2

Motion of the conducting rigid body in alternating magnetic field

2.1 Asymptotic transformation of the equations of motion for conducting rigid body in alternating magnetic field

It is known that the alternating magnetic forces can destabilize a stable equilibrium position which results in self-oscillations. Mathematically, this effect is related to change in the structure of the Lagrange-Maxwell equations. In addition to the gyroscopic forces the circulational generalized forces due to the alternating external field appear in these equations.

Let us consider in detail the problem of the slow motion of a conducting rigid body in the alternating magnetic field. We assume that the distribution of eddy currents in the rigid body can be presented as an expansion in terms of a complete system of solenoidal vectorial functions $\mathbf{S_r}$, the latter being constant in the coordinate system fixed in the rigid body [90]

$$\mathbf{j}(t,h,u,z)=\sum_{r=1}^{\infty} i_r(t)\mathbf{S}_r(h,u,z). \tag{2.1.1}$$

The expression for the energy of magnetic field can be presented as the sum

$$W=\frac{1}{2}i^{\mathrm{T}}Li+J(t)^{\mathrm{T}}M(q)i. \tag{2.1.2}$$

Here the infinite-dimensional vector i of the coefficients of decomposition (2.1.1) is equivalent to the vector of currents in the “fictitious” contours of the rigid body. Thus, the rigid body is presented as a countable or finite system of conducting contours whose mutual position does not change under the motion. This corresponds to a matrix L of constant coefficients of self-induction and mutual induction. Next, $J(t)$ denotes the vector of currents prescribing the alternating external field and $M(q)$ is a rectangular matrix of the coefficients of mutual induction of the contours in the rigid body and the contours of external sources of the alternating field.

Let our consideration be limited by the case of periodic or quasi-periodic function $J(t)$, that is,

$$J(t) = \sum_k (I_k \cos \nu_k t + J_k \sin \nu_k t) \tag{2.1.3}$$

and let us assume that the differences in frequencies $\nu_i - \nu_j$ have approximately the order of the smallest frequency ν_1. The electromagnetic forces $F_e = J^{\mathrm{T}} \partial M/\partial q$ have the components with frequencies of order ν_1 and higher. These components cause mechanical oscillations with the amplitudes of the order of $[F_{er}]/\left(\nu_1^2 [A_{rr}]\right), r = 1, \ldots, n$, where the square brackets denote the characteristic value of the corresponding quantity. In the case of high-frequency current J these amplitudes are small in comparison with the characteristic value $[q_r]$ of the coordinate q_r.

Let us introduce a small parameter ε by means of the following relation: $\varepsilon^2 \sim [F_{er}]/(\nu_1^2 [A_{rr}][q_r])$. In the problem of slow motions the values of momenta are assumed to be of the order of ε in comparison to value $\nu_1 [A_{rr}][q_r]$. Hence, the generalized forces Q $\left([Q_r] = \varepsilon^2 \nu_1^2 [A_{rr}][q_r]\right)$ should be considered as being small, i.e. of the order of forces $[F_{er}]$.

Let us now introduce the dimensionless time which is equal to the product of the dimensional time and ν_1, the dimensionless coordinates and currents as the ratios of the dimensional values to their characteristic values and the dimensionless momenta which are equal to the ratios of the momenta to $\varepsilon \nu_1 [A_{rr}][q_r]$.

Keeping the previous denotation for the dimensionless momenta we arrive at the equations

$$\begin{gathered} \dot{q} = \varepsilon A^{-1} p, \quad \dot{p} = \varepsilon \left(-\frac{1}{2} p^{\mathrm{T}} \frac{\partial A^{-1}}{\partial q} p + J \frac{\partial M}{\partial q} i + Q \right), \\ Li^{\cdot} + Ri + (JM^{\mathrm{T}})^{\cdot} = 0. \end{gathered} \tag{2.1.4}$$

The above assumptions and equations (2.1.4) are used in a number of technical applications: orientation of details by magnetic field, suspension of a rigid body for the non-crucible process of melting, magnetic suspensions etc. The problem of rapid rotation of a rigid body in the magnetic field is studied in [79]. In this case, the equations are similar to eq. (2.1.4) and the small parameter appears only in front of the electromagnetic forces.

According to Section A.1 we seek the solution of eq. (2.1.4) in the form of an asymptotic series

$$q = \xi + \varepsilon u_1(\xi, \eta, t) + \ldots, \quad r = \eta + \varepsilon v_1(\xi, \eta, t) + \ldots,$$
$$i = i_0(\xi, t) + \varepsilon i_1(\xi, \eta, t) + \ldots. \tag{2.1.5}$$

Functions u_1, v_1 are assumed to have zero mean value

$$\langle u_1 \rangle = \lim_{T\to\infty} \frac{1}{T} \int\limits_{t_0}^{t_0+T} u_1(\xi, \eta, t) dt = 0, \tag{2.1.6}$$

which ensures the uniqueness of second approximations in the averaged equations. We construct the equations for ξ, η in the form

$$\dot{\xi} = \varepsilon \Xi_1(\xi, \eta) + \varepsilon^2 \Xi_2(\xi, \eta) + \ldots,$$
$$\dot{\eta} = \varepsilon H_1(\xi, \eta) + \varepsilon^2 H_2(\xi, \eta) + \ldots, \tag{2.1.7}$$

and limit the analysis by the terms of order $O(\varepsilon^2)$. It is sufficient to find i_0, i_1 satisfying the equations

$$L\ddot{i}_0 + Ri_0 + M\dot{J} = 0,$$
$$L\ddot{i}_1 + Ri_1 + L\frac{\partial i_0}{\partial \xi} A^{-1}\eta + J\frac{\partial M}{\partial \xi} A^{-1}\eta = 0. \tag{2.1.8}$$

Here $\xi, \eta = \text{const}$ and it was taken into account that $\Xi_1 = A^{-1}\eta$. To obtain eq. (2.1.7) it is sufficient to determine only the periodic or quasi-periodic solutions of equations (2.1.8)

$$i_0 = \sum_{k=1}^{n} (I_{0k} \cos \nu_k t + J_{0k} \sin \nu_k t),$$
$$i_1 = \sum_{k=1}^{n} (I_{1k} \cos \nu_k t + J_{1k} \sin \nu_k t). \tag{2.1.9}$$

Inserting eq. (2.1.9) into eq. (2.1.8) yields

$$-\nu_k L I_{0k} + R J_{0k} = \nu_k M I_k, \quad \nu_k L J_{0k} + R I_{0k} = -\nu_k M J_k,$$
$$-\nu_k L I_{1k} + R J_{1k} + L\frac{\partial J_{0k}}{\partial \xi} A^{-1}\eta + J_k \frac{\partial M}{\partial \xi} A^{-1}\eta = 0,$$
$$\nu_k L J_{1k} + R I_{1k} + L\frac{\partial I_{0k}}{\partial \xi} A^{-1}\eta + I_k \frac{\partial M}{\partial \xi} A^{-1}\eta = 0. \tag{2.1.10}$$

Resolving the system of linear equations (2.1.10) for the unknowns I_{0k}, J_{0k}, I_{1k}, J_{1k} we obtain

$$\begin{aligned} I_{0k} &= -\nu_k^2 I_k U_k M - \nu_k J_k V_k M, \quad J_{0k} = -\nu_k^2 J_k U_k M + \nu_k I_k V_k M, \\ I_{1k} &= \left(\nu_k U_k L \frac{\partial J_{0k}}{\partial \xi} + \nu_k J_k U_k \frac{\partial M}{\partial \xi} - V_k L \frac{\partial I_{0k}}{\partial \xi} - I_k V_k \frac{\partial M}{\partial \xi}\right) A_{-1}\eta, \\ J_{1k} &= -\left(\nu_k U_k L \frac{\partial I_{0k}}{\partial \xi} + \nu_k I_k U_k \frac{\partial M}{\partial \xi} + V_k L \frac{\partial J_{0k}}{\partial \xi} + J_k V_k \frac{\partial M}{\partial \xi}\right) A^{-1}\eta. \end{aligned} \tag{2.1.11}$$

Here the positive definite symmetric matrices U_k, V_k are given by the formulae

$$\begin{gathered} U_k = (\nu_k^2 L + RL^{-1}R)^{-1}, \quad V_k = (R + \nu_k^2 LR^{-1}L)^{-1}, \\ LV_k = RU_k, \quad V_k L = U_k R. \end{gathered} \tag{2.1.12}$$

The averaged equations of second approximation are as follows

$$\begin{aligned} \dot{\xi} &= \varepsilon A^{-1}(\xi)\eta, \\ \dot{\eta} &= \varepsilon \left(-\frac{1}{2}\eta^{\mathrm{T}} \frac{\partial A^{-1}}{\partial \xi}\eta + \left\langle J \left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} i_0 \right\rangle + \langle Q_1 \rangle\right) + \\ &\qquad + \varepsilon^2 \left(\left\langle J \left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} i_1 \right\rangle + \langle Q_2 \rangle\right). \end{aligned} \tag{2.1.13}$$

Equations (2.1.13) are the equations of motion of the original mechanical system subjected to the forces

$$\varepsilon\langle Q_1\rangle + \varepsilon^2\langle Q_2\rangle, \quad \varepsilon\langle J(\partial M/\partial\xi)^{\mathrm{T}} i_0\rangle + \varepsilon^2\langle J(\partial M/\partial\xi)^{\mathrm{T}} i_1\rangle.$$

We will study the properties of these forces. Taking into account eqs. (2.1.10) and (2.1.12) we obtain after some transformations

$$\begin{aligned} P_1(\xi) &= \left\langle J \left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} i_0 \right\rangle = \frac{1}{2}\sum_k I_k \left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} I_{0k} + \frac{1}{2}\sum_k J_k \left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} J_{0k} \\ &= -\frac{1}{2}\sum_k \left\{ \left(\frac{\partial I_{0k}}{\partial \xi}\right)^{\mathrm{T}} L I_{0k} + \left(\frac{\partial J_{0k}}{\partial \xi}\right)^{\mathrm{T}} L J_{0k} \right\} + \\ &\quad + \frac{1}{2}\sum_k \frac{1}{\nu_k} \left\{ \left(\frac{\partial J_{0k}}{\partial \xi}\right)^{\mathrm{T}} R I_{0k} - \left(\frac{\partial I_{0k}}{\partial \xi}\right)^{\mathrm{T}} R J_{0k} \right\}. \end{aligned} \tag{2.1.14}$$

It follows from eq. (2.1.11) that the second sum in eq. (2.1.14) is equal to zero. Then $P_1(\xi)$ can be set in the form

$$P_1(\xi) = -\frac{\partial}{\partial \xi}\langle W_0\rangle, \quad W_0 = \frac{1}{2} i_0^{\mathrm{T}} L i_0. \tag{2.1.15}$$

Thus, the averaged electromagnetic forces calculated in the first approximation are potential and the potential is the mean value of the energy of magnetic field of Foucault currents provided that the energy is obtained by the same (first) approximation.

Using eq. (2.1.11) we transform the second approximation to the expressions for the mean electromagnetic forces to the form

$$
\begin{aligned}
P_2(\xi,\eta) &= \left\langle J\left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} i_1 \right\rangle = \\
&= \frac{1}{2}\sum_k I_k \left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} I_{1k} + \frac{1}{2}\sum_k J_k \left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} J_{1k} = \\
&= -\frac{1}{2}\sum_k (I_k^2 + J_k^2)\left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} (V_k - 2\nu_k^2 U_k R U_k)\frac{\partial M}{\partial \xi} A^{-1}\eta.
\end{aligned}
\tag{2.1.16}
$$

Taking into account that $\varepsilon A^{-1}\eta = \dot{\xi}$ and the matrices

$$
\Phi_k = \frac{1}{2}(I_k^2 + J_k^2)\left(\frac{\partial M}{\partial \xi}\right)^{\mathrm{T}} (V_k - 2\nu_k^2 U_k R U_k)\frac{\partial M}{\partial \xi} \tag{2.1.17}
$$

are symmetric, we can set the expression $\varepsilon^2 P_2(\xi,\dot{\xi})$ in the following form

$$
\varepsilon^2 P_2(\xi,\dot{\xi}) = -\frac{\partial}{\partial \dot{\xi}}\left(\frac{\varepsilon}{2}\dot{\xi}^{\mathrm{T}}\sum_k \Phi_k \dot{\xi}\right) = -\frac{\partial \Psi}{\partial \dot{\xi}}. \tag{2.1.18}
$$

The expression for the dissipative function Ψ can be written as a quadratic form of the rates of change of the induction coefficients

$$
\Psi = \frac{\varepsilon}{2}\dot{M}^{\mathrm{T}} D \dot{M}, \quad D = \frac{1}{2}\sum_k (I_k^2 + J_k^2)(V_k - 2\nu_k^2 U_k R U_k). \tag{2.1.19}
$$

Thus the second approximations to the averaged electromagnetic forces describe the “formally dissipative” forces. The sign of the dissipative function depends on the relationship between the time constant of attenuation of the proper field of the conducting body and the period of change of the external field.

By virtue of relationships (2.1.15), (2.1.18) and (2.1.19) the equations for second approximation (2.1.13) can be written down in the form

$$
\begin{aligned}
\dot{\xi} &= \varepsilon A^{-1}(\xi)\eta, \\
\dot{\eta} &= \varepsilon\left(-\frac{1}{2}\eta^{\mathrm{T}}\frac{\partial A^{-1}}{\partial \xi}\eta - \frac{\partial \Lambda}{\partial \xi} + \langle Q_1\rangle\right) + \varepsilon^2\left(P_2(\xi,\dot{\xi}) + \langle Q_2\rangle\right),
\end{aligned}
\tag{2.1.20}
$$

where $\Lambda = \langle W(i_0)\rangle$.

In the most important case in which forces $\varepsilon\langle Q_1\rangle$ are potential with the potential function $\Pi(\xi)$ (for example, the moments due to the gravity force) the first approximation in the averaged equations (2.1.13) describes the conservative system whose motions can be qualitatively changed by forces of the next order of smallness. Therefore the second approximations are necessary in order to have the averaged equations for qualitative description of some motions, e.g. attenuating or increasing oscillations. In this case the oscillations decrease to small amplitudes within the time $t \sim T/\varepsilon^2$.

However it follows from the general theorems of the averaging method that the expressions $q = \xi + \varepsilon u_1, r = \eta + \varepsilon v_1$ approximate the solution of the original systems (3.1.5) with the error $O(\varepsilon^2)$ in the interval $t \sim T/\varepsilon$. Therefore these theorems are not sufficient for the qualitative analysis of motion of the considered system, for example, these do not allow one to assert, that the system tends to quasi-stationary motions, when the forces $\varepsilon^2(P_2 + \langle Q_2\rangle)$ are dissipative.

If the frictional forces in $\langle Q_1\rangle$ are sufficiently large and can be deemed as quantities of the order of ε, the first approximation is sufficient for analysis of stability of quasi-equilibria and motions. However in a number of applications [86, 104] one observes increasing oscillations of a rigid body which can be explained only by instability due to swinging character of forces $\varepsilon^2 P_2$. In these cases the forces of external friction are naturally associated with the forces of order ε^2 and higher. Because the forces are potential in the first approximation, the motion of the system depends drastically on the forces of next order of smallness, in particular $\varepsilon^2 P_2$.

By virtue of eqs. (2.1.18) and (2.1.19) we have

$$\varepsilon^2 P_2(\xi, \dot{\xi}) = \varepsilon B(\xi)\dot{\xi}, \tag{2.1.21}$$

where, generally speaking, the symmetric matrix B of the "formally dissipative" forces P_2 depends on the generalized coordinates. Thus the influence of magnetic field on slow motions results in the appearance of potential and "dissipative" forces in the first and second approximations, respectively.

Let us consider system (2.1.20) in the first approximation, i.e. without terms $O(\varepsilon^2)$. This system is conservative and has the Hamilton function

$$H = \frac{\varepsilon}{2}\eta^{\mathrm{T}} A^{-1}(\xi)\eta + \varepsilon\Lambda + \varepsilon\Pi\,. \tag{2.1.22}$$

Following Appendix D we take function V in the form $V = H/\varepsilon$.

Let the system of first approximation have a stable equilibrium position. It is encircled by closed surfaces $V = C, C = \text{const}$. Let $\Lambda = \Pi = 0$ in the equilibrium position and let D_1 denote the region bounded by two surfaces $S(C_1), S(C_2), C_1 > C_2, C_1, C_2 = O(1)$, the first enclosing the second. By virtue of eq. (2.1.20) the derivative $\dot{V}$ is equal to

$$\dot{V} = -\varepsilon^2(A^{-1}\eta)^{\mathrm{T}}(B + G)(A^{-1}\eta)\,, \tag{2.1.23}$$

where G denotes the matrix of coefficients of forces of external viscous damping. We will consider the case when matrix B is positive definite. It is possible to show that the inequality

$$V(\xi(t^{(0)}+T/\varepsilon)) - V(\xi(t^{(0)})) \leq -\varepsilon^{m-1}W_0 \tag{2.1.24}$$

holds true. Indeed, if $|\eta(t^{(0)})| = O(1)$, then by virtue of eq. (2.1.22) an increase in function V caused by the solution $\xi(t), \eta(t)$ in any time interval Δt is not greater than $-\varepsilon^2(\mathrm{const}\,\Delta t)$. If $|\eta(t^{(0)}|$ is a small value, we have $|\eta(t^{(0)})| = O(1)$ after the time span of the order

$$\Delta t = \mathrm{const}\left(\varepsilon \inf_{D_1}\left|\frac{\partial(\Lambda+\Pi)}{\partial\xi}\right|\right)^{-1}$$

according to the second equation in (2.1.20). Choosing $T = \varepsilon k \Delta t, k > 1$ which affects only the constant c_m in the estimate $|\xi_m - \xi^{(m)}| \leq c_m \varepsilon^m$ we obtain the required inequality.

Inequality (2.1.24) and relationship (2.1.22) are satisfied for any non-small value of C_2. This allows us to apply Theorem 3 of Appendix D. As a result we obtain that for sufficient small ε any phase trajectory $\xi(t), \eta(t)$ in the above region D_1 eventually get in a small neighbourhood of the equilibrium position and will remain there. This corresponds to oscillations of the original system which are qualitatively similar to damped oscillations tending to quasi-statical ones.

The divergent oscillations are also observed in the case of the negative definite matrix of the total friction $B + G$.

Matrix B depends on ξ. Therefore it is possible that the total non-potential forces $\varepsilon^2 Q_2 - \varepsilon B\dot{\xi}$ are destabilizing near the equilibrium position and dissipative far away from it. In these cases a limit cycle is possible. At least, for a system with one mechanical degree of freedom it is possible to show in the similar manner that $\xi(t)$ reaches a small neighbourhood of this cycle. For a periodic function $J(t)$ and large t the oscillations are qualitatively similar to quasi-periodic ones and the motions of the system look like decreasing or increasing oscillations tending to quasi-periodic ones. However it seems that the existence of exact quasi-periodic solutions has not been proved for the cases when a limit cycle is revealed in the higher approximations.

Let us consider the dependence of forces $\varepsilon^2 P_2$ on the field frequency and values of electric resistances. We will assume a harmonic current $J(t)$, that is, $J(t) = I\cos\nu t + J\sin\nu t$. There is an orthogonal transformation of vector i which reduces matrices L and R to the unit and diagonal form, i.e. $R = \mathrm{const}(\lambda_1, \lambda_2 \ldots)$, where $\lambda_1 < \lambda_2 < \ldots$. Matrices U and V also become diagonal and matrix D expressed in terms of them is as follows

$$D = \mathrm{const}\left[\frac{(\lambda_1^2 - \nu^2)\lambda_1}{(\lambda_1^2 + \nu^2)^2}, \ldots\right]. \tag{2.1.25}$$

One can see that for $\nu < \lambda_1$ matrix D is positive definite, i.e. if the field frequency is smaller than the first eigenvalue of the problem of attenuating electromagnetic processes in the body, the average electromagnetic forces in the second approximation are dissipative. Thus the dissipation is complete when the coefficients of the matrix of mutual inductions M depend on all generalized coordinates.

If $\nu > \lambda_1$, force $\varepsilon^2 P_2$ can be of the divergent character. These forces can be dissipative or destabilising since they depend on the coordinates. This fact indicates that self-oscillations are possible. Because of a small external friction, the self-oscillations are possible also if forces $\varepsilon^2 P_2$ are destabilising for all values of coordinates.

It follows from eq. (2.1.20) that the destabilizing effect of forces $\varepsilon^2 P_2$ can be eliminated by decreasing the body conductivity. Indeed, the coefficients of matrix R and the absolute values of eigenvalues $|\lambda_1|, |\lambda_2|, \ldots$ increase as conductivity reduces in a similar way to increasing eigenfrequencies as rigidity increases. In a number of cases the body heating can eliminate instability of the considered type.

Having reduced the matrices R and L to the diagonal form one can set the expression for forces εP_1 in the form

$$\begin{aligned} P_1^T(\xi) &= -\frac{\partial}{\partial \xi}[\nu_1^2(I^2+J^2)M^{\mathrm{T}}(\nu^2 ULU + VLV)M] = \\ &= -(I^2+J^2)\frac{\partial}{\partial \xi}\left[M^{\mathrm{T}}\,\mathrm{const}\left(\frac{\nu^2}{\lambda_1^2+\nu^2},\ldots\right)M\right] = \\ &= -(I^2+J^2)\frac{\partial}{\partial \xi}(M^{\mathrm{T}}NM). \end{aligned} \tag{2.1.26}$$

Comparing eqs. (2.1.25) and (2.1.26) one can find a way of destroying instability caused by forces $\varepsilon^2 P_2$ and keeping a stable equilibrium in the first (potential) approximation. For the bodies like rings, plane plate etc. [104, 40] the dependence of forces εP_1 and $\varepsilon^2 P_2$ on frequency is qualitatively the same as in the case of a single non-zero first member of the diagonal matrices N and M. In this case the first members N_{11} and M_{11} of the diagonal matrices N and M are respectively monotonically increasing and decreasing functions of ν.

Let an equilibrium of the body be stable in the potential approximation for some value of ν and be unstable in the higher approximation because of force $\varepsilon^2 P_2$. Let us assume that the field-generating-contour conducts a basic current of frequency ν and an additional current of a lower frequency ν_1. The additional force $\varepsilon \Delta P_1(\nu_1)$ is considerably smaller than the basic force $\varepsilon P_1(\nu)$ and if ν_1 is properly chosen the new equilibrium position will be stable in the first approximation and close to the original one. The additional dissipation will be greater than the original one, can have another sign and stabilize the equilibrium.

In the case of superconductivity, one can prove the potentiality of the mean electromagnetic forces in the first approximation even for the system with several superconducting bodies and any number of the independent prescribed currents creating the field. Instead of eq. (2.1.4) we have

$$\begin{gathered}\dot{q} = \varepsilon A^{-1}(q)r,\\ \dot{r} = \varepsilon\left[-\frac{1}{2}r^{\mathrm{T}}\frac{\partial A^{-1}}{\partial q}r + \sum_s J_s\left(\frac{\partial M_s}{\partial q}\right)^{\mathrm{T}} i + \frac{1}{2}i^{\mathrm{T}}\frac{\partial L}{\partial q}i + Q\right],\\ (\dot{Li}) + \left(\sum_s J_s M_s\right)^{\cdot} = 0 \end{gathered} \tag{2.1.27}$$

Here i denotes the infinite-dimensional vector-column constructed arbitrarily from the vectors of eddy currents in the separate bodies, $L(q)$ is the corresponding matrix of the induction coefficients (in the case a few interactive bodies it depends on their coordinates) and index s marks the prescribed currents creating the field.

For the sake of simplicity we consider the case when the external field is switched on after the bodies have reached the superconducting state. The induction fluxes through the conditional contours in the bodies are equal to zero and system (2.1.27) has a countable number of the integrals which are similar to the cyclic integrals

$$Li + \sum_s J_s M_s = 0\,. \tag{2.1.28}$$

Using eq. (2.1.28) we eliminate the currents

$$i = -\sum_s L^{-1} M_s J_s \tag{2.1.29}$$

from the equations of motion in eq. (2.1.26) and arrive at the standard form of the system. Averaging this system we obtain the expression for the mean electromagnetic force in the first approximation

$$\begin{aligned} P_1(\xi) &= \left\langle \frac{1}{2}i^{\mathrm{T}}\frac{\partial L}{\partial \xi}i + \sum_s J_s\left(\frac{\partial M_s}{\partial \xi}\right)^{\mathrm{T}} i\right\rangle =\\ &= \left\langle \frac{1}{2}i^{\mathrm{T}}\frac{\partial L}{\partial \xi}i - i^{\mathrm{T}}\frac{\partial L}{\partial \xi}i - \left(\frac{\partial i}{\partial \xi}\right)^{\mathrm{T}} Li\right\rangle =\\ &= -\left\langle \frac{1}{2}i^{\mathrm{T}}\frac{\partial L}{\partial \xi}i + \left(\frac{\partial i}{\partial \xi}\right)^{\mathrm{T}} Li\right\rangle = -\frac{\partial}{\partial \xi}\langle W_0\rangle\,. \end{aligned} \tag{2.1.30}$$

This proves the potentiality of the mean electromagnetic forces in the first approximation. As before, the potential is the mean value of energy of the field of the whirling currents (in this case only on bodies' surface).

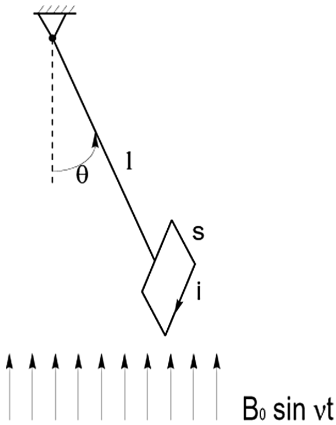

FIGURE 2.1.

In particular, the conclusion about the potentiality of the mean electromagnetic forces in the first approximation allows us to simplify the analysis of the problem of existence and stability of quasi-static motions. In the first approximation these motions correspond to the values of ξ satisfying the equation $P_1(\xi)+\langle Q_1\rangle = 0$. Let us consider the case in which the mechanical system consists only of a rigid body having a fixed point and forces Q are a sum of the gravity force (or other potential forces) and the frictional forces. In the general case the potential $\langle W_0\rangle$ depends on three generalized coordinates (angles). Both $\langle W_0\rangle$ and the potential for the "potential component" $\langle Q_1\rangle$ are the periodic functions of generalized coordinates. Therefore, generally speaking, the total potential has a minimum, that is, the rigid body in a rapidly changing magnetic field has, in the first approximation in ε, a position of the quasi-equilibrium. The bodies with different types of symmetry (i.e. when $\langle W_0\rangle$ does not depend on all generalized coordinates) are an exception.

2.2 Pendulum in the alternating magnetic field

As an example of the electromechanical system in the alternating magnetic field we consider a pendulum, the role of a rigid body being played by a closed contour of current rigidly connected to the suspension by a rigid bar, see Fig. 2.1.

Let us assume that the pendulum is placed in the alternating homogeneous magnetic field whose frequency ν is much higher than the natural frequency of small oscillations of the pendulum. We designate the angle of deviation of the pendulum from the vertical axis by θ and assume that $\theta = 0$ corresponds to the lower position of the pendulum. The expressions for the kinetic energy T, the magnetic field energy W and the potential energy Π have the form

$$
\begin{aligned}
&T = \frac{1}{2} I \dot{\theta}^2, \quad W = \frac{1}{2} L i^2 + B_0 S \sin \nu t \sin \theta \, i, \\
&\Pi = mgl(1 - \cos\theta),
\end{aligned} \tag{2.2.1}
$$

where I denotes the moment of inertia of the contour about the axis passing through the suspension point, m is contour's mass, l is bar's length; L is the coefficient of self-induction of the contour of current i and B_0 is the amplitude of external field.

The Lagrange–Maxwell equations for the considered electromechanical system are

$$
\begin{aligned}
I\ddot{\theta} - B_0 S \sin \nu t \cos\theta \, i + mgl \sin\theta = 0, \\
L\dot{i} + B_0 S \sin \nu t \cos\theta \, \dot{\theta} + B_0 S \nu \cos \nu t \sin\theta + Ri = 0.
\end{aligned} \tag{2.2.2}
$$

We assume that the frequency of magnetic field is much higher than the natural frequency of the pendulum $k = \sqrt{mgl/I} \ll \nu$ and enter a small parameter $\varepsilon^2 = k^2/\nu^2$. Introducing a dimensionless (fast) time $\tau = \nu t$ and a dimensionless current $i_u = i/i_*$ ($i_* = B_0 S/L$ denotes a basis value of the current) we can write eq. (2.2.2) in terms of the dimensionless variables

$$
\begin{aligned}
\ddot{\theta} - \varepsilon^2 \gamma \sin\tau \cos\theta i_u + \varepsilon^2 \sin\theta = 0, \\
\dot{i}_u + \sin\tau \cos\theta \; \dot{\theta} + \cos\tau \sin\theta + r i_u = 0.
\end{aligned} \tag{2.2.3}
$$

In these equations the dimensionless parameters $\gamma = \dfrac{(B_0 S)^2}{Lmgl}$ and $r = \dfrac{R}{L\nu}$ are introduced. In what follows the dimensionless variables use the denotations of dimensional variables. Intending to study the slow motions which are close to free oscillations of the pendulum we rewrite the system (2.2.3) as

$$
\begin{aligned}
\dot{\theta} &= \varepsilon \omega, \\
\dot{\omega} &= \varepsilon \gamma \sin\tau \cos\theta \; i - \varepsilon \sin\theta, \\
\dot{i} &+ \sin\tau \cos\theta \; \dot{\theta} + \cos\tau \sin\theta + ri = 0.
\end{aligned} \tag{2.2.4}
$$

System (2.2.4) is a quasi-linear system with a single non-critical fast variable. As shown in Section 2.1 the first approximation of the asymptotic approach describes a conservative system. Electromagnetic forces in the first approximation have a potential which is a mean energy of the eddy currents calculated in the same approximation. Conservatism can be lost in the second approximation to the electromagnetic forces. As shown in Section 2.1, if the field frequency exceeds the value equal to the inverse of the time constant of the conducting contour, that is $\nu > R/L$, the second approximation of electromagnetic forces can be of destabilizing character. Therefore, we derive the averaged equations for slow motions in the second approximation. Assuming $\omega, \theta = \text{const}$ we obtain the second approximation for the current

$$i = -\frac{\sin\theta}{1+r^2}(\sin\tau + r\cos\tau) + \varepsilon\frac{\omega r\cos\theta}{(1+r^2)^2}((1-r^2)\sin\tau + 2r\cos\tau)\,. \quad (2.2.5)$$

Substituting this expression into the first two equations of system (2.2.4) and averaging over the fast time τ we obtain the second order autonomous differential equation

$$\ddot{\theta} - \varepsilon\alpha\cos^2\theta\;\dot{\theta} + (\beta\cos\theta + 1)\sin\theta = 0\,, \quad (2.2.6)$$

where

$$\alpha = \frac{\gamma r(1-r^2)}{2(1+r^2)^2}, \quad \beta = \frac{\gamma}{2(1+r^2)}$$

and a dot denotes a differentiation with respect to slow time $t' = kt$. The term $\beta\cos\theta\sin\theta$ describes the potential electromagnetic forces in the first approximation whilst the term $-\varepsilon\alpha\cos^2\theta\,\dot{\theta}$ stands for "formally dissipative" forces which according to Section 2.1 are of the destabilizing character for $r < 1$. In what follows we consider only the case $r < 1$ and in order to make the approach applicable for analysis of self-oscillations we add a dissipative term in eq. (2.2.6) describing a external viscous damping. As a result the equation of slow oscillations of the pendulum takes the form

$$\ddot{\theta} + \varepsilon(n - \alpha\cos^2\theta)\dot{\theta} + (\beta\cos\theta + 1)\sin\theta = 0. \quad (2.2.7)$$

Depending on the value of β the pendulum can have either two or four equilibrium positions. If $\beta \le 1$ there are two equilibrium position $\theta = 0$ and $\theta = \pi$, the lower being stable and the upper being unstable. In the case of $\beta > 1$ there appear two additional saddle equilibrium positions $\theta = \pi \pm \arccos(1/\beta)$. If $n > \alpha$ both upper and lower equilibria are stable, i.e. the unstable upper position becomes stable due to the oscillating electromagnetic forces. For $n = \alpha$ the stable focus in the equilibrium positions $\theta = 0, \pi$ passes to a complex focus of the first order. When n decreases, these equilibrium positions become unstable giving rise to a soft birth of the limit cycles which means originating self-oscillations with frequencies$\sqrt{1+\beta}$ and $\sqrt{1-\beta}$ near the lower and upper equilibrium positions respectively.

A further reduce in damping results in a collapse of self-oscillations. The limit cycle near the upper equilibrium position collapses because it first merges with the separatrix from the saddle $\pi - \arccos(1/\beta)$ to saddle $\pi + \arccos(1/\beta)$ and then, in just the same way, it merges with the "own" separatrix near the lower equilibrium. The boundaries of the region of parameters causing the collapse of limit cycles can be found by using

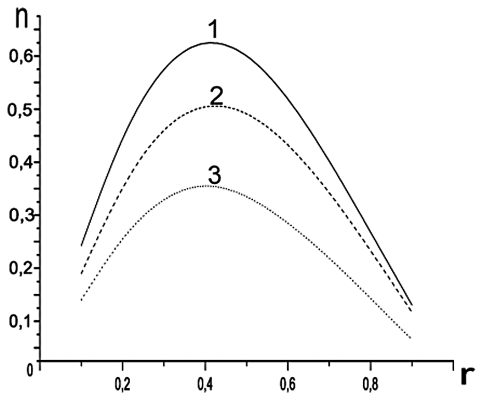

FIGURE 2.2.

the method of small parameter for the quasi-conservative system (2.2.7). Merging the limit cycle with the separatrix of the corresponding conservative systems corresponds to a zero of Pontryagin's function [10]

$$\Psi(h, \phi_1, \phi_2) = 2 \int_{\phi_1}^{\phi_2} (n - \alpha \cos^2 \theta) \sqrt{\beta \cos^2 \theta + 2 \cos \theta + 2h} d\theta, \qquad (2.2.8)$$

where ϕ_1, ϕ_2 denote the coordinates of saddle points connected by the corresponding separatrix, $h = 1/2\beta$ denotes the energy level corresponding to the motion along the separatrix. Thus, for determining $n = n_1(\alpha, \beta)$ for which the limit cycle around $\theta = \pi$ disappears we obtain the relationship

$$\Psi\left(\frac{1}{2\beta}, \pi - \arccos\frac{1}{\beta}, \pi + \arccos\frac{1}{\beta}\right) = 0. \qquad (2.2.9)$$

By analogy we find the expression for determining the parameters $n = n_2(\alpha, \beta)$ for which the cycle around $\theta = 0$ disappears

$$\Psi\left(\frac{1}{2\beta}, -\pi + \arccos\frac{1}{\beta}, \pi - \arccos\frac{1}{\beta}\right) = 0. \qquad (2.2.10)$$

Transforming relationships (2.2.9) and (2.2.10) to parameters γ and r we obtain the implicit dependences $n_1(r, \gamma)$ and $n_2(r, \gamma)$. All bifurcation curves $n(r)$ are shown in Fig. 2.2 for $\gamma = 5$.

In this figure the boundaries of stability of the equilibrium positions, existence of the limit cycles around $\theta = \pi$ and $\theta = 0$ are marked by 1,2 and 3 respectively.

In addition to slow oscillations, the electromechanical system exhibits also fast rotatory motions. These can be investigated by change of variables

$\psi = \theta - \Omega\nu t = \theta - \Omega\tau$ in eq. (2.2.3)

$$\ddot{\psi} - \varepsilon\gamma\cos(\psi + \Omega\tau)\sin\tau i + \varepsilon^2\sin(\psi + \Omega\tau) + \varepsilon^2 n(\dot{\psi} + \Omega) = 0,$$
$$\dot{i} + \cos(\psi + \Omega\tau)\sin\tau(\dot{\psi} + \Omega) + \sin(\psi + \Omega\tau)\cos\tau + ri = 0. \qquad (2.2.11)$$

Here in contrast to eq. (2.2.3) the mechanical dissipation is already taken into account. The physical meaning of ψ is the deviation of the trajectory of motion of the pendulum from the rotation with a constant angular speed $\Omega\nu$. Assuming that the rate of change of ψ is small, eq. (2.2.11) can be rewritten in the following form

$$\dot{\psi} = \varepsilon\omega,$$
$$\dot{\omega} = \varepsilon\gamma i\cos(\psi + \Omega\tau)\sin\tau - \varepsilon\sin(\psi + \Omega\tau) - \varepsilon n(\dot{\psi} + \Omega),$$
$$\dot{i} + (\dot{\psi} + \Omega)\cos(\psi + \Omega\tau)\sin\tau + \sin(\psi + \Omega\tau)\cos\tau + ri = 0. \qquad (2.2.12)$$

Similar to the case of slow oscillations system (2.2.12) contains two slow variables and one non-critical fast variable. Using the previous procedure we find the first approximation for the current

$$\begin{aligned} i_0 = &-\frac{1-\Omega}{2((1-\Omega)^2 + r^2)}\left[((1-\Omega)\sin\psi - r\cos\psi)\sin(1-\Omega)\tau + \right.\\ &\left. + (r\sin\psi + (1-\Omega)\cos\psi)\cos(1-\Omega)\tau\right] - \\ &- \frac{1+\Omega}{2((1+\Omega)^2 + r^2)}\left[((1+\Omega)\sin\psi + r\cos\psi)\sin(1+\Omega)\tau + \right.\\ &\left. + r(\sin\psi - (1+\Omega)\cos\psi)\cos(1+\Omega)\tau\right]. \end{aligned} \qquad (2.2.13)$$

Inserting eq. (2.2.13) into eq. (2.2.12) and averaging the obtained expressions over time we arrive at the equation for the first approximation to ψ

$$\ddot{\psi} - \varepsilon^2\frac{\gamma r}{8}\left(\frac{1-\Omega}{(1-\Omega)^2 + r^2} - \frac{1+\Omega}{(1+\Omega)^2 + r^2}\right) + \varepsilon^2 n\Omega = 0. \qquad (2.2.14)$$

Obviously, the solutions of system (2.2.12) which are close to stationary rotations describe the solution $\psi =$ const of the averaged equation. This solution exists if the following condition

$$\frac{\gamma r}{8}\left(\frac{1-\Omega}{(1-\Omega)^2 + r^2} - \frac{1+\Omega}{(1+\Omega)^2 + r^2}\right) - n\Omega = 0 \qquad (2.2.15)$$

holds. This equation provides one with the dependence of the frequency of rotation of the pendulum on the system parameters γ, r, n. Obviously, Ω can not exceed the unity since eq. (2.2.15) has no solution for any r, γ and n if $\Omega > 1$. Consequently, the frequency of stationary rotation of the pendulum can not exceed the frequency of oscillation of the magnetic field.

For analysis of stability of stationary rotations one must consider the second approximation as it adds an additional term proportional to $\dot{\psi}$ in eq. (2.2.14). As it will be shown below in this case the stability of rotations can be judged from qualitative reasoning.

The algebraic equation (2.2.15) can be rewritten in the form of a polynomial of fifth degree for Ω without free term. Hence it has a root $\Omega = 0$ for any parameters. This particular solution makes no sense since the very procedure of deriving eq. (2.2.14) is violated for $\Omega = 0$. In fact, this root corresponds to slow oscillations obtained above.

In order to determine the possible frequencies of stationary rotation of the pendulum we obtain the biquadratic equation with the roots

$$\Omega_{1,2}^2 = \frac{-(r\gamma - 8n + 8nr^2) \pm \sqrt{r^2\gamma^2 - 256n^2r^2}}{8n}. \qquad (2.2.16)$$

One can see from this expression that $\Omega \to 0$ if $n \to \dfrac{\gamma r(1-r^2)}{4(1+r^2)^2}$. It follows from eq. (2.2.16) that if $\gamma/16 < n$ and

$$\frac{\gamma r(1-r^2)}{4(1+r^2)^2} < n < \frac{\gamma r}{8(1-r^2)} \quad \left(r > \sqrt{2}-1\right),$$

then no rotation exists. If $\dfrac{\gamma r(1-r^2)}{4(1+r^2)^2} > n$ there are two rotations with frequencies equal in value but opposite in sign which corresponds to rotations in opposite directions. For $\dfrac{\gamma r(1-r^2)}{4(1+r^2)^2} > n$ slow motion of the pendulum is a monotonic motion from the equilibrium position. No self-oscillations appear and thus the obtained rotations must be stable. When the condition

$$\frac{\gamma r(1-r^2)}{4(1+r^2)^2} < n < \frac{\gamma}{16} r < \sqrt{2}-1$$

holds true the system assumes four rotations. In the phase space the trajectories of these rotations are symmetric about axis $\dot{\theta} = 0$. In the region of these parameters, slow motions of the system are stable, i.e. there exist limit cycles or stable equilibrium positions and all slow motions tend to them. Thus the rotations whose angular velocities are smaller (greater) in absolute value will be unstable (stable). The regions of parameters in the plane r, n with the qualitatively different types of motions are shown in Fig. 2.3.

Numeral analysis of integral (2.2.8) shows that the values $n = n_2(r, \gamma)$ describing the slow outgoing motions (i.e. the limit cycle near the equilibrium position $\theta = 0$ disappears) are close to $n = \dfrac{\gamma r(1-r^2)}{4(1+r^2)^2}$. Thus, the transition from four rotations to two rotations is realized by merging the

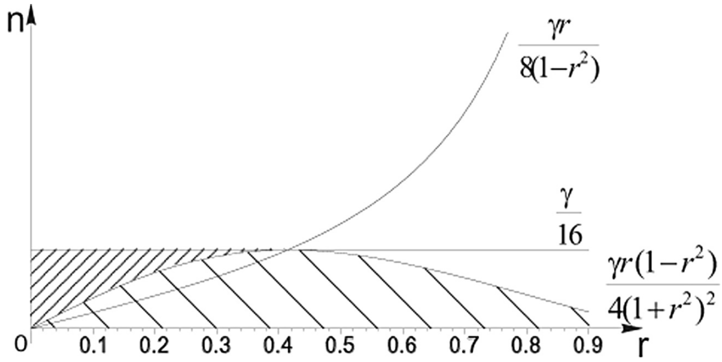

FIGURE 2.3.

unstable rotation and the limit cycle near $\theta = 0$ with the corresponding separatrix.

We can conclude that, depending on the relationship between the parameters and the initial conditions, the pendulum with a single conducting contour tends either to one of the equilibrium positions $\theta = 0, \pi$ or to the limit cycles near these equilibrium positions (that corresponds to self-oscillations) or to rotations with a frequency smaller than the frequency of external magnetic field.

2.3 Magnetic suspension in the field of alternating currents

It is shown in this section that the action of fast oscillating electromagnetic forces can be used for creating a new passive magnetic suspension of inductor type. In a simple case this suspension is the system of two rings with a joint vertical axis. One of the rings is fixed whereas the other is free. The rings conduct fast-oscillating alternating currents and this results in some stable equilibrium positions. Such suspensions can serve as an alternative to the superconducting suspension in a constant magnetic field known as the "magnetic potential well" [64].

A simple model of the contactless passive magnetic suspension can be presented in the form of a system of two aligned thin closed conducting rings (one above the other) in the gravity field, see Fig. 2.4. One of the rings is fixed and conducts an alternating current of amplitude i_* and frequency ω. The second ring is free and a voltage of the same frequency $U_c \cos \omega t + U_s \sin \omega t$ is applied to it. Let us assume that the geometrical sizes (the radius a and the cross-sectional diameter $2b$), the electric and magnetic parameters (inductance L and resistance R) of both rings are coincident. Let us first consider the case in which the free ring can move only vertically,

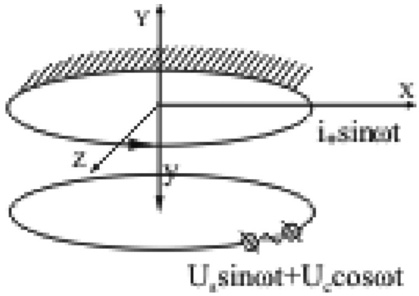

FIGURE 2.4.

i.e. parallel to the upper ring. Let y denote the distance between the rings, i be the current in the free ring and $L_e(y)$ be the coefficient of mutual induction of the rings. The Lagrange-Maxwell equations for the free ring have the form

$$m\ddot{y} - ii_* \sin \omega t \frac{dL_e}{dy} \pm mg = 0\,,$$

$$Li^{\cdot} + Ri + L_e i_* \omega \cos \omega t + i_* \sin \omega t \frac{dL_e}{dy} \dot{y} = U_s \sin \omega t + U_s \cos \omega t\,. \quad (2.3.1)$$

In the first equation, the sign $+$ $(-)$ in front of the gravity force corresponds to the case when the free ring is above (below) the fixed one, that is, axis y is directed upward (downward).

The coefficient of mutual induction of two conductors modelled by the linear currents is determined by the formula

$$L_e = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_a \cdot d\mathbf{l}_b}{R_*} = \frac{\mu_0}{4\pi} \oint \oint \frac{a^2 \cos(\phi - \psi) d\phi d\psi}{\sqrt{2a^2(1 - \cos(\phi - \psi)) + y^2}}\,. \quad (2.3.2)$$

Here $d\mathbf{l}_a$ and $d\mathbf{l}_b$ designate the elementary vectors directed along the tangent to the contours of the conductors, R_* is the distance between the corresponding elements.

Evaluating the double integral along the length of the rings we obtain

$$L_e(y) = \frac{\mu_0 a}{k} \left[(2 - k^2)K(k) - 2E(k)\right], \qquad k^2 = \frac{4a^2}{4a^2 + y^2}\,, \quad (2.3.3)$$

where $K(k)$ and $E(k)$ denote the complete elliptic Legendre integrals of the first and second kind respectively.

Introducing the dimensionless time $\tau = \omega t$ and the small parameter $\varepsilon^2 = \dfrac{g}{a\omega^2}$ $(\varepsilon << 1)$ in eq. (2.3.1) we obtain the equations for the current and motion of the free ring in terms of the dimensionless variables

$$
\begin{aligned}
\dot{i}_u + r i_u + L_{eu}\cos\tau + \frac{dL_{eu}}{dy_u}\sin\tau\,\dot{y}_u &= u_s\sin\tau + u_c\cos\tau\,,\\
\ddot{y}_u - i_u\gamma\varepsilon^2\sin\tau\frac{dL_{eu}}{dy_u} \pm \varepsilon^2 &= 0\,,
\end{aligned}
\tag{2.3.4}
$$

where

$$\gamma = \frac{Li_*^2}{mga}, u_s = \frac{U_s}{L\omega i_*}, u_c = \frac{U_s}{L\omega i_*}, r = \frac{R}{L\omega}, i_u = \frac{i}{i_*}, y_u = \frac{y}{a}, L_{eu} = \frac{L_e}{L}.$$

In what follows we omit the subscript u (which denotes the dimensionless quantities).

The solution of the problem is carried out by a modified method of averaging for quasi-linear systems, see Appendix A. Assuming $y = y_0 =$ const in eq. (2.3.4) we obtain that the first approximation for the current satisfies the equation

$$\dot{i_0} + r i_0 = u_s\sin\tau + (u_c - L_e)\cos\tau\,. \tag{2.3.5}$$

Hence, the first approximation to the stationary current in the free ring is given by

$$i_0 = \frac{1}{1+r^2}\left[(u_c - L_e + r u_s)\sin\tau + ((u_c - L_e)r - u_s)\cos\tau\right]\,. \tag{2.3.6}$$

Inserting eq. (2.3.6) into the first equation of system (2.3.4) and averaging over τ we obtain the first approximation to the averaged equation of motion for the free ring

$$\ddot{y} - \varepsilon^2\gamma\langle i\sin\tau\rangle\frac{dL_e}{dy} \pm \varepsilon^2 = 0\,, \tag{2.3.7}$$

where the denotation $\langle f(t)\rangle$ means the averaging over the period, that is,

$$\langle f(t)\rangle = \frac{1}{T}\int\limits_T f(t)dt.$$

The mean electromagnetic force acting on the free ring is equal to

$$P_W = \gamma\langle i\sin\tau\rangle\frac{dL_e}{dy} = \frac{\gamma}{2(1+r^2)}(u_c + r u_s - L_e)\frac{dL_e}{dy}\,. \tag{2.3.8}$$

It follows from eq. (2.3.8) that in the first approximation the electromagnetic force has the following potential

$$\Pi_W = -\frac{\gamma}{2(1+r^2)}\left[(u_c + r u_s)L_e - \frac{{L_e}^2}{2}\right]\,. \tag{2.3.9}$$

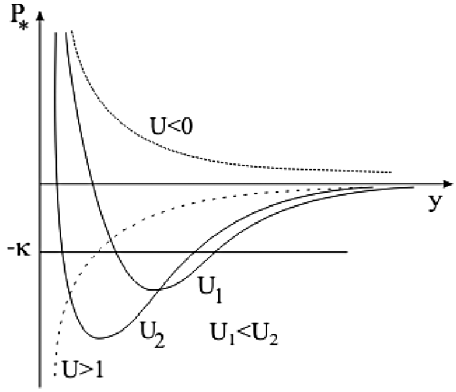

FIGURE 2.5.

In the equilibrium position the following relationship must hold

$$\frac{d\Pi_W}{dL_e}\frac{dL_e}{dy} \pm 1 = 0 \quad \Rightarrow \quad -(U - L_e)\frac{dL_e}{dy} \pm \kappa = 0\,. \tag{2.3.10}$$

Here parameter $U = u_c + ru_s$ characterizes the voltage in the free ring and parameter $\kappa = \dfrac{2(1+r^2)}{\gamma}$ is proportional to the ring weight.

As the derivative $\dfrac{dL_e}{dy} < 0$ for any y, for existence of the equilibrium position it is necessary that $\dfrac{d\Pi_W}{dL_e} > 0$ (i.e. $U - L_e < 0$) in the case of the upper free ring. The necessary and sufficient conditions for stability of this position are as follows

$$\frac{d^2\Pi_W}{dy^2} = \frac{\partial^2\Pi_W}{\partial L_e^2}\left(\frac{\partial L_e}{\partial y}\right)^2 + \frac{\partial\Pi_W}{\partial L_e}\frac{\partial^2 L_e}{\partial y^2} > 0\,. \tag{2.3.11}$$

This inequality holds for any y and parameters U and κ inasmuch as $\dfrac{\partial^2\Pi_W}{\partial L_e^2} = \dfrac{1}{\kappa} > 0$ and $\dfrac{\partial^2 L_e}{\partial y^2} > 0$ for any y. Thus, if equilibrium position above the fixed ring exists it is always stable with respect to y. Additionally, it follows from eq. (2.3.10) and inequality (2.3.11) that a stable equilibrium position of the upper free ring exists even in the case $U = 0$, i.e. if only the fixed ring conducts a current.

Let us consider the case when the free ring is under the fixed one. For existence of the equilibrium position the condition $\dfrac{\partial\Pi_W}{\partial L_e} < 0$ must hold, that is $U - L_e > 0$. The stability of this position is determined by inequality

(2.3.11) whose explicit form has a sufficiently complex structure

$$\frac{\mu_0 a}{L}\left[(2-k^2)E-2(1-k^2)K\right]^2 - \\ -(U-L_e)k^3\left[(1-k^2)K-(1-2k^2)E\right] > 0\,. \quad (2.3.12)$$

However, the existence and stability of the equilibrium positions of the free ring can be judged by means of the graphic approach. Figure 2.5 displays the graphs of the dependence $P_*(y) = 2P_W(1+r^2)/\gamma$ (dimensionless electromagnetic force) for several values of parameter U. For $0 < U < 1$ the points of intersection of the graph and the line $P_* = -\kappa$ determine two positions of equilibrium, the first (closer to the fixed ring) being stable and the second (distant) being unstable. One can see from the graphs that the closer U to 1, the greater weight can be stably suspended.

If $U < 0$, the average magnetic force is positive (i.e. is directed downward) and consequently, there is no equilibrium position under the fixed ring. If $U > 1$, the magnetic force is negative (is directed upwards) and increases monotonically with the growth of y therefore there exists only one equilibrium position which is unstable.

The characteristic phase trajectories obtained by means of the numeral integration of the averaged equation (2.3.7) are shown in Fig. 2.6 in the case of two equilibrium positions.

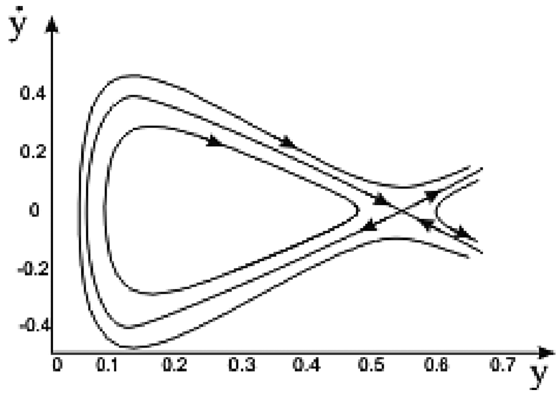

FIGURE 2.6.

The system of the first approximation (2.3.7) is conservative, thus in order to judge the actual stability of the equilibrium it is necessary to construct the second approximation to the current.

We assume the slow motion along axis y and consider the system

$$\dot{y} = \varepsilon x \,,$$
$$\dot{x} = \varepsilon \left(\frac{\gamma i}{2(1+r^2)} \frac{\partial L_e}{\partial y} \sin\tau + 1 \right) ,$$
$$\ddot{i} + ri + L_e \cos\tau + \frac{dL_e}{dy} \sin\tau \varepsilon x = u_s \sin\tau + u_c \cos\tau \,. \tag{2.3.13}$$

Applying the averaging method to eq. (2.3.13) (Appendix A) we obtain

$$i_1 = \frac{r(1-r^2)}{(1+r^2)^2} \frac{dL_e}{dy} \sin\tau + \frac{2r^2}{(1+r^2)^2} \frac{dL_e}{dy} \cos\tau \,. \tag{2.3.14}$$

The second approximation to the equation of motion for the free ring has the form

$$\ddot{y} + \varepsilon^2 \frac{\gamma r(1-r^2)}{2(1+r^2)^2} \left(\frac{dL_e}{dy} \right)^2 \dot{y} - \varepsilon^2 \left((U - L_e) \frac{\gamma}{2(1+r^2)} \frac{dL_e}{dy} + 1 \right) = 0 \,. \tag{2.3.15}$$

It follows from eq. (2.3.15) that for $r < 1$ the considered system has a dissipative element. In this case the equilibrium position becomes a stable focus (Fig. 2.7a). There is no stable suspension of the free ring if $r > 1$, that is, both equilibrium positions are unstable, Fig. 2.7b.

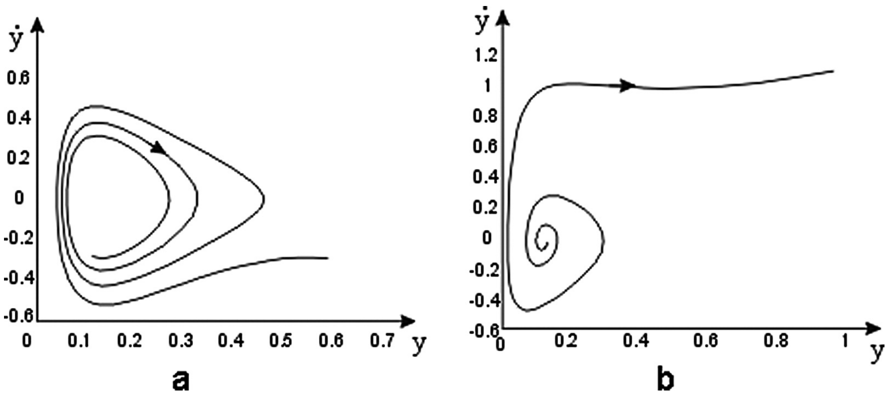

FIGURE 2.7.

Let us now consider the case of free motions of the suspended ring. It has six degrees of freedom described in terms of the coordinates ρ, α_*, y of the center of the free ring in the cylindrical system of coordinate with the origin at the center of the fixed ring (Fig. 2.8) and Euler's angles θ, ψ, ϕ for the orientation of the rings. The kinetic and potential energies and the

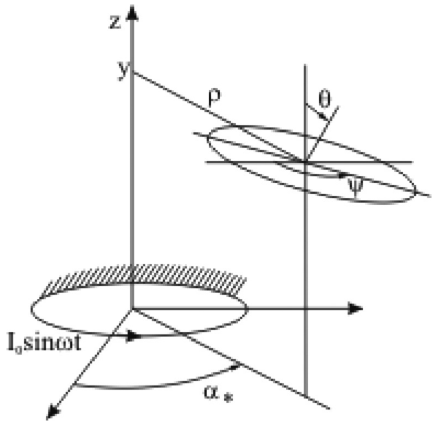

FIGURE 2.8.

energy of magnetic field have the form

$$T = \frac{1}{2}\left(m\dot{\rho}^2 + m\rho^2\dot{\alpha}_*^2 + m\dot{y}^2 + I_1\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + I_2\left(\dot{\phi} + \dot{\psi}\cos\theta^2\right)\right),$$

$$\Pi = \pm mgy, \quad W = \frac{1}{2}Li^2 + iL_e(\rho, \theta, y, \alpha_* - \psi)I_0\sin\omega t, \tag{2.3.16}$$

where I_1 and I_2 denote the moments of inertia about the vertical and horizontal axes of symmetry respectively.

The coefficient of mutual induction describing the mutual orientation of the rings is determined by the integral

$$L_e = \frac{\mu_0 a}{\pi}\int_0^{2\pi}\frac{1}{k\rho_*^{3/2}}\left[\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right](\cos\theta +$$
$$+ \rho(\sin\alpha\sin(\alpha_* - \psi) + \cos\alpha\cos\theta\cos(\alpha_* - \psi)))d\alpha\,. \tag{2.3.17}$$

Here

$$k^2 = \frac{4\rho_*}{(1+\rho_*)^2 + z^2}, \qquad z = \sin\alpha\sin\theta + \frac{y}{a},$$
$$\rho_* = 1 - \sin^2\alpha\sin^2\theta + \frac{\rho}{a}\left(\frac{\rho}{a} + 2\cos\alpha\cos(\alpha_* - \psi) +\right.$$
$$\left. + 2\sin\alpha\cos\theta\sin(\alpha_* - \psi)\right)^{1/2}. \tag{2.3.18}$$

The first approximation of the vector of dimensionless electromagnetic forces acting on the free ring is given by the following formula $P_W = \gamma\left\langle\sin\tau\left(\partial L_e/\partial\xi\right)^T i_0\right\rangle$, where $\xi = \left[\frac{\rho}{a}\theta, \frac{y}{a}, \psi, \phi, \alpha_*\right]^T$. Substituting expression (2.3.6) for the current i_0 we obtain

$$P_W = \frac{\gamma}{2(1+r^2)}(U - L_e)\left(\frac{\partial L_e}{\partial\xi}\right)^T. \tag{2.3.19}$$

These forces also have the potential

$$\Pi_W = \Pi_W(\rho, \theta, y, \alpha_* - \psi) = -\frac{\gamma}{2(1+r^2)}\left(UL_e - \frac{L_e^2}{2}\right). \qquad (2.3.20)$$

Since the combined potential energy $\Pi_\Sigma = \Pi + mga\Pi_W$ depends only on the difference of angles α_* and ψ it is reasonable to enter a new variable $\sigma = \alpha_* - \psi$. The system of Lagrange equations that govern motions of the free ring in the field of the fixed ring conducting the prescribed alternating current has the form

$$\begin{gathered}
I_2(\dot{\phi} + \dot{\psi}\cos\theta)^{\dot{}} = -\frac{\partial \Pi_\Sigma}{\partial \phi} = 0, \\
m\ddot{\rho} - m\rho(\dot{\sigma} + \dot{\psi})^2 = -\frac{\partial \Pi_\Sigma}{\partial \rho}, \\
m\rho^2(\ddot{\sigma} + \ddot{\psi}) + 2m(\dot{\sigma} + \dot{\psi})\rho\dot{\rho} = -\frac{\partial \Pi_\Sigma}{\partial \sigma}, \\
m\ddot{y} = -\frac{\partial \Pi_\Sigma}{\partial y}, \\
I_1\ddot{\theta} - I_1\dot{\psi}^2\sin\theta\cos\theta + I_2(\dot{\phi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta = -\frac{\partial \Pi_\Sigma}{\partial \theta}, \\
(m\rho^2(\dot{\sigma} + \dot{\psi}) + I_1\dot{\psi}\sin^2\theta + I_2(\dot{\phi} + \dot{\psi}\cos\theta)\cos\theta)^{\dot{}} = -\frac{\partial \Pi_\Sigma}{\partial \psi} = 0, \qquad (2.3.21)
\end{gathered}$$

These equations have two cyclic variables: ϕ and ψ. Entering two corresponding generalized momenta $p_1 = I_2(\dot{\phi} + \dot{\psi}\cos\theta)$ and $p_2 = m\rho^2(\dot{\sigma} + \dot{\psi}) + I_1\dot{\psi}\sin^2\theta + I_2(\dot{\phi} + \dot{\psi}\cos\theta)\cos\theta$ we obtain the dimensionless potential energy in the Routh form

$$\Pi_R = -\frac{\gamma}{2(1+r^2)}\left[UL_e - \frac{L_e^2(\rho,\theta,y,\sigma)}{2}\right] \pm y - \frac{1}{2}\frac{(r_1 - r_2\cos\theta)^2}{(ma^2\rho^2 + I_1\sin^2\theta)mga}, \qquad (2.3.22)$$

where y and ρ are the dimensionless sizes attributed to the ring radii. The equilibrium positions are determined from the condition $\dfrac{\partial \Pi_R}{\partial \xi} = 0$. Reasoning from symmetry of the problem one can conclude that the positions of equilibrium are possible only on axis z (Fig. 2.8) and hence this condition can be rewritten in the form $\left.\dfrac{\partial \Pi_R}{\partial y}\right|_{\rho=0,\theta=0,\sigma=0} = 0$. As shown above, this condition determines two positions of equilibrium under the fixed ring if $0 < U < 1$ and one equilibrium position above the fixed ring if $U < 1$. Let us find the parametric boundaries of stability of the equilibrium positions with respect to all coordinates.

A stable equilibrium position is given by a minimum of the potential energy which is characterized by the positive definiteness of the matrix of second derivatives of the energy in Routh's form which holds under the condition

$$\frac{d\Pi_R^2}{d\xi^2} = \frac{\partial \Pi_R^2}{\partial L_e^2}\left(\frac{\partial L_e}{\partial \xi}\right)^2 + \frac{\partial \Pi_R}{\partial L_e}\frac{\partial^2 L_e}{\partial \xi^2} > 0\,.$$

It is necessary to determine the values $\frac{\partial L_e}{\partial \xi}$ and $\frac{\partial^2 L_e}{\partial \xi^2}$ at the investigated equilibrium positions. Using expression (2.3.17) we find

$$\begin{aligned}
&\left.\frac{\partial L_e}{\partial \rho}\right|_0 = \left.\frac{\partial L_e}{\partial \theta}\right|_0 = \left.\frac{\partial L_e}{\partial \sigma}\right|_0 = 0,\\
&\left.\frac{\partial^2 L_e}{\partial \sigma^2}\right|_0 = \left.\frac{\partial^2 L_e}{\partial \sigma \partial \rho}\right|_0 = \left.\frac{\partial^2 L_e}{\partial \sigma \partial \theta}\right|_0 = \left.\frac{\partial^2 L_e}{\partial \sigma \partial y}\right|_0 = \left.\frac{\partial^2 L_e}{\partial \rho \partial y}\right|_0 = \left.\frac{\partial^2 L_e}{\partial \theta \partial y}\right|_0 = 0,\\
&\left.\frac{\partial L_e}{\partial y}\right|_0 = -\frac{\lambda}{2(1-k^2)^{1/2}}[(2-k^2)E - 2(1-k^2)K],\\
&\left.\frac{\partial^2 L_e}{\partial \rho^2}\right|_0 = \frac{\lambda k^3}{8(1-k^2)}[(1-2k^2)E - (1-k^2)K],\\
&\left.\frac{\partial^2 L_e}{\partial \rho \partial \theta}\right|_0 = \frac{\lambda \sin\sigma}{8(1-k^2)^{1/2}}[(4-k^2-2k^4)E + (-4+3k^2+k^4)K],\\
&\left.\frac{\partial^2 L_e}{\partial y^2}\right|_0 = \frac{\lambda k^3}{4(1-k^2)}[(1-k^2)K - (1-2k^2)E],\\
&\left.\frac{\partial^2 L_e}{\partial \theta^2}\right|_0 = \frac{\lambda k}{8(1-k^2)}[(2-3k^2+2k^4)E + (-2+3k^2-k^4)K].
\end{aligned} \tag{2.3.23}$$

Here $\lambda = \mu_0 a/L$ $k^2|_0 = 4/(4+y^2)$ and the denotation $|_0$ means that the function is taken at $\rho = 0, \theta = 0, \sigma = 0$.

Let us apply Routh's theorem on stability for $p_1 = p_2 = 0$. Taking into account eq. (2.3.23) we can set the second derivatives of the modified potential energy at zero generalized momenta in the form

$$\begin{aligned}
&\left.\frac{\partial^2 \Pi_R}{\partial \rho^2}\right|_0 = \left.\frac{\partial \Pi_R}{\partial L_e}\frac{\partial^2 L_e}{\partial \rho^2}\right|_0, \quad \left.\frac{\partial^2 \Pi_R}{\partial \rho \partial \theta}\right|_0 = \left.\frac{\partial \Pi_R}{\partial L_e}\frac{\partial^2 L_e}{\partial \rho \partial \theta}\right|_0,\\
&\left.\frac{\partial^2 \Pi_R}{\partial \theta^2}\right|_0 = \left.\frac{\partial \Pi_R}{\partial L_e}\frac{\partial^2 L_e}{\partial \theta^2}\right|_0, \quad \left.\frac{\partial^2 \Pi_R}{\partial y^2}\right|_0 = \left.\frac{\partial^2 \Pi_R}{\partial L_e^2}\left(\frac{\partial L_e}{\partial y}\right)^2\right|_0 + \left.\frac{\partial \Pi_R}{\partial L_e}\frac{\partial^2 L_e}{\partial y^2}\right|_0.
\end{aligned} \tag{2.3.24}$$

Since the equilibrium positions are "indifferent" with respect to σ the following conditions

$$1.\ \left.\frac{\partial^2 \Pi_R}{\partial \rho^2}\right|_0 > 0, \quad 2.\ \left.\frac{\partial^2 \Pi_R}{\partial \rho^2}\frac{\partial^2 \Pi_R}{\partial \theta^2} - \left(\frac{\partial^2 \Pi_R}{\partial \rho \partial \theta}\right)^2\right|_0 > 0, \quad 3.\ \left.\frac{\partial^2 \Pi_R}{\partial y^2}\right|_0 > 0. \tag{2.3.25}$$

are necessary and sufficient for stability of the suspended ring.

As already shown above, the third condition states that the equilibrium position above the fixed ring is stable with respect to y for all parameters ensuring $U < 1$ and stability of the equilibrium position under the fixed ring is achieved only for $0 < U < 1$.

As $\dfrac{\partial L_e}{\partial \rho} = 0$ the first condition in (2.3.25) can be rewritten as follows $\dfrac{\partial \Pi_R}{\partial L_e} \dfrac{\partial^2 L_e}{\partial \rho^2}\Big|_0 > 0$. For identical rings $\dfrac{\partial^2 L_e}{\partial \rho^2}\Big|_0 < 0$ for all values of y. Since $\dfrac{\partial \Pi_R}{\partial L_e} > 0$ in the equilibrium position above the fixed ring, the first condition does not hold for all parameters and hence the global stability is not possible.

Contrary to the previous case, $\dfrac{\partial \Pi_R}{\partial L_e} < 0$ in the lower equilibrium position, hence it is stable with respect to y, and in turn it is stable "with respect to ρ" for all parameters.

The second condition in eq. (2.3.25) is to be rewritten in the following form $\dfrac{\partial \Pi_R}{\partial L_e} \dfrac{\partial^2 L_e}{\partial \rho^2} \dfrac{\partial \Pi_R}{\partial L_e} \dfrac{\partial^2 L_e}{\partial \theta^2} - \left(\dfrac{\partial \Pi_R}{\partial L_e} \dfrac{\partial^2 L_e}{\partial \rho \partial \theta} \right)^2 > 0$, and consequently, the condition $F_{lim}(y_*) = \dfrac{\partial^2 L_e}{\partial \rho^2} \dfrac{\partial^2 L_e}{\partial \theta^2} - \left(\dfrac{\partial^2 L_e}{\partial \rho \partial \theta} \right)^2 > 0$ must hold. This inequality does not depend explicitly on parameters U and κ however it depends on y_* which is the coordinate of equilibrium position. This dependence is displayed in Fig. 2.9. As calculations show the critical value of y determining the zone of stability "with respect to θ" is $y_{lim} = 1.9029$. For $y_* > y_{lim}$ the equilibrium is stable. Figure 2.10 shows the parametric boundaries for the existing stable position of equilibrium of the suspended ring, namely the boundary for existing equilibrium position $\max \left[(U - L_e) \dfrac{\partial L_e}{\partial y} \right] = \kappa$ and the condition $y_* = y_{lim}$ are marked as 1 and 2 respectively.

Let us consider a numerical example. Let the radius of each ring be $a = 10$cm and the diameter of the cross-section be $b = 1$mm. We take the parameters ensuring the stability of the maximum weight with respect to all coordinates, i.e. $U = 0.02$ and $\kappa = 5 \cdot 10^{-5}$. Let us determine the strength of current needed for hanging a weight of 10g. Assuming that parameter r is close to 1 we obtain the strength of current $I \approx 10$kA. One can see from these calculations that a great alternating current is required for a stable suspending of such a small weight.

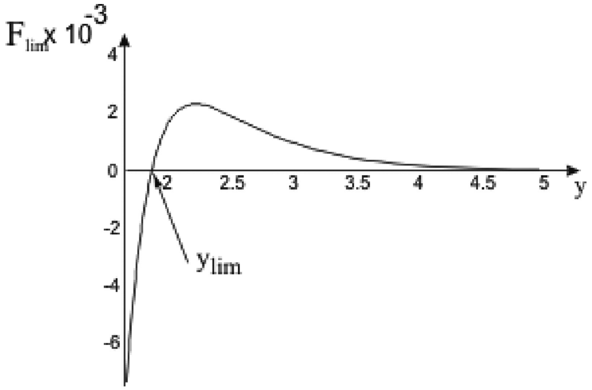

FIGURE 2.9.

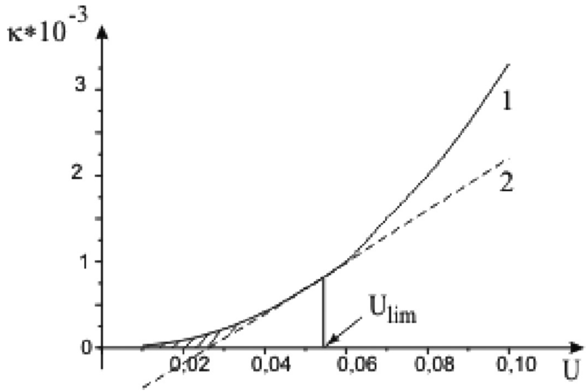

FIGURE 2.10.

Since the physical instability "with respect to θ" is related to the substantial heterogeneity of magnetic field near the fixed rings one can try to decrease the influence of this heterogeneity by considering a system in which the fixed ring has a greater radius than that of the suspended ring. Let us designate the ratio of rings' radii as $\eta = a_2/a_1 > 1$ where a_1 and a_2 denote radius of the free and fixed ring respectively. In this case the coefficient of mutual induction has the form

$$L_e = \frac{\mu_0 a_1 \sqrt{\eta}}{\pi} \int_0^{2\pi} \frac{1}{k\rho_*^{3/2}} \left[(1 - \frac{k^2}{2}) K(k) - E(k) \right]$$
$$\times (\cos\theta + \rho(\sin\alpha \sin\sigma + \cos\alpha \cos\theta \cos\sigma)) d\alpha, \quad (2.3.26)$$

where $k^2 = \dfrac{4\eta\rho_*}{(\eta+\rho_*)^2+z^2}$ (all dimensional coordinates are related to radius a_1). The derivatives of L_e are given by

$$\left.\frac{\partial L_e}{\partial y}\right|_0 = -\frac{\lambda(1-k^2)^{3/2}}{2\eta}[(2-k^2)E-2(1-k^2)K],$$

$$\left.\frac{\partial^2 L_e}{\partial \rho^2}\right|_0 = \frac{\lambda k^3}{8\sqrt{\eta}}\Big[(1-k^2)K-(1+k^2)E+$$
$$+\frac{(1+\eta)^2}{4\eta}\big[(1-k^2)(2-k^2)(E-K)+k^2(1+k^2)E\big]\Big],$$

$$\left.\frac{\partial^2 L_e}{\partial \rho\partial\theta}\right|_0 = \frac{\lambda y k\sin\sigma}{32\sqrt{\eta}}\Big[(8-16k^2+7k^4+k^6)K-(8-12k^2+2k^6)E-$$
$$\frac{k^2}{\eta}[2(1-k^2+k^4)E-(2-3k^2+k^4)K]\Big],$$

$$\left.\frac{\partial^2 L_e}{\partial y^2}\right|_0 = \frac{\lambda k^3}{4\sqrt{\eta}}\Big[(1+k^2)E-(1-k^2)K- \tag{2.3.27}$$
$$-\frac{(1+\eta)^2}{4\eta}\big[(1-k^2)(2-k^2)(E-K)+k^2(1+k^2)E\big]\Big],$$

$$\left.\frac{\partial^2 L_e}{\partial \theta^2}\right|_0 = \frac{\lambda\sqrt{\eta}k^3}{8}\left[(1-k^2)E-\frac{y^2}{4\eta^2}\big[2(1-k^2+k^4)E-(2-3k^2+k^4)K\big]\right]$$

where $\lambda = \mu_0 a_1/(1-k^2)^2$.

Figure 2.11 displays the dependences L_e on ρ, θ and y for different values of η. One can see that $\left.\dfrac{\partial L_e}{\partial y}\right|_0 = 0$ for any y (in contrast to the case of identical rings in which $\left.\dfrac{\partial L_e}{\partial y}\right|_0 = -\infty$). Using this property, one can construct the qualitative dependences of the dimensionless electromagnetic force ($P_* = P_W\kappa$) on y for different U (Fig. 2.12). The curve 1 corresponds to the case $U<0$, thus there is no equilibrium positions whilst the curve 2 describes the case $0<U<L_{e*}$ ($L_{e*} = \left.\dfrac{L_e}{L_1}\right|_0$). The curve 3 ($U > L_{e*}$) corresponds to the case of two equilibrium positions under the fixed ring, the position close to (far from) the fixed ring being stable (unstable).

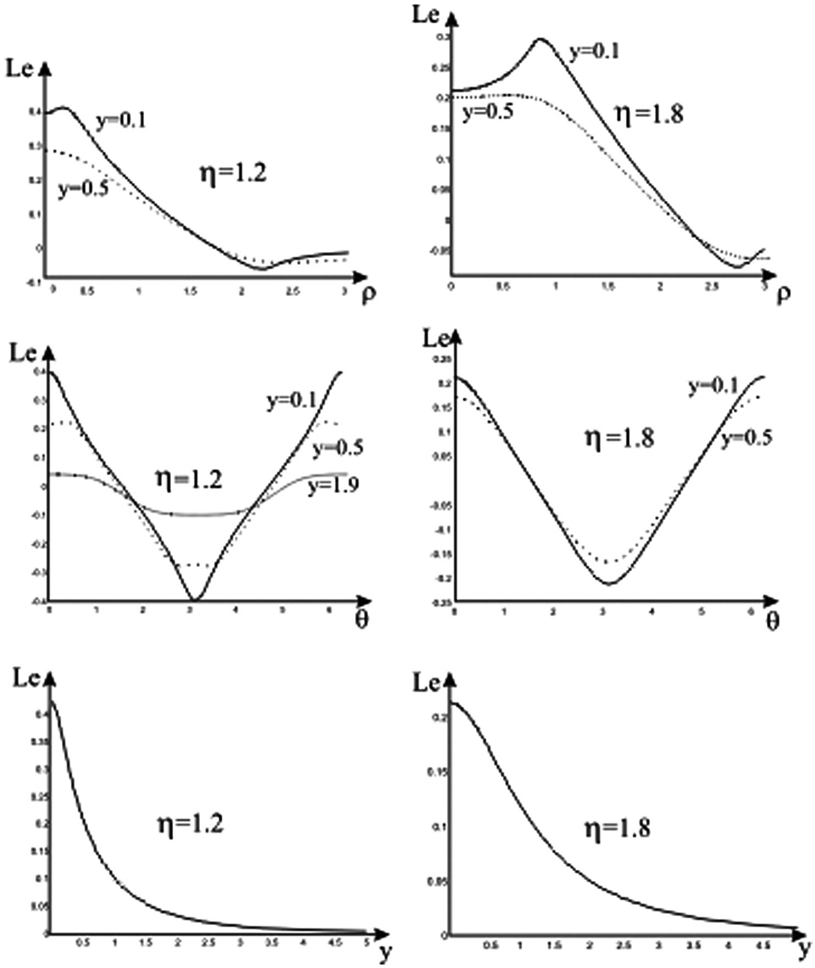
Le
y=0.1
y=0.5
η=1.2
ρ
Le
y=0.1
y=0.5
η=1.8
ρ
Le
y=0.1
y=0.5
y=1.9
η=1.2
θ
Le
y=0.1
y=0.5
η=1.8
θ
Le
η=1.2
y
Le
η=1.8
y

FIGURE 2.11.

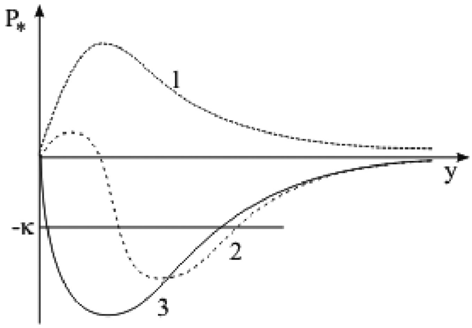

FIGURE 2.12.

The first condition of stability in eq. (2.3.25) holds for $\dfrac{\partial^2 L_e}{\partial \rho^2} < 0$. It determines the critical value of $y_\rho(U, \kappa, \eta)$. For $y_* > y_\rho$ the lower equilibrium position is stable "with respect to ρ" and unstable for $y_* < y_\rho$. The second condition determines another critical value y_θ, which corresponds to the boundary of stability "with respect to θ". The numeral analysis shows that $y_\theta > y_\rho$ for any η from the interval $1 \leq \eta \leq 2$.

The boundaries of existence of the stable position are shown in Fig. 2.13 for different values of η. One can see than the greater η, the broader the parametric region of stability, however the suspended weight becomes smaller for the same value of U. In addition to this, the stability region increases insignificantly as η grows and exceeds some critical value.

A further increase in the difference in radii of the rings does not improve the situation because the maximum electromagnetic force (attainable in the area of stability) exhibits an essentially nonlinear dependence on η and begins to decrease for $\eta > 1.8$. In other words, a smaller weight can be stably suspended for larger values of η, see Fig. 2.14.

Let us consider a numerical example. Let the suspended ring have the above geometrical sizes and the ratio of radii be equal to $\eta = 1.8$. We take the parameters which are needed for a stable suspension of the ring, that is $U = 0.3$ and $\kappa \approx 0.02$. The strength of current which is necessary for holding a weight of $10g$ is equal to $I \approx 1kA$ and the voltage in the free ring is $U_c \sim U_s \approx 0.15V$. Thus an increase in radius of the fixed ring leads to a tenfold decrease in the strength of current nevertheless the current remains too large. Since an equilibrium position (stable with respect to y) exists for large values of U and different radii of the rings one can stably hang any weight with a single degree of freedom on a shorter distance by means of a smaller current. For example, to fix a weight of 10g in the distance of 6cm from the fixed ring the required strength of current should be about 100A at voltage 5V or 20A at voltage 10V.

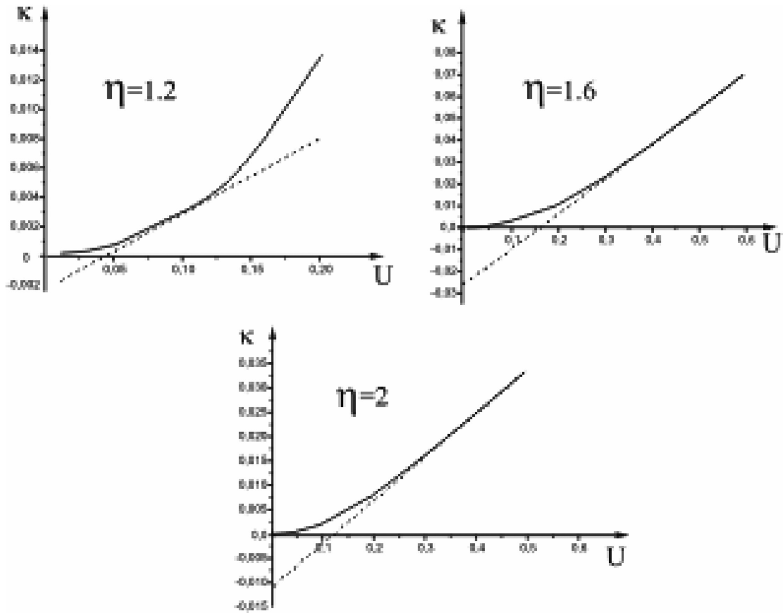

FIGURE 2.13.

In contrast to the case of two identical rings, the derivative $\dfrac{\partial^2 L_e}{\partial \rho^2}$ changes its sign for different values of y, hence for $y_* < y_{\rho g}$ the equilibrium position above the fixed ring is stable with respect to ρ. However the second condition of stability with respect to θ in eq. (2.3.25) holds only for $y_* > y_{\theta g}$, therefore a total stability of the overhead equilibrium position is not possible $(y_{\theta g} > y_{\rho g})$.

A practical realization of a ring suspension with a prescribed alternating current is rather difficult. In practice, an alternating voltage in the fixed ring is simpler to achieve.

Let us consider two identical rings (one fixed and one free) with the given voltages $U_1 \sin \omega t + V_1 \cos \omega t$ and $U_2 \sin \omega t + V_2 \cos \omega t$ respectively. The Lagrange-Maxwell equations for the free ring with a single vertical degree of freedom take the form

$$\begin{aligned} m\ddot{y} - i_1 i_2 \frac{dL_e}{dy} \pm mg &= 0, \\ L\dot{i}_1 + Ri_1 + L_e\dot{i}_2 + \frac{dL_e}{dy}\dot{y}ti_2 &= u_1 \sin \omega t + v_1 \cos \omega t, \\ L\dot{i}_2 + Ri_2 + L_e\dot{i}_1 + \frac{dL_e}{dy}\dot{y}ti_1 &= u_2 \sin \omega t + v_2 \cos \omega t. \end{aligned} \tag{2.3.28}$$

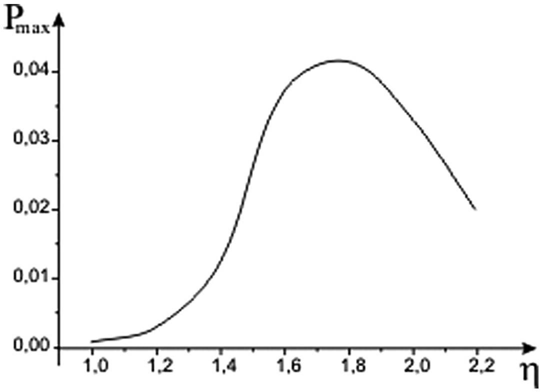

FIGURE 2.14.

Entering the dimensionless time $\tau = \omega t$ and the small parameter $\varepsilon = g/(a\omega^2)$ we obtain a dimensionless form of equations (2.3.28). Then applying the above method of averaging we obtain in the first approximation

$$i_{01} = c_1 \sin\tau + d_1 \cos\tau, \quad i_{02} = c_2 \sin\tau + d_2 \cos\tau, \tag{2.3.29}$$

where the coefficients are as follows

$$\begin{aligned}
s_1 &= \frac{L_e^3 v_2 + L_e^2(ru_1 - v_1) + L_e(v_2(r^2-1) - 2u_2 r) + (ru_1 + v_1)(1+r^2)}{\Delta}, \\
s_2 &= \frac{L_e^3 v_1 + L_e^2(ru_2 - v_2) + L_e(v_1(r^2-1) - 2u_1 r) + (ru_2 + v_2)(1+r^2)}{\Delta}, \\
d_1 &= \frac{-L_e^3 u_2 + L_e^2(rv_1 + u_1) + L_e(u_2(1-r^2) - 2v_2 r) + (rv_1 - u_1)(1+r^2)}{\Delta}, \\
d_2 &= \frac{-L_e^3 u_1 + L_e^2(rv_2 + u_2) + L_e(u_1(1-r^2) - 2v_1 r) + (rv_2 - u_2)(1+r^2)}{\Delta}, \\
&\Delta = (L_e^2 + r^2 - 1)^2 + 4r^2.
\end{aligned} \tag{2.3.30}$$

The denotation for all parameters in eq. (2.3.30) coincides with the previous ones except for i_* which denotes a basic current.

Using the obtained expressions for the currents we can determine the average electromagnetic force acting on the free ring:

$$\begin{aligned}
P &= \gamma \left\langle i_1 i_2 \frac{dL_e}{dy} \right\rangle = \frac{\gamma}{2}\frac{dL_e}{dy}[s_1 s_2 + d_1 d_2] = \\
&= \frac{\gamma}{2\Delta}\left(pL_e^2 - sL_e + r(1+r^2)\right)\frac{dL_e}{dy},
\end{aligned} \tag{2.3.31}$$

where $p = u_1 u_2 + v_1 v_2$ and $s = u_1^2 + u_2^2 + v_1^2 + v_2^2$. Depending on the relationship between the parameters p,s and r this force can have a qualitatively

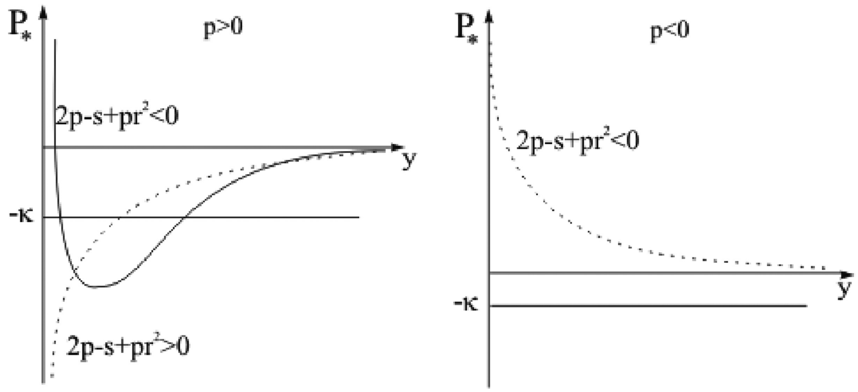

FIGURE 2.15.

different character, see Fig. 2.15. The positions of equilibrium are given by

$$-(pL_e^2 - sL_e + r(1+r^2))\frac{1}{\Delta}\frac{dL_e}{dy} \pm \kappa = 0, \quad \kappa = \frac{2mga}{Li_*^2}. \tag{2.3.32}$$

The necessary condition for the equilibrium position of the free ring above the fixed ring is $pL_e^2 - sL_e + p(1+r^2) < 0$ which limits the value of the voltage. This is possible when the following conditions hold true: $s^2 - 4p^2(1+r^2) > 0$ (the discriminant is positive, i.e. L_e is real-valued) and $0 < \dfrac{s - \sqrt{s^2 - 4r^2(1+r^2)}}{2p} < 1$. These two conditions reduce to a single condition, namely $\dfrac{s}{2p} > 1 + \dfrac{r^2}{2}$ for $p > 0$ and $\dfrac{s}{2p} < 1 + \dfrac{r^2}{2}$ for $p < 0$. The first inequality (for $p > 0$) is a necessary condition for existence of two positions of equilibrium in the case when the suspended ring is beneath. If this condition is not satisfied the suspended ring has only one equilibrium position which is unstable. The dependence of κ/p on s/p is shown in Fig. 2.16 and determines the boundary for the equilibrium position under the fixed ring, this position being stable with respect to y.

For analysis of the total stability of the suspended ring (i.e. with respect to all coordinates) one can make use of eq. (2.3.21) with the only difference that the potential of electromagnetic forces Π_W has a more difficult character of the dependence on L_e

$$\Pi_W = -\frac{\gamma}{8r}\left((2r-s)\arctan\frac{L_e-1}{r} + (2r+s)\arctan\frac{L_e+1}{r}\right). \tag{2.3.33}$$

Since in the stable equilibrium position $pL_e^2 - sL_e + p(1+r^2) > 0$ and $\dfrac{\partial^2 L_e}{\partial \rho^2} < 0$ (by virtue of the property of the coefficient of mutual induction

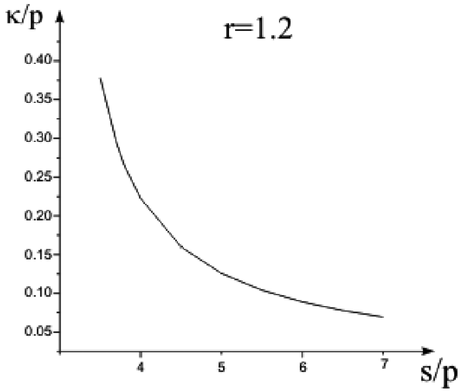

FIGURE 2.16.

of identical rings) this position is stable with respect to ρ for all parameters. The second condition in eq. (2.3.25) determines the critical value y_{lim} which, with the help of eq. (2.3.32), yields the dependence κ/p on s/p describing the border of total stability (in Fig. 2.17 the region of stability is marked by a shading).

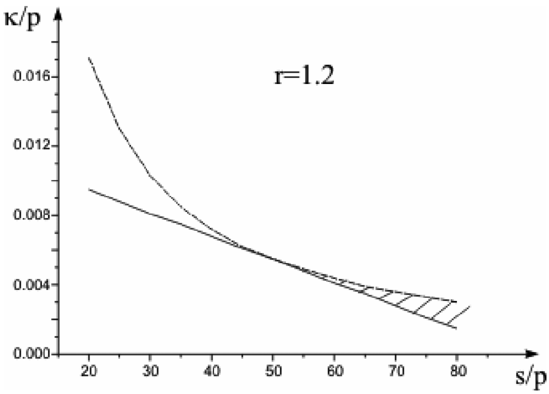

FIGURE 2.17.

The computations show that, in spite of small voltages needed for a stable suspension of the weight, the currents in the rings are too great. However similar to the case of rings with the prescribed current, elimination of the rotational degree of freedom leads to a considerable decrease in the strength of current and makes such a construction feasible. Additionally, an increase of the electromagnetic power can be achieved if a system of electromagnetic rings in series is used.

Springer

http://www.springer.com/978-3-540-25139-2

Non-Linear Electromechanics
Skubov, D.; Khodzhaev, K.S.
2008, XIV, 399 p., Hardcover
ISBN: 978-3-540-25139-2