

# Preface

Interpolation of functions is one of the basic part of *Approximation Theory*. There are many books on approximation theory, including interpolation methods that appeared in the last fifty years, but a few of them are devoted only to interpolation processes. An example is the book of J. Szabados and P. Vértesi: *Interpolation of Functions*, published in 1990 by World Scientific. Also, two books deal with a special interpolation problem, the so-called Birkhoff interpolation, written by G.G. Lorentz, K. Jetter, S.D. Riemenschneider (1983) and Y.G. Shi (2003).

The classical books on interpolation address numerous negative results, i.e., results on divergent interpolation processes, usually constructed over some equidistant system of nodes. The present book deals mainly with new results on convergent interpolation processes in uniform norm, for algebraic and trigonometric polynomials, not yet published in other textbooks and monographs on approximation theory and numerical mathematics. Basic tools in this field (orthogonal polynomials, moduli of smoothness,  $K$ -functionals, etc.), as well as some selected applications in numerical integration, integral equations, moment-preserving approximation and summation of slowly convergent series are also given.

The first chapter provides an account of basic facts on approximation by algebraic and trigonometric polynomials introducing the most important concepts on approximation of functions. Especially, in Sect. 1.4 we give basic results on interpolation by algebraic polynomials, including representations and computation of interpolation polynomials, Lagrange operators, interpolation errors and uniform convergence in some important classes of functions, as well as an account on the Lebesgue function and some estimates for the Lebesgue constant.

The second chapter is devoted to orthogonal polynomials on the real line and weighted polynomial approximation. For polynomials orthogonal on the real line we give the basic properties and introduce and discuss the associated polynomials, functions of the second kind, Stieltjes polynomials, as well as the Christoffel functions and numbers. The *classical orthogonal polynomials* as the most important class of orthogonal polynomials on the real line are treated in Sect. 2.3, and new results on *nonclassical orthogonal polynomials*, including methods for their numerical construction, are studied in Sect. 2.4. Introducing the weighted functional spaces, moduli of smoothness and  $K$ -functionals, the weighted best polynomial approximations on  $(-1, 1)$ ,  $(0, +\infty)$  and  $(-\infty, +\infty)$  are treated in Sect. 2.5, as well as the weighted polynomial approximation of functions having interior isolated singularities.

Trigonometric approximation is considered in Chap. 3. Approximations by sums of Fourier and Fejér and de la Vallée Poussin means are given. Their discrete versions and the Lagrange trigonometric operator are also investigated. As a basic tool for studying approximating properties of the Lagrange and de la Vallée Poussin operators we consider the so-called Marcinkiewicz inequalities. Beside the uniform

approximation we also investigate the Lagrange interpolation error in  $L^p$ -norm ( $1 < p < +\infty$ ) and give some estimates in the  $L^1$ -Sobolev norm, including some weighted versions.

Chapter 4 treats algebraic interpolation processes  $\{L_n(\mathcal{X})\}_{n \in \mathbb{N}}$  in the uniform norm, starting with the so-called optimal system of nodes  $\mathcal{X}$ , which provides Lebesgue constants of order  $\log n$  and the convergence of the corresponding interpolation processes. Moreover, the error of such an approximation is near to the error of the best uniform approximation. Beside two classical examples of the well-known optimal system of nodes (zeros of the Jacobi polynomials  $P_n^{(\alpha, \beta)}(x)$  ( $-1 < \alpha, \beta \leq -1/2$ ) and the so-called Clenshaw's abscissas), we introduce more general results for constructing interpolation processes at nodes with an *arc sine distribution* having Lebesgue constants of order  $\log n$ . The so-called *additional nodes method with Jacobi zeros* is presented in Sect. 4.2.2. Some other optimal interpolation processes are analyzed in 4.2.3. The third section of this chapter is devoted to the weighted interpolation in the corresponding weighted spaces (Jacobi, Laguerre and Hermite cases). In addition, we consider the weighted interpolation of functions with internal isolated singularities.

The final chapter provides some selected applications in numerical analysis. In the first section on quadrature formulae we present some special Newton–Cotes rules, the Gauss–Christoffel, Gauss–Radau and Gauss–Lobatto quadratures, the so-called product integration rules, as well as a method for the numerical integration of periodic functions on the real line with respect to a rational weight function. Also, we include the error estimates of Gaussian rules for some classes of functions. The second section is devoted to methods for solving the Fredholm integral equations of the second kind. The methods are based on the so-called *Approximation and Polynomial Interpolation Theory* and lead to the construction of polynomial sequences converging to the exact solutions in some weighted uniform norms. In the third section we consider some kinds of moment-preserving approximations by polynomials and splines. In the last section of this chapter we consider two recent methods of summation of slowly convergent series based on integral representations of series and an application of the Gaussian quadratures. In the first method we assume that the general term of the series is expressible in terms of the Laplace transform (or its derivative) of a known function. It leads to the Gaussian quadrature formulas with respect to the Einstein and Fermi weight functions on  $(0, +\infty)$ . The second method is based on a contour integration over a rectangle in the complex plane, reducing then the summation of a slowly convergent series to a problem of Gaussian quadrature rules on  $(0, +\infty)$  with respect to some hyperbolic weight functions.

Notation of this book is standard. If it is not defined in another way, throughout this book  $C, C_0, C_1, C_2, \dots$  denote positive constants, which can take different values even in subsequent formulae. It will always be clear what indices and variables the constants are independent of. If we use the notation  $C_p$ , it means that this constant always depends on a parameter  $p$ . Sometimes, we will write  $C \neq C(a, b, \dots)$  in order to denote that the constant  $C$  is independent only of  $a, b, \dots$ , but it can depend on parameters which are not mentioned in the list  $(a, b, \dots)$ .

If  $A$  and  $B$  are two expressions depending on certain indices and variables, then we write

$$A \sim B \quad \text{if and only if} \quad 0 < C_1 \leq \left| \frac{A}{B} \right| \leq C_2$$

uniformly for the indices and variables considered.

Some five hundred references have been cited here. As a rule, we have studied the original sources and in some cases have retrieved some forgotten but useful results. At the end of the book we included an index, combined with subjects and names.

The book addresses researchers and students in mathematics, physics, and other computational and applied sciences.

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We dedicate this book to our wives, Ida and Dobrila, in appreciation of their patience and unwavering support.

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