Chapter 2

The laws of some quadratic functionals of BM

In Chapter 1, we studied a number of properties of the Gaussian space of Brownian motion; this space may be seen as corresponding to the first level of complexity of variables which are measurable with respect to $\mathcal{F}_\infty \equiv \sigma\{B_s, s \geq 0\}$, where $(B_s, s \geq 0)$ denotes Brownian motion. Indeed, recall that N. Wiener proved that every $L^2(\mathcal{F}_\infty)$ variable X may be represented as:

$$X = E(X) + \sum_{n=1}^{\infty} \int_0^\infty dB_{t_1} \int_0^{t_1} dB_{t_2} \dots \int_0^{t_{n-1}} dB_{t_n} \varphi_n(t_1, \dots, t_n)$$

where φ_n is a deterministic Borel function which satisfies:

$$\int_0^\infty dt_1 \dots \int_0^{t_{n-1}} dt_n \varphi_n^2(t_1, \dots, t_n) < \infty \quad .$$

In this Chapter, we shall study the laws of some of the variables X which correspond to the second level of complexity, that is: which satisfy $\varphi_n = 0$, for $n \geq 3$. In particular, we shall obtain the Laplace transforms of certain quadratic functionals of B, such as:

$$\alpha B_t^2 + \beta \int_0^t ds\ B_s^2 \ , \quad \int_0^t d\mu(s) B_s^2 \quad , \quad \text{and so on...}$$

2.1 Lévy's area formula and some variants

(2.1.1) We consider $(B_t, t \geq 0)$ a δ-dimensional BM starting from $a \in \mathbb{R}^\delta$. We write $x = |a|^2$, and we look for an explicit expression of the quantity:

$$I_{\alpha,b} \stackrel{\text{def}}{=} E\left[\exp\left(-\alpha|B_t|^2 - \frac{b^2}{2}\int_0^t ds|B_s|^2\right)\right] \quad .$$

We now show that, as a consequence of Girsanov's transformation, we may obtain the following formula[1] for $I_{\alpha,b}$:

$$I_{\alpha,b} = \left(\mathrm{ch}(bt) + 2\frac{\alpha}{b}\mathrm{sh}(bt)\right)^{-\delta/2} \exp -\frac{xb}{2}\frac{\left(1 + \frac{2\alpha}{b}\coth bt\right)}{\left(\coth(bt) + \frac{2\alpha}{b}\right)} \tag{2.1}$$

Proof: We may assume that $b \geq 0$. We consider the new probability $P^{(b)}$ defined by:

$$P^{(b)}_{|\mathcal{F}_t} = \exp\left\{-\frac{b}{2}\left(|B_t|^2 - x - \delta t\right) - \frac{b^2}{2}\int_0^t ds|B_s|^2\right\} \cdot P_{|\mathcal{F}_t} \quad .$$

Then, under $P^{(b)}$, $(B_u, u \leq t)$ satisfies the following equation,

$$B_u = a + \beta_u - b\int_0^u ds\ B_s \quad , \qquad u \leq t \quad ,$$

where $(\beta_u, u \leq t)$ is a $(P^{(b)}, \mathcal{F}_t)$ Brownian motion.

Hence, $(B_u, u \leq t)$ is an Ornstein-Uhlenbeck process with parameter $-b$, starting from a. Consequently, $(B_u, u \leq t)$ may be expressed explicitly in terms of β, as

$$B_u = e^{-bu}\left(a + \int_0^u e^{bs}d\beta_s\right) \quad , \tag{2.2}$$

a formula from which we can immediately compute the mean and the variance of the Gaussian variable B_u (considered under $P^{(b)}$). This clearly solves the problem, since we have:

[1] Throughout the volume, we use the French abbreviations ch, sh, th for, respectively, cosh, sinh, tanh,...

$$I_{\alpha,b} = E^{(b)}\left[\exp\left(-\alpha|B_t|^2 + \frac{b}{2}\left(|B_t|^2 - x - \delta t\right)\right)\right] \quad ,$$

and formula (2.1) now follows from some straightforward, if tedious, computations. $\square$

Exercise 2.1: Show that $\exp\left\{\frac{b}{2}\left(|B_t|^2 - x - \delta t\right) - \frac{b^2}{2}\int_0^t ds|B_s|^2\right\}$ is also a $(P, (\mathcal{F}_t))$ martingale, and that we might have considered this martingale as a Radon-Nikodym density to arrive to the same formula (2.1).

(2.1.2) The same method allows to compute the joint Fourier-Laplace transform of the pair: $\left(\int_0^t f(u)dB_u, \ \int_0^t du\ B_u^2\right)$ where for simplicity, we take here the dimension δ to be 1.

Indeed, to compute:

$$E\left[\exp\left(i\int_0^t f(u)dB_u - \frac{b^2}{2}\int_0^t du\ B_u^2\right)\right] \quad , \tag{2.3}$$

all we need to know, via the above method, is the joint distribution of $\int_0^t f(u)dB_u$ and B_t, under $P^{(b)}$.

This is clearly equivalent to being able to compute the mean and variance of $\int_0^t g(u)dB_u$, for any $g \in L^2([0,t], du)$.

However, thanks to the representation (2.2), we have:

$$\begin{aligned}
\int_0^t g(u)dB_u &= \int_0^t g(u)\left\{-be^{-bu}du \cdot \left(a + \int_0^u e^{bs}d\beta_s\right) + e^{-bu}(e^{bu}d\beta_u)\right\} \\
&= -ba\int_0^t g(u)e^{-bu}du + \int_0^t d\beta_u\left(g(u) - e^{bu}b\int_u^t e^{-bs}g(s)ds\right) \quad .
\end{aligned}$$

Hence, the mean of $\int_0^t g(u)dB_u$ under $P^{(b)}$ is: $-ba\int_0^t g(u)e^{-bu}du$, and its variance is: $\int_0^t du\left(g(u)-be^{bu}\int_u^t e^{-bs}g(s)ds\right)^2$.

We shall not continue the discussion at this level of generality, but instead, we indicate one example where the computations have been completely carried out.

The next formulae will be simpler if we work in a two-dimensional setting; therefore, we shall consider $Z_u = X_u + iY_u$, $u \geq 0$, a $\mathbb{C}$-valued BM starting from 0, and we define $G = \int_0^1 ds\, Z_s$, the barycenter of Z over the time-interval [0,1].

The above calculations lead to the following formula (taken with small enough $\rho, \sigma \geq 0$):

$$\begin{aligned} &E\left[\exp -\frac{\lambda^2}{2}\left(\int_0^1 ds|Z_s|^2 - \rho|G|^2 - \sigma|Z_1|^2\right)\right] \\ &= \left\{(1-\rho)\mathrm{ch}\lambda + \rho\frac{\mathrm{sh}\lambda}{\lambda} + \sigma\left[(\rho-1)\lambda\mathrm{sh}\lambda - 2\rho(\mathrm{ch}\lambda - 1)\right]\right\}^{-1} \end{aligned} \tag{2.4}$$

which had been obtained by a different method by Chan-Dean-Jansons-Rogers [26].

(2.1.3) Before we continue with some consequences of formulae (2.1) and (2.4), let us make some remarks about the above method:

it consists in changing probability so that the quadratic functional disappears, and the remaining problem is to compute the mean and variance of a Gaussian variable. Therefore, this method consists in transfering some computational problem for a variable belonging to (the first and) the second Wiener chaos to computations for a variable in the first chaos; in other words, it consists in a *linearization* of the original problem.

In the last paragraph of this Chapter, we shall use this method again to deal with the more general problem, when $\int_0^t ds|B_s|^2$ is replaced by $\int_0^t d\mu(s)|B_s|^2$.

(2.1.4) A number of computations found in the literature can be obtained very easily from the formulae (2.1) and (2.4).

a) The following formula is easily deduced from formula (2.1):

$$E_a\left[\exp -\frac{b^2}{2}\int_0^t ds|B_s|^2 \mid B_t = 0\right] = E_0\left[\exp -\frac{b^2}{2}\int_0^t ds|B_s|^2 \mid B_t = a\right]$$
$$= \left(\frac{bt}{\mathrm{sh}(bt)}\right)^{\delta/2} \exp -\frac{|a|^2}{2t}\left(bt\coth(bt) - 1\right) \tag{2.5}$$

which, in the particular case $a = 0$, yields the formula:

$$E_0\left[\exp\left(-\frac{b^2}{2}\int_0^t ds|B_s|^2\right) \mid B_t = 0\right] = \left(\frac{bt}{\mathrm{sh}(bt)}\right)^{\delta/2} \tag{2.6}$$

Lévy's formula for the stochastic area

$$\mathcal{A}_t \stackrel{\mathrm{def}}{=} \int_0^t (X_s dY_s - Y_s dX_s)$$

of planar Brownian motion $B_t = (X_t, Y_t)$ may now be deduced from formula (2.5); precisely, one has:

$$E_0\left[\exp(ib\mathcal{A}_t) \mid B_t = a\right] = \left(\frac{bt}{\mathrm{sh}\, bt}\right) \exp -\frac{|a|^2}{2t}(bt\coth bt - 1) \tag{2.7}$$

To prove formula (2.7), first remark that, thanks to the rotational invariance of the law of Brownian motion (starting from 0), we have:

$$E_0\left[\exp(ib\mathcal{A}_t) \mid B_t = a\right] = E_0\left[\exp(ib\mathcal{A}_t) \mid |B_t| = |a|\right] \quad ,$$

and then, we can write:

$$\mathcal{A}_t = \int_0^t |B_s| d\gamma_s \quad ,$$

where $(\gamma_t, t \geq 0)$ is a one dimensional Brownian motion independent from $(|B_s|, s \geq 0)$. Therefore, we obtain:

$$E_0\left[\exp(ib\mathcal{A}_t) \mid |B_t| = |a|\right] = E_0\left[\exp\left(-\frac{b^2}{2}\int_0^t ds|B_s|^2\right) \mid B_t = a\right]$$

and formula (2.7) is now deduced from formula (2.5).

b) Similarly, from formula (2.4), one deduces:

$$\begin{aligned} &E\left[\exp\left(-\frac{\mu^2}{2}\int_0^1 ds|Z_s - G|^2\right) \mid Z_1 = z\right] \\ &= \left(\frac{\mu/2}{\mathrm{sh}\mu/2}\right)^2 \exp -\frac{|z|^2}{2}\left(\frac{\mu}{2}\coth\frac{\mu}{2} - 1\right) \end{aligned} \tag{2.8}$$

c) As yet another example of application of the method, we now derive the following formula obtained by M. Wenocur [91] (see also, in the same vein, [92]):
consider $(W(t), t \geq 0)$ a 1-dimensional BM, starting from 0, and define: $X_t = W_t + \mu t + x$, so that $(X_t, t \geq 0)$ is the Brownian motion with drift μ, starting from x.

Then, M. Wenocur [91] obtained the following formula:

$$E\left[\exp\left(-\frac{\lambda^2}{2}\int_0^1 ds\ X_s^2\right)\right] = \frac{1}{(\mathrm{ch}\lambda)^{1/2}}\exp(H(x,\mu,\lambda)) \quad , \tag{2.9}$$

where

$$H(x,\mu,\lambda) = -\frac{\mu^2}{2}\left(1 - \frac{\mathrm{th}\lambda}{\lambda}\right) - x\mu\left(1 - \frac{1}{\mathrm{ch}\lambda}\right) - \frac{x^2}{2}\lambda\mathrm{th}\lambda \quad .$$

We shall now sketch a proof of this formula, by applying twice Girsanov's theorem. First of all, we may "get rid of the drift μ", since:

$$\begin{aligned} &E\left[\exp\left(-\frac{\lambda^2}{2}\int_0^1 ds\ X_s^2\right)\right] \\ &= E_x\left[\exp\left(\mu(X_1 - x) - \frac{\mu^2}{2}\right)\exp -\frac{\lambda^2}{2}\int_0^1 ds\ X_s^2\right] \end{aligned}$$

where P_x denotes the law of Brownian motion starting from x. We apply Girsanov's theorem a second time, thereby replacing P_x by $P_x^{(\lambda)}$, the law of the Ornstein-Uhlenbeck process, with parameter λ, starting from x. We then obtain:

$$E_x\left[\exp\left(\mu X_1 - \frac{\lambda^2}{2}\int_0^1 ds\, X_s^2\right)\right]$$
$$= E_x^{(\lambda)}\left[\exp\left(\mu X_1 + \frac{\lambda}{2}X_1^2\right)\exp\left(-\frac{\lambda}{2}(x^2+1)\right)\right] \quad ,$$

and it is now easy to finish the proof of (2.9), since, as shown at the beginning of this paragraph, the mean and variance of X_1 under $P_x^{(\lambda)}$ are known.

Exercise 2.2: 1) Extend formula (2.9) to a δ-dimensional Brownian motion with constant drift.

2) Derive formula (2.1) from this extended formula (2.9).

Hint: Integrate both sides of the extended formula (2.9) with respect to $d\mu \exp -\left(c|\mu|^2\right)$ on $\mathbb{R}^\delta$.

Exercise 2.3: Let $(B_t, t \geq 0)$ be a 3-dimensional Brownian motion starting from 0.

1. Prove the following formula:
for every $m \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$ with $|\xi| = 1$, and $\lambda \in \mathbb{R}^*$,

$$E\left[\exp\left(i\lambda\xi \cdot \int_0^1 B_s \times dB_s\right) \mid B_1 = m\right]$$
$$= \left(\frac{\lambda}{\mathrm{sh}\lambda}\right)\exp\left(\frac{|m|^2 - (\xi\cdot m)^2}{2}(1-\lambda\coth\lambda)\right) \quad ,$$

where $x \cdot y$, resp.: $x \times y$, denotes the scalar product, resp.: the vector product, of x and y in $\mathbb{R}^3$.

Hint: Express $\xi \cdot \int_0^1 B_s \times dB_s$ in terms of the stochastic area of the 2-dimensional Brownian motion: $(\eta \cdot B_s; (\xi \times \eta) \cdot B_s;\ s \geq 0)$ where η is a suitably chosen unit vector of $\mathbb{R}^3$, which is orthogonal to ξ.

2. Prove that, for any $\lambda \in \mathbb{R}^*$, $z \in \mathbb{R}^3$, and $\xi \in \mathbb{R}^3$, with $|\xi| = 1$, one has:

$$E\left[\exp i \left(z \cdot B_1 + \lambda \xi \cdot \int_0^1 B_s \times dB_s \right)\right]$$
$$= \frac{1}{(\mathrm{ch}\lambda)} \exp -\frac{1}{2}\left(|z|^2 \frac{\mathrm{th}\lambda}{\lambda} + (z \cdot \xi)^2 \left(1 - \frac{\mathrm{th}\lambda}{\lambda}\right)\right) \quad .$$

2.2 Some identities in law and an explanation of them via Fubini's theorem

(2.2.1) We consider again formula (2.4), in which we take $\rho = 1$, and $\sigma = 0$. We then obtain:

$$E\left[\exp\left(-\frac{\lambda^2}{2} \int_0^1 ds |Z_s - G|^2 \right)\right] = \frac{\lambda}{\mathrm{sh}\lambda} \quad ,$$

but, from formula (2.6), we also know that, using the notation $(\tilde{Z}_s, s \le 1)$ for the complex Brownian bridge of length 1:

$$E\left[\exp\left(-\frac{\lambda^2}{2} \int_0^1 ds |\tilde{Z}_s|^2 \right)\right] = \frac{\lambda}{\mathrm{sh}\lambda} \quad ;$$

hence, the following identity in law holds:

$$\int_0^1 ds |Z_s - G|^2 \overset{\text{(law)}}{=} \int_0^1 ds |\tilde{Z}_s|^2 \; , \tag{2.10}$$

an identity which had been previously noticed by several authors (see, e.g., [33]).

Obviously, the fact that, in (2.10), Z, resp. $\tilde{Z}$, denotes a complex valued BM, resp. Brownian bridge,instead of a real-valued process, is of no importance, and (2.10) is indeed equivalent to:

$$\int_0^1 dt (B_t - G)^2 \overset{\text{(law)}}{=} \int_0^1 dt \tilde{B}_t^2 \quad , \tag{2.11}$$

where $(B_t, t \le 1)$, resp. $(\tilde{B}_t, t \le 1)$ now denotes a 1-dimensional BM, resp. Brownian bridge, starting from 0.

(2.2.2) Our first aim in this paragraph is to give a simple explanation of (2.11) via Fubini's theorem.

Indeed, if B and C denote two independent Brownian motions and $\varphi \in L^2([0,1], du\, ds)$, we have:

$$\int_0^1 dB_u \int_0^1 dC_s \varphi(u,s) \overset{\text{a.s.}}{=} \int_0^1 dC_s \int_0^1 dB_u \varphi(u,s) \quad ,$$

which, as a corollary, yields:

$$\int_0^1 du \left(\int_0^1 dC_s \varphi(u,s) \right)^2 \overset{\text{(law)}}{=} \int_0^1 du \left(\int_0^1 dC_s \varphi(s,u) \right)^2 \tag{2.12}$$

(in the sequel, we shall refer to this identity as to the "Fubini-Wiener identity in law").

The identity (2.11) is now a particular instance of (2.12), as the following Proposition shows.

Proposition 2.1 *Let $f : [0,1] \to \mathbb{R}$ be a C^1-function such that $f(1) = 1$. Then, we have:*

$$\int_0^1 ds \left(B_s - \int_0^1 dt\, f'(t) B_t \right)^2 \overset{\text{(law)}}{=} \int_0^1 ds (B_s - f(s) B_1)^2 \ . \tag{2.13}$$

In particular, in the case $f(s) = s$, we obviously recover (2.11).

Proof: It follows from the identity in law (2.12), where we take:

$$\varphi(s,u) = \left(1_{(u \le s)} - (f(1) - f(u))\right) 1_{((s,u)\in[0,1]^2)} \ . \qquad \square$$

Here is another variant, due to Shi Zhan, of the identity in law (2.13).

Exercise 2.4: Let $\mu(dt)$ be a probability on $\mathbb{R}_+$. Then, prove that:

$$\int_0^\infty \mu(dt) \left(B_t - \int_0^\infty \mu(ds) B_s \right)^2 \stackrel{\text{(law)}}{=} \int_0^\infty \tilde{B}^2_{\mu[0,t]} dt \quad ,$$

where $(\tilde{B}_u, u \leq 1)$ is a standard Brownian bridge.

As a second application of (2.12), or rather of a discrete version of (2.12), we prove a striking identity in law (2.14), which resembles the integration by parts formula.

Theorem 2.1 *Let $(B_t, t \geq 0)$ be a 1-dimensional BM starting from 0. Let $0 \leq a \leq b < \infty$, and $f, g : [a,b] \to \mathbb{R}_+$ be two continuous functions, with f decreasing, and g increasing.*

$$\int_a^b -df(x) B^2_{g(x)} + f(b) B^2_{g(b)} \stackrel{\text{(law)}}{=} g(a) B^2_{f(a)} + \int_a^b dg(x) B^2_{f(x)} \quad . \tag{2.14}$$

In order to prove (2.14), it suffices to show that the identity in law:

$$\begin{aligned} & - \sum_{i=1}^n (f(t_{i+1}) - f(t_i)) B^2_{g(t_i)} + f(t_n) B^2_{g(t_n)} \\ \stackrel{\text{(law)}}{=} \; & g(t_1) B^2_{f(t_1)} + \sum_{i=2}^n (g(t_i) - g(t_{i-1})) B^2_{f(t_i)} \quad , \end{aligned} \tag{2.15}$$

where $a = t_1 < t_2 < \cdots < t_n = b$, holds, and then to let the mesh of the subdivision tend to 0.

Now, (2.15) is a particular case of a discrete version of (2.12), which we now state.

Theorem 2.2 *Let $\mathbf{X}_n = (X_1, \ldots, X_n)$ be an n-dimensional Gaussian vector, the components of which are independent, centered, with variance 1. Then, for any $n \times n$ matrix A, we have:*

$$|A\mathbf{X}_n| \stackrel{\text{(law)}}{=} |A^* \mathbf{X}_n| \quad ,$$

where A^* is the transpose of A, and, if $\mathbf{x}_n = (x_1, \text{——}, x_n) \in \mathbb{R}^n$, we denote: $|\mathbf{x}_n| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$.

Corollary 2.2.1 *Let $(Y_1, \dots, Y_n)$ and $(Z_1, \dots, Z_n)$ be two n-dimensional Gaussian vectors such that*

i) $Y_1, Y_2 - Y_1, \dots, Y_n - Y_{n-1}$ *are independent;*

ii) $Z_n, Z_n - Z_{n-1}, \dots, Z_2 - Z_1$ *are independent.*

Then, we have

$$-\sum_{i=1}^{n} Y_i^2 \left(E(Z_{i+1}^2) - E(Z_i^2)\right) \overset{(\text{law})}{=} \sum_{i=1}^{n} Z_i^2 \left(E(Y_i^2) - E(Y_{i-1}^2)\right) \qquad (*)$$

where we have used the convention: $E(Z_{n+1}^2) = E(Y_0^2) = 0$.

The identity in law (2.15) now follows as a particular case of $(*)$.

2.3 The laws of squares of Bessel processes

Consider $(B_t, t \geq 0)$ a δ-dimensional ($\delta \in \mathbb{N}$, for the moment...) Brownian motion starting from a, and define: $X_t = |B_t|^2$. Then, $(X_t, t \geq 0)$ satisfies the following equation

$$X_t = x + 2\int_0^t \sqrt{X_s}\, d\beta_s + \delta t \quad , \qquad (2.16)$$

where $x = |a|^2$, and $(\beta_t, t \geq 0)$ is a 1-dimensional Brownian motion. More generally, from the theory of 1-dimensional stochastic differential equations, we know that for any pair $x, \delta \geq 0$, the equation (2.16) admits one strong solution, hence, a fortiori, it enjoys the uniqueness in law property.

Therefore, we may define, on the canonical space $\Omega_+^* \equiv C(\mathbb{R}_+, \mathbb{R}_+)$, Q_x^δ as the law of a process which satisfies (2.16).

The family $(Q_x^\delta, x \geq 0, \delta \geq 0)$ possesses the following additivity property, which is obvious for integer dimensions.

Theorem 2.3 *(Shiga-Watanabe [83]) For any $\delta, \delta', x, x' \geq 0$, the identity:*

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'}$$

holds, where $*$ denotes the convolution of two probabilities on Ω_+^*.

Now, for any positive, σ-finite, measure μ on $\mathbb{R}_+$, we define:

$$I_\mu(\omega) = \int_0^\infty d\mu(s) X_s(\omega) \quad ,$$

and we deduce from the theorem that there exist two positive constants $A(\mu)$ and $B(\mu)$ such that:

$$Q_x^\delta \left(\exp -\frac{1}{2} I_\mu \right) = (A(\mu))^x (B(\mu))^\delta \ .$$

The next theorem allows to compute $A(\mu)$ and $B(\mu)$.

Theorem 2.4 *For any ≥ 0 Radon measure μ on $[0, \infty)$, one has:*

$$Q_x^\delta \left(\exp -\frac{1}{2} I_\mu \right) = (\phi_\mu(\infty))^{\delta/2} \exp \left(\frac{x}{2} \phi_\mu^+(0) \right) \quad ,$$

where ϕ_μ denotes the unique solution of:

$$\phi'' = \mu\phi \ \ on \ \ (0, \infty) \ , \quad \phi_\mu(0) = 1, 0 \leq \phi \leq 1 \ ,$$

and $\phi_\mu^+(0)$ is the right derivative of ϕ_μ at 0.

Proof: For simplicity, we assume that μ is diffuse, and that its support is contained in $(0, 1)$.

Define: $F_\mu(t) = \dfrac{\phi'_\mu(t)}{\phi_\mu(t)}$, and $\hat{F}_\mu(t) = \displaystyle\int_0^t \frac{\phi'_\mu(s) ds}{\phi_\mu(s)} = \log \phi_\mu(t)$.

Then, remark that:

$$Z_t^\mu \stackrel{\text{def}}{=} \exp \left\{ \frac{1}{2} \left[F_\mu(t) X_t - F_\mu(0) x - \delta \hat{F}_\mu(t) \right] - \frac{1}{2} \int_0^t X_s d\mu(s) \right\}$$

is a Q_x^δ-martingale, since it may be written as:

$$\exp\left\{\int_0^t F_\mu(s)dM_s - \frac{1}{2}\int_0^t F_\mu^2(s)d\langle M\rangle_s\right\} ,$$

where: $M_t = \frac{1}{2}(X_t - \delta t)$, and $\langle M\rangle_t = \int_0^t ds\, X_s$.

It now remains to write: $Q_x^\delta(Z_1^\mu) = 1$, and to use the fact that $F_\mu(1) = 0$ to obtain the result stated in the theorem. □

Exercise 2.5: 1) Prove that the integration by parts formula (2.14) can be extended as follows:

$$(*) \qquad \int_a^b -df(x)X_{g(x)} + f(b)X_{g(b)} \stackrel{\text{(law)}}{=} g(a)X_{f(a)} + \int_a^b dg(x)X_{f(x)} ,$$

where X is a BESQ process, with any strictly positive dimension, starting from 0.

2) Prove the following convergence in law result:

$$\left(\sqrt{n}\left(\frac{1}{n}X_t^{(n)} - t\right), t \geq 0\right) \xrightarrow[n\to\infty]{\text{(law)}} (c\beta_{t^2}; t \geq 0) ,$$

for a certain constant $c > 0$, where $(X_t^{(n)}, t \geq 0)$ denotes a BESQ^n process, starting from 0, and $(\beta_t, t \geq 0)$ denotes a real-valued BM, starting from 0.

3) Prove that the process $(X_t \equiv \beta_{t^2}, t \geq 0)$ satisfies $(*)$.

Comments on Chapter 2

For many reasons, a number of computations of the Laplace or Fourier transform of the distribution of quadratic functionals of Brownian motion, or related processes, are being published almost every year; the origins of the interests in such functionals range from Bismut's proof of the Atiyah-Singer theorem, to polymer studies (see Chan-Dean-Jansons-Rogers [26] for the latter).

Duplantier [36] presents a good list of references to the literature.

The methods used by the authors to obtain closed formulae for the corresponding characteristic functions or Laplace transforms fall essentially into one of the three following categories:

i) P. Lévy's *diagonalisation procedure,* which has a strong functional analysis flavor; this method may be applied very generally and is quite powerful; however, the characteristic functions or Laplace transforms then appear as infinite products, which have to be recognized in terms of, say, hyperbolic functions...

ii) the *change of probability method* which, in effect, linearizes the problem, i.e.: it allows to transform the study of a quadratic functional into the computation of the mean and variance of an adequate Gaussian variable; paragraph 2.1 above gives an important example of this method.

iii) finally, the *reduction method,* which simply consists in trying to reduce the computation for a certain quadratic functional to similar computations which have already been done. Exercise 2.3, and indeed the whole paragraph 2.2 above give some examples of application. The last formula in Exercise 2.3 is due to Foschini and Shepp [44] and the whole exercise is closely related to the work of Berthuet [6] on the stochastic volume of $(B_u, u \leq 1)$.

Paragraph 2.3 is closely related to Pitman-Yor ([73], [74]).

Some extensions of the integration by parts formula (2.14) to stable processes and some converse studies have been made by Donati-Martin, Song and Yor [31].

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