)$$

are split-injections for all p and n . Using the flatness of C^\bullet , we obtain

$$\begin{aligned} \text{Dis}(F)^p A^n &= \ker \left(F^{p+n} A^n \rightarrow F^{p+n} A^{n+1} / F^{p+n+1} A^{n+1} \right), \\ &= \left(\bigoplus_{i > p+n} C^i \otimes_R D^{n-i} \right) \oplus C^{p+n} \otimes_R \ker(d_D^{-p}) \\ &= \text{tot} \left(C^\bullet \otimes_R \tau_{\leq -p}(D^\bullet) \right)^n. \end{aligned}$$

As the filtration on $H^n(A^\bullet)$ is finite, it therefore remains to show that for all $n, m \in \mathbb{Z}$ the natural map

$$H^n \left(\text{tot}(C^\bullet \otimes_R \tau_{\leq m}(D^\bullet)) \right) \longrightarrow H^n \left(\text{tot}(C^\bullet \otimes_R \tau_{\leq m+1}(D^\bullet)) \right)$$

is a split-injection. Let X^\bullet be a complex consisting of projective R -modules together with a quasi-isomorphism $X^\bullet \rightarrow \tau_{\leq m+1}(D^\bullet)$. Using the flatness of C^\bullet , we obtain a commutative diagram

$$\begin{array}{ccc} \text{tot}(C^\bullet \otimes_R \tau_{\leq m}(X^\bullet)) & \xrightarrow{\text{quasi-iso}} & \text{tot}(C^\bullet \otimes_R \tau_{\leq m}(D^\bullet)) \\ \downarrow & & \downarrow \\ \text{tot}(C^\bullet \otimes_R X^\bullet) & \xrightarrow{\text{quasi-iso}} & \text{tot}(C^\bullet \otimes_R \tau_{\leq m+1}(D^\bullet)). \end{array}$$

By (2.2.5)(iii) the horizontal maps are quasi-isomorphisms. Since X^\bullet is a complex consisting of projective R -modules, the inclusion of complexes

$$\tau_{\leq m}(X^\bullet) \hookrightarrow X^\bullet$$

has a section for each $m \in \mathbb{Z}$. Indeed, by our assumption on R , $d(X^m) \subseteq X^{m+1}$ is a projective R -module, and the short exact sequence

$$0 \longrightarrow \ker(d^m) \longrightarrow X^m \longrightarrow d(X^m) \longrightarrow 0$$

splits. Using any section $s: X^m \rightarrow \ker(d^m)$ in dimension m , and the obvious maps in the other dimensions, we obtain a splitting of $\tau_{\leq m}(X^\bullet) \hookrightarrow X^\bullet$.

Therefore the left vertical complex homomorphism in the diagram above has a section, which finishes the proof. \square

The technique of displacing, as well as the results (2.3.1) and (2.3.3), are due to *P. DELIGNE*, cf. [34]. The statement of (2.3.4) is implicitly contained in the article [98] by *U. JANNSEN*, and we adopted his idea of proof.

Exercise (Künneth-formula): Let R be a Dedekind domain and let C^\bullet and D^\bullet be complexes of R -modules. Assume that C^\bullet consists of flat R -modules. Consider the biregular filtration on $A^\bullet = \text{tot}(C^\bullet \otimes_R D^\bullet)$ defined by

$$F^p(A^n) = \begin{cases} \bigoplus_{i+j=n} C^i \otimes_R D^j, & \text{for } p \leq -1, \\ \bigoplus_{i+j=n} Z^i(C^\bullet) \otimes_R D^j, & \text{for } p = 0, \\ 0, & \text{for } p \geq 1, \end{cases}$$

and show the following assertions regarding the associated spectral sequence:

(i) $E_2^{p,q} = \bigoplus_{i+j=q} \text{Tor}_{-p}^R(H^i(C^\bullet), H^j(D^\bullet))$; in particular $E_2^{pq} = 0$ for $p \neq 0, -1$.

(ii) For each n we obtain a short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H^i(C^\bullet) \otimes_R H^j(D^\bullet) \rightarrow H^n(A^\bullet) \rightarrow \bigoplus_{i+j=n+1} \text{Tor}_1^R(H^i(C^\bullet), H^j(D^\bullet)) \rightarrow 0.$$

Moreover, these sequences split (non-canonically).

§4. The Hochschild-Serre Spectral Sequence

Relevant for the cohomology of profinite groups is the following

(2.4.1) Theorem. *Let G be a profinite group, H a closed normal subgroup and A a G -module. Then there exists a first quadrant spectral sequence*

$$E_2^{pq} = H^p(G/H, H^q(H, A)) \Rightarrow H^{p+q}(G, A).$$

*It is called the **Hochschild-Serre spectral sequence**.*

Proof: To the standard resolution $0 \rightarrow A \rightarrow X^\bullet$ of the G -module A , we apply the functor $H^0(H, -)$, and get the complex

$$H^0(H, X^0) \longrightarrow H^0(H, X^1) \longrightarrow H^0(H, X^2) \longrightarrow \dots$$

of G/H -modules. For each $H^0(H, X^q)$, we consider the cochain complex

$$H^0(H, X^q)^{G/H} \longrightarrow C^\bullet(G/H, H^0(H, X^q))$$

and we put

$$C^{pq} = C^p(G/H, H^0(H, X^q)) = X^p(G/H, X^q(G, A)^{H})^{G/H}, \quad p, q \geq 0.$$

We make $C^{\bullet\bullet}$ into a (anti-commutative) double complex by using the following differentials: We let

$$d'_{pq} : C^{pq} \longrightarrow C^{p+1,q}$$

be the differential of the complex $X^\bullet(G/H, X^q(G, A)^H)^{G/H}$ at the place p . Further, we define

$$d''_{pq} : C^{pq} \longrightarrow C^{p, q+1}$$

as $(-1)^p$ times the differential of the complex $X^p(G/H, X^\bullet(G, A)^H)^{G/H}$ at the place q . Then $(C^{\bullet\bullet}, d', d'')$ is a double complex and we define the Hochschild-Serre spectral sequence as the associated spectral sequence

$$E_2^{pq} \Rightarrow E^n.$$

We compute the terms E_2^{pq} and E^n . By definition, $E_2^{pq} = H^p(H^q(C^{\bullet\bullet}))$. We have $H^q(H^0(H, X^\bullet)) = H^q(H, A)$ (see p.34). The functor $C^p(G/H, -)$ is exact (I §3, ex.1). Therefore

$$\begin{aligned} H^q(C^p) &= H^q(C^p(G/H, H^0(H, X^\bullet))) = C^p(G/H, H^q(H^0(H, X^\bullet))) \\ &= C^p(G/H, H^q(H, A)), \end{aligned}$$

hence

$$E_2^{pq} = H^p(C^\bullet(G/H, H^q(H, A))) = H^p(G/H, H^q(H, A)).$$

As for the limit terms, we note that for every $q \geq 0$ the complexes $C^{\bullet q} = C^\bullet(G/H, H^0(H, X^q))$ are exact. In fact, every X^q is an induced G -module, hence $H^0(H, X^q)$ is an induced, and thus acyclic, G/H -module by (1.3.6) and (1.3.7), i.e. $H^p(C^{\bullet q}) = H^p(G/H, H^0(H, X^q)) = 0$ for $p > 0$. By lemma (2.2.4), we obtain $E^n = H^n(B^\bullet)$, where B^\bullet is the complex

$$B^\bullet = \ker(C^0(G/H, (X^\bullet)^H) \rightarrow C^1(G/H, (X^\bullet)^H)) = ((X^\bullet)^H)^{G/H} = (X^\bullet)^G.$$

Therefore

$$E^n = H^n((X^\bullet)^G) = H^n(G, A). \quad \square$$

From (2.1.4) follows the

(2.4.2) Corollary. *If $H^q(H, A) = 0$ for $q > 0$, then*

$$H^n(G/H, A^H) \cong H^n(G, A).$$

Another consequence is the five term exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(G/H, A^H) \longrightarrow H^1(G, A) \longrightarrow H^1(H, A)^{G/H} \\ &\longrightarrow H^2(G/H, A^H) \longrightarrow H^2(G, A), \end{aligned}$$

which we proved in I §6 in an elementary way, but with some difficulty. It still requires, however, careful checking to show that the maps are the inflation, restriction and transgression respectively.

For *inf* and *res*, this identification is given in the available literature (e.g. [128]). For the transgression, we give the proof here.

(2.4.3) Theorem. *The differential*

$$d_2^{0,1} : H^1(H, A)^{G/H} \longrightarrow H^2(G/H, A^H)$$

is the transgression tg as defined in (1.6.6).

Proof (*TH. MOSER, J. STIX*): In order to calculate $H^1(H, A)$ we use the acyclic resolution $A \rightarrow X^\bullet(G, A)$ of the G -module A . Thus an element $z \in H^1(H, A)^{G/H}$ is represented by an H -invariant 1-cocycle $x : G \times G \rightarrow A$. The invariance of z under G/H implies that for $\rho, \sigma \in G$ the cocycles ρx and σx differ by a 1-coboundary, i.e. there is a map

$$b : G \times G \longrightarrow X^0(G, A)^H, \quad (\sigma, \rho) \mapsto b_{\sigma, \rho},$$

such that

$$d(b_{\sigma, \rho}) = \rho x - \sigma x,$$

where d denotes the differential of $X^\bullet(G, A)$. We obtain

$$(*) \quad b_{\sigma, \rho}(\tau_1) - b_{\sigma, \rho}(\tau_0) = (\rho x)_{\tau_0, \tau_1} - (\sigma x)_{\tau_0, \tau_1}$$

for all $\sigma, \rho, \tau_0, \tau_1 \in G$. Therefore we may assume that b is G -invariant and $b_{1,1} = 0$. Furthermore, since $\tau x = x$ for $\tau \in H$, we may also assume that b factors through $G/H \times G/H$. Then, for all $\sigma, \rho, \gamma \in G/H$,

$$b_{\rho, \gamma}(\tau) - b_{\sigma, \gamma}(\tau) + b_{\sigma, \rho}(\tau)$$

is an element of the module A^H independent of $\tau \in G$, and so the 2-cocycle $\partial b : (G/H)^3 \rightarrow X^0(G, A)^H$, given by

$$\partial b_{\sigma, \rho, \gamma} = b_{\rho, \gamma} - b_{\sigma, \gamma} + b_{\sigma, \rho},$$

is constant with value in A^H . By the remark on page 105 it represents the image of z under $d_2^{0,1}$. The associated inhomogeneous 2-cocycle, which also represents $d_2^{0,1}(z)$, is

$$a : G/H \times G/H \longrightarrow A^H, \quad (\sigma, \rho) \mapsto a_{\sigma, \rho},$$

where

$$(**) \quad a_{\sigma, \rho} = \partial b_{1, \sigma, \sigma \rho}(\zeta) = b_{\sigma, \sigma \rho}(\zeta) - b_{1, \sigma \rho}(\zeta) + b_{1, \sigma}(\zeta)$$

with $\zeta \in G$ arbitrary.

We now represent $tg(z)$ as follows. We restrict x to $H \times H$, and pass to the associated inhomogeneous 1-cocycle $x_0 : H \rightarrow A$, $x_0(\tau) = x_{1, \tau}$, which also represents z . Consider then the function

$$y : G \longrightarrow A, \quad \sigma \mapsto y_\sigma = x_{1, \sigma} + b_{1, \sigma}(\sigma).$$

We will show that y satisfies the properties (i), (ii), (iii) of the proof of proposition (1.6.6),

- (i) $y|_H = x_0$,
- (ii) $y_{\sigma\tau} = y_\sigma + \sigma y_\tau$ for $\sigma \in G, \tau \in H$,
- (iii) $y_{\tau\sigma} = y_\tau + \tau y_\sigma$ for $\sigma \in G, \tau \in H$,

hence $tg(z) = [\partial y]$. Since $b_{1,\tau} = b_{1,1} = 0$ for $\tau \in H$, property (i) follows. Let $\sigma, \rho \in G$. Then, by (*) and (**) and since x is a cocycle, we obtain

$$\begin{aligned} (\partial y)_{\sigma,\rho} &= (dx)_{1,\sigma,\sigma\rho} + (\sigma x)_{\sigma,\sigma\rho} - x_{\sigma,\sigma\rho} \\ &\quad + b_{1,\sigma}(\sigma) - b_{1,\sigma}(\sigma\rho) + (\partial b)_{1,\sigma,\sigma\rho}(\sigma\rho) \\ &= (\partial b)_{1,\sigma,\sigma\rho}(\sigma\rho) = a_{\sigma,\rho}. \end{aligned}$$

If $\sigma \in H$ or $\rho \in H$, then the expression above is zero and so (ii) and (iii) follow. Now, for arbitrary $\sigma, \rho \in G$, the equality above shows that $tg(z) = [\partial y] = [a] = d_2^{0,1}$. This proves the theorem. \square

A subtle and useful relation of the Hochschild-Serre spectral sequence to the *cup-product* is obtained as follows. Let G be a profinite group, H an open normal subgroup of G and H' the closure of the commutator subgroup of H . Let A be a G -module on which H acts trivially. We then have, for $p > 0$, two canonical homomorphisms

$$(*) \quad d_2, u\cup : H^{p-1}(G/H, H^1(H, A)) \longrightarrow H^{p+1}(G/H, A)$$

which are defined as follows. The first map d_2 is the differential $d_2^{p-1,1}$ of the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G/H, H^q(H, A)) \Rightarrow H^{p+q}(G, A).$$

On the other hand, the group extension

$$1 \longrightarrow H^{ab} \longrightarrow G/H' \longrightarrow G/H \longrightarrow 1$$

defines a cohomology class $u \in H^2(G/H, H^{ab})$. Using the equality $H^1(H, A) = \text{Hom}(H^{ab}, A)$, we obtain a canonical pairing

$$H^{ab} \times H^1(H, A) \longrightarrow A,$$

which induces a cup-product

$$H^2(G/H, H^{ab}) \times H^{p-1}(G/H, H^1(H, A)) \xrightarrow{\cup} H^{p+1}(G/H, A).$$

The second map $u\cup$ is given by $x \mapsto u \cup x$.

(2.4.4) Theorem. *Let A be a G -module and H an open normal subgroup of G which acts trivially on A . Then, for $p > 0$, the maps*

$$d_2, u\cup : H^{p-1}(G/H, H^1(H, A)) \longrightarrow H^{p+1}(G/H, A)$$

are the same up to sign, i.e. $d_2(x) = -u \cup x$.

In particular, the transgression $tg : H^1(H, A)^{G/H} \rightarrow H^2(G/H, A)$ is given by $tg(x) = -u \cup x$.

Remark: The statement of (2.4.4) remains true for an arbitrary closed normal subgroup H of G , but then one has to use the continuous cohomology class $u \in H_{cts}^2(G/H, H^{ab})$ representing the group extension $1 \rightarrow H^{ab} \rightarrow G/H' \rightarrow G/H \rightarrow 1$ (cf. (2.7.7)).

Proof: Suppose first that G is a finite group. Then the projection $H \rightarrow H^{ab}$ may be regarded as a G/H -invariant 1-cocycle of H , i.e. as an element $\varepsilon \in H^0(G/H, H^1(H, H^{ab}))$. From the spectral sequence

$$E_2^{pq} = H^p(G/H, H^q(H, H^{ab})) \Rightarrow H^{p+q}(G, H^{ab}),$$

we obtain the differential

$$d_2^{0,1} : H^1(H, H^{ab})^{G/H} \longrightarrow H^2(G/H, H^{ab}),$$

which by (2.4.3) is the transgression tg as defined in (1.6.6). We prove $tg(\varepsilon) = -u$ (in the additive notation of $H^2(G/H, H^{ab})$).

Let $s : G/H \rightarrow G$ be a section of the projection $G \rightarrow G/H$, $\sigma \mapsto \bar{\sigma}$. Define the 1-cochain $y : G \rightarrow H^{ab}$ by

$$y(\sigma) = \sigma(s\bar{\sigma})^{-1} \mod H'.$$

Then $y|_H = \varepsilon$ and

$$\begin{aligned} (\partial y)(\sigma_1, \sigma_2) &\equiv y(\sigma_1\sigma_2)^{-1}\sigma_1 y(\sigma_2)y(\sigma_1) \\ &\equiv s(\bar{\sigma}_1\bar{\sigma}_2)\sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2(s\bar{\sigma}_2)^{-1}\sigma_1^{-1}\sigma_1(s\bar{\sigma}_1)^{-1} \\ &\equiv [(s\bar{\sigma}_1)(s\bar{\sigma}_2)s(\bar{\sigma}_1\bar{\sigma}_2)^{-1}]^{-1} \mod H', \end{aligned}$$

showing that ∂y depends only on the classes $\bar{\sigma}_1, \bar{\sigma}_2 \in G/H$, and hence $tg(\varepsilon) = [\partial y]$ by definition of tg . But the function in square brackets is a 2-cocycle which represents the class u , hence $tg(\varepsilon) = -u$.

The differential

$$d_2^{p,1} : H^p(G/H, H^1(H, A)) \longrightarrow H^{p+2}(G/H, A^H)$$

is obtained from the part

$$\begin{array}{ccc} C^p(G/H, X^1(G, A)^H) & \xrightarrow{\partial'} & C^{p+1}(G/H, X^1(G, A)^H) \\ & \uparrow \partial'' & \\ C^{p+1}(G/H, X^0(G, A)^H) & \xrightarrow{\partial'} & C^{p+2}(G/H, X^0(G, A)^H) \end{array}$$

of the double complex $C^{pq}(A) = C^p(G/H, X^q(G, A)^H)$ as follows. Let $z \in H^p(G/H, H^1(H, A))$. z is given by an element $\alpha \in C^{p,1}(A)$ such that $\partial''\alpha = 0$ and that the induced function $\bar{\alpha} : (G/H)^{p+1} \rightarrow H^1(X^\bullet(G, A)^H) = H^1(H, A)$ is a p -cocycle representing z . This means that there exists a $\beta \in C^{p+1,0}$ such that $\partial'\alpha = \partial''\beta$. The image $d_2^{p,1}(z)$ is then represented by the cocycle $\partial'\beta \in C^{p+2,0}$.

This process may also be interpreted as follows. From the complex $0 \rightarrow A^H \rightarrow X^0(G, A)^H \rightarrow X^1(G, A)^H \rightarrow X^2(G, A)^H$, we obtain the exact sequence of G/H -modules

$$(1) \quad 0 \rightarrow A^H \rightarrow X^0(G, A)^H \rightarrow Z \rightarrow H^1(H, A) \rightarrow 0$$

with $Z = Z^1(X^\bullet(G, A)^H)$. Splitting it up into two short exact sequences

$$(2) \quad 0 \rightarrow I(A) \rightarrow Z \rightarrow H^1(H, A) \rightarrow 0,$$

$$(3) \quad 0 \rightarrow A^H \rightarrow X^0(G, A)^H \rightarrow I(A) \rightarrow 0,$$

we obtain two δ -homomorphisms

$$H^p(G/H, H^1(H, A)) \xrightarrow{\delta} H^{p+1}(G/H, I(A)) \xrightarrow{\delta} H^{p+2}(G/H, A^H),$$

with $d_2^{p,1} = \delta \circ \delta$. In fact, the element α above is a lifting $\alpha : (G/H)^{p+1} \rightarrow Z^1(G, A)^H$ of the p -cocycle $\bar{\alpha}$ representing z , hence δz is represented by the $(p+1)$ -cocycle $\partial' \alpha$. Since $\partial' \alpha = \partial'' \beta$, $\beta : (G/H)^{p+2} \rightarrow X^0(G, A)^H$ is a cochain which lifts $\partial' \alpha$, hence $\partial' \beta$ represents $\delta([\partial' \alpha]) = \delta \delta(z)$, showing that $\delta \delta(z) = d_2^{p,1}(z)$.

Let us abbreviate the sequences (2) and (3) by

$$0 \rightarrow S'(A) \rightarrow S(A) \rightarrow S''(A) \rightarrow 0,$$

$$0 \rightarrow T'(A) \rightarrow T(A) \rightarrow T''(A) \rightarrow 0,$$

with $S' = T''$. Replacing A by the G -module H^{ab} , we obtain exact sequences of G/H -modules

$$0 \rightarrow S'(H^{ab}) \rightarrow S(H^{ab}) \rightarrow S''(H^{ab}) \rightarrow 0,$$

$$0 \rightarrow T'(H^{ab}) \rightarrow T(H^{ab}) \rightarrow T''(H^{ab}) \rightarrow 0.$$

Setting $B = \text{Hom}(H^{ab}, A)$, we have the pairings

$$S(H^{ab}) \times B \rightarrow S(A), \quad T(H^{ab}) \times B \rightarrow T(A),$$

which induce pairings $S'(H^{ab}) \times B \rightarrow S'(A)$, $S''(H^{ab}) \times B \rightarrow S''(A)$, and similarly for T . We now assume that H acts trivially on A , i.e. $\text{Hom}(H^{ab}, A) = H^1(H, A)$. We apply proposition (1.4.3) twice and obtain the commutative diagram

$$\begin{array}{ccccc} H^0(G/H, S''(H^{ab})) \times H^p(G/H, B) & \xrightarrow{\cup} & H^p(G/H, S''(A)) \\ \delta \downarrow & \parallel & \downarrow \delta \\ H^1(G/H, S'(H^{ab})) \times H^p(G/H, B) & \xrightarrow{\cup} & H^{p+1}(G/H, S'(A)) \\ \delta \downarrow & \parallel & \downarrow \delta \\ H^2(G/H, T'(H^{ab})) \times H^p(G/H, B) & \xrightarrow{\cup} & H^{p+2}(G/H, T'(A)). \end{array}$$

In the upper pairing, we have $S''(H^{ab}) = H^1(H, H^{ab})$, $S''(A) = H^1(H, A)$ and $\varepsilon \cup z = z$ (recall that $z \in H^p(G/H, \text{Hom}(H^{ab}, A)) = H^p(G/H, H^1(H, A))$). Since $\delta\delta\varepsilon = d_2^{0,1}(\varepsilon) = tg(\varepsilon) = -u$, it follows that

$$d_2^{p,1}(z) = \delta\delta(z) = \delta\delta(\varepsilon \cup z) = \delta\delta\varepsilon \cup z = -u \cup z.$$

This proves the theorem for a finite group G .

Now let G be an arbitrary profinite group. We denote by G' the commutator subgroup of G . We let W run through the open normal subgroups of G which are contained in H . Setting $\bar{G} = G/W$, $\bar{H} = H/W$, we have a commutative exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & H^{ab} & \longrightarrow & G/H' & \longrightarrow & G/H \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \bar{H}^{ab} & \longrightarrow & \bar{G}/\bar{H}' & \longrightarrow & G/H \longrightarrow 1. \end{array}$$

The lower group extension defines an element $u_W \in H^2(G/H, \bar{H}^{ab})$ which is the image of u under $H^2(G/H, H^{ab}) \xrightarrow{\pi} H^2(G/H, \bar{H}^{ab})$ (see (3.6.1)). On the other hand, we have a commutative diagram

$$\begin{array}{ccc} H^p(G/H, H^1(\bar{H}, A)) & \xrightarrow{d_2} & H^{p+2}(G/H, A) \\ \downarrow \text{inf} & & \parallel \\ H^p(G/H, H^1(H, A)) & \xrightarrow{d_2} & H^{p+2}(G/H, A). \end{array}$$

We have shown that $d_2 x_W = -u_W \cup x_W$ for $x_W \in H^p(G/H, H^1(\bar{H}, A))$. The diagram

$$\begin{array}{ccc} H^2(G/H, \bar{H}^{ab}) \times H^p(G/H, H^1(\bar{H}, A)) & \xrightarrow{\cup} & H^{p+2}(G/H, A) \\ \pi \uparrow & & \downarrow \text{inf} \quad \parallel \\ H^2(G/H, H^{ab}) \times H^p(G/H, H^1(H, A)) & \xrightarrow{\cup} & H^{p+2}(G/H, A) \end{array}$$

is commutative because of the commutativity of the cup-product with the inflation. So, $d_2 \text{inf } x_W = d_2 x_W = -u_W \cup x_W = -\pi u \cup x_W = -u \cup \text{inf } x_W$. Since $H^1(H, A) = \varinjlim_W H^1(H/W, A)$, each $x \in H^p(G/H, H^1(H, A))$ is of the form $x = \text{inf } x_W$ for some W . This completes the reduction to the case of a finite group G . \square

If $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ is a split group extension and A is a G -module on which H acts trivially, then (2.4.4) shows that the differentials $d_2^{*,1}$ are trivial. However, the Hochschild-Serre spectral sequence does, in general, *not* degenerate at E_2 , even if A is a trivial G -module, see [46], example on page 83. But, we have the following

(2.4.5) Proposition. *Let $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ be a split exact sequence of profinite groups, and let A be a discrete G -module on which H acts trivially. Then*

$$\inf : H^*(G/H, A) \longrightarrow H^*(G, A)$$

is a monomorphism onto a direct summand, and all differentials into the horizontal edge of the Hochschild-Serre spectral sequence for A vanish, i.e. $d_r^{,r-1} = 0$ for all $r \geq 2$.*

Proof: Let $\iota: G/H \hookrightarrow G$ be a splitting of the projection $\pi: G \twoheadrightarrow G/H$, i.e. $\pi\iota = \text{id}$. Since $A^H = A$, the edge homomorphisms

$$H^*(G/H, A) = E_2^{*,0} \longrightarrow E_\infty^{*,0} \subseteq H^*(G, A)$$

are split injections, so $E_2^{*,0} = E_r^{*,0} = E_\infty^{*,0}$ for $r \geq 2$. Since $E_{r+1}^{*,0} = E_r^{*,0} / \text{im} d_r^{*,r-1}$, we obtain $d_r^{*,r-1} = 0$. \square

In case of a *direct* product, the Hochschild-Serre spectral sequence degenerates at E_2 . We follow the proof given by U. JANNSEN, see [98].

(2.4.6) Theorem. *Let G and H be profinite groups and let A be a discrete H -module, regarded as a $(G \times H)$ -module via trivial action of the group G . Then the Hochschild-Serre spectral sequence*

$$E_2^{pq} = H^p(G, H^q(H, A)) \Rightarrow H^n(G \times H, A)$$

degenerates at E_2 . Furthermore, it splits in the sense that there is a decomposition

$$H^n(G \times H, A) \cong \bigoplus_{p+q=n} H^p(G, H^q(H, A)).$$

Proof: For a trivial G -module B , we have a natural isomorphism of complexes

$$C^\bullet(G, B) \cong C^\bullet(G, \mathbb{Z}) \otimes B.$$

This is easily verified if G is finite, and the result for general G follows by a straightforward limit process. Therefore, under the assumptions of the theorem, the Hochschild-Serre spectral sequence is the spectral sequence associated to the double complex

$$C^\bullet(G, \mathbb{Z}) \otimes X^\bullet(G \times H, A)^H.$$

As the complex $C^\bullet(G, \mathbb{Z})$ consists of flat \mathbb{Z} -modules, the result follows from (2.3.4). \square

Remark: The decomposition in the theorem is non-canonical: it cannot be made functorial in A . However, for a fixed H -module A , it can be made functorial in G . For a proof of these assertions see [98].

Exercise 1. Show that in the Hochschild-Serre spectral sequence, the edge morphisms

$$H^n(G/H, A^H) \longrightarrow H^n(G, A) \longrightarrow H^n(H, A)^{G/H}$$

are the inflation and the restriction.

Exercise 2. From the exact sequence $0 \rightarrow A \rightarrow \text{Ind}_G(A) \rightarrow A_1 \rightarrow 0$, one obtains the four term exact sequence

$$0 \longrightarrow A^H \longrightarrow \text{Ind}_G(A)^H \longrightarrow A_1^H \longrightarrow H^1(H, A) \longrightarrow 0$$

of G/H -modules, and hence a homomorphism

$$\delta^2 : H^p(G/H, H^1(H, A)) \longrightarrow H^{p+2}(G/H, A^H).$$

Show that δ^2 is the differential $d_2^{p,1} : E_2^{p,1} \rightarrow E_2^{p+2,0}$.

Exercise 3. Let $E(G, H, A)$ denote the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G/H, H^q(H, A)) \Rightarrow H^n(G, A).$$

If G' is an open subgroup of G and $H' = H \cap G'$, then we have two morphisms of spectral sequences $E(G, H, A) \xrightleftharpoons[\text{cor}]{\text{res}} E(G', H', A)$.

Exercise 4. (i) Assume that $H^q(H, A) = 0$ for $q > 1$. Then we have an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(G/H, H^0(H, A)) &\longrightarrow H^1(G, A) \longrightarrow H^0(G/H, H^1(H, A)) \\ &\longrightarrow H^2(G/H, H^0(H, A)) \longrightarrow H^2(G, A) \longrightarrow H^1(G/H, H^1(H, A)) \\ &\longrightarrow H^3(G/H, H^0(H, A)) \longrightarrow H^3(G, A) \longrightarrow H^2(G/H, H^1(H, A)) \longrightarrow \dots \end{aligned}$$

(ii) Assume that $cd\, G/H \leq 1$. Then we have exact sequences

$$0 \longrightarrow H^1(G/H, H^{n-1}(H, A)) \longrightarrow H^n(G, A) \longrightarrow H^n(H, A)^{G/H} \longrightarrow 0$$

for all $n \geq 1$ and all discrete G -modules A .

Exercise 5. If $A \times B \rightarrow C$ is a pairing of G -modules, then for the terms E_r^{pq} of the Hochschild-Serre spectral sequence, we have a cup-product

$$E_r^{pq}(A) \times E_r^{p'q'}(B) \xrightarrow{\cup} E_r^{p+p', q+q'}(C)$$

such that

$$d_r(\alpha \cup \beta) = (d_r \alpha) \cup \beta + (-1)^{p+q} \alpha \cup d_r(\beta).$$

Exercise 6. (Künneth-formula) Let G and H be profinite groups and let A be a discrete H -module, regarded as a $(G \times H)$ -module via trivial action of the group G . Show that there exist natural short exact sequences for all n

$$0 \rightarrow \bigoplus_{i+j=n} H^i(G, \mathbb{Z}) \otimes H^j(H, A) \rightarrow H^n(G \times H, A) \rightarrow \bigoplus_{i+j=n+1} \text{Tor}_1^{\mathbb{Z}}(H^i(G, \mathbb{Z}), H^j(H, A)) \rightarrow 0.$$

Moreover, these sequences split.

Hint: Consider the double complex $C^\bullet(G, \mathbb{Z}) \otimes X^\bullet(G \times H, A)^H$, which, by the proof of (2.4.6), calculates the cohomology $H^*(G \times H, A)$. Then use the result of the exercise at the end of §3.

Exercise 7. Let G and H be profinite groups and let F be a field, considered as a trivial module. Show that

$$H^n(G \times H, F) \cong \bigoplus_{i+j=n} H^i(G, F) \otimes_F H^j(H, F).$$

§5. The Tate Spectral Sequence

Another spectral sequence, due to *J. TATE*, which in a sense is dual to the Hochschild-Serre spectral sequence, is obtained as follows. Let G be a profinite group. For an abelian group M we set

$$M^* = \text{Hom}(M, \mathbb{Q}/\mathbb{Z}),$$

which is equipped with the natural G -structure $g(\phi)(m) = \phi(g^{-1}(m))$, if M is a G -module. Let A be a G -module. For two open subgroups $V \subseteq U$ of G , we have the maps

$$\text{cor}^* : H^n(U, A)^* \longrightarrow H^n(V, A)^*,$$

dual to the corestriction, by which the family $(H^n(U, A)^*)$ becomes a direct system of abelian groups.

(2.5.1) Definition. Let G be a profinite group, H a closed subgroup and A a G -module. Then, for every $n \geq 0$, we set

$$D_n(H, A) = \varinjlim_{U \supseteq H} H^n(U, A)^*,$$

where U runs through the open subgroups of G containing H .

If H is open, then $D_n(H, A) = H^n(H, A)^*$. If H is a normal closed subgroup of G , then it suffices to let U run through the normal open subgroups of G containing H . $D_n(H, A)$ is then a G/H -module. As the functors $X \mapsto X^*$ and \varinjlim are exact, we see that the family $(D_n(H, -))_{n \geq 0}$ is a contravariant $^*)$ δ -functor on $\text{Mod}(G)$. We set

$$D_n(A) = D_n(\{1\}, A).$$

(2.5.2) Definition. For a G -module A and an integer $n \geq 0$, we write

$$cd(G, A) \leq n$$

if $H^q(H, A) = 0$ for all $q > n$ and all closed subgroups H of G . (The letters “ cd ” stand for “cohomological dimension”.)

We call the following spectral sequence the **Tate spectral sequence**.

$^*)$ This means that the arrows in the usual exact cohomology sequence are reversed.

(2.5.3) Theorem. *If $cd(G, A) \leq n$, then for every closed normal subgroup H , there exists a first quadrant spectral sequence*

$$E_2^{pq} = H^p(G/H, D_{n-q}(H, A)) \Rightarrow H^{n-(p+q)}(G, A)^*.$$

In particular, for $H = 1$, we have a spectral sequence

$$E_2^{pq} = H^p(G, D_{n-q}(A)) \Rightarrow H^{n-(p+q)}(G, A)^*.$$

Proof: We consider the standard resolution $A \rightarrow X^\bullet$ of the G -module A and set $Z^i = \ker(X^i \rightarrow X^{i+1})$. Splitting up the complex

$$\tilde{X}^\bullet : 0 \rightarrow X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow \cdots \rightarrow X^{n-1} \rightarrow Z^n \rightarrow 0$$

into short exact sequences and, recalling that the G -modules X^i are cohomologically trivial, we obtain for $r > 0$

$$H^r(H, Z^n) \cong H^{r+1}(H, Z^{n-1}) \cong \cdots \cong H^{r+n}(H, A).$$

In particular, $H^r(H, Z^n) = 0$ for $r > 0$, since $cd(G, A) \leq n$, i.e. Z^n is a cohomologically trivial G -module.

Let U be an open normal subgroup of G . We apply to the complex \tilde{X}^\bullet first the functor $H^0(U, -)$ and then the functor $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$, and obtain a complex

$$0 \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow Y^2 \longrightarrow \cdots \longrightarrow Y^n \longrightarrow 0,$$

where $Y^q = H^0(U, \tilde{X}^{n-q})^*$ for $q \geq 0$, and in particular, $Y^0 = H^0(U, Z^n)^*$. This is a complex of G/U -modules, which by (1.8.2) and (1.8.5) are cohomologically trivial, since X^{n-q} and Z^n are cohomologically trivial G -modules. Since the functor $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact, we obtain

$$H^q(Y^\bullet) = H^q(H^0(U, \tilde{X}^{n-\bullet})^*) = H^q(H^0(U, \tilde{X}^{n-\bullet}))^* = H^{n-q}(U, A)^*$$

for all q .

For each Y^q , we consider the cochain complex $C^\bullet(G/U, Y^q)$ and obtain a double complex $C^{pq} = C^p(G/U, Y^q)$, $p, q \geq 0$. In order to make it anti-commutative, we multiply the differentials $(p, q) \rightarrow (p, q+1)$ by $(-1)^p$, cf. the proof of (2.4.1). As described in §2, this double complex yields a spectral sequence

$$E_2^{pq} \Rightarrow E^{p+q}.$$

We compute the initial terms $E_2^{pq} = H^p(H^q(C^{\bullet\bullet}))$. The functor $C^p(G/U, -)$ is exact (I §3, ex.1), so that

$$\begin{aligned} H^q(C^{\bullet\bullet}) &= H^q(C^\bullet(G/U, Y^\bullet)) = C^\bullet(G/U, H^q(Y^\bullet)) \\ &= C^\bullet(G/U, H^{n-q}(U, A)^*), \end{aligned}$$

hence

$$E_2^{pq} = H^p(C^\bullet(G/U, H^{n-q}(U, A)^*)) = H^p(G/U, H^{n-q}(U, A)^*).$$

As for the limit terms E^{p+q} , we note that for each $q \geq 0$ the complex

$$C^0(G/U, Y^q) \longrightarrow C^1(G/U, Y^q) \longrightarrow C^2(G/U, Y^q) \longrightarrow \cdots$$

is exact; its homology groups are the groups $H^p(G/U, Y^q)$, which are zero for $p > 0$, since Y^q is cohomologically trivial.

Therefore, setting $B^\bullet = \ker(C^{0\bullet} \rightarrow C^{1\bullet})$, we have by (2.2.4)

$$E^{p+q} = H^{p+q}(B^\bullet) = H^{p+q}(H^0(G/U, Y^\bullet)).$$

Since Y^q is cohomologically trivial, we have $\hat{H}^i(G/U, Y^q) = 0$ for all i . By (1.2.6), we obtain

$$\begin{aligned} H^0(G/U, Y^q) &= \text{Hom}((\tilde{X}^{n-q})^U, \mathbb{Q}/\mathbb{Z})^{G/U} = \text{Hom}(((\tilde{X}^{n-q})^U)_{G/U}, \mathbb{Q}/\mathbb{Z}) \\ &\xleftarrow[N_{G/U}^*]{} \text{Hom}(((\tilde{X}^{n-q})^U)^{G/U}, \mathbb{Q}/\mathbb{Z}) = H^0(G, \tilde{X}^{n-q})^* \end{aligned}$$

and consequently

$$\begin{aligned} E^{p+q} &= H^{p+q}(H^0(G, \tilde{X}^{n-\bullet})^*) = H^{p+q}(H^0(G, \tilde{X}^{n-\bullet}))^* \\ &= H^{n-(p+q)}(G, A)^*. \end{aligned}$$

We thus obtain a spectral sequence

$$E_2^{pq} = H^p(G/U, H^{n-q}(U, A)^*) \Rightarrow H^{n-(p+q)}(G, A)^*.$$

If we now let U run through the open subgroups of G containing H and take direct limits, we get the spectral sequence

$$E_2^{pq} = H^p(G/H, D_{n-q}(H, A)) \Rightarrow H^{n-(p+q)}(G, A)^*. \quad \square$$

Remark: Tate gave a proof of this spectral sequence using the cohomology groups in negative dimensions (see [230]), which we have avoided here.

The Tate spectral sequence

$$E(G, H, A) : E_2^{pq} = H^p(G/H, D_{n-q}(H, A)) \Rightarrow H^{n-(p+q)}(G, A)^*$$

is functorial in G and H in the following sense. Let G' be an open subgroup of G and $H' = H \cap G'$. We then have two morphisms of spectral sequences

$$(*) \quad E(G, H, A) \xrightleftharpoons[\text{cor } res^*]{\text{cor}^* \text{ res}} E(G', H', A),$$

such that the maps on the initial terms and the limit terms are given as follows.

The $E_2^{pq} \xrightleftharpoons{\text{cor}^* \text{ res}} E_2^{pq}$ are given as the composites of

$$H^p(G/H, D_{n-q}(H, A)) \xrightleftharpoons[\text{cor}]{\text{res}} H^p(G'/H', D_{n-q}(H, A))$$

$$\xrightleftharpoons[\text{res}^*]{\text{cor}^*} H^p(G'/H', D_{n-q}(H', A)),$$

where cor^* and res^* are induced by the direct limit of the maps

$$H^{n-q}(U, A)^* \xrightleftharpoons[\text{res}^*]{\text{cor}^*} H^{n-q}(U', A)^*,$$

U running through the open normal subgroups of G containing H and $U' = U \cap G'$. The maps on the limit terms are

$$H^{n-(p+q)}(G, A)^* \xrightleftharpoons[\text{res}^*]{\text{cor}^*} H^{n-(p+q)}(G', A)^*.$$

In particular, for $H = \{1\}$, we obtain for the edge morphisms a commutative diagram

$$\begin{array}{ccccc} H^p(G, D_n(A)) & \longrightarrow & H^{n-p}(G, A)^* & \longrightarrow & H^0(G, D_{n-p}(A)) \\ \text{res} \updownarrow \text{cor} & & \text{cor}^* \updownarrow \text{res}^* & & \text{incl} \updownarrow N_{G/G'} \\ H^p(G', D_n(A)) & \longrightarrow & H^{n-p}(G', A)^* & \longrightarrow & H^0(G', D_{n-p}(A)). \end{array}$$

All this results from the following consideration. Assume that H is an open subgroup of G . Then the spectral sequence $E(G, H, A)$ is obtained from the double complex

$$C^{pq}(G, H, A) = C^p(G/H, \tilde{X}^{n-q}(G, A)^{H*}),$$

where $\tilde{X}^i = X^i$ for $i = 0, \dots, n-1$ and $\tilde{X}^n = \ker(X^n \rightarrow X^{n+1})$ as in the proof of (2.5.3). We have on the one hand the homomorphisms

$$C^p(G/H, \tilde{X}^{n-q}(G, A)^{H*}) \xrightleftharpoons[\text{cor}]{\text{res}} C^p(G'/H', \tilde{X}^{n-q}(G', A)^{H'*}).$$

On the other hand, the duals of the maps

$$\tilde{X}^{n-q}(G, A) \xrightleftharpoons[\text{cor}]{\text{res}} \tilde{X}^{n-q}(G', A)$$

yield homomorphisms

$$\tilde{X}^{n-q}(G, A)^{H*} \xrightleftharpoons[\text{res}^*]{\text{cor}^*} \tilde{X}^{n-q}(G', A)^{H'*}. \quad *)$$

After composing, we obtain two morphisms of double complexes

$$C^{pq}(G, H, A) \xrightleftharpoons[\text{cor res}^*]{\text{cor}^* \text{res}} C^{pq}(G', H', A),$$

and these induce the above morphisms $(*)$ of spectral sequences. The effects on the cohomology groups mentioned are obtained in a straightforward manner by recalling our identifications

$$E_2^{pq} \cong H^p(G/H, H^{n-q}(H, A)^*), \quad E^{p+q} \cong H^n(G, A)^*.$$

We finish this section on spectral sequences by explicitly determining the edge morphisms $E^q \rightarrow E_2^{0,q}$ and $E_2^{p,0} \rightarrow E^p$ of the Tate spectral sequence.

*) The corestriction cor is defined on all of $\tilde{X}^{n-q}(G', A)$ after choosing a section $s: G/G' \rightarrow G$ of $G \rightarrow G/G'$. It induces a homomorphism $\text{cor}: \tilde{X}^{n-q}(G', A)^{H'} \rightarrow \tilde{X}^{n-q}(G, A)^H$ if this choice is taken in such a way that $s(c)\sigma = \tau_\sigma s(c\sigma)$ for all $c \in G' \setminus G$, $\sigma \in H$ and some $\tau_\sigma \in H'$.

(2.5.4) Theorem. *The edge morphism*

$$H^{n-q}(G, A)^* \longrightarrow \left(\varinjlim_{U \supseteq H} H^{n-q}(U, A)^* \right)^{G/H}$$

in the Tate spectral sequence is the direct limit of the maps cor^ , dual to the corestriction maps*

$$cor : H^{n-q}(U, A) \longrightarrow H^{n-q}(G, A).$$

Proof: We use the notation of the proof of (2.5.3). We may assume that H is open. The spectral sequence is obtained from the double complex

$$C^{pq} = C^p(G/H, Y^q),$$

where $Y^q = H^0(H, \tilde{X}^{n-q})^*$ and $\tilde{X}^i = X^i$ for $i < n$ and $\tilde{X}^n = Z^n$. Let $C^\bullet = \text{tot}(C^{\bullet\bullet})$, $B^\bullet = \ker(C^{0\bullet} \rightarrow C^{1\bullet})$ and $K^\bullet = C^{0\bullet}$. Then B^\bullet is a subcomplex of C^\bullet and $H^q(B^\bullet) = H^q(C^\bullet) = E^q$ by (2.2.4). Therefore the edge morphism $E^q \rightarrow E_2^{0,q}$ is the map

$$(1) \quad H^q(B^\bullet) \longrightarrow H^q(K^\bullet)$$

induced by the composite $B^\bullet \rightarrow C^\bullet \xrightarrow{\pi} K^\bullet$, which is the inclusion. Identifying $C^{0\bullet} = C^0(G/H, Y^\bullet)$ with Y^\bullet , this is the inclusion

$$(2) \quad (Y^\bullet)^{G/H} \hookrightarrow Y^\bullet.$$

The image of (1) is contained in

$$E_2^{0,q} = \ker(H^q(C^{0\bullet}) \xrightarrow{\partial} H^q(C^{1\bullet})) = H^q(Y^\bullet)^{G/H},$$

and the edge morphism becomes the map

$$edge : H^q((Y^\bullet)^{G/H}) \longrightarrow H^q(Y^\bullet)^{G/H},$$

induced by the inclusion (2). From what we have seen in the proof of (2.5.3), this map is identified with a map

$$H^q(G, A)^* \longrightarrow [H^q(H, A)^*]^{G/H}$$

as follows. We have the canonical isomorphism

$$\text{Hom}((\tilde{X}^{n-\bullet})^G, \mathbb{Q}/\mathbb{Z}) \cong H^0(G/H, Y^\bullet),$$

which is the same as the dual $N_{G/H}^*$ of

$$(\tilde{X}^{n-\bullet})^H \xrightarrow{N_{G/H}} (\tilde{X}^{n-\bullet})^G.$$

We obtain a commutative diagram

$$\begin{array}{ccc} H^q(H^0(G/H, Y^\bullet)) & \xrightarrow{edge} & H^q(Y^\bullet)^{G/H} \\ N_{G/H}^* \uparrow & & \parallel \\ H^q(H^0(G, \tilde{X}^{n-\bullet})^*) & \xrightarrow{N_{G/H}^*} & H^q(H^0(H, \tilde{X}^{n-\bullet})^*)^{G/H} \\ \parallel & & \parallel \\ H^{n-q}(G, A)^* & \xrightarrow{cor^*} & [H^{n-q}(H, A)^*]^{G/H} \end{array}$$

which identifies the edge homomorphism with the dual of the corestriction. \square

We now consider the edge morphism $E_2^{p,0} \rightarrow E^p$ in the Tate spectral sequence for the case $H = \{1\}$. It is a homomorphism

$$H^p(G, D_n(A)) \xrightarrow{\text{edge}} H^{n-p}(G, A)^*.$$

In particular, for $p = n$ and $A = \mathbb{Z}$, we have a canonical homomorphism, called the **trace map**,

$$\text{tr} : H^n(G, D_n(\mathbb{Z})) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

provided $cd(G, \mathbb{Z}) \leq n$. On the other hand, for every pair $V \subseteq U$ of open normal subgroups of G and each $i \geq 0$, we have the canonical pairing

$$H^i(V, A)^* \times A^U \longrightarrow H^i(V, \mathbb{Z})^*, \quad (\chi, a) \longmapsto f(x) = \chi(ax).$$

Taking first the direct limit over V and then over U , we obtain a canonical bilinear map $D_i(A) \times A \longrightarrow D_i(\mathbb{Z})$, which gives us a cup-product

$$H^p(G, D_i(A)) \times H^{n-p}(G, A) \xrightarrow{\cup} H^n(G, D_i(\mathbb{Z})).$$

For $i = n$ this yields, together with the map tr , a homomorphism

$$H^p(G, D_n(A)) \xrightarrow{\text{cup}} H^{n-p}(G, A)^*.$$

(2.5.5) Theorem. *Suppose that $cd(G, \mathbb{Z}) \leq n$ and let $A \in \text{Mod}(G)$ be finitely generated as a \mathbb{Z} -module. If $cd(G, A) \leq n$, then the two maps*

$$H^p(G, D_n(A)) \xrightarrow[\text{cup}]{\text{edge}} H^{n-p}(G, A)^*$$

coincide for all $p \in \mathbb{Z}$.

Proof: The Tate spectral sequence arises from the double complex

$$C^{pq}(A) = C^p(G/U, H^0(U, \tilde{X}^{n-q}(G, A))^*)$$

and the application of $\lim_{\rightarrow U}$. We have a canonical pairing

$$\varphi : (\tilde{X}^{n-q}(G, A)^U)^* \times A \longrightarrow (\tilde{X}^{n-q}(G, \mathbb{Z})^U)^*, \quad (\chi, a) \longmapsto \varphi(\chi, a) = f,$$

where $f : \tilde{X}^{n-q}(G, \mathbb{Z})^U \longrightarrow \mathbb{Q}/\mathbb{Z}$ is defined by $f(x) = \chi(ax)$.

If $z(\sigma_0, \dots, \sigma_{p-j})$ is a $(p-j)$ -cochain with coefficients in $(\tilde{X}^{n-q}(G, A)^U)^*$ and $t(\sigma_0, \dots, \sigma_j)$ is an j -cocycle in $Z^j(G, A)$, then

$$(z \cup t)(\sigma_0, \dots, \sigma_p) = \varphi(z(\sigma_0, \dots, \sigma_{p-j}), t(\sigma_{p-j}, \dots, \sigma_p))$$

is a p -cochain with coefficients in $(\tilde{X}^{n-q}(G, \mathbb{Z})^U)^*$. Thus we get a map

$$C^{p-j,q}(A) \times Z^j(G, A) \longrightarrow C^{p,q}(\mathbb{Z}),$$

i.e. for fixed $t \in Z^j(G, A)$, we have a morphism of double complexes

$$\cup t : C^{\bullet-j, \bullet}(A) \longrightarrow C^{\bullet, \bullet}(\mathbb{Z})$$

of degree $(j, 0)$ and hence a transformation of the associated edge morphisms. Applying \varinjlim_U , we obtain a map

$$\cup t : H^{n-j}(G, D_n(A)) \longrightarrow H^n(G, D_n(\mathbb{Z})),$$

which only depends on the cohomology class of the cocycle t . As the map tr is defined via the edge morphism, we obtain a commutative diagram

$$\begin{array}{ccc} H^{n-j}(G, D_n(A)) \times H^j(G, A) & \xrightarrow{\cup} & H^n(G, D_n(\mathbb{Z})) \\ \text{edge} \downarrow & \parallel & \downarrow tr \\ H^j(G, A)^* \times H^j(G, A) & \longrightarrow & \mathbb{Q}/\mathbb{Z}, \end{array}$$

where the upper arrow is the cup-product with respect to the pairing

$$D_n(A) \times A \longrightarrow D_n(\mathbb{Z}),$$

and the lower arrow is the evaluation map $(\chi, t) \mapsto \chi(t)$. From this diagram we get the commutative diagram

$$\begin{array}{ccc} H^{n-j}(G, D_n(A)) & \xrightarrow{cup} & \text{Hom}(H^j(G, A), H^n(G, D_n(\mathbb{Z}))) \\ \text{edge} \downarrow & & \downarrow tr \\ H^j(G, A)^* & \xlongequal{\quad} & \text{Hom}(H^j(G, A), \mathbb{Q}/\mathbb{Z}), \end{array}$$

which shows that the edge morphism coincides with the composite $tr \circ cup$. Setting $j = n - p$, we obtain the assertion of the theorem. \square

We conclude this section with a vanishing criterion for the terms $D_i(A)$.

(2.5.6) Lemma. *Suppose that $D_i(\mathbb{Z}) = 0$. Assume that $A \in \text{Mod}(G)$ is finitely generated as a \mathbb{Z} -module and has torsion divisible only by prime numbers ℓ for which $D_{i+1}(\mathbb{Z})$ is ℓ -divisible. Then $D_i(A) = 0$.*

Proof: If A is finitely generated as a \mathbb{Z} -module, then $A = A^U$ for some open subgroup U of G . Since in the definition of the D_i the group G may be replaced by U , we may assume that A is a trivial G -module. If A is torsion-free, then $A \cong \mathbb{Z}^n$ as a G -module, hence $D_i(A) \cong D_i(\mathbb{Z})^n = 0$. It remains to consider the case $A = \mathbb{Z}/m\mathbb{Z}$. From the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0$, we obtain the exact sequence

$$D_{i+1}(\mathbb{Z}) \xrightarrow{m} D_{i+1}(\mathbb{Z}) \longrightarrow D_i(\mathbb{Z}/m\mathbb{Z}) \longrightarrow D_i(\mathbb{Z}) = 0,$$

hence our assumptions imply $D_i(\mathbb{Z}/m\mathbb{Z}) = 0$. \square

Exercise 1. Compute the Tate spectral sequence for the case $G \cong \hat{\mathbb{Z}}$.

Exercise 2. If $0 < cd(G, A) \leq n$, then, for all open normal subgroups U of G , we have

$$D_n(A)^U = H^n(U, A)^*.$$

Hint: Use (3.3.11).

Exercise 3. Let G be a profinite group and suppose we are given a direct limit $A = \varinjlim A_\alpha$ of G -modules. Show that there exist natural homomorphisms

$$D_i(G, A) \longrightarrow \varprojlim D_i(A_\alpha)$$

for all i . What are the images of these maps?

§6. Derived Functors

We have constructed the cohomology groups $H^n(G, A)$ in a direct and natural way from the diagram

$$\cdots \rightrightarrows G \times G \times G \rightrightarrows G \times G \rightrightarrows G.$$

The advantage of this definition is that it is concrete, elementary and down to earth. Its disadvantage is that it is difficult to generalize and to get deeper insights. There is another much more general definition of cohomology which we describe now.

We have explained in I§3 the notion of δ -functor $H = (H^n)_{n \geq 0}$ between abelian categories \mathcal{A} and \mathcal{A}' . It is a family of additive functors $H^n : \mathcal{A} \rightarrow \mathcal{A}'$, which turns a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{A} functorially into a long exact sequence

$$\cdots \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \xrightarrow{\delta} H^{n+1}(A) \longrightarrow \cdots$$

in \mathcal{A}' . A morphism between two δ -functors H and H' from \mathcal{A} to \mathcal{A}' is a system $f = (f^n)_{n \geq 0}$ of functorial morphisms

$$f^n : H^n \longrightarrow H'^n,$$

which commute with δ . That is, for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in \mathcal{A} , the diagram

$$\begin{array}{ccc} H^n(C) & \xrightarrow{\delta} & H^{n+1}(A) \\ f^n(C) \downarrow & & \downarrow f^{n+1}(A) \\ H^n(C) & \xrightarrow{\delta} & H^{n+1}(A) \end{array}$$

commutes.

(2.6.1) Definition. A δ -functor $H = (H^n)_{n \geq 0}$ from \mathcal{A} to \mathcal{A}' is called **universal** if, for every other δ -functor $H' = (H'^n)_{n \geq 0}$ from \mathcal{A} to \mathcal{A}' , each morphism $f^0 : H^0 \rightarrow H'^0$ of functors extends uniquely to a morphism $f : H \rightarrow H'$ of δ -functors.

We have the following criterion for the universality of a δ -functor. An additive functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ is called **effaceable** if, for each object A in \mathcal{A} , there is a monomorphism $u : A \rightarrow I$ in \mathcal{A} such that $F(u) = 0$.

(2.6.2) Theorem. A δ -functor $H = (H^n)_{n \geq 0}$ from \mathcal{A} to \mathcal{A}' is universal if the functors H^n are effaceable for $n > 0$.

For the proof we refer to [66], chap.I. The idea is the following. Let $H' = (H'^n)_{n \geq 0}$ be an arbitrary δ -functor from \mathcal{A} to \mathcal{A}' and let $f^0 : H^0 \rightarrow H'^0$ be a morphism of functors. Assume that we have shown that there exists a uniquely determined morphism of functors $f^i : H^i \rightarrow H'^i$, $i = 1, \dots, n$, which commute with δ . Let $A \in \mathcal{A}$ and let $0 \rightarrow A \xrightarrow{u} I \rightarrow J \rightarrow 0$ be an exact sequence such that $H^{n+1}(u) = 0$. Then we obtain a uniquely determined morphism $f^{n+1} : H^{n+1}(A) \rightarrow H'^{n+1}(A)$ using the exact commutative diagram

$$\begin{array}{ccccccc} H^n(I) & \longrightarrow & H^n(J) & \xrightarrow{\delta} & H^{n+1}(A) & \longrightarrow & 0 \\ \downarrow f^n & & \downarrow f^n & & \downarrow f^{n+1} & & \\ H^n(I) & \longrightarrow & H^n(J) & \xrightarrow{\delta} & H^{n+1}(A) & & \end{array}$$

It remains to show that f^{n+1} is functorial and commutes with δ .

If G is a profinite group, then the functors $H^n(G, -)$, $n > 0$, are effaceable, since every G -module A embeds into the induced G -module $\text{Ind}_G(A)$, which is acyclic, i.e. has trivial cohomology. We therefore have the

(2.6.3) Theorem. The δ -functor $(H^n(G, -))_{n \geq 0}$ is universal.

With this theorem many proofs of isomorphism, uniqueness etc. are obtained automatically.

Example 1. Let H be a closed normal subgroup. We then have the following proof of Shapiro's lemma (1.6.4)

$$sh : H^n(G, \text{Ind}_G^H(A)) \cong H^n(H, A).$$

Noting that Ind_G^H is exact and that $\text{Ind}_G^H(\text{Ind}_H(B)) = \text{Ind}_G(B)$, we see that $(H^n(G, \text{Ind}_G^H(-)))$ and $(H^n(H, -))$ are effaceable δ -functors on $\text{Mod}(H)$, and are thus universal. They are functorially isomorphic in dimension $n = 0$. Hence they are isomorphic as δ -functors. By the uniqueness assertion, the isomorphism is the composite

$$H^n(G, \text{Ind}_G^H(A)) \xrightarrow{\text{res}} H^n(H, \text{Ind}_G^H(A)) \xrightarrow{\pi_*} H^n(H, A),$$

where π_* is induced by $\pi : \text{Ind}_G^H(A) \rightarrow A$, $f \mapsto f(1)$. In fact, this composite is a morphism of δ -functors, which in dimension $n = 0$ coincides with the initial isomorphism.

Example 2. For every G -module A , we have the commutative diagram

$$\begin{array}{ccc} H^n(G, \text{Ind}_G^H(A)) & & \\ \swarrow sh & & \searrow \nu_* \\ H^n(H, A) & \xrightarrow{cor} & H^n(G, A) \end{array}$$

as claimed in (1.6.5). In fact, ν_* and $cor \circ sh$ are morphisms of universal δ -functors, which coincide in dimension $n = 0$ by the definition of sh , cor , ν_* . Hence they coincide for all $n \geq 0$ by the uniqueness assertion of (2.6.1).

If $F : \mathcal{A} \rightarrow \mathcal{A}'$ is an additive functor, there exists up to a canonical isomorphism at most one universal δ -functor H from \mathcal{A} to \mathcal{A}' with $H^0 = F$. This δ -functor, if it exists, is then called the **right derived functor** of F and is denoted by $R^\bullet F = (R^n F)_{n \geq 0}$. Obviously, it is defined up to canonical isomorphism. The question is, when does it exist?

By theorem (2.6.3), the universal δ -functor $H^\bullet(G, -)$ on $\text{Mod}(G)$ is the right derived functor

$$H^\bullet(G, -) = R^\bullet \Gamma$$

of the functor

$$\Gamma(-) = H^0(G, -) : \text{Mod}(G) \longrightarrow \mathcal{A}b, \quad A \longmapsto A^G.$$

Suppose \mathcal{M} is a full abelian subcategory of $\text{Mod}(G)$ which has the property that for a discrete G -module M , the induced module $\text{Ind}_G(M)$ is also in \mathcal{M} . Then the same reasoning as above shows that the restriction of $H^\bullet(G, -)$ to \mathcal{M} is the right derived functor $R^\bullet \Gamma$ of the functor $\Gamma(-) : \mathcal{M} \longrightarrow \mathcal{A}b$, $A \mapsto A^G$. Examples of such subcategories \mathcal{M} are the category $\text{Mod}_t(G)$ of discrete G -modules which are torsion groups, or the category $\text{Mod}_{(p)}(G)$ of discrete G -modules which are p -torsion groups, where p is a prime number.

Recall that an additive functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ is called **left exact** if for each exact sequence $0 \rightarrow A \rightarrow B \rightarrow C$ the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is also exact.

The left exactness of F is clearly a necessary condition for the existence of the right derived functor $R^\bullet F$, since H^0 is left exact. This condition is already sufficient if \mathcal{A} has *sufficiently many injectives*:

An object A in \mathcal{A} is **injective** if for every monomorphism $B \rightarrow C$ in \mathcal{A} the map $\text{Hom}(C, A) \rightarrow \text{Hom}(B, A)$ is surjective. \mathcal{A} is said to have **sufficiently many injectives** if, for any object A , there exists a monomorphism $A \rightarrow I$ into an injective object.

(2.6.4) Theorem. *Let \mathcal{A} have sufficiently many injectives. Then for each left exact additive functor $F : \mathcal{A} \rightarrow \mathcal{A}'$, the right derived functor $R^\bullet F = (R^n F)_{n \geq 0}$ exists.*

For the proof we refer to [21], chap. V, §3, but we explain the idea of it. Since \mathcal{A} has sufficiently many injectives, each object $A \in \mathcal{A}$ has an **injective resolution**, i.e. there is an exact complex

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

with injective objects I^n in \mathcal{A} . We apply the functor F and get a complex

$$0 \longrightarrow F(A) \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \dots$$

We define

$$R^n F(A) = H^n(F(I^\bullet)), \quad n \geq 0;$$

in particular, $R^0 F(A) = \ker(F(I^0) \rightarrow F(I^1)) = F(A)$.

The independence of this definition from the injective resolution chosen is seen as follows. If $A \rightarrow I^\bullet$ and $A' \rightarrow I'^\bullet$ are injective resolutions of A and A' , then, because of the injectivity property of the I^n , every morphism $u : A \rightarrow A'$ extends to a morphism of complexes

$$\begin{array}{ccc} A & \longrightarrow & I^\bullet \\ \downarrow u & & \downarrow u \\ A' & \longrightarrow & I'^\bullet, \end{array}$$

and every two such extensions are homotopic (cf. I §3, exercise 6). This means that the induced maps from $F(I^\bullet)$ to $F(I'^\bullet)$ are homotopic, hence induce the same homomorphism $H^n(F(I^\bullet)) \rightarrow H^n(F(I'^\bullet))$ on the homology. In particular, if $A = A'$, we find extensions $u : I^\bullet \rightarrow I'^\bullet$, $v : I'^\bullet \rightarrow I^\bullet$, such that $u \circ v$ and $v \circ u$ are homotopic to the identity, hence induce mutually inverse isomorphisms $H^n(F(I^\bullet)) \xrightarrow{\sim} H^n(F(I'^\bullet))$. This shows the independence.

The property of being a δ -functor is seen as follows. Any exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} may be extended to an exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I_A^\bullet & \longrightarrow & I_B^\bullet & \longrightarrow & I_C^\bullet \longrightarrow 0
\end{array}$$

of injective resolutions. Since I_A^n is injective, all exact sequences

$$0 \longrightarrow I_A^n \longrightarrow I_B^n \longrightarrow I_C^n \longrightarrow 0$$

split and therefore

$$0 \longrightarrow F(I_A^n) \longrightarrow F(I_B^n) \longrightarrow F(I_C^n) \longrightarrow 0$$

remains exact. The exact sequence

$$0 \longrightarrow F(I_A^\bullet) \longrightarrow F(I_B^\bullet) \longrightarrow F(I_C^\bullet) \longrightarrow 0$$

of complexes yields, in the same way as in I §2, a long exact sequence

$$\cdots \longrightarrow R^n F(A) \longrightarrow R^n F(B) \longrightarrow R^n F(C) \xrightarrow{\delta} R^{n+1} F(A) \longrightarrow \cdots$$

We have obtained a δ -functor $R^\bullet F = (R^n F)_{n \geq 0}$. For an injective object I , we have $R^n F(I) = 0$ for $n > 0$ since $0 \rightarrow I \xrightarrow{id} I \rightarrow 0$ is an injective resolution of I . Since \mathcal{A} has sufficiently many injectives, the $R^n F$, $n > 0$, are effaceable, hence $R^\bullet F$ is universal.

(2.6.5) Lemma. *If G is a profinite group, then the category $Mod(G)$ of discrete G -modules has sufficiently many injectives.*

Proof: To every abstract G -module M we can associate the submodule

$$M^\delta := \bigcup_{U \subseteq G} M^U,$$

where U runs through the open subgroups of G . If we endow M with the discrete topology, then M^δ is the maximal submodule on which G acts continuously (compare with the remark after (1.1.8)). One easily verifies that every G -homomorphism from a discrete G -module N to M factors through M^δ . In particular, we see that the discrete module I^δ is an injective object in $Mod(G)$ provided the (abstract) G -module I is injective. The category of abstract G -modules has sufficiently many injective objects (see [79], chap. IV: it is canonically equivalent to the category of modules over the group ring $\mathbb{Z}[G]$). Therefore we can embed a given discrete G -module M into an injective abstract module I and then M is automatically contained in the injective discrete module I^δ . \square

The Hochschild-Serre spectral sequence (2.4.1) becomes a special case of the following general result.

(2.6.6) Theorem. *Let \mathcal{A} and \mathcal{A}' be abelian categories with sufficiently many injectives and let \mathcal{A}'' be another abelian category. Let*

$$\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}''$$

be left exact additive functors. Assume that F maps injective objects from \mathcal{A} to F' -acyclic objects, i.e. those annihilated by $R^n F'$ for $n > 0$. Then there is a cohomological spectral sequence

$$E_2^{pq} = R^p F'(R^q F(A)) \Rightarrow R^{p+q}(F' \circ F)(A),$$

*which is called the **Grothendieck spectral sequence**.*

This spectral sequence is obtained as follows. There exists a homomorphism of the complex $F(I^\bullet)$ into a double complex of \mathcal{A}' -injective objects $I^{\bullet\bullet}$ which induces injective resolutions of all groups $F(I^q)$ and also for all cocycle, coboundary and cohomology groups of the complex $F(I^\bullet)$ (a so-called *Cartan-Eilenberg resolution*, cf. [21], chap.XVII). Applying to the double complex $I^{\bullet\bullet} = (I^{pq})_{p,q \geq 0}$ the functor F' , we obtain a double complex

$$(A^{pq})_{p,q \geq 0} = (F'(I^{pq}))_{p,q \geq 0}.$$

The spectral sequence $E^{pq} \Rightarrow E^n$ associated with this double complex is the maintained spectral sequence

$$E_2^{pq} = R^p F'(R^q F(A)) \Rightarrow R^n(F' \circ F)(A).$$

For the proof we refer to [66] and [21].

If G is a profinite group and H a closed subgroup, then we have the additive left exact functors

$$\begin{aligned} F = H^0(H, -) : \text{Mod}(G) &\rightarrow \text{Mod}(G/H), & A &\mapsto A^H, \\ F' = H^0(G/H, -) : \text{Mod}(G/H) &\rightarrow \mathcal{A}b, & B &\mapsto B^{G/H}, \\ F' \circ F = H^0(G, -) : \text{Mod}(G) &\rightarrow \mathcal{A}b, & A &\mapsto A^G. \end{aligned}$$

In this case, the Grothendieck spectral sequence

$$E_2^{pq} = R^p F'(R^q F(A)) \Rightarrow R^{p+q}(F' \circ F)(A)$$

coincides with the Hochschild-Serre spectral sequence. We easily see that the E_2 -terms and the limit terms are the same, since

$$R^q F = H^q(H, -), \quad R^p F' = H^p(G/H, -), \quad R^n(F' \circ F) = H^n(G, -).$$

That both spectral sequences actually coincide follows from the fact that the functor ‘homogeneous cochain complex’ is a ‘resolving functor’, see [66].

So far we have dealt with the right derivation of a left exact, covariant functor. In the later applications it will be often useful to work with certain modifications of this concept.

Assume that we are given abelian categories \mathcal{B} and \mathcal{B}' . We say that a family $H = (H_n)_{n \in \mathbb{Z}}$ of functors $H_n : \mathcal{B} \rightarrow \mathcal{B}'$ is a **homological δ -functor** if the family $K = (K^n)_{n \in \mathbb{Z}}$, defined by $K^n := H_{-n}$, is a (cohomological) δ -functor as defined before. The following notions and statements are dual to those given before for cohomological δ -functors and we leave their verification to the reader. We also note that, up to the obvious modifications, one can also work with contravariant functors.

We say that a homological δ -functor $H = (H_n)_{n \geq 0}$ is **universal** if, for every other homological δ -functor $H' = (H'_n)_{n \geq 0}$, each morphism $f_0 : H'_0 \rightarrow H_0$ of functors extends uniquely to a morphism $f : H' \rightarrow H$ of homological δ -functors. A functor $G : \mathcal{B} \rightarrow \mathcal{B}'$ is called **coeffaceable** if, for every object $B \in \mathcal{B}$, there is an epimorphism $\phi : P \rightarrow B$ with $G(\phi) = 0$. A homological δ -functor $H = (H_n)_{n \geq 0} : \mathcal{B} \rightarrow \mathcal{B}'$ is universal if the functors H_n are coeffaceable for $n > 0$. If $G : \mathcal{B} \rightarrow \mathcal{B}'$ is an additive functor, there exists up to canonical isomorphism at most one universal homological δ -functor H from \mathcal{B} to \mathcal{B}' with $H_0 = G$. This δ -functor, if it exists, is then called the **left derived functor** of G and is denoted by $L_\bullet G = (L_n G)_{n \geq 0}$.

An object P of \mathcal{B} is called **projective** if for every epimorphism $A \rightarrow B$ in \mathcal{B} the map $\text{Hom}(P, A) \rightarrow \text{Hom}(P, B)$ is surjective. We say that \mathcal{B} has **sufficiently many projectives** if for every object B there exists an epimorphism $P \rightarrow B$ with projective P .

The left derived functor of $G : \mathcal{B} \rightarrow \mathcal{B}'$ exists if G is right exact and \mathcal{B} has sufficiently many projectives.

Now we introduce the **homology of profinite groups**. The homology groups are compact abelian groups and they have compact G -modules as coefficients. In order to prevent confusion, we use the notation $\mathcal{D} = \mathcal{D}(G)$ for the category of discrete G -modules, which so far has been denoted by $\text{Mod}(G)$. The category of compact G -modules will be denoted by $\mathcal{C} = \mathcal{C}(G)$.

(2.6.7) Definition. Let G be a profinite group and let $A \in \mathcal{C}$ be a compact G -module. The **cofixed module** (or **module of coinvariants**) A_G of A is the largest Hausdorff quotient of A on which G acts trivially, i.e. A_G is the quotient of A by the closed subgroup generated by the elements $(ga - a)$, $g \in G$, $a \in A$.

We denote the category of compact abelian groups by $\mathcal{A}b^c$ and in order to stress the difference we write $\mathcal{A}b^d$ for the category of (discrete) abelian groups. One easily verifies that $(-)_G$ is a right exact functor from \mathcal{C} to $\mathcal{A}b^c$. Furthermore, the category \mathcal{C} is dual to \mathcal{D} by Pontryagin duality and therefore it has sufficiently many projectives by (2.6.5).

(2.6.8) Definition. For a profinite group G and a compact module A , the **homology groups** are defined as the left derivatives of the cofixed-module functor

$$H_n(G, A) := L_n(-)_G(A).$$

In particular, we have $H_0(G, A) = A_G$, and if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence in \mathcal{C} , then we get a long exact homology sequence

$$\cdots \rightarrow H_{n+1}(G, C) \rightarrow H_n(G, A) \rightarrow H_n(G, B) \rightarrow H_n(G, C) \rightarrow \cdots.$$

The homology theory for profinite groups is dual to the cohomology theory: every cohomological result has its homological analogue. Fortunately we do not have to prove everything twice because of the following

(2.6.9) Theorem. Let G be a profinite group and A be a compact G -module. Then there are functorial isomorphisms for all $i \geq 0$

$$H_i(G, A)^\vee \cong H^i(G, A^\vee),$$

where $^\vee$ denotes the Pontryagin dual.

Proof: The theorem is true for $i = 0$ by the definition of the fixed and the cofixed module. Now the following diagram of categories and functors is commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\vee} & \mathcal{D} \\ \downarrow (-)_G & & \downarrow (-)^G \\ \mathcal{A}b^c & \xrightarrow{\vee} & \mathcal{A}b^d. \end{array}$$

Furthermore, Pontryagin duality is an exact, contravariant functor that transfers \mathcal{C} -projectives to \mathcal{D} -injectives. The statement of the theorem now follows from the universal property for the derived functors. \square

We see from the last theorem that, in principle, one can avoid the use of homology groups, working only with cohomology. Indeed, the decision whether to work with cohomology or homology, is more or less a question of personal taste.

We finish this section with a spectral sequence for Ext-groups. Let R be a ring (with unit). Then the functor $\text{Hom}_R(-, -)$ is a bifunctor from the category of R -modules to abelian groups, which is contravariant in the first and covariant in the second variable. Its derivations

$$\text{Ext}_R^i(-, -)$$

may likewise be computed using projective resolutions of the first, or using injective resolutions of the second, variable. (See [79] or any textbook about homological algebra for this basic fact.) The Hom-acyclic objects in the first resp. second variable are projective resp. injective R -modules. The following spectral sequence connects the Ext-groups for modules over a group ring $R[G]$ with that over R .

(2.6.10) Theorem. *Let R be a commutative ring with unit, let G be a finite group and let M and N be $R[G]$ -modules. Then there exists a natural spectral sequence*

$$E_2^{pq} = H^p(G, \text{Ext}_R^q(M, N)) \Rightarrow \text{Ext}_{R[G]}^{p+q}(M, N).$$

Proof: First we observe that

$$\text{Hom}_{R[G]}(M, N) \cong \text{Hom}_R(M, N)^G,$$

thus the left exact functor $\text{Hom}_{R[G]}(M, -)$ is the composition of the left exact functors $\text{Hom}_R(M, -)$ and $H^0(G, -)$. Now assume that N is injective. Then N is a direct summand of $\text{Ind}_G N$. By (1.3.6)(iii), the G -module $\text{Hom}_R(M, \text{Ind}_G N)$ is induced. Thus $\text{Hom}_R(M, N)$ is cohomologically trivial because it is a direct summand of an induced module. Therefore theorem (2.6.6) gives us the desired spectral sequence. \square

(2.6.11) Corollary. *Let G be a finite group whose order is invertible in the commutative ring R . Then an $R[G]$ -module M is projective if and only if it is R -projective.*

Proof: A free $R[G]$ -module is free as an R -module. If M is a projective $R[G]$ -module, then it is a direct summand of a free $R[G]$ -module and therefore also projective as an R -module. In order to show the other implication, assume that M is an $R[G]$ -module which is R -projective. The cohomology groups $H^i(G, M)$ are R -modules and annihilated by $\#G$ for $i \geq 1$ by (1.6.1). Hence they are trivial for $i \geq 1$ and for an arbitrary $R[G]$ -module N , the spectral sequence (2.6.10) degenerates to a sequence of isomorphisms

$$\text{Ext}_{R[G]}^i(M, N) \longrightarrow \text{Ext}_R^i(M, N)^G = 0$$

for $i \geq 1$. Hence M is $R[G]$ -projective. \square

We obtain the following result, which is known as **Maschke's Theorem**.

(2.6.12) Corollary (*MASCHKE*). *Let G be a finite group and let K be a field whose characteristic does not divide the order of G . Then the category of $K[G]$ -modules is semi-simple.*

Exercise 1. Define for an abstract group G the homology with values in a G -module as the left derivation of the cofixed module functor on the category of abstract G -modules. Assume that G is a finite group and let A be a finite G -module. Then we can view G as an abstract group and A as an abstract module or we can view G as a profinite group and A as a compact module. Show that the corresponding homology groups are the same and that both coincide with the homology groups introduced in I §9.

Exercise 2. Let R be a commutative ring with unit, let G be a finite group and let M and N be $R[G]$ -modules such that M is R -projective and N is cohomologically trivial. Show that

$$\mathrm{Ext}_{R[G]}^1(M, N) = 0.$$

§7. Continuous Cochain Cohomology

In chapter I, we started considering the simplicial diagram

$$\cdots \rightrightarrows G \times G \times G \rightrightarrows G \times G \rightrightarrows G,$$

which gave rise to a standard complex $C^\bullet(G, A)$, and defined the cohomology of the profinite group G with values in the discrete module A . In the last section we learned that this cohomology can be characterized by a universal property which, in particular, explains its good functorial behaviour. This functorial approach via universal constructions is extremely useful and clarifies the principles behind classical homological notions.

Sometimes, however, the simplicial approach seems to reach further. For instance, the categories of finite, compact, locally compact or of all topological G -modules lack the existence of sufficiently many “good” objects, and the existence of derived functors on these categories is not guaranteed. But we can still define cohomology using the standard complex.

Let G be a profinite group and let A be any *topological* G -module (see I §1). We form the **continuous homogeneous cochain complex of G with coefficients in A** , denoted by $C_{cts}^\bullet(G, A)$, by taking the G -invariants of the continuous standard resolution $X_{cts}^\bullet(G, A)$, which is defined in exactly the same way as in I §2 for discrete modules.

(2.7.1) Definition. We call the n -th cohomology group of the complex $C_{cts}^\bullet(G, A)$ the n -th **continuous cochain cohomology group** of G with coefficients in A . We denote this group by $H_{cts}^n(G, A)$.

If A is discrete, then clearly $H_{cts}^n(G, A) = H^n(G, A)$, but for an arbitrary topological G -module A , the right-hand side of the equation is not defined.

The same arguments as used in I §3 show the following

(2.7.2) Lemma. Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of topological G -modules such that the topology of A is induced by that of B and such that β has a continuous section (just a continuous map, not necessarily a homomorphism). Then there exist canonical boundary homomorphisms

$$\delta: H_{cts}^n(G, C) \longrightarrow H_{cts}^{n+1}(G, A)$$

and we obtain an exact sequence

$$\cdots \rightarrow H_{cts}^n(G, A) \rightarrow H_{cts}^n(G, B) \rightarrow H_{cts}^n(G, C) \xrightarrow{\delta} H_{cts}^{n+1}(G, A) \rightarrow \cdots$$

which begins $0 \rightarrow A^G \rightarrow B^G \rightarrow C^G \xrightarrow{\delta} H_{cts}^1(G, A)$.

Note that we can apply this lemma in the particular case that A is an open submodule of B and $C = B/A$ is the quotient module with the quotient topology, which is discrete.

Suppose

$$\mathcal{B}: A \times B \longrightarrow C$$

is a continuous G -pairing, i.e. a continuous biadditive map such that $\sigma(a \cdot b) = (\sigma a \cdot \sigma b)$ for $a \in A, b \in B, \sigma \in G$, where $a \cdot b$ denotes $\mathcal{B}(a, b)$. Then \mathcal{B} induces biadditive maps

$$C_{cts}^p(G, A) \times C_{cts}^q(G, B) \longrightarrow C_{cts}^{p+q}(G, C)$$

via the formula

$$(f \cup g)(\sigma_0, \dots, \sigma_{p+q}) = f(\sigma_0, \dots, \sigma_p) \cdot g(\sigma_p, \dots, \sigma_{p+q}).$$

This cup-product of cochains satisfies the identity

$$\partial(f \cup g) = (\partial f) \cup g + (-1)^p f \cup (\partial g)$$

and consequently induces pairings

$$H_{cts}^p(G, A) \times H_{cts}^q(G, B) \longrightarrow H_{cts}^{p+q}(G, C),$$

which enjoy the same properties as the cup-products considered in I §4.

The maps *inf*, *res*, *cor* with respect to a change of groups are defined and have the same properties as the maps introduced in I §5. (Note that we exclusively made calculations there on the chain level!)

In some important special situations, we can still relate the continuous cochain cohomology to derived functors. For this we need some preparations.

Let \mathcal{A} be an abelian category and let $\mathcal{A}^{\mathbb{N}}$ be the category of inverse systems in \mathcal{A} indexed by the set \mathbb{N} of natural numbers with the natural order. Thus objects in $\mathcal{A}^{\mathbb{N}}$ are inverse systems

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_n} A_n \longrightarrow \cdots \longrightarrow A_1 \xrightarrow{d_0} A_0$$

in \mathcal{A} and morphisms are commutative diagrams

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & \cdots \longrightarrow A_1 \longrightarrow A_0 \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_1 & \downarrow f_0 \\ \cdots & \longrightarrow & B_{n+1} & \longrightarrow & B_n & \longrightarrow & \cdots \longrightarrow B_1 \longrightarrow B_0. \end{array}$$

The category \mathcal{A} is abelian, with kernels and cokernels taken componentwise. One can show that $\mathcal{A}^{\mathbb{N}}$ has sufficiently many injectives, provided that this is the case for \mathcal{A} . Moreover, an object (A_n, d_n) in $\mathcal{A}^{\mathbb{N}}$ is injective if and only if all A_n are injective and all d_n are split surjections (see [96], prop. 1.1). A left exact functor

$$F: \mathcal{A} \longrightarrow \mathcal{B}$$

into another abelian category induces a functor $F^{\mathbb{N}}: \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{B}^{\mathbb{N}}$ in the obvious way. If \mathcal{A} has sufficiently many injectives, then the right derivatives $R^i F^{\mathbb{N}}$ exist and we have (loc.cit. prop. 1.2)

$$R^i F^{\mathbb{N}} = (R^i F)^{\mathbb{N}} \text{ for } i \geq 0, \text{ i.e. } R^i F^{\mathbb{N}}(A_n, d_n) = (R^i F A_n, R^i F(d_n)).$$

If inverse limits over \mathbb{N} exist in \mathcal{B} , we can define the functor

$$\varprojlim_n F: \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{B},$$

which is by definition the composition of the functor $F^{\mathbb{N}}$ with the limit functor $\varprojlim: \mathcal{B}^{\mathbb{N}} \longrightarrow \mathcal{B}$.

Assume that \mathcal{A} and \mathcal{B} have sufficiently many injectives, F is left exact and $F^{\mathbb{N}}$ sends injectives to \varprojlim -acyclic objects. Then, by (2.6.6), we have a spectral sequence

$$E_2^{pq} = \varprojlim_n^p (R^q F A_n) \Rightarrow R^{p+q}(\varprojlim_n F)(A_n, d_n),$$

where \varprojlim^\bullet denotes the right derived functor of \varprojlim .

We say that a system (A_n, d_n) satisfies the **Mittag-Leffler property** if for each n the images of the transition maps $A_{n+m} \rightarrow A_n$ are the same for all sufficiently large m . Observe that every inverse system of finite abelian

groups automatically has the Mittag-Leffler property. We call a system (A_n, d_n) **ML-zero** if for each n there is an $m = m(n)$ such that the transition map $A_{n+m} \rightarrow A_n$ is zero.

(2.7.3) Proposition. *Let (A_n, d_n) be a ML-zero system in \mathcal{A} . Then*

$$R^p(\varprojlim_n F)(A_n, d_n) = 0$$

for all $p \geq 0$.

Proof: Given a system $A = (A_n, d_n)$, we define a new system $A[1]$ by $A[1] = (A_{n+1}, d_{n+1})$. There is a canonical map $A[1] \xrightarrow{\text{can}} A$ given by the composition of the transition maps. If $A \hookrightarrow I^\bullet$ is an injective resolution, then $A[1] \hookrightarrow I[1]^\bullet$ is too, and we obtain a commutative diagram

$$\begin{array}{ccc} A[1] & \hookrightarrow & I[1]^\bullet \\ \text{can} \downarrow & & \downarrow \text{can} \\ A & \hookrightarrow & I^\bullet. \end{array}$$

Since $\varprojlim I[1]^m \xrightarrow{\sim} \varprojlim I^m$ for all $m \geq 0$, we obtain natural isomorphisms for all $p \geq 0$

$$R^p(\varprojlim_n F)A[1] \xrightarrow{\sim} R^p(\varprojlim_n F)A.$$

If all transition maps of A are zero, then this canonical isomorphism is the zero map, and hence $R^p(\varprojlim_n F)A = 0$ for all $p \geq 0$ in this case.

Let (A_n) be a ML-zero system and let $J = \{n_1, n_2, \dots\} \subset \mathbb{N}$ be a cofinal subset such that all transition maps in the inverse system $(A_{n_j})_{j \in \mathbb{N}}$ are zero. Now $\varprojlim F$ is also the composition of the exact forgetful functor

$$V_J: \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}, \quad (A_n) \mapsto (A_{n_j}),$$

which preserves injectives, with $\varprojlim F$. The spectral sequence implies

$$R^p(\varprojlim_n F)(A_n) = R^p(\varprojlim_j F)(A_{n_j}) = 0$$

for all $p \geq 0$. □

The following well-known proposition was formulated in [182], prop. 1, with too much generality (see [150] for a counter-example). Therefore we give a complete proof here in the case that \mathcal{B} is the category of modules over a ring.

(2.7.4) Proposition. *Let \mathcal{B} be the category of R -modules for some ring R . Then $\varprojlim^p = 0$ for $p \geq 2$, and there is a canonical exact sequence*

$$0 \longrightarrow \varprojlim_n (B_n) \longrightarrow \prod_n B_n \xrightarrow{id-d_n} \prod_n B_n \longrightarrow \varprojlim_n^1 (B_n) \longrightarrow 0.$$

If $B \in \text{Ob}(\mathcal{B}^{\mathbb{N}})$ satisfies the Mittag-Leffler property, then $\varprojlim^1 B = 0$.

Proof: By the definition of \varprojlim , we have an exact sequence

$$0 \longrightarrow \varprojlim_n (B_n) \longrightarrow \prod_n B_n \xrightarrow{id-d_n} \prod_n B_n,$$

which is also exact on the right, if all transition maps of (B_n) are surjective (for any element $(b_n) \in \prod_n B_n$ a pre-image under $(id - d_n)$ is easily constructed recursively). Let

$$0 \longrightarrow (A_n) \longrightarrow (B_n) \longrightarrow (C_n) \longrightarrow 0$$

be an exact sequence in $\mathcal{B}^{\mathbb{N}}$ such that all transition maps of the systems (A_n) and (B_n) are surjective. Then the transition maps of (C_n) are also surjective and the snake lemma shows that the sequence of inverse limits

$$0 \longrightarrow \varprojlim_n (A_n) \longrightarrow \varprojlim_n (B_n) \longrightarrow \varprojlim_n (C_n) \longrightarrow 0$$

is exact. Since injective objects in $\mathcal{B}^{\mathbb{N}}$ have surjective transition maps, we obtain $\varprojlim^p (B_n) = 0$ for all $p \geq 1$ and any system (B_n) with surjective transition maps.

If (B_n) is an arbitrary inverse system, we consider the inverse systems (PB_n) and (QP_n) given by $PB_n = \prod_{i=1}^n B_i$ and $QP_n = \prod_{i=1}^{n-1} B_i$, and with the natural projections $p_n: \prod_{i=1}^{n+1} B_i \rightarrow \prod_{i=1}^n B_i$ as transition maps. We obtain an exact sequence of inverse systems

$$0 \longrightarrow (B_n) \longrightarrow (PB_n) \longrightarrow (QP_n) \longrightarrow 0,$$

given by the commutative exact diagrams

$$\begin{array}{ccccc} 0 \rightarrow B_{n+1} & \xrightarrow{(id, d_n, d_n d_{n-1}, \dots)} & \prod_{i=1}^{n+1} B_i & \xrightarrow{p_n - (d_n \text{pr}_{n+1}, d_{n-1} \text{pr}_n, \dots)} & \prod_{i=1}^n B_i \rightarrow 0 \\ \downarrow d_n & & \downarrow p_n & & \downarrow p_{n-1} \\ 0 \rightarrow B_n & \xrightarrow{(id, d_{n-1}, d_{n-2} d_{n-1}, \dots)} & \prod_{i=1}^n B_i & \xrightarrow{p_{n-1} - (d_{n-1} \text{pr}_n, d_{n-2} \text{pr}_{n-1}, \dots)} & \prod_{i=1}^{n-1} B_i \rightarrow 0, \end{array}$$

where pr_i denotes the projection onto B_i . Since $\varprojlim^p (PB_n) = 0 = \varprojlim^p (QB_n)$ for $p \geq 1$, we obtain $\varprojlim^p (B_n) = 0$ for $p \geq 2$ and the exact sequence stated in the proposition.

Finally, let (B_n) be a Mittag-Leffler system. We consider the subsystem (B'_n) of (B_n) , where, for each n , B'_n is the image of the transition map $B_{n+m} \rightarrow B_n$, for sufficiently large m , which is independent of m . Then the transition maps of (B'_n) are surjective and the cokernel (B''_n) of the natural inclusion $(B'_n) \hookrightarrow (B_n)$ is ML-zero. Therefore we obtain isomorphisms

$$0 = \varprojlim_n^p (B'_n) \xrightarrow{\sim} \varprojlim_n^p (B_n)$$

for all $p \geq 1$. □

Suppose we are given a profinite group G and a compact topological G -module A whose underlying topology is profinite. It is not difficult to see that such a module is the inverse limit of finite G -modules and we call such modules **profinite modules** for short. The continuous cochain cohomology of profinite modules can also be defined by means of derived functors, namely as an Ext-group in the category of profinite G -modules.*) We will not make use of this in the following, but the reader should notice that this is the conceptual reason behind the next theorem.

(2.7.5) Theorem. *Suppose that the compact G -module A has a presentation*

$$A = \varprojlim_{n \in \mathbb{N}} A_n$$

as a countable inverse limit of finite, discrete G -modules. Then there exists a natural exact sequence

$$0 \longrightarrow \varprojlim_n^1 H^{i-1}(G, A_n) \longrightarrow H_{cts}^i(G, A) \longrightarrow \varprojlim_n H^i(G, A_n) \longrightarrow 0.$$

Proof: Consider the categories $(G\text{-Mod})^{\mathbb{N}}$ and $\mathcal{A}b^{\mathbb{N}}$ of inverse systems of discrete G -modules and of abelian groups, respectively. Both categories are abelian and have sufficiently many injective objects. Furthermore, the functor $H^0(G, -)$ sends injective G -modules to injective (i.e. divisible) abelian groups: indeed, this is obvious for modules of the form $\text{Ind}_G M$ for a divisible module M . If I is injective, then it is divisible and it is a direct summand of $\text{Ind}_G I$. Let us consider the functor $\varprojlim H^0(G, -)$. By (2.7.4), applied to $\mathcal{B} = \mathcal{A}b$, the spectral sequence for the composition of derived functors degenerates to a series of short exact sequences

$$0 \rightarrow \varprojlim_n^1 H^{i-1}(G, A_n) \rightarrow R^i(\varprojlim_n H^0(G, -))(A_n) \rightarrow \varprojlim_n H^i(G, A_n) \rightarrow 0.$$

*) $H_{cts}^i(G, A) \cong \text{Ext}_{\text{profinite } G\text{-modules}}^i(\hat{\mathbb{Z}}, A)$, where G acts trivially on $\hat{\mathbb{Z}}$. This can easily be deduced from (5.2.14).

It therefore remains to show that there are isomorphisms for all i :

$$R^i(\varprojlim_n H^0(G, -))(A_n) \cong H_{cts}^i(G, A).$$

To prove this, let us first replace the system (A_n) by its subsystem (A'_n) with

$$A'_n = \bigcap_m \text{im}(A_{n+m})$$

(since the A_i are finite, all intersections are finite too). The system A'_n has surjective transition maps and $A \cong \varprojlim A'_n$ as a topological G -module. The cokernel of the natural inclusion map $(A'_n) \hookrightarrow (A_n)$ is ML-zero. By (2.7.3), we have isomorphisms

$$R^i(\varprojlim H^0(G, -))(A'_n) \simeq R^i(\varprojlim H^0(G, -))(A_n)$$

for all $i \geq 0$. Therefore, replacing (A_n) by (A'_n) , we may assume that all transition maps of the system (A_n) are surjective. By definition, $H_{cts}^i(G, A)$ is the i -th cohomology of the continuous homogeneous cochain complex $C_{cts}^\bullet(G, A)$, which can be identified with the inverse limit over n of $C^\bullet(G, A_n)$. The complexes $C^\bullet(G, A_n)$ are the G -invariants of the complexes $X^\bullet(G, A_n)$, which are acyclic resolutions of A_n for all n , see I §3. Moreover, these complexes are functorial, hence we get a $H^0(G, -)^{\mathbb{N}}$ -acyclic resolution

$$(A_n) \hookrightarrow (X^\bullet(G, A_n)),$$

from which $H_{cts}^i(G, A)$ is obtained by applying $\varprojlim H^0(G, -)$ and taking cohomology. Hence it remains to show that the systems $(X^\bullet(G, A_n))$ are $\varprojlim H^0(G, -)$ -acyclic, which follows easily from the fact that the systems of abelian groups $(C^i(G, A_n)) = (X^i(G, A_n)^G)$ have surjective transition maps for all i (cf. I §3, ex.1). \square

(2.7.6) Corollary. *Let A be a compact G -module having a presentation*

$$A = \varprojlim_{n \in \mathbb{N}} A_n$$

as a countable inverse limit of finite, discrete G -modules. If $H^i(G, A_n)$ is finite for all n , then

$$H_{cts}^{i+1}(G, A) = \varprojlim_n H^{i+1}(G, A_n).$$

If $A = \varprojlim_{n \in \mathbb{N}} A_n$, where A_n is finite, then the corollary applies, for example, to $H_{cts}^1(G, A)$ and, if the group G is finitely generated, also to $H_{cts}^2(G, A)$.

As we did in I §2, p.17, in the case of a finite A , we consider for a profinite G -module A the set $EXT(G, A)$ of equivalence classes of exact sequences of

profinite groups

$$1 \longrightarrow A \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

such that the action of G on A is given by

$$\sigma a = \hat{\sigma} a \hat{\sigma}^{-1},$$

where $\hat{\sigma} \in \hat{G}$ is a pre-image of $\sigma \in G$. The same proof as that of (1.2.4) shows also in this case the

(2.7.7) Theorem. *We have a canonical bijection of pointed sets*

$$H_{cts}^2(G, A) \cong EXT(G, A).$$

Let ℓ be a prime number and T a topological G -module which, as a topological group, is a finitely generated \mathbb{Z}_ℓ -module with the natural topology, and on which G acts \mathbb{Z}_ℓ -linearly.

(2.7.8) Proposition. *Let Y be a finitely generated \mathbb{Z}_ℓ -submodule of $H_{cts}^n(G, T)$. Then the quotient group $H_{cts}^n(G, T)/Y$ contains no nontrivial ℓ -divisible subgroup.*

Proof: (cf. [232], prop. 2.1) Suppose $x_i \in H_{cts}^n(G, T)$, $0 \leq i < \infty$, such that $x_i \equiv \ell x_{i+1} \pmod{Y}$ for all i . We must show $x_0 \in Y$. Let y_j , $1 \leq j \leq m$, be a finite set of generators for Y . For each i , let f_i be an n -cocycle representing x_i , and for each j , let g_j be an n -cocycle representing y_j . Then there are $(n-1)$ -cochains h_i and elements $a_{ij} \in \mathbb{Z}_\ell$ such that

$$f_i = \ell f_{i+1} + \sum_{j=1}^m a_{ij} g_j + \partial h_i.$$

Hence

$$f_0 = \sum_{j=1}^m a_j g_j + \partial h$$

with $a_j = \sum_{i \geq 0} \ell^i a_{ij}$ and $h = \sum_{i \geq 0} \ell^i h_i$. The use of infinite sums here is formally justified by the fact that T is the inverse limit of its quotients $T/\ell^i T$ and consequently

$$C_{cts}^\bullet(G, T) = \varprojlim_i C^\bullet(G, T/\ell^i T)$$

is a projective limit of modules, each of which is killed by a fixed power of ℓ . This completes the proof. \square

(2.7.9) Corollary. *The \mathbb{Z}_ℓ -module $H_{cts}^n(G, T)$ is finitely generated if and only if $H_{cts}^n(G, T)/\ell H_{cts}^n(G, T)$ is finite.*

Proof: In order to show the nontrivial assertion, assume that y_1, \dots, y_m generate $H_{cts}^n(G, T)$ modulo ℓ . Putting $Y = \langle y_1, \dots, y_m \rangle$, we conclude that the group $H_{cts}^n(G, T)/Y$ is ℓ -divisible, hence trivial by the last proposition. \square

(2.7.10) Corollary. *Assume that the cohomology groups of G with coefficients in finite ℓ -primary modules are finite. Then $H_{cts}^n(G, T)$ is a finitely generated \mathbb{Z}_ℓ -module for all n and the canonical map*

$$H_{cts}^n(G, T) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \longrightarrow H^n(G, T \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell)$$

is an isogeny, i.e. has finite kernel and finite cokernel.

Proof: Replacing, if necessary, T by an open submodule, we may assume that T is torsion-free. Then the exact sequence $0 \rightarrow T \xrightarrow{\ell^m} T \rightarrow T/\ell^m \rightarrow 0$ implies the exact sequence

$$0 \longrightarrow H_{cts}^n(G, T)/\ell^m \longrightarrow H^n(G, T/\ell^m) \longrightarrow \ell^m H_{cts}^n(G, T) \longrightarrow 0.$$

Now the statements follow from (2.7.9). \square

Suppose now that T is torsion-free. Tensoring it, over \mathbb{Z}_ℓ , by the exact sequence $0 \rightarrow \mathbb{Z}_\ell \rightarrow \mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell / \mathbb{Z}_\ell \rightarrow 0$ gives an exact sequence

$$(*) \quad 0 \longrightarrow T \longrightarrow V \longrightarrow W \longrightarrow 0,$$

in which V is a finite dimensional \mathbb{Q}_ℓ -vector space, T is an open compact subgroup and W is a discrete divisible ℓ -primary torsion group.

(2.7.11) Proposition. *There are isomorphisms for all n*

$$H_{cts}^n(G, V) \cong H_{cts}^n(G, T) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

In the exact cohomology sequence associated with $()$, the kernel of the boundary homomorphism*

$$\delta: H^{n-1}(G, W) \longrightarrow H_{cts}^n(G, T)$$

is the maximal divisible subgroup of $H^{n-1}(G, W)$, and its image is the torsion subgroup of $H_{cts}^n(G, T)$.

Proof: Since V is a vector space over \mathbb{Q}_ℓ , so is $H_{cts}^n(G, V)$ for all n . Furthermore, $H^n(G, W)$ is an ℓ -torsion group for all n . By (2.7.2), we have a long

exact sequence associated to (*). Tensoring over \mathbb{Z}_ℓ with \mathbb{Q}_ℓ implies the first statement. Clearly,

$$\ker(H^{n-1}(G, W) \rightarrow H_{cts}^n(G, T)) = \text{im}(H^{n-1}(G, V) \rightarrow H^{n-1}(G, W))$$

is ℓ -divisible. On the other hand, by (2.7.8), each divisible subgroup of $H^{n-1}(G, W)$ must be contained in the kernel. Since W is torsion, the group $\text{im}(H^{n-1}(G, W) \rightarrow H_{cts}^n(G, T))$ is a torsion group. On the other hand, it is equal to the kernel of the map $H_{cts}^n(G, T) \rightarrow H_{cts}^n(G, V)$ and therefore must contain all torsion elements of $H_{cts}^n(G, T)$. \square

(2.7.12) Corollary. *Assume that the cohomology groups of G with coefficients in finite ℓ -primary modules are finite. Then*

$$\begin{aligned} \text{rank}_{\mathbb{Z}_\ell} H_{cts}^n(G, T) &= \dim_{\mathbb{Q}_\ell} H_{cts}^n(G, V) \\ &= \text{corank}_{\mathbb{Z}_\ell} H^n(G, T \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell) \\ &= \text{rank}_{\mathbb{Z}_\ell} H_n(G, \text{Hom}(T, \mathbb{Z}_\ell)). \end{aligned}$$

Proof: The first two equalities follow from (2.7.11) and (2.7.10). Since T is finitely generated, we have

$$\text{Hom}(T \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell, \mathbb{Q}_\ell / \mathbb{Z}_\ell) = \text{Hom}(T, \text{Hom}(\mathbb{Q}_\ell / \mathbb{Z}_\ell, \mathbb{Q}_\ell / \mathbb{Z}_\ell)) = \text{Hom}(T, \mathbb{Z}_\ell),$$

thus (2.6.9) implies the third equality. \square

Continuous cochain cohomology was introduced by *J. TATE* in [232]. In addition, we have taken several arguments from the paper [96] of *U. JANNSEN*.



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