Chapter 2
Orthogonal Polynomials and Weighted Polynomial Approximation

2.1 Orthogonal Systems and Polynomials

2.1.1 Inner Product Space and Orthogonal Systems

Suppose that X is a complex linear space of functions with an inner product $(f, g)\colon X^2 \to \mathbb{C}$ such that

(a)	$(f+g,h) = (f,h) + (g,h)$	(Linearity),
(b)	$(\alpha f, g) = \alpha(f,g)$	(Homogeneity),
(c)	$(f,g) = \overline{(g,f)}$	(Hermitian Symmetry),
(d)	$(f,f) \geq 0,\ (f,f) = 0 \iff f = 0$	(Positivity),

where $f, g, h \in X$ and α is a complex scalar. The bar in the above line denotes the complex conjugate. The space X is called an *inner product space*.

If X is a real linear space, then the inner product $(f, g)\colon X^2 \to \mathbb{R}$ is such that the condition (c) is reduced to

(c′) $(f,g) = (g,f)$ (Symmetry).

An important inequality for the inner product is the *Cauchy-Schwarz-Buniakowsky inequality* (cf. [328, p. 87])

$$|(f,g)| \leq \|f\|\,\|g\| \qquad (f, g \in X), \tag{2.1.1}$$

where the *norm* of f is defined by $\|f\| = \sqrt{(f,f)}$.

A system S of elements of an inner product space is called *orthogonal* if $(f, g) = 0$ for every $f \neq g$ $(f, g \in S)$. An orthogonal system S is called *orthonormal* if $(f, f) = 1$ for all $f \in S$.

Suppose that $U = \{g_0, g_1, g_2, \ldots\}$ is a system of linearly independent functions in a complex inner product space X. Starting from this system of elements and using the well-known *Gram-Schmidt orthogonalizing process* we can construct the corresponding orthogonal (orthonormal) system $S = \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$, where φ_n is, in fact, a linear combination of the functions $g_0, g_1, \ldots, g_n$, such that $(\varphi_n, \varphi_k) = 0$ for $n \neq k$.

Using the functions g_n and *Gram matrix* of order $n+1$,

$$G_{n+1} = \begin{bmatrix} (g_0, g_0) & (g_0, g_1) & \cdots & (g_0, g_n) \\ (g_1, g_0) & (g_1, g_1) & & (g_1, g_n) \\ \vdots & & & \\ (g_n, g_0) & (g_n, g_1) & & (g_n, g_n) \end{bmatrix}$$

an explicit expression for the orthogonal functions φ_n can be obtained. Notice that this matrix is non-singular. Namely, it is well-known that $\Delta_{n+1} = \det G_{n+1} \neq 0$ if and only if the system of functions $\{g_0, g_1, g_2, \ldots, g_n\}$ is linearly independent. Moreover, we can prove that $\det G_{n+1} > 0$.

Firstly, the matrix G_{n+1} is Hermitian, because of the property (c) of the inner product, i.e., $(g_i, g_j) = \overline{(g_j, g_i)}$. Putting $\mathbf{x} = [x_0\ x_1\ \cdots\ x_n]^{\mathrm{T}}$ and $\psi_n = \sum_{k=0}^{n} \overline{x}_k g_k$, we can see that the Gram matrix is also positive definite. Namely, then

$$\mathbf{x}^* G_{n+1} \mathbf{x} = \sum_{i=0}^{n} \sum_{j=0}^{n} (g_i, g_j) \overline{x}_i x_j$$

can be expressed in the form

$$\mathbf{x}^* G_{n+1} \mathbf{x} = (\psi_n, \psi_n) = \|\psi_n\|^2,$$

which is positive, except if $\psi_n = 0$ (i.e., $\mathbf{x} = \mathbf{0}$). This means that $\Delta_{n+1} = \det G_{n+1} > 0$.

Theorem 2.1.1 *The orthonormal functions φ_n are given by*

$$\varphi_n(z) = \frac{1}{\sqrt{\Delta_n \Delta_{n+1}}} \begin{vmatrix} (g_0, g_0) & (g_0, g_1) & \cdots & (g_0, g_{n-1}) & g_0(z) \\ (g_1, g_0) & (g_1, g_1) & & (g_1, g_{n-1}) & g_1(z) \\ \vdots & & & & \\ (g_n, g_0) & (g_n, g_1) & & (g_n, g_{n-1}) & g_n(z) \end{vmatrix} \tag{2.1.2}$$

where $\Delta_n = \det G_n$ and $\Delta_0 = 1$.

Proof For the proof of this statement it is enough to prove that φ_n given by (2.1.2) satisfies the orthogonality condition $(\varphi_n, g_k) = 0$ for each $k = 0, 1, \ldots, n-1$.

Since

$$(\varphi_n, g_k) = \frac{1}{\sqrt{\Delta_n \Delta_{n+1}}} \begin{vmatrix} (g_0, g_0) & (g_0, g_1) & \cdots & (g_0, g_{n-1}) & (g_0, g_k) \\ (g_1, g_0) & (g_1, g_1) & & (g_1, g_{n-1}) & (g_1, g_k) \\ \vdots & & & & \\ (g_n, g_0) & (g_n, g_1) & & (g_n, g_{n-1}) & (g_n, g_k) \end{vmatrix},$$

we see immediately that this determinant is equal to zero for each $k = 0, 1, \ldots, n-1$, and for $k = n$ we have $(\varphi_n, g_n) = \sqrt{\Delta_{n+1}/\Delta_n}$. Expanding the determinant from (2.1.2) along the last column, we obtain an expansion

$$\varphi_n(z) = \frac{1}{\sqrt{\Delta_n \Delta_{n+1}}} \big(c_0 g_0(z) + c_1 g_1(z) + \cdots + c_n g_n(z)\big),$$

in terms of g_k, where $c_n = \Delta_n$. Therefore,

$$(\varphi_n, \varphi_n) = \frac{c_n}{\sqrt{\Delta_n \Delta_{n+1}}} (g_n, \varphi_n) = 1. \qquad \square$$

2.1.2 Fourier Expansion and Best Approximation

Taking an orthonormal system of functions $S = \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$, it is easy to construct the corresponding *Fourier expansion* for a given function $f \in X$,

$$f(z) \sim \sum_{k=0}^{+\infty} f_k \varphi_k(z). \tag{2.1.3}$$

The *Fourier coefficients* f_k are given by

$$f_k = (f, \varphi_k) \qquad (k = 0, 1, \ldots), \tag{2.1.4}$$

which follows directly from (2.1.3). This sequence of coefficients is bounded. Indeed, applying the Cauchy-Schwarz-Buniakowsky inequality (2.1.1) to (2.1.4), we obtain

$$|f_k| = |(f, \varphi_k)| \le \|f\| \|\varphi_k\| = \|f\|.$$

The partial sums of (2.1.3), i.e.,

$$s_n(z) = \sum_{k=0}^{n} f_k \varphi_k(z), \tag{2.1.5}$$

play a very important role in approximation theory.

Let X_n be a subspace of X spanned by $S_n = \{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ ($\dim X_n = n + 1$), i.e., $X_n = \operatorname{span} S_n$. The following theorem shows that s_n is the closest element to $f \in X$ among all elements of the subspace X_n with respect to the metric induced by the given norm. Thus, the partial sum s_n is the *best approximation* to $f \in X$ in the subspace X_n.

Theorem 2.1.2 *Let $f \in X$ and X_n be a subspace of X spanned by $\{\varphi_0, \varphi_1, \ldots\}$. Then*

$$\min_{\varphi \in X_n} \|f - \varphi\|^2 = \|f - s_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} |f_k|^2, \tag{2.1.6}$$

where s_n is given by (2.1.5).

Proof Let $f \in X$ and let s_n be given by (2.1.5). An arbitrary element of X_n can be expressed as a linear combination of the orthonormal functions $\varphi_0, \varphi_1, \dots, \varphi_n$, i.e., $\varphi = \sum_{k=0}^{n} a_k \varphi_k$. Then

$$\|f - \varphi\|^2 = (f - \varphi, f - \varphi) = (f, f) - (f, \varphi) - (\varphi, f) + (\varphi, \varphi).$$

Since

$$(f, \varphi) = \sum_{k=0}^{n} \overline{a}_k (f, \varphi_k) = \sum_{k=0}^{n} \overline{a}_k f_k, \quad (\varphi, f) = \sum_{k=0}^{n} a_k \overline{f}_k, \quad (\varphi, \varphi) = \sum_{k=0}^{n} \overline{a}_k a_k,$$

we get

$$\begin{aligned} \|f - \varphi\|^2 &= \|f\|^2 - \sum_{k=0}^{n} |f_k|^2 + \sum_{k=0}^{n} \bigl(f_k \overline{f}_k - \overline{a}_k f_k - a_k \overline{f}_k + a_k \overline{a}_k\bigr) \\ &= \|f\|^2 - \sum_{k=0}^{n} |f_k|^2 + \sum_{k=0}^{n} |f_k - a_k|^2. \end{aligned}$$

This expression attains the minimal value for $a_k = f_k$ $(k = 0, 1, \dots, n)$, i.e., when $\varphi = s_n$, and the minimum is given by (2.1.6). □

Since $(f, s_n) = (s_n, s_n) = \sum_{k=0}^{n} |f_k|^2$, we see that the error $e_n = f - s_n$ in the best approximation is orthogonal to s_n, i.e.,

$$(f - s_n, s_n) = 0.$$

Also, for each $k = 0, 1, \dots, n$,

$$(f - s_n, \varphi_k) = \Bigl(f - \sum_{\nu=0}^{n} f_\nu \varphi_\nu, \varphi_k\Bigr) = (f, \varphi_k) - \sum_{\nu=0}^{n} f_\nu (\varphi_\nu, \varphi_k) = 0, \tag{2.1.7}$$

In other words, the error e_n is orthogonal to each φ_k, i.e., e_n is orthogonal to the subspace X_n.

Based on (2.1.6) we conclude that

$$\|f - s_0\| \ge \|f - s_1\| \ge \|f - s_2\| \ge \cdots,$$

which also follows directly from the fact that $X_0 \subset X_1 \subset X_2 \subset \cdots$.

Notice also that (2.1.6) implies the *Bessel inequality*

$$\sum_{k=0}^{n} |f_k|^2 \le \|f\|^2,$$

which holds for every $n \in \mathbb{N}$. When $n \to +\infty$, this becomes

$$\sum_{k=0}^{+\infty} |f_k|^2 \le \|f\|^2.$$

Thus, the series on the left hand side in this limit inequality converges, which implies that

$$\lim_{k\to+\infty} f_k = \lim_{k\to+\infty} (f, \varphi_k) = 0.$$

Therefore, we conclude that the Fourier coefficients of any function $f \in X$ approach zero.

The limit Bessel inequality reduces to an equality (*Parseval's equality*) in the case when $\operatorname{span}\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ is dense in X.

2.1.3 Examples of Orthogonal Systems

In this section we mention several interesting orthogonal systems.

2.1.3.1 Trigonometric System

In Sect. 1.2.2 we have seen that the trigonometric system (1.2.18), i.e.,

$$\bigl\{1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots,\ \cos nx,\ \sin nx, \ldots\bigr\}$$

is orthogonal with respect to the inner product defined by (1.2.19). According to (1.2.16) and (1.2.17), the corresponding orthonormal system is

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \ldots\right\}.$$

2.1.3.2 Chebyshev Polynomials

Let

$$(f, g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{\lambda-1/2}dx, \quad \lambda > -1/2. \tag{2.1.8}$$

The Chebyshev polynomials of the first kind $\{T_n\}_{n\in\mathbb{N}_0}$ and of the second kind $\{U_n\}_{n\in\mathbb{N}_0}$ are orthogonal on $[-1, 1]$ with respect to the inner product (2.1.8) for $\lambda = 0$ and $\lambda = 1$, respectively (see Sect. 1.1.4). The corresponding orthonormal systems are

$$\left\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\,T_1, \sqrt{\frac{2}{\pi}}\,T_2, \ldots\right\} \quad \text{and} \quad \left\{\sqrt{\frac{2}{\pi}}, \sqrt{\frac{2}{\pi}}\,U_1, \sqrt{\frac{2}{\pi}}\,U_2, \ldots\right\}, \tag{2.1.9}$$

respectively.

2.1.3.3 Orthogonal Polynomials on the Unit Circle

The system of monomials $\{z^n\}_{n\in\mathbb{N}_0}$ is orthonormal with respect to the inner product

$$(f,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,v(\theta)\,d\theta, \tag{2.1.10}$$

where $v(\theta)=1$. This is the simplest case of polynomials orthogonal on the unit circle with respect to (2.1.10). Such polynomials were introduced and studied by Szegő ([468, 469]) and Smirnov ([445, 446]). A more general case was considered by Achieser and Kreĭn [7], Geronimus ([177, 178]), Nevai ([378, 379]), Simon [436], etc. (see also surveys [9] and [334], as well as in a very impressive new book of Barry Simon in two volumes [437, 438]). These polynomials are linked to many questions in the theory of time series, digital filters, statistics, image processing, scattering theory, control theory, etc.

The general theory of orthogonality on a union of circular arcs was also considered by several authors (cf. Peherstorfer and Steinbauer [394–396], Simon [437, 438]). For the zeros of such polynomials see Lukashov and Peherstorfer [277], Simon [439, 440], etc.

2.1.3.4 Orthogonal Polynomials on the Unit Disk

The system of monomials $p_n(z)=\sqrt{(n+1)/\pi}\,z^n$, $n=0,1,\ldots,$ is orthonormal with respect to the inner product defined by the following double integral

$$(f,g)=\iint\limits_{|z|\le 1} f(z)\overline{g(z)}\,dx\,dy.$$

2.1.3.5 Orthogonal Polynomials on the Ellipse

Let E_r $(r>1)$ denote the ellipse with its foci at ± 1 and such that the sum of its semi-axes is r.

The Chebyshev polynomials of the first kind $\{T_n\}_{n\in\mathbb{N}_0}$ are orthogonal with respect to the inner product defined by the following contour integral

$$(f,g)=\oint\limits_{E_r}\frac{f(z)\overline{g(z)}}{|1-z^2|^{1/2}}\,ds \qquad (ds^2=dx^2+dy^2).$$

Namely, we have

$$(T_n,T_n)=\begin{cases}\dfrac{\pi}{2}\left(r^{2n}+r^{-2n}\right), & n>0,\\ 2\pi, & n=0.\end{cases}$$

However, the polynomials

$$p_n(z) = 2\sqrt{\frac{n+1}{\pi}}\left(r^{2n+2} - r^{-2n-2}\right)U_n(z), \quad n = 0, 1, \ldots,$$

where $U_n(z)$ are the Chebyshev polynomials of the second kind, are orthonormal with respect to the inner product

$$(f, g) = \iint\limits_{\operatorname{int} E_r} f(z)\overline{g(z)}\,dx\,dy.$$

2.1.3.6 Malmquist-Takenaka System of Rational Functions

Let $\{a_\nu\}_{\nu\in\mathbb{N}_0}$ be a sequence of complex numbers such that $|a_\nu| < 1$ $(\nu = 0, 1, \ldots)$. The system of rational functions

$$w_n(z) = \frac{\prod_{\nu=0}^{n-1}(z - a_\nu)}{\prod_{\nu=0}^{n}(z - 1/\overline{a}_\nu)}, \quad n \in \mathbb{N}_0, \tag{2.1.11}$$

considered by Malmquist [281], Takenaka [472], Walsh [501, Sects. 9.1 and 10.7], Djrbashian [102], etc., is orthogonal with respect to the inner product defined by (2.1.10), where again $v(\theta) = 1$. Thus, the orthogonal functions (2.1.11), which are known as the *Malmquist-Takenaka functions* (*basis*), generalize Szegő's orthogonal polynomials. Here,

$$\|w_n\|^2 = (w_n, w_n) = \frac{|a_0 a_1 \cdots a_n|^2}{1 - |a_n|^2}$$

(cf. [345, 362]).

2.1.3.7 Polynomials Orthogonal on the Radial Rays

The following sequence of polynomials

$$1,\ z,\ z^2,\ z^3,\ z^4 - \frac{1}{3},\ z^5 - \frac{5}{11}z,\ z^6 - \frac{7}{13}z^2,\ z^7 - \frac{3}{5}z^3,\ z^8 - \frac{14}{17}z^4 + \frac{21}{221},\ \ldots \tag{2.1.12}$$

is orthogonal with respect to the inner product

$$(f, g) = \int_0^1 \Big[f(x)\overline{g(x)} + f(ix)\overline{g(ix)} + f(-x)\overline{g(-x)} + f(-ix)\overline{g(-ix)}\Big]\omega(x)\,dx,$$

where $\omega(x) = (1 - x^4)^{1/2} x^2$. This is a special case of polynomials orthogonal on the radial rays introduced by Milovanović in [336]. For details and several properties of such polynomials see [333, 337, 341, 360, 361]. It is interesting that the polynomial sequence (2.1.12) contains a subsequence which was obtained from a physical problem connected to a non-linear diffusion equation (cf. [448]).

2.1.3.8 Müntz Orthogonal Polynomials

Let $\Lambda = \{\lambda_0, \lambda_1, \ldots\}$ be a given sequence of complex numbers such that $\mathrm{Re}(\lambda_k) > -1/2$, $k \in \mathbb{N}_0$, and let $\Lambda_n = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$. As we mentioned in Chap. 1 (Remark 1.3.1), Müntz systems can be considered also for the complex sequences. According to [23, 473], and [48], we can introduce the so-called *Müntz-Legendre polynomials* on (0, 1] by

$$P_n(x) = P_n(x; \Lambda_n) = \frac{1}{2\pi i} \oint_{\Gamma} W_n(z) x^z \, dz, \qquad n = 0, 1, \ldots, \tag{2.1.13}$$

where the simple contour Γ surrounds all poles of the rational function

$$W_n(z) = \prod_{\nu=0}^{n-1} \frac{z + \overline{\lambda}_\nu + 1}{z - \lambda_\nu} \cdot \frac{1}{z - \lambda_n} \qquad (n \in \mathbb{N}_0).$$

An empty product for $n = 0$ should be taken to be equal to 1.

The polynomials (2.1.13) are orthogonal with respect to the inner product $(f, g) = \int_0^1 f(x)\overline{g(x)}\, dx$.

The corresponding orthonormal polynomials are $P_n^*(x) = (1 + \lambda_n + \overline{\lambda}_n)^{1/2} P_n(x)$. In the simplest case when $\lambda_\nu \neq \lambda_\mu$ $(\nu \neq \mu)$ it is easy to show that the polynomials $P_n(x)$ can be expressed in a power form $P_n(x) = \sum_{k=0}^{n} c_{n,k} x^{\lambda_k}$, where

$$c_{n,k} = \frac{\displaystyle\prod_{\nu=0}^{n-1} (1 + \lambda_k + \overline{\lambda}_\nu)}{\displaystyle\prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} (\lambda_k - \lambda_\nu)}, \qquad k = 0, 1, \ldots, n.$$

An important special case of the Müntz-Legendre polynomials when

$$\lambda_{2k} = \lambda_{2k+1} = k, \qquad k = 0, 1, \ldots,$$

was considered in [338]. Namely, we put $\lambda_{2k} = k$ and $\lambda_{2k+1} = k + \varepsilon$, $k = 0, 1, \ldots,$ where ε decreases to zero. The corresponding limit process leads to the orthogonal Müntz polynomials with logarithmic terms,

$$P_n(x) = R_n(x) + S_n(x) \log x, \qquad n = 0, 1, \ldots, \tag{2.1.14}$$

where $R_n(x)$ and $S_n(x)$ are algebraic polynomials of degree $\left[\frac{n}{2}\right]$ and $\left[\frac{n-1}{2}\right]$, respectively, i.e.,

$$R_n(x) = \sum_{\nu=0}^{[n/2]} a_\nu^{(n)} x^\nu, \qquad S_n(x) = \sum_{\nu=0}^{[(n-1)/2]} b_\nu^{(n)} x^\nu. \tag{2.1.15}$$

Notice that $P_n(1) = R_n(1) = 1$. The first few Müntz polynomials (2.1.14) are:

$$\begin{aligned}
P_0(x) &= \quad 1, \\
P_1(x) &= \quad 1 + \log x, \\
P_2(x) &= -3 + 4x - \log x, \\
P_3(x) &= \quad 9 - 8x + 2(1 + 6x)\log x, \\
P_4(x) &= -11 - 24x + 36x^2 - 2(1 + 18x)\log x, \\
P_5(x) &= \quad 19 + 276x - 294x^2 + 3(1 + 48x + 60x^2)\log x, \\
P_6(x) &= -21 - 768x + 390x^2 + 400x^3 - 3(1 + 96x + 300x^2)\log x.
\end{aligned}$$

The explicit expressions for the coefficients of the polynomials (2.1.15) for arbitrary n are given in [338]. These Müntz polynomials can be used in the proof of the irrationality of $\zeta(3)$ and of other familiar numbers (see [47, pp. 372–381] and [486]).

Similarly, if we take

$$\lambda_{3k} = \lambda_{3k+1} = \lambda_{3k+2} = k, \qquad k = 0, 1, \ldots,$$

i.e., $\lambda_{3k} = k - \varepsilon$, $\lambda_{3k+1} = k$, $\lambda_{3k+2} = k + \varepsilon$, $k = 0, 1, \ldots$, where ε tends to zero, we get the corresponding orthogonal Müntz polynomials:

$$\begin{aligned}
P_0(x) &= \quad 1, \\
P_1(x) &= \quad 1 + \log x, \\
P_2(x) &= \quad 1 + 2\log x + \frac{1}{2}\log^2 x, \\
P_3(x) &= -7 + 8x - 4\log x - \frac{1}{2}\log^2 x, \\
P_4(x) &= \quad 29 - 28x + (11 + 24x)\log x + \log^2 x, \\
P_5(x) &= -97 + 98x - 4(7 + 15x)\log x + (36x - 2)\log^2 x, \\
P_6(x) &= \quad 127 - 342x + 216x^2 + (32 - 108x)\log x + (2 - 108x)\log^2 x.
\end{aligned}$$

These polynomials have the form

$$P_n(x) = R_n(x) + S_n(x)\log x + T_n(x)\log^2 x,$$

where $R_n(x)$, $S_n(x)$, and $T_n(x)$ are algebraic polynomials of degree $\left[\frac{n}{3}\right]$, $\left[\frac{n-1}{3}\right]$, and $\left[\frac{n-2}{3}\right]$, respectively. Notice that $P_n(1) = R_n(1) = 1$.

A direct evaluation of the Müntz polynomials $P_n(x)$ in the power form can be problematic in finite arithmetic, especially when n is a large number and x is close to 1. A numerical method for a stable evaluation of Müntz polynomials and some applications were given in [338] (see also [339]).

2.1.3.9 Müntz Orthogonal Polynomials of the Second Kind

In [72] and [362] an external operation for the Müntz polynomials from $M(\Lambda)$ and the corresponding inner product were defined. Namely, at first an operation $\odot$ for monomials was introduced in the following way:

$$x^\alpha \odot x^\beta = x^{\alpha\beta} \qquad (x \in (0, +\infty),\ \alpha, \beta \in \mathbb{C}),$$

and then it was extended to the Müntz polynomials $P \in M_n(\Lambda)$ and $Q \in M_m(\Lambda)$, i.e.,

$$P(x) = \sum_{i=0}^{n} p_i x^{\lambda_i} \qquad \text{and} \qquad Q(x) = \sum_{j=0}^{m} q_j x^{\lambda_j},$$

as

$$(P \odot Q)(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_i q_j x^{\lambda_i \lambda_j}.$$

Under the restrictions that for each i and j, $|\lambda_i| > 1$ and $\mathrm{Re}(\lambda_i \overline{\lambda}_j - 1) > 0$, the following inner product can be defined ([362])

$$[P, Q] = \int_0^1 (P \odot \overline{Q})(x) \frac{dx}{x^2}, \tag{2.1.16}$$

as well as the Müntz polynomials

$$Q_n(x) = \frac{1}{2\pi i} \oint_{\Gamma} W_n(z) x^z \, dz, \qquad n = 0, 1, \ldots, \tag{2.1.17}$$

where

$$W_n(s) = \frac{\prod_{\nu=0}^{n-1} (s - 1/\overline{\lambda}_\nu)}{\prod_{\nu=0}^{n} (s - \lambda_\nu)}, \qquad n = 0, 1, \ldots, \tag{2.1.18}$$

and the simple contour Γ surrounds all the points λ_ν, $\nu = 0, 1, \ldots, n$.

We note that the rational functions (2.1.18) form a Malmquist-Takenaka system. Indeed, putting $a_\nu = 1/\overline{\lambda}_\nu$ $\nu = 0, 1, \ldots$, these functions reduce to (2.1.11).

Under the previous conditions on the sequence Λ, the Müntz polynomials $Q_n(x)$, $n = 0, 1, \ldots,$ defined by (2.1.17), are orthogonal with respect to the inner product (2.1.16). Furthermore, this orthogonality is connected to the orthogonality of the Malmquist-Takenaka system (2.1.11) (see ([362]) and

$$[Q_n, Q_m] = \frac{\delta_{n,m}}{(|\lambda_n|^2 - 1)|\lambda_0 \lambda_1 \cdots \lambda_{n-1}|^2}.$$

2.1.3.10 Generalized Exponential Polynomials

Let $A = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ be a complex sequence such that $\operatorname{Re} \alpha_k > 0$. For each k ($k \geq 0$) denote by $m_k \geq 1$ the multiplicity of the appearance of the numbers α_k in the set $A_k = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. With the sequence A we associate the sequence of functions $\left\{t^{m_k - 1} e^{-\alpha_k t}\right\}_{k \in \mathbb{N}_0}$, which can be orthogonalized with respect to the inner product

$$(f, g) = \int_0^{+\infty} f(t) \overline{g(t)} \, dt, \tag{2.1.19}$$

for example, using the well-known Gram-Schmidt method. Such an orthonormal system $\{g_k(t)\}_{k \in \mathbb{N}_0}$ is unique up to a multiplicative constant of the form $e^{i\gamma_k}$, with $\operatorname{Im} \gamma_k = 0$.

For example, if we take $A = \{1/2, 1, 1, 2, 5/2, \ldots\}$, for which $m_0 = m_1 = 1$, $m_2 = 2$, $m_3 = m_4 = 1, \ldots,$ using an orthogonalizing process we get the exponential functions (generalized exponential polynomials)

$$\begin{aligned}
g_0(t) &= e^{-t/2}, \\
g_1(t) &= \sqrt{2}\, e^{-t} \left(3 - 2e^{t/2}\right), \\
g_2(t) &= \sqrt{2}\, e^{-t} \left(5 - 6e^{t/2} + 6t\right), \\
g_3(t) &= 2e^{-2t} \left(15 - 8e^t - 6e^{3t/2} + 12te^t\right), \\
g_4(t) &= \tfrac{1}{2} \sqrt{5}\, e^{-5t/2} \left(147 - 240e^{t/2} + 80e^{3t/2} + 15e^{2t} - 48te^{3t/2}\right), \ \ldots,
\end{aligned}$$

which are orthonormal with respect to the inner product (2.1.19) (cf. [339]).

2.1.3.11 Discrete Chebyshev Polynomials

Let

$$(f, g) = f(0)g(0) + f(1)g(1) + f(2)g(2) + f(3)g(3).$$

Starting from monomials and using the Gram-Schmidt orthogonalizing process we get orthogonal polynomials with respect to this inner product:

$$g_0(x) = 1, \ g_1(x) = x - \frac{1}{3}, \ g_2(x) = x^2 - 3x + 1, \ g_3(x) = x^3 - \frac{9}{2}x^2 + \frac{47}{10}x - \frac{3}{10}.$$

The corresponding orthonormal polynomials are

$$\frac{1}{2}, \quad \frac{2x-3}{2\sqrt{5}}, \quad \frac{x^2-3x+1}{2}, \quad \frac{10x^3-45x^2+47x-3}{6\sqrt{5}}.$$

Continuing this process we find $g_4(x) = x(x-1)(x-2)(x-3)$ and, as we can see, $g_4(x) = 0$ for $x = 0, 1, 2, 3$. Therefore, $\|g_4\| = 0$.

This is a special case of the so-called *discrete Chebyshev orthogonal polynomials* (cf. Szegő [470, pp. 33–34]) which can be expressed (except for constant factors) in the form

$$t_n(x) = n!\Delta^n \left\{ \binom{x}{n}\binom{x-N}{n} \right\}, \qquad n = 0, 1, \ldots, N-1, \tag{2.1.20}$$

where the inner product is given by

$$(f, g) = \sum_{k=0}^{N-1} f(k)g(k) \tag{2.1.21}$$

and N is a given natural number. In (2.1.20) the symbol Δ is the forward difference operator with unit spacing acting on the variable x. The inner product (2.1.21) can be rewritten in the integral form

$$(f, g) = \int_{\mathbb{R}} f(x)g(x)d\mu(x),$$

with the distribution $d\mu(x) = \sum_{k=0}^{N-1} \delta(x-k)dx$, where $\mu(x)$ is a step function with jumps of one unit at the points $x = k$, $k = 0, 1, \ldots, N-1$. Here, δ is the Dirac delta function. Notice that

$$\|t_n\|^2 = (t_n, t_n) = \frac{N(N^2-1^2)(N^2-2^2)\cdots(N^2-n^2)}{2n+1}.$$

Chebyshev also considered the case when the set of equidistant points $\{0, 1, \ldots, N-1\}$ is replaced by an arbitrary set of N distinct points. Finally, we mention that there are several other cases of discrete polynomials when the jumps in $\mu(x)$ are different from one unit (see [29, pp. 221–227], [382], [470, pp. 34–37]).

2.1.3.12 Formal Orthogonal Polynomials with Respect to a Moment Functional

Let a complex valued linear functional $\mathcal{L}$ be given on the linear space of all algebraic polynomials $\mathcal{P}$. The values of the linear functional $\mathcal{L}$ at the set of monomials are called moments and they are denoted by μ_k. Thus, $\mathcal{L}[x^k] = \mu_k$, $k \in \mathbb{N}_0$. A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is called a *formal orthogonal polynomial sequence* with respect to a *moment functional* $\mathcal{L}$ provided for all nonnegative integers k and n,

1° $P_n(x)$ is a polynomial of degree n,
2° $\mathcal{L}[P_n(x)P_k(x)] = 0$ for $k \neq n$,
3° $\mathcal{L}[P_n(x)^2] \neq 0$.

If a sequence of orthogonal polynomial exists for a given linear functional $\mathcal{L}$, then $\mathcal{L}$ is called *quasi-definite linear functional*. Under the condition $\mathcal{L}[P_n^2(x)] > 0$, the functional $\mathcal{L}$ is called positive definite. For details see Chihara [60, pp. 6–17].

The necessary and sufficient conditions for the existence of a sequence of orthogonal polynomials with respect to the linear functional $\mathcal{L}$ are that for each $n \in \mathbb{N}$ the Hankel determinants

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & & \mu_n \\ \vdots & & & \\ \mu_{n-1} & \mu_n & & \mu_{2n-2} \end{vmatrix} \neq 0. \tag{2.1.22}$$

When $\mathcal{L}$ is positive definite, we can define $(p, q) := \mathcal{L}[p(x)\overline{q(x)}]$ for all algebraic polynomials $p(x)$ and $q(x)$, so that the orthogonality with respect to the moment functional $\mathcal{L}$ is consistent with the standard definition of orthogonality with respect to an inner product. On the other hand, there are several interesting quasi-definite cases when the moments are non-real. We mention here two cases:

(a) *Orthogonality with respect to an oscillatory weight*. Let $w(x) = xe^{im\pi x}$, where $x \in [-1, 1]$ and $m \in \mathbb{Z}$. Putting

$$\mathcal{L}[p] := \int_{-1}^{1} p(x)w(x)\,dx \quad (p \in \mathcal{P}), \tag{2.1.23}$$

i.e.,

$$\mu_k = \frac{(-1)^{m+k}(k+1)!}{(im\pi)^{k+1}} \sum_{\nu=0}^{k} \frac{(1+(-1)^\nu)(-im\pi)^\nu}{(\nu+1)!},$$

in [346] it was proved that for every integer m ($\neq 0$), the sequence of formal orthogonal polynomials with respect to the functional $\mathcal{L}$ exists uniquely. Such orthogonal polynomials have several interesting properties and they can be applied in numerical integration of highly oscillatory functions (see [346]). The corresponding case with the Chebyshev weight, i.e., when $w(x) = x(1-x^2)^{-1/2}e^{i\zeta x}$, $\zeta \in \mathbb{R}$, was recently investigated in [347].

According to (2.1.23) we can define

$$(p, q) := \int_{-1}^{1} p(x)q(x)w(x)\,dx \quad (p, q \in \mathcal{P}),$$

but, as we can see, (p, q) is not Hermitian and not positive definite. Namely, the properties (c) and (d) of the inner product do not hold!

(b) *Orthogonality on the semicircle*. Polynomials orthogonal on the semicircle $\Gamma = \{z \in \mathbb{C} \mid z = e^{i\theta},\ 0 \le \theta \le \pi\}$ with respect to the moment functional

$$\mathcal{L}[z^k] = \mu_k = \int_0^\pi e^{ik\theta} d\theta = \begin{cases} \pi, & k = 0, \\ \dfrac{2i}{k}, & k \text{ odd}, \\ 0, & k \neq 0 \text{ even}, \end{cases}$$

have been introduced by Gautschi and Milovanović (see [170] and [171]). The corresponding non-Hermitian inner product is given by

$$(p,q) = \int_\Gamma p(z)q(z)(iz)^{-1}\,dz = \int_0^\pi p(e^{i\theta})q(e^{i\theta})\,d\theta.$$

A few first polynomials of this orthogonal system are

$$\begin{aligned}
\pi_0(x) &= 1, \\
\pi_1(x) &= x - \frac{2i}{\pi}, \\
\pi_2(x) &= x^2 - \frac{\pi i}{6}x - \frac{1}{3}, \\
\pi_3(x) &= x^3 - \frac{8i}{5\pi}x^2 - \frac{3}{5}x + \frac{8i}{15\pi}, \\
\pi_4(x) &= x^4 - \frac{9\pi i}{56}x^3 - \frac{6}{7}x^2 + \frac{27\pi i}{280}x + \frac{3}{35}, \\
\pi_5(x) &= x^5 - \frac{128i}{81\pi}x^4 - \frac{10}{9}x^3 + \frac{256i}{189\pi}x^2 + \frac{5}{21}x - \frac{128i}{945\pi}.
\end{aligned}$$

The general case of complex polynomials orthogonal with respect to a *complex weight function* was considered by Gautschi, Landau and Milovanović [175]. For some properties and applications see [153, 326, 327, 329]. A generalization of such polynomials on a circular arc was given by de Bruin [50], and further investigations were done by Milovanović and Rajković [354].

In this chapter we mainly consider those polynomial orthogonal systems, when an inner product is defined on some lines or on a curve in the complex plane $\mathbb{C}$. Furthermore, starting from Sect. 2.2, we only consider polynomials orthogonal on the real line.

By $\mathcal{P}_n$ we denote the set of all algebraic polynomials (with complex coefficients) of degree at most n. Further, let $\hat{\mathcal{P}}_n$ be the set of all monic polynomials of degree n, i.e.,

$$\hat{\mathcal{P}}_n = \left\{ z^n + q(z) \mid q(z) \in \mathcal{P}_{n-1} \right\}.$$

A system of polynomials $\{p_n\}$, where

$$\begin{aligned} &p_n(z) = \gamma_n z^n + \text{ lower degree terms}, \quad \gamma_n > 0, \\ &(p_n, p_m) = \delta_{nm}, \quad n, m \geq 0, \end{aligned} \tag{2.1.24}$$

is called a system of *orthonormal polynomials* with respect to the inner product $(\cdot\,,\,\cdot)$. In many considerations and applications we use the *monic orthogonal polynomials*

$$\pi_n(z) = \frac{p_n(z)}{\gamma_n} = z^n + \text{ lower degree terms}. \tag{2.1.25}$$

Sometimes, we also use the notation $\hat{p}_n(z)$ for monic orthogonal polynomials instead of $\pi_n(z)$.

2.1.4 Basic Facts on Orthogonal Polynomials and Extremal Problems

Let $d\mu$ be a finite positive Borel measure in the complex plane $\mathbb{C}$, with an infinite set as its support, and let $L^2(d\mu)$ denote the Hilbert space of measurable functions f for which $\int |f(z)|^2\, d\mu(z) < +\infty$. Finally, let the inner product $(\cdot\,,\,\cdot)$ be defined by

$$(f, g) = \int f(z)\overline{g(z)}\, d\mu(z) \qquad (f, g \in L^2(d\mu)). \tag{2.1.26}$$

Starting from the system of monomials $U = \{1, z, z^2, \ldots\}$, by the Gram-Schmidt orthogonalization process, we can obtain the unique orthonormal polynomials (2.1.24). In order to emphasize the orthogonality with respect to the given measure $d\mu$, we write

$$p_n(z) = p_n(d\mu; z) = \gamma_n z^n + \text{ lower degree terms}, \quad \gamma_n = \gamma_n(d\mu) > 0.$$

Also, the monic orthogonal polynomials (2.1.25) are unique.

The following extremal property characterizes orthogonal polynomials:

Theorem 2.1.3 *The polynomial $\pi_n(z) = p_n(z)/\gamma_n = z^n + \cdots$ is the unique monic polynomial of degree n of minimal $L^2(d\mu)$-norm, i.e.,*

$$\min_{p \in \hat{\mathcal{P}}_n} \int |p(z)|^2 d\mu(z) = \int |\pi_n(z)|^2 d\mu(z) = \frac{1}{\gamma_n^2}. \tag{2.1.27}$$

Proof Using the polynomials $\{p_k(z)\}$, orthonormal with respect to the measure $d\mu$, an arbitrary monic polynomial $p(z) \in \hat{\mathcal{P}}_n$ can be expressed in the form

$$p(z) = \sum_{k=0}^{n-1} c_k p_k(z) + \frac{1}{\gamma_n} p_n(z).$$

Then, we have

$$\|p\|^2 = \sum_{k=0}^{n-1} |c_k|^2 + \frac{1}{\gamma_n^2} \geq \frac{1}{\gamma_n^2},$$

with equality if and only if $c_0 = c_1 = \cdots = c_{n-1} = 0$, i.e.,

$$p(z) = p^*(z) = \frac{1}{\gamma_n} p_n(z) = \pi_n(z). \qquad \square$$

This extremal property is completely equivalent to orthogonality. Namely, many questions regarding orthogonal polynomials can be answered by using only this extremal property (cf. [477] and [452]).

Notice also that the previous theorem gives the polynomial of the best approximation to the monomial z^n in the class $\mathcal{P}_{n-1}$. Indeed, according to Theorem 2.1.2, the best L^2-approximation to $f(z) = z^n$ is given by $s_{n-1}(z) = \sum_{k=0}^{n-1} f_k p_k(z)$, where $f_k = (z^n, p_k)$, $k = 0, 1, \ldots, n-1$. But, by Theorem 2.1.3, $f(z) - s_{n-1}(z) = \pi_n(z)$, so that the polynomial of the best approximation in this case can be expressed in the form $s_{n-1}(z) = z^n - \pi_n(z)$.

Now, we define the function $(z, t) \mapsto K_n(z, t)$ by

$$K_n(z,t) = \sum_{k=0}^{n} p_k(z)\overline{p_k(t)} \qquad (n \geq 0), \tag{2.1.28}$$

which plays a fundamental role in the integral representation of the partial sums of the orthogonal expressions.

For a function $f \in L^2(d\mu)$ we can determine its Fourier coefficients f_k with respect to the inner product (2.1.26). Thus, using (2.1.4) and the orthonormal polynomials $\{p_k(z)\}$, we have

$$f_k = (f, p_k) = \int f(t)\overline{p_k(t)}\, d\mu(t).$$

Then, the partial sums (2.1.5) can be expressed in an integral form

$$s_n(z) = \sum_{k=0}^{n} f_k p_k(z) = \sum_{k=0}^{n} (f, p_k) p_k(z) = \int f(t) K_n(z,t)\, d\mu(t).$$

Suppose that f is an arbitrary polynomial of degree at most n, i.e., $f(z) = P(z)$ $(P(z) \in \mathcal{P}_n)$. Then, the corresponding partial sum s_n coincides with f and we obtain

$$P(z) = \int P(t) K_n(z,t)\, d\mu(t) \qquad (P(z) \in \mathcal{P}_n). \tag{2.1.29}$$

Because of that, the function K_n is very often called the *reproducing kernel*. Notice that $\overline{K_n(z,t)} = K_n(t,z)$ and

$$K_n(z,z) = \sum_{k=0}^{n} |p_k(z)|^2 \geq |p_0(z)|^2 = \gamma_0^2 > 0$$

for each $z \in \mathbb{C}$ and $n \geq 0$.

The reciprocal of this function is known as the *Christoffel function*,

$$\lambda_n(z) = \lambda_n(d\mu; z) = \frac{1}{K_{n-1}(z,z)} = \left(\sum_{k=0}^{n-1} |p_k(z)|^2 \right)^{-1}. \tag{2.1.30}$$

The following extremal problem is related to the reproducing kernel (cf. [359] and [485]):

Theorem 2.1.4 *For every $P(z) \in \mathcal{P}_n$ such that $P(t) = 1$, we have*

$$\int |P(z)|^2 \, d\mu(z) \geq \lambda_{n+1}(d\mu; t), \tag{2.1.31}$$

with equality only for

$$P(z) = P^*(z) = \frac{K_n(z,t)}{K_n(t,t)}.$$

Proof Let t be a fixed complex number and $P(z) \in \mathcal{P}_n$. In order to find the minimum of the integral on the left hand side in (2.1.31) under the constraint $P(t) = 1$, we represent $P(z)$ as a linear combination of the orthonormal polynomials $p_k(z) = p_k(d\mu; z)$, i.e., $P(z) = \sum_{k=0}^{n} c_k p_k(z)$. Then, we have

$$F(P) = \int |P(z)|^2 \, d\mu(z) = \sum_{k=0}^{n} |c_k|^2.$$

Since $P(t) = \sum_{k=0}^{n} c_k p_k(t) = 1$, using the Cauchy inequality for the complex sequences $\mathbf{c} = \{c_k\}_{k=0}^{n}$ and $\mathbf{p} = \{p_k(t)\}_{k=0}^{n}$ (see Mitrinović [364, p. 32]), we have

$$1 = \left| \sum_{k=0}^{n} c_k p_k(t) \right|^2 \leq \left(\sum_{k=0}^{n} |c_k|^2 \right) \left(\sum_{k=0}^{n} |p_k(t)|^2 \right) = F(P) K_n(t,t), \tag{2.1.32}$$

which implies $F(P) \geq 1/K_n(t,t) = \lambda_{n+1}(d\mu; t)$, i.e., (2.1.31).

In the case of equality in (2.1.32), which is attained only if the sequences $\mathbf{c}$ and $\overline{\mathbf{p}}$ are proportional, i.e., when $c_k = \gamma \, \overline{p_k(t)}$ $(k = 0, 1, \ldots, n)$, with some complex constant λ, we find that

$$P(t) = \gamma \sum_{k=0}^{n} |p_k(t)|^2 = \gamma K_n(t,t) = 1.$$

Thus, $\gamma = 1/K_n(t,t)$ and the extremal polynomial is given by

$$P(z) = P^*(z) = \gamma \sum_{k=0}^{n} \overline{p_k(t)}\, p_k(z) = \frac{K_n(z,t)}{K_n(t,t)}\,. \qquad \square$$

According to this theorem, the Christoffel function can also be expressed in the form

$$\lambda_n(d\mu;t) = \min_{\substack{P\in\mathcal{P}_{n-1}\\ P(t)=1}} \int |P(z)|^2\, d\mu(z). \tag{2.1.33}$$

Using the previous theorem we can also prove:

Theorem 2.1.5 *Let t be a fixed complex constant, and let $P(z)$ be an arbitrary polynomial of degree at most n, normalized by the condition*

$$\|P\|^2 = \int |P(z)|^2\, d\mu(z) = 1.$$

The maximum of $|P(t)|^2$ taken over all such polynomials is attained for

$$P(z) = \gamma\, \frac{K_n(z,t)}{\sqrt{K_n(t,t)}} \qquad (|\gamma| = 1).$$

The maximum itself is $K_n(t,t)$.

Proof Taking $Q(z) = P(z)/P(t)$ we see that $Q(t) = 1$ and, according to Theorem 2.1.4,

$$\int |Q(z)|^2\, d\mu(z) = \frac{1}{|P(t)|^2} \ge \lambda_{n+1}(d\mu;t),$$

with equality case for $Q(z) = P(z)/P(t) = K_n(z,t)/K_n(t,t)$.

Thus, $|P(t)|^2 \le K_n(t,t)$, with equality only for

$$P(z) = P(t)Q(z) = \gamma\sqrt{K_n(t,t)}\,\frac{K_n(z,t)}{K_n(t,t)} = \gamma\,\frac{K_n(z,t)}{\sqrt{K_n(t,t)}},$$

where $|\gamma| = 1$. $\square$

Under certain conditions, the extremal property from Theorem 2.1.3, can be extended to $L^r(d\mu)$-norm $(1 < r < +\infty)$, so that the unique monic polynomial $p_n^*(z) = z^n + \cdots$ of the minimal $L^r(d\mu)$-norm exists, i.e.,

$$\min_{p\in\hat{\mathcal{P}}_n} \int |p(z)|^r\, d\mu(z) = \int |p_n^*(z)|^r\, d\mu(z). \tag{2.1.34}$$

For measures with support on the real line, an interesting special case $r = 2s + 2$, where $s \in \mathbb{N}_0$, leads to a case of the *power orthogonality*. Then, the extremal (monic) polynomials in (2.1.34), denoted by $p_n^*(x) = \pi_{n,s}(x) = \pi_{n,s}(x; d\mu)$, exist uniquely and they are known as *s-orthogonal polynomials* (for more details see [146, 179, 180, 340, 387]). These polynomials must satisfy the "orthogonality conditions" (cf. Ghizzetti and Ossicini [180], Milovanović [340])

$$\int_{\mathbb{R}} \pi_{n,s}(x)^{2s+1} x^k \, d\mu(x) = 0, \quad k = 0, 1, \ldots, n-1. \tag{2.1.35}$$

In the case $s = 0$, the s-orthogonal polynomials reduce to the standard orthogonal polynomials, $\pi_{n,0} = \pi_n$.

Also, the *generalized Christoffel function* can be defined for $0 < r < +\infty$, by

$$\lambda_n(d\mu, r; t) = \min_{\substack{P \in \mathcal{P}_{n-1} \\ P(t)=1}} \int |P(z)|^r \, d\mu(z). \tag{2.1.36}$$

Notice that (2.1.36) for $r = 2$ reduces to (2.1.33), i.e., $\lambda_n(d\mu, 2; t) = \lambda_n(d\mu; t)$.

Several properties of the generalized Christoffel functions for measures on $\mathbb{R}$ can be found in Nevai [375, pp. 106–123].

2.1.5 Zeros of Orthogonal Polynomials

Now, we study some basic properties of the zeros of orthogonal polynomials. According to the fundamental theorem of algebra, we know that any polynomial of degree n has exactly n zeros, counting multiplicities. The zeros of orthogonal polynomials play a very important role in interpolation theory, quadrature formulas, etc.

Using Theorem 2.1.3 it is easy to prove a general result on the location of zeros. This result is connected with the support of the measure supp $(d\mu)$, which is a closed set. Firstly, we need some definitions:

Definition 2.1.1 A set $A \subset \mathbb{C}$ is *convex* if for each pair of points $z, t \in A$ the line connecting z and t is a subset of A.

Definition 2.1.2 The *convex hull* $\operatorname{Co}(B)$ of a set $B \subset \mathbb{C}$ is the smallest convex set containing B.

Definition 2.1.3 Let D_∞ be the connected component of the complement of E that contains the point ∞, then D_∞ is open and

$$\operatorname{Pc}(E) = \mathbb{C} \setminus D_\infty$$

is the *polynomial convex hull* of E.

It is clear that $\mathrm{Co}\,(\mathrm{supp}\,(d\mu))$ is the intersection of all closed half-planes containing $\mathrm{supp}\,(d\mu)$. Also,

$$\mathrm{supp}\,(d\mu) \subset \mathrm{Pc}\,(\mathrm{supp}\,(d\mu)) \subset \mathrm{Co}\,(\mathrm{supp}\,(d\mu)).$$

The following result is due to Fejér (see [422] and [485]).

Theorem 2.1.6 *All the zeros of the* (*monic*) *polynomial* $\pi_n(d\mu; z)$ *lie in the convex hull of the support* $E = \mathrm{supp}\,(d\mu)$.

Proof Suppose that ζ is a zero of $\pi_n(d\mu; z) = \pi_n(z)$ such that $\zeta \notin \mathrm{Co}\,(E)$, where $E = \mathrm{supp}\,(d\mu)$. Then

$$\pi_n(z) = (z - \zeta)q(z) \qquad (q(z) \in \hat{\mathcal{P}}_n).$$

Since $\zeta \notin \mathrm{Co}\,(E)$, there exists a line L separating E and ζ. Let $\hat{\zeta}$ be the orthogonal projection of ζ on L. Then, for every $z \in E$, we have $|z - \hat{\zeta}| < |z - \zeta|$, i.e.,

$$|(z - \hat{\zeta})q(z)| < |(z - \zeta)q(z)| = |\pi_n(z)|,$$

from which it follows that

$$\int |(z - \hat{\zeta})q(z)|^2\, d\mu(z) < \int |\pi_n(z)|^2\, d\mu(z),$$

which is a contradiction to Theorem 2.1.3. Thus, we conclude that there are no zeros outside $\mathrm{Co}\,(E)$. $\square$

An improvement of this theorem was given by Saff [422]:

Theorem 2.1.7 *If* $\mathrm{Co}\,(\mathrm{supp}\,(d\mu))$ *is not a line segment*, *then all the zeros of the polynomial* $\pi_n(d\mu; z)$ *lie in the interior of* $\mathrm{Co}\,(\mathrm{supp}\,(d\mu))$.

For example, if C is the unit circle $|z| = 1$ and $\mathrm{supp}\,(d\mu) \subset C$, then Theorem 2.1.7 asserts that all the zeros of $\pi_n(d\mu; z)$ must lie in the open unit disk $|z| < 1$. This is a classical result of Szegő [470, p. 292] for orthogonal polynomials on the unit circle.

An interesting question is related with a number of zeros of $\pi_n(d\mu; z)$ which are outside $E = \mathrm{supp}\,(d\mu)$ (i.e., in $\mathrm{Co}\,(E) \setminus E$). If the set E has holes, it is possible that all the zeros are in the holes, as in the case of polynomials orthogonal on the unit circle. Here, we mention a result of Widom [505] (see Saff [422] for the proof).

Theorem 2.1.8 *Let* $E = \mathrm{supp}\,(d\mu)$ *and* A *be a closed set such that* $A \cap \mathrm{Pc}\,(E) = \emptyset$. *Then the number of zeros of* $\pi_n(d\mu; z)$ *on* A *is uniformly bounded in* n.

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