2. Boundary Integral Equations

In Chapter 1 we presented basic ideas for the reduction of boundary value problems of the Laplacian to various forms of boundary integral equations based on the direct approach. This reduction can be easily extended to more general partial differential equations. Here we will consider, in particular, the Helmholtz equation, the Lamé system, the Stokes equations and the biharmonic equation.

For the Helmholtz equation, we also investigate the solution's asymptotic behavior for small wave numbers and the relation to solutions of the Laplace equation by using the boundary integral equations.

For the Lamé system of elasticity, we first present the boundary integral equations of the first kind as well as of the second kind. Furthermore, we study the behavior of the solution and the boundary integral equations for incompressible materials. As will be seen, this has a close relation to the Stokes system and its boundary integral equations. In the two–dimensional case, both the Stokes and the Lamé problems can be reduced to solutions of biharmonic boundary value problems which, again, can be solved by using boundary integral equations based on the direct approach.

In this chapter we consider these problems for domains whose boundaries are smooth enough, mostly Lyapounov boundaries, and the boundary charges belonging to Hölder spaces. Later on we shall consider the boundary integral equations again on Sobolev trace spaces which is more appropriate for stability and convergence of corresponding discretization procedures.

2.1 The Helmholtz Equation

A slight generalization of the Laplace equation is the well–known *Helmholtz equation*

$$-(\Delta + k^2)u = 0 \quad \text{in} \;\; \Omega \text{ (or } \Omega^c) \,. \tag{2.1.1}$$

This equation arises in connection with the propagation of waves, in particular in acoustics (Filippi [78], Kupradze [175] and Wilcox [321]) and electromagnetics (Ammari [6], Cessenat [38], Colton and Kress [47], Jones [152], Müller [221] and Neledec [234]). In acoustics, k with Im $k \geq 0$ denotes the complex *wave number* and u corresponds to the *acoustic pressure* field.

The reduction of boundary value problems for (2.1.1) to boundary integral equations can be carried out in the same manner as for the Laplacian in Chapter 1. For (2.1.1), in the exterior domain, one requires at infinity the *Sommerfeld radiation conditions*,

$$u(x) = \mathrm{O}(|x|^{-(n-1)/2}) \quad \text{and} \quad \frac{\partial u}{\partial |x|}(x) - iku(x) = o\left(|x|^{-(n-1)/2}\right) \tag{2.1.2}$$

where i is the imaginary unit. (See the book by Sommerfeld [287] and the further references therein; see also Neittaanmäki and Roach [236] and Wilcox [321]). These conditions select the outgoing waves; they are needed for uniqueness of the exterior Dirichlet problem as well as for the Neumann problem.

The pointwise condition (2.1.2) can be replaced by a more appropriate and weaker version of the radiation condition given by Rellich [261, 262],

$$\lim_{R\to\infty} \int\limits_{|x|=R} |\frac{\partial u}{\partial n}(x) - iku(x)|^2 ds = 0\,. \tag{2.1.3}$$

This form is to be used in the variational formulation of exterior boundary value problems.

The fundamental solution $E(x,y)$ to (2.1.1), subject to the radiation condition (2.1.2) for fixed $y \in R^n$ is given by

$$E_k(x,y) = \begin{cases} \dfrac{i}{4} H_0^{(1)}(kr) & \text{in } \mathbb{R}^2, \\ & \qquad \text{with } r = |x-y| \\ \dfrac{e^{ikr}}{4\pi r} & \text{in } \mathbb{R}^3, \end{cases} \tag{2.1.4}$$

where $H_0^{(1)}$ denotes the modified Bessel function of the first kind. We note, that for $n = 2, E(x,y)$ has a branch point for $\mathbb{C} \ni k \to 0$. Therefore, in the following we confine ourselves first to the case $k \neq 0$. In terms of these fundamental solutions, which obviously are symmetric, the representation of solutions to (2.1.1) in Ω or in Ω^c with (2.1.2) assumes the same forms as (1.1.7) and (1.4.5), namely

$$\begin{aligned} u(x) &= \pm\Big\{ \int\limits_{y\in\Gamma} E_k(x,y)\frac{\partial u}{\partial n}(y) ds_y - \int\limits_{y\in\Gamma} u(y)\frac{\partial E_k(x,y)}{\partial n_y} ds_y \Big\} \\ &= \pm\Big\{ V_k \frac{\partial u}{\partial n}(x) - W_k u(x) \Big\} \end{aligned} \tag{2.1.5}$$

for all $x \in \Omega$ or Ω^c respectively, where the $\pm$sign corresponds to the interior and the exterior domain. Here, V_k is defined by

$$\begin{aligned} V_k\sigma(x) :=& \int\limits_{y\in\Gamma} E_k(x,y)\sigma(y) ds_y \\ &= V\sigma(x) + S_k\sigma(x) + \{\delta_{n3}\tfrac{ik}{4\pi} - \delta_{n2}(\log k + \gamma_0)\} \int\limits_{\Gamma} \sigma ds\,, \end{aligned} \tag{2.1.6}$$

where V is given by (1.2.1) with (1.1.2) and S_k is a k–dependent remainder defined by

$$S_k\sigma(x) = \begin{cases} \int\limits_{y\in\Gamma} \Big\{\frac{i}{4}\, H_0^{(1)}(k|x-y|) + \frac{1}{2\pi}\log|x-y|\Big\}\sigma(y)ds_y, \\ -\frac{1}{4\pi}\int\limits_{y\in\Gamma}\Big\{\sum\limits_{m=2}^{\infty}\frac{(m-1)}{m!}(ik|x-y|)^m\Big\}\frac{1}{|x-y|}\sigma(y)ds_y \end{cases} \tag{2.1.7}$$

for $n = 2$ or 3, respectively. The potential W_k is defined by

$$W_k\varphi(x) := \int\limits_{y\in\Gamma} \frac{\partial E_k(x,y)}{\partial n_y}\varphi(y)ds_y = W\varphi(x) + R_k\varphi(x) \tag{2.1.8}$$

where W is given by (1.2.2) with (1.1.2) and R_k is a k–dependent remainder defined by

$$R_k\varphi(x) = \begin{cases} \int\limits_{y\in\Gamma\setminus\{x\}} \frac{\partial}{\partial n_y}\{\frac{i}{4}H_0^{(1)}(k|x-y|) + \frac{1}{2\pi}\log|x-y|\}\varphi(y)ds_y\,, \\ -\frac{1}{4\pi}\int\limits_{y\in\Gamma\setminus\{x\}}\Big\{\sum\limits_{m=2}^{\infty}\frac{(m-1)}{m!}(ik|x-y|)^m\Big\}\Big(\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\Big)\varphi(y)ds_y \end{cases} \tag{2.1.9}$$

for $n = 2$ or 3, respectively. Note that S_k and R_k have bounded kernels for all $y \in \Gamma$ and $x \in \mathbb{R}^n$ and for $k \neq 0$ in the case $n = 2$. Hence, $S_k\sigma(x)$ and $R_k\varphi(x)$ are well defined for all $x \in \mathbb{R}^n$. Moreover, the properties of the operators V and W, as given by Lemmata 1.2.1, 1.2.2, 1.2.4 and Theorem 1.2.3 for the Laplacian, remain valid for V_k and W_k. Since the kernels of V_k and W_k, S_k and V_k depend analytically on $k \in \mathbb{C}\setminus\{0\}$ for $n = 2$ and $k \in \mathbb{C}$ for $n = 3$, the solutions of the corresponding boundary integral equations will depend analytically on the wave number k, as well.

Here again we consider the interior and *exterior Dirichlet problem* for (2.1.1), where

$$u_{|\Gamma} = \varphi \quad \text{on } \Gamma \text{ is given.} \tag{2.1.10}$$

In acoustics, (2.1.10) models a "soft" boundary for the interior and a "soft scatterer" for the exterior problem. As in Section 1.3.1, here the missing Cauchy datum on Γ is $\frac{\partial u}{\partial n}_{|\Gamma} = \sigma$.

The boundary integral equation for the interior Dirichlet problem reads

$$V_k\sigma(x) = \frac{1}{2}\varphi(x) + K_k\varphi(x), \quad x \in \Gamma\,, \tag{2.1.11}$$

where

$$K_k\varphi(x) := \int\limits_{y\in\Gamma\setminus\{x\}} \frac{\partial E_k(x,y)}{\partial n_y}\varphi(y)ds_y = K\varphi(x) + R_k\varphi(x) \quad \text{for } x\in\Gamma \tag{2.1.12}$$

with K given by (1.2.8) with (1.2.12) or (1.2.13), respectively. As before, (2.1.11) is a Fredholm boundary integral equation of the first kind for σ on Γ. In the classical Hölder continuous function space $C^\alpha, 0 < \alpha < 1$, this integral equation has been studied by Colton and Kress in [47, Chap.3]. In particular, for $k \neq 0$ and $\varphi \in C^{1+\alpha}(\Gamma)$, (2.1.11) is uniquely solvable with $\sigma \in C^\alpha(\Gamma)$, except for certain values of $k \in \mathbb{C}$ which are the *exceptional* or *irregular frequencies* of the boundary integral operator V_k. For any irregular frequency k_0, the operator V_{k_0} has a nontrivial nullspace *ker* V_{k_0} = *span* $\{\sigma_{0j}\}$. The eigensolutions σ_{0j} are related to the eigensolutions $\widetilde{u}_{0j}$ of the *interior Dirichlet problem for the Laplacian*,

$$\begin{aligned} -\Delta\widetilde{u}_0 &= k_0^2\widetilde{u}_0 \ \text{ in } \ \Omega\,, \\ \widetilde{u}_{0_{|\Gamma}} &= 0 \ \text{ on } \ \Gamma\,. \end{aligned} \tag{2.1.13}$$

That is,

$$\sigma_{0j} = \frac{\partial\widetilde{u}_{0j}}{\partial n|_\Gamma}. \tag{2.1.14}$$

Moreover, the solutions are real–valued and

$$\dim kerV_{k_0} = \text{ dimension of the eigenspace of (2.1.13)}\,.$$

As is known, see e.g. Hellwig [123, p.229], the eigenvalue problem (2.1.13) admits denumerably infinitely many eigenvalues $k_{0\ell}^2$. They are all real and have at most finite multiplicity. Moreover, they can be ordered according to size $0 < k_{01}^2 < k_{02}^2 < \cdots$ and have $+\infty$ as their only limit point. For any of the corresponding eigensolutions $\widetilde{u}_{0j}$, (2.1.14) can be obtained from (2.1.5) applied to $\widetilde{u}_{0j}$. In this case, when k_0 is an eigenvalue, the interior Dirichlet problem (2.1.1), (2.1.10) admits solutions in $C^\alpha(\Gamma)$ if and only if the given boundary values $\varphi \in C^{1+\alpha}(\Gamma)$ satisfy the *orthogonality conditions*

$$\int\limits_\Gamma \varphi\sigma_0 ds = \int\limits_\Gamma \varphi\frac{\partial\widetilde{u}_0}{\partial n}ds = 0 \quad \text{for all } \ \sigma_0 \in kerV_{k_0}. \tag{2.1.15}$$

Correspondingly, for $\varphi \in C^{1+\alpha}(\Gamma)$, the boundary integral equation (2.1.11) has solutions $\sigma \in C^\alpha(\Gamma)$ if and only if (2.1.15) is satisfied.

For the *exterior Dirichlet problem*, i.e. (2.1.1) in Ω^c with the Sommerfeld radiation conditions (2.1.2) and boundary condition (2.1.10), from (2.1.5) again we obtain a boundary integral equation of the first kind,

$$V_k\sigma(x) = -\tfrac{1}{2}\varphi(x) + K_k\varphi(x)\,,\ x\in\Gamma\,, \tag{2.1.16}$$

which differs from (2.1.11) only by a sign in the right–hand side. Hence, the exceptional values k_0 are the same as for the interior Dirichlet problem, namely the eigenvalues of (2.1.13). If $k \neq k_0$, (2.1.16) is always uniquely solvable for $\sigma \in C^\alpha(\Gamma)$ if $\varphi \in C^{1+\alpha}(\Gamma)$. For $k = k_0$, in contrast to the interior Dirichlet problem, the exterior Dirichlet problem remains uniquely solvable. However, (2.1.16) now has eigensolutions, and the right–hand side always satisfies the orthogonality conditions

$$\int\limits_{x\in\Gamma} \big(-\tfrac{1}{2}\varphi(x) + K_{k_0}\varphi(x)\big)\sigma_0(x)ds_x$$
$$= \int\limits_{x\in\Gamma} \varphi(x)\big\{-\tfrac{1}{2}\sigma_0(x) + K'_{k_0}\sigma_0(x)\big\}ds_x = 0 \;\text{ for all }\; \sigma_0 \in \ker V_{k_0}\,,$$

since σ_0 is real valued and the simple layer potential $V_{k_0}\sigma_0(x)$ vanishes identically for $x \in \Omega^c$. The latter implies

$$\frac{\partial}{\partial n_x} V_{k_0}\sigma_0(x) = -\frac{1}{2}\sigma_0(x) + K'_{k_0}\sigma_0(x) = 0 \;\text{ for }\; x \in \Gamma\,,$$

where

$$K'_k\sigma(x) := \int\limits_{y\in\Gamma\setminus\{x\}} \Big(\frac{\partial}{\partial n_x}E_k(x,y)\Big)\sigma(y)ds_y = K'\sigma(x) + R'_k\sigma(x)$$

with

$$R'_k\sigma(x) = \begin{cases} \displaystyle\int\limits_{y\in\Gamma\setminus\{x\}} \frac{\partial}{\partial n_x}\Big\{\frac{i}{4}H_0^{(1)}(k|x-y|) + \frac{1}{2\pi}\log|x-y|\Big\}\sigma(y)ds_y\,, \\ \displaystyle -\frac{1}{4\pi}\int\limits_{y\in\Gamma\setminus\{x\}} \Big\{\sum_{m=2}^{\infty}\frac{(m-1)}{m!}(ik|x-y|)^m\Big\}\Big(\frac{\partial}{\partial n_x}\frac{1}{|x-y|}\Big)\sigma(y)ds_y\,, \end{cases}$$

for $n = 2$ and $n = 3$, respectively. Accordingly, the representation formula (2.1.5) with $u_{|\Gamma} = \varphi$ and $\frac{\partial u}{\partial n}_{|\Gamma} = \sigma$ will generate a *unique* solution for any σ solving (2.1.16).

Alternatively, both, the interior and exterior Dirichlet problem can also be solved by the Fredholm integral equations of the second kind as (1.3.7) and (1.4.14) for the Laplacian. In order to avoid repetition we summarize the different direct formulations of the interior and exterior Dirichlet and Neumann problems which will be abbreviated by (IDP), (EDP), (INP) and (ENP), accordingly, in Table 2.1.1. The Neumann data in (INP) and (ENP) will be denoted by ψ.

In addition to the previously defined integral operators V_k, K_k, K'_k we also introduce the hypersingular integral operator, D_k for the Helmholtz equation, namely

Table 2.1.1. Summary of the boundary integral equations for the Helmholtz equation and the related eigenvalue problems

BVP	BIE	Eigensolutions $\tilde{u}_0$ or $\tilde{u}_1$ for BVP and Exceptional values k_0, k_1	Eigensolutions for BIE, σ_0, σ_1 or u_0, u_1	Solvability Conditions for given φ, ψ
IDP	$(1) V_k\sigma = (\frac{1}{2}I + K_k)\varphi$	(D_0):		
	$(2) (\frac{1}{2}I - K'_k)\sigma = D_k\varphi$	$-(\Delta + k_0^2)\tilde{u}_0 = 0$ in Ω,	$\sigma_0 = \frac{\partial \tilde{u}_0}{\partial n}_{\mid \Gamma}$	$\int_\Gamma \sigma_0 \varphi ds = 0$
EDP	$(1) V_k\sigma = (-\frac{1}{2}I + K_k)\varphi$	$\tilde{u}_{0_{\mid \Gamma}} = 0$ on Γ		
	$(2) (\frac{1}{2}I + K'_k)\sigma = -D_k\varphi$	(N_0):	$V_{k_1}\sigma_1 = \tilde{u}_1$ on Γ	None
INP	$(1) D_k u = (\frac{1}{2}I - K'_k)\psi$	$-(\Delta + k_1^2)\tilde{u}_1 = 0$ in Ω		
	$(2) (\frac{1}{2}I + K_k)u = V_k\psi$	$\frac{\partial \tilde{u}_1}{\partial n}_{\mid \Gamma} = 0$ on Γ	$u_1 = \tilde{u}_{1_{\mid \Gamma}}$ on Γ	$\int_\Gamma u_1 \psi ds = 0$
ENP	$(1) D_k u = -(\frac{1}{2}I + K'_k)\psi$			
	$(2) (\frac{1}{2}I - K_k)u = -V_k\psi$	(D_0)	$D_{k_0}u_0 = \frac{\partial \tilde{u}_0}{\partial n}_{\mid \Gamma}$	None

$$D_k\varphi(x) := -\frac{\partial}{\partial n_x} \int\limits_{\Gamma\setminus\{x\}} \frac{\partial E_k(x,y)}{\partial n_y}\varphi(y)ds_y = D\varphi(x) + \frac{\partial}{\partial n_x} R_k\varphi(x) \ \text{ on } \Gamma\,, \tag{2.1.17}$$

where D is the hypersingular integral operator (1.4.10) of the Laplacian and R_k is the remainder in (2.1.9).

Note that the relations between the eigensolutions of the BIEs and the interior eigenvalue problems of the Laplacian are given explicitly in column three of Table 2.1.1. We also observe that for the exterior boundary value problems, the exceptional values k_0 and k_1 of the corresponding boundary integral operators depend on the type of boundary integral equations derived by the direct formulation. For instance, we see that for (EDP), k_0 are the exceptional values for V_k whereas k_1 are those for $(\frac{1}{2}I + K_k')$. Similar relations hold for (ENP).

In Table 2.1.1, the second column contains all of the boundary integral equations (BIE) obtained by the direct approach. As we mentioned earlier, for the *exterior* boundary value problems, the solvability conditions of the corresponding boundary integral equations at the exceptional values are always satisfied due to the special forms of the corresponding right-hand sides. For the indirect approach, this is not the case anymore; see Colton and Kress [47, Chap. 3]. There are various ways to modify the boundary integral equations so that some of the exceptional values will not belong to the spectrum of the boundary integral operator anymore. In this connection, we refer to the work by Brakhage and Werner [22] Colton and Kress [47], Jones [152], Kleinman and Kress [158] and Ursell [309], to name a few.

2.1.1 Low Frequency Behaviour

Of particular interest is the case $k \to 0$ which corresponds to the low–frequency behaviour. This case also determines the large–time behaviour of the solution to time–dependent problems if (2.1.1) is obtained from the wave equation by the Fourier–Laplace transformation (see e.g. MacCamy [194, 195] and Werner [319]). As will be seen, some of the boundary value problems will exhibit a singular behaviour for $k \to 0$. The main results are summarized in the Table 2.1.2 below.

To illustrate the singular behaviour we begin with the explicit asymptotic expansions of the boundary integral equations in Table 2.1.1. Our presentation here follows [140]. In particular, we begin with the fundamental solution for small kr having the series expansions:

For $n = 2$:

$$E_k(x,y) = \frac{i}{4} H_0^{(1)}(kr) = E(x,y) - \frac{1}{2\pi}(\log k + \gamma_0) + S_k(x,y)\,, \tag{2.1.18}$$

where

$$\gamma_0 = c_0 - \log 2 - i\frac{\pi}{2} \quad \text{with } \; c_0 \approx 0.5772\,, \quad \text{Euler's constant,}$$

and

$$\begin{aligned}
S_k(x,y) &= \frac{i}{4}H_0^{(1)}(kr) + \frac{1}{2\pi}(\log(kr) + \gamma_0) \\
&= -\frac{1}{2\pi}\{\log(kr)\sum_{m=1}^{\infty} a_m (kr)^{2m} + \sum_{m=1}^{\infty} b_m (kr)^{2m}\}\,, \\
a_m &= \frac{(-1)^m}{2^{2m}(m!)^2}\,, \qquad b_m = (\gamma_0 - 1 - \frac{1}{2} - \cdots - \frac{1}{m})a_m\,.
\end{aligned}$$

For $n = 3$:

$$E_k(x,y) = E(x,y) + \frac{ik}{4\pi} + S_k(x,y)\,,$$

where

$$S_k(x,y) = \frac{1}{4\pi r}(e^{ikr} - 1 - ikr) = \frac{1}{4\pi r}\sum_{m=2}^{\infty}\frac{(ikr)^m}{m!}\,. \tag{2.1.19}$$

Correspondingly, for the double layer potential kernel we obtain

$$\frac{\partial}{\partial n_y}E_k(x,y) = \frac{\partial}{\partial n_y}E(x,y) + R_k(x,y)\,,$$

where

$$R_k(x,y) = \Big\{\sum_{m=1}^{\infty}\left(a_m(1 + 2m\log(kr)) + 2mb_m\right)(rk)^{2m}\Big\}\frac{\partial}{\partial n_y}E(x,y) \quad \text{for } \; n = 2\,, \tag{2.1.20}$$

and

$$R_k(x,y) = -\Big\{\sum_{m=2}^{\infty}\frac{(m-1)}{m!}(ikr)^m\Big\}\frac{\partial}{\partial n_y}E(x,y) \quad \text{for } \; n = 3\,. \tag{2.1.21}$$

The kernel of the adjoint operator R'_k can be obtained by interchanging the variables x and y. Hence, R'_k has the same asymptotic behaviour as R_k for $k \to 0$.

Similarly, for the hypersingular kernel we have, for $n = 2$,

$$\begin{aligned}
\frac{\partial R_k}{\partial n_x}(x,y) = -4\pi\sum_{m=2}^{\infty}(m-1)c_m(k)(rk)^{2m}\frac{\partial E}{\partial n_x}\frac{\partial E}{\partial n_y}(x,y) \qquad (2.1.22)\\
+ \frac{1}{2\pi}\sum_{m=1}^{\infty}c_m(k)(rk)^{2m}\frac{n_x \cdot n_y}{r^2}\,, \\
+ \log r\sum_{m=1}^{\infty}2ma_m(rk)^{2m}\Big\{\frac{1}{2\pi}\;\frac{\boldsymbol{n}_x \cdot \boldsymbol{n}_y}{r^2} - 4\pi(m-1)\frac{\partial E}{\partial n_x}\frac{\partial E}{\partial n_y}\Big\}
\end{aligned}$$

where

$$c_m(k) = (1 + 2m \log k)a_m + 2mb_m\,,$$

and, for $n = 3$,

$$\begin{aligned}\frac{\partial R_k}{\partial n_x}(x,y) = &-\frac{1}{4\pi}\sum_{m=2}^{\infty}\frac{(m-1)}{m!}(ikr)^m\frac{\boldsymbol{n}_x\cdot\boldsymbol{n}_y}{r^3}\\ &-4\pi\sum_{m=2}^{\infty}(3+m)\frac{(m-1)}{m!}r(ikr)^m\frac{\partial E}{\partial n_x}\frac{\partial E}{\partial n_y}(x,y)\,.\end{aligned} \tag{2.1.23}$$

Note that the kernel $\dfrac{\partial R_k}{\partial n_x}(x,y)$ is symmetric.

As can be seen from the above expansions, the term $\log k$ appears in (2.1.18) explicitly which shows that V_k is a singular perturbation of V whereas the other operators are regular perturbations of the corresponding operators of the Laplacian as $k \to 0$.

IDP: Let us consider first the simplest case, i.e. Equation (2) for (IDP) in Table 2.1.1,

$$\left(\tfrac{1}{2}I - K' - R_k'\right)\sigma = D_k\varphi \quad\text{on}\quad \Gamma\,. \tag{2.1.24}$$

Since, for given $\varphi \in C^{1+\alpha}(\Gamma)$, the equation

$$(\tfrac{1}{2}I - K')\widetilde{\sigma} = D\varphi$$

has a unique solution $\widetilde{\sigma} \in C^{\alpha}(\Gamma)$, we may rewrite (2.1.24) as

$$\begin{aligned}\sigma &= (\frac{1}{2}I - K')^{-1}R_k'\sigma + (\frac{1}{2}I - K')^{-1}D_k\varphi\,,\\ &= \widetilde{\sigma} + (\frac{1}{2}I - K')^{-1}R_k'\sigma + (\frac{1}{2}I - K')^{-1}\frac{\partial}{\partial n_x}R_k\varphi\,,\\ &= \widetilde{\sigma} + \begin{cases}O(k^2\log k) & \text{for } n=2,\\ O(k^2) & \text{for } n=3,\end{cases}\end{aligned} \tag{2.1.25}$$

where the last expressions can be obtained from the expansions (2.1.20) and (2.1.22) in case $n = 2$ and from (2.1.21) and (2.1.23) in case $n = 3$ (see MacCamy [194]).

The analysis for the integral equation of the first kind (1) for (IDP) in Table 2.1.1 is more involved, depending on $n = 2$ or 3. For $n = 2$, from (2.1.11) with the expansion (2.1.18) of (2.1.7) we have

$$V\sigma + \omega + S_k\sigma = \frac{1}{2}\varphi + K\varphi + R_k\varphi \tag{2.1.26}$$

where

$$\omega = -\frac{1}{2\pi}(\log k + \gamma_0)\int\limits_{\Gamma}\sigma ds\,. \tag{2.1.27}$$

Similar to (2.1.25), we seek the solution of (2.1.26) and (2.1.27) in the asymptotic form,

$$\begin{aligned} \sigma &= \widetilde{\sigma} + \alpha_1(k)\widetilde{\sigma}_1 + \sigma_R\,, \\ \omega &= \widetilde{\omega} + \alpha_1(k)\widetilde{\omega}_1 + \omega_R\,, \end{aligned} \tag{2.1.28}$$

where $\widetilde{\sigma}, \widetilde{\omega}$ correspond to the solution of the interior Dirichlet problem for the Laplacian (1.1.6), (1.3.1). Hence, $\widetilde{\sigma}, \widetilde{\omega}$ satisfy (1.3.3), namely

$$V\widetilde{\sigma} + \widetilde{\omega} = \frac{1}{2}\varphi + K\varphi \tag{2.1.29}$$

subject to the constraints

$$\int_\Gamma \widetilde{\sigma} ds = 0 \quad \text{and} \quad \widetilde{\omega} = 0\,.$$

The first perturbation terms $\widetilde{\sigma}_1, \widetilde{\omega}_1$ are independent of k with the coefficient $\alpha_1(k) = o(1)$ for $k \to 0$. The functions σ_R, ω_R are the remainders which are of order $o(\alpha_1(k))$. To construct $\widetilde{\sigma}_1$ and $\widetilde{\omega}_1$, we employ equation (2.1.26) inserting (2.1.28). For $k \to 0$ we arrive at

$$\begin{aligned} V\widetilde{\sigma}_1 + \widetilde{\omega}_1 &= 0\,, \\ \int_\Gamma \widetilde{\sigma}_1 ds &= 1\,, \end{aligned} \tag{2.1.30}$$

where we appended the last normalizing condition for $\widetilde{\sigma}_1$, since from the previous results in Section 1.3 we know that $\int_\Gamma \widetilde{\sigma}_1 ds = 0$ would yield the trivial solution $\widetilde{\sigma}_1 = 0, \widetilde{\omega}_1 = 0$. Inserting (2.1.28) into (2.1.27), it follows from $\int_\Gamma \widetilde{\sigma}_1 ds = 1$ with $\widetilde{\omega} = 0$ that $\alpha_1(k) = O(\sigma_R)$. Hence, without loss of generality, we may set $\alpha_1(k) = 0$ in (2.2.28). Now (2.1.26) and (2.1.27) with (2.1.28) imply that the remaining terms σ_R, ω_R satisfy the equations

$$\begin{aligned} V\sigma_R + \omega_R + S_k\sigma_R = R_k\varphi - S_k\widetilde{\sigma}\,, \\ \int_\Gamma \sigma_R ds + \omega_R 2\pi(\log k + \gamma_0)^{-1} = 0\,; \end{aligned} \tag{2.1.31}$$

which are *regular* perturbations of equations (1.3.5), (1.3.6) due to the expansions for S_k in (2.1.18). The right–hand side of (2.1.31) is of order $O(k^2 \log k)$ because of the expansions of S_k and of R_k in (2.1.20). Therefore, σ_R and ω_R are, indeed, of order $O(k^2 \log k)$ as in (2.1.25), which was already obtained with the integral equations (2.1.24) of the second kind.

For $n = 3$, the integral equation of the first kind takes the form

$$V\sigma + ik\frac{1}{4\pi}\int_\Gamma \sigma ds + S_k\sigma = \frac{1}{2}\varphi + K\varphi + R_k\varphi\,, \tag{2.1.32}$$

where the kernels of the integral operators S_k and R_k are given by (2.1.19) and (2.1.21), respectively. Both are of order $O(k^2)$. Hence, (2.1.32) is a regular perturbation of equation (1.3.3). If we insert (2.1.28) with $\alpha_1(k) = k$ then the function $\widetilde{\sigma}$ is given by the solution of

$$V\widetilde{\sigma} = \tfrac{1}{2}\varphi + K\varphi\,,$$

which corresponds to the interior Dirichlet problem of the Laplacian. Hence,

$$\int_\Gamma \widetilde{\sigma} ds = 0 \quad \text{and} \quad \widetilde{\sigma}_1 \equiv 0$$

since it is the solution of

$$V\widetilde{\sigma}_1 + \frac{i}{4\pi}\int_\Gamma \widetilde{\sigma} ds = 0\,.$$

Therefore, the solution of (2.1.32) is of the form $\sigma = \widetilde{\sigma} + O(k^2)$.

<u>EDP:</u> By using the indirect formulation of boundary integral equations, this case was also analyzed by Hariharan and MacCamy in [120]. In case $n = 2$, for the exterior Dirichlet problem (EDP), Equation (1) in Table 2.1.1 has the form

$$V\sigma + \omega + S_k\sigma = -\tfrac{1}{2}\varphi + K\varphi + R_k\varphi \tag{2.1.33}$$

with ω again defined by (2.1.27). Again, the solution admits the asymptotic expansion (2.1.28) with $\widetilde{\sigma}, \widetilde{\omega}$ being the solution of

$$V\widetilde{\sigma} + \widetilde{\omega} = -\tfrac{1}{2}\varphi + K\varphi\,, \int_\Gamma \widetilde{\sigma} ds = 0\,. \tag{2.1.34}$$

Hence, $\widetilde{\sigma}, \widetilde{\omega}$ correspond to the exterior Dirichlet problem in Section 1.4.1. Therefore, in contrast to the (IDP), here $\widetilde{\omega} \neq 0$, in general. Again, $\widetilde{\sigma}_1, \widetilde{\omega}_1$ are solutions of equations (2.1.30). In contrast to the (IDP), the coefficient $\alpha_1(k)$ is explicitly given by

$$\alpha_1(k) = -\widetilde{\omega}\left\{\frac{1}{\pi}(\log k + \gamma_0) + \widetilde{\omega}_1\right\}^{-1}\,,$$

which is not identically equal to zero, in general. The remainders σ_R, ω_R satisfy equations similar to (2.1.31) and are of order $O(k^2 \log k)$.

For the case $n = 3$, the integral equation(1) for (EDP) is of the form

$$V\sigma + ik\frac{1}{4\pi}\int_\Gamma \sigma ds + S_k\sigma = -\tfrac{1}{2}\varphi + K\varphi + R_k\varphi$$

and σ is of the form

$$\sigma = \widetilde{\sigma} + k\widetilde{\sigma}_1 + \sigma_R\,.$$

This yields

$$V\widetilde{\sigma} = -\tfrac{1}{2}\varphi + K\varphi$$

for $\widetilde{\sigma}$ corresponding to the (EDP) for the Laplacian . For the next term $\widetilde{\sigma}_1$ we have

$$V\widetilde{\sigma}_1 = -\frac{i}{4\pi}\int\limits_{\Gamma}\widetilde{\sigma}ds\,.$$

Therefore, $\widetilde{\sigma}_1$ is proportional to the *natural charge* q which is the unique eigensolution of

$$\tfrac{1}{2}q + K'q = 0 \ \text{ on } \Gamma \ \text{ with } \int\limits_{\Gamma} qds = 1\,.$$

Note that the corresponding simple layer potential satisfies

$$Vq(x) = c_0 = \text{ const. for all } x \in \overline{\Omega}\,. \tag{2.1.35}$$

For $n = 3$, a simple contradiction argument shows $c_0 \neq 0$. Hence,

$$\widetilde{\sigma}_1(x) = -\Big(\frac{i}{4\pi c_0}\int\limits_{\Gamma}\widetilde{\sigma}ds\Big)q(x)\,.$$

The remainder term σ_R is of order k^2 which follows easily from

$$V\sigma_R + S_k\sigma_R = -S_k(\widetilde{\sigma} + k\widetilde{\sigma}_1) - k^2\frac{i}{4\pi}\int\limits_{\Gamma}\widetilde{\sigma}_1 ds - R_k\varphi\,,$$

which is a regular perturbation of (1.4.12).

For the integral equation of the second kind (2) of the (EDP) and $k = 0$, the homogeneous reduced integral equation reads

$$\frac{1}{2}q + K'q = 0 \tag{2.1.36}$$

with the natural charge q as an eigensolution. We therefore modify boundary integral equation (2) by using a method by Wielandt [320] (see also Werner [318]). Using Equation (1), we obtain a normalization condition for σ,

$$\int\limits_{\Gamma}\sigma ds = l_k(\varphi) + \widetilde{l}_k(\sigma)\,,$$

where the linear functionals l_k and $\widetilde{l}_k$ are given by

$$l_k(\varphi) = \begin{cases} \left(\frac{1}{2\pi}(\log k + \gamma_0) - c_0\right)^{-1} \left\{ \int\limits_\Gamma \varphi q ds + \int\limits_\Gamma (R_k\varphi) q ds \right\} & \text{for } n = 2, \\ (c_0 + k\frac{i}{4\pi})^{-1} \left\{ \int\limits_\Gamma \varphi q ds + \int\limits_\Gamma (R_k\varphi) q ds \right\} & \text{for } n = 3; \end{cases}$$

and

$$\widetilde{l}_k(\sigma) = \begin{cases} \left(\frac{1}{2\pi}(\log k + \gamma_0) - c_0\right)^{-1} \int\limits_\Gamma (S_k\sigma) q ds & \text{for } n = 2, \\ \left(c_0 + k\frac{i}{4\pi}\right)^{-1} \int\limits_\Gamma (S_k\sigma) q ds & \text{for } n = 3. \end{cases}$$

The modified boundary integral equation of the second kind reads

$$(\tfrac{1}{2}I + K')\sigma + q \int\limits_\Gamma \sigma ds + R'_k\sigma - q\widetilde{l}_k(\sigma) = -D_k\varphi + q l_k(\varphi)\,, \tag{2.1.37}$$

which is a regular perturbation of the equation

$$(\tfrac{1}{2}I + K')\widetilde{\sigma} + q \int\limits_\Gamma \widetilde{\sigma} ds = -D\varphi + q l_0(\varphi)\,.$$

The latter is uniquely solvable. The next term $\widetilde{\sigma}_1$ satisfies the equation

$$(\tfrac{1}{2}I + K')\widetilde{\sigma}_1 + q \int\limits_\Gamma \widetilde{\sigma}_1 ds = \int\limits_\Gamma \varphi q ds.$$

It is not difficult to see that both boundary integral equations (2.1.33) and (2.1.37) provide the same asymptotic solutions

$$\sigma = \begin{cases} \widetilde{\sigma} + \alpha_1(k)\widetilde{\sigma}_1 + O(k^2 \log k) & \text{for } n = 2, \\ \widetilde{\sigma} + k\widetilde{\sigma}_1 + O(k^2) & \text{for } n = 3. \end{cases}$$

INP: Since the boundary integral equation of the second kind (see (2) in Table 2.1.1) for the (INP) and $k = 0$ has the constant functions as eigensolutions, we need an appropriate modification of (2). This modification can be derived from the Helmholtz equation (2.1.1) in Ω together with the Neumann boundary condition

$$\frac{\partial u}{\partial n}|_\Gamma = \psi \quad \text{on } \Gamma\,, \tag{2.1.38}$$

namely from the solvability condition (Green's formula for the Laplacian)

$$\int\limits_\Omega u(x) dx = -\frac{1}{k^2} \int\limits_\Gamma \psi ds\,. \tag{2.1.39}$$

This condition can be rewritten in terms of the representation formula (2.1.5),

$$\int\limits_{\Omega} W_k u(x)dx = \int\limits_{\Omega} V_k \psi(x)dx + \frac{1}{k^2}\int\limits_{\Gamma} \psi ds\,, \tag{2.1.40}$$

the left–hand side of which actually depends on the boundary values $u_{|\Gamma}$. It is not difficult to replace the domain integration on the left–hand side by appropriate boundary integrals. With the help of (2.1.40), the boundary integral equation of the second kind (see (2) in Table 2.1.1) for (INP) can be modified,

$$\begin{aligned} \frac{1}{2}u + Ku - \frac{1}{|\Omega|}\int\limits_{\Omega} Wu(x)dx + R_k u - \frac{1}{|\Omega|}\int\limits_{\Omega} R_k u(x)dx \\ = -\frac{1}{|\Omega|k^2}\int\limits_{\Gamma}\psi ds + V\psi - \frac{1}{|\Omega|}\int\limits_{\Omega} V\psi(x)dx \\ + S_k\psi - \frac{1}{|\Omega|}\int\limits_{\Omega} S_k\psi(x)dx \quad \text{on } \Gamma \end{aligned} \tag{2.1.41}$$

where $|\Omega| = meas(\Omega)$. Note, that the constant term of order $\log k$ in V_k for $n = 2$ and of order k for $n = 3$, respectively, has been canceled in (2.1.41) due to (2.1.40).

The boundary integral operator on the left–hand side in (2.1.41) is a regular perturbation of the reduced modified operator

$$A := \tfrac{1}{2}I + K - \frac{1}{|\Omega|}\int\limits_{\Omega} W \bullet dx \quad \text{on } \Gamma$$

due to (2.1.20) and (2.1.21). Moreover, we will see that the reduced homogeneous equation

$$Av_0 = 0 \;\text{ on } \Gamma$$

admits only the trivial solution $v_0 = 0$, since for the equivalent equation,

$$\tfrac{1}{2}v_0 + Kv_0 = \frac{1}{|\Omega|}\int\limits_{\Omega} Wv_0 dx = \kappa = \text{ const}, \tag{2.1.42}$$

the orthogonality condition for the original reduced operator on the left–hand side reads

$$\int\limits_{\Gamma} q\kappa ds = \kappa = 0\,.$$

This implies from (2.1.42) that $v_0 = const$ as the eigensolution discussed in Section 1.3.2. Hence, with (1.1.7), we have

$$0 = \kappa = \frac{1}{|\Omega|}\int\limits_{\Omega} Wv_0 dx = -v_0\,.$$

Consequently, A^{-1} exists in the classical function space $C^\alpha(\Gamma)$ due to the Fredholm alternative.

As for the asymptotic behaviour of u, it is suggested from (2.1.40) or (2.1.41) that u admits the form

$$u = \frac{1}{k^2}\alpha + \widetilde{u} + u_R\,. \tag{2.1.43}$$

The first term α satisfies the equation

$$A\alpha = \tfrac{1}{2}\alpha + K\alpha - \frac{1}{|\Omega|}\int\limits_\Omega W\alpha dx = -\frac{1}{|\Omega|}\int\limits_\Gamma \psi ds$$

which has the unique solution

$$\alpha = -\frac{1}{|\Omega|}\int\limits_\Gamma \psi ds = \text{ const.} \tag{2.1.44}$$

The second term $\widetilde{u}$ is the unique solution of the equation

$$A\widetilde{u} = V\psi - \frac{1}{|\Omega|}\int\limits_\Omega V\psi dx - \alpha \lim_{k\to 0}\frac{1}{k^2}\Big\{R_k 1 - \frac{1}{|\Omega|}\int\limits_\Omega R_k 1 dx\Big\}\,. \tag{2.1.45}$$

Now we investigate the last term in (2.1.45).

For $n = 2$, we have

$$R_k 1 = \frac{|\Omega|}{2\pi}(\log k + \gamma_0)k^2 + \frac{k^2}{2\pi}\int\limits_\Omega \log|x-y|dy + O(k^4\log k).$$

This implies

$$\begin{aligned}&\frac{1}{k^2}\Big\{R_k 1 - \frac{1}{|\Omega|}\int\limits_\Omega R_k 1 dx\Big\}\,,\\ &\quad = \frac{1}{2\pi}\Big\{\int\limits_\Omega \log|x-y|dy - \frac{1}{|\Omega|}\int\limits_\Omega\int\limits_\Omega \log|x-y|dydx\Big\} + O(k^2\log k)\,.\end{aligned}$$

Hence, the limit on the right–hand side in (2.1.45) exists.

For $n = 3$, the limit is a well defined function in $C^\alpha(\Gamma)$ due to (2.1.21).

The remainder u_R satisfies the equation

$$Au_R + R_k u_R - \frac{1}{|\Omega|}\int\limits_\Omega R_k u_R(x)dx = f_R \;\text{ on }\; \Gamma \tag{2.1.46}$$

where

$$f_R = \begin{cases} O(k^2 \log k) & \text{for } n = 2, \\ O(k^2) & \text{for } n = 3. \end{cases}$$

Hence, u_R is uniquely determined by (2.1.46) and is of the same order as f_R.

For the first kind hypersingular equation (1) in (INP) we use the same normalization condition (2.1.40) by subtracting it from the equation and obtain the modified hypersingular equation

$$\begin{aligned} Du - \int_\Omega Wu(z)dz - \frac{\partial}{\partial n_x} R_k u - \int_\Omega R_k u dz \\ = \; (\tfrac{1}{2} - K')\psi - R_k'\psi - \int_\Omega V\psi dz - \int_\Gamma S_k \psi dz \qquad (2.1.47) \\ + |\Omega| \Big(\delta_{n2} \frac{1}{2\pi} (\log k + \gamma_0) - \delta_{n3} \frac{ik}{4\pi} \Big) \int_\Gamma \psi ds - \frac{1}{k^2} \int_\Gamma \psi ds \,. \end{aligned}$$

Again, the operator on the left-hand side is a regular perturbation of the reduced operator

$$Bu := Du - \int_\Omega Wu(z)dz,$$

which can be shown to be invertible, in the same manner as for A. From the previous analysis, we write the solution in the form (2.1.43), namely

$$u = \frac{\alpha}{k^2} + \widetilde{u} + u_R \,.$$

We note that

$$B1 = |\Omega|,$$

and therefore (2.1.47) for $k \to 0$ again yields (2.1.44) for α. We further note that for $n = 2$, it can be verified from (2.1.20) that

$$\int_\Omega R_k \left(\frac{\alpha}{k^2} \right) dz = \frac{1}{2\pi} |\Omega|^2 \alpha \log k + O(1) \;\text{ for }\; k \to 0 \,,$$

which shows that the choice of α from (2.1.44) cancels the term $\frac{1}{2\pi}|\Omega| \log k \int_\Gamma \psi ds$ on the right–hand side of (2.1.47).

Consequently in both cases $n = 2$ and $n = 3$, $\widetilde{u}$ can be obtained as the unique solution of the equation

$$B\widetilde{u} = \frac{1}{2}\psi - K'\psi - \int_\Omega V\psi dz + \alpha \chi_n \,,$$

where

$$\chi_n = \begin{cases} -\dfrac{\gamma_0}{2\pi}|\Omega|^2 + \lim\limits_{k\to 0}\left\{\dfrac{1}{k^2}\int\limits_{\Omega} R_k 1 dz - \dfrac{|\Omega|^2}{2\pi}\log k\right\}, & n=2, \\ \lim\limits_{k\to 0}\dfrac{1}{k^2}\{\dfrac{\partial}{\partial n_x}R_k 1 + \int\limits_{\Omega} R_k 1 dz\}, & n=3. \end{cases}$$

Similarly, again we can show that u_R is the unique solution of the equation

$$Bu_R - \frac{\partial}{\partial n_x}R_k u_R - \int\limits_{\Omega} R_k u_R(z)dz = f_R$$

with

$$\begin{aligned} f_R &= -R_k'\psi - \int\limits_{\Omega} S_k\psi dx + \frac{\partial}{\partial n_x}R_k\widetilde{u} + \int\limits_{\Omega} R_k\widetilde{u}dx \\ &\quad + \frac{\alpha}{k^2}\Big\{\frac{\partial}{\partial n_x}R_k 1 + \int\limits_{\Omega} R_k 1 dx - \delta_{n2}\frac{|\Omega|^2}{2\pi}k^2\log k + \delta_{n3}\frac{|\Omega|^2}{4\pi}ik^3\Big\} \\ &\quad -\alpha\lim_{k\to 0}\frac{1}{k^2}\Big\{\frac{\partial}{\partial n_x}R_k 1 + \int\limits_{\Omega} R_k 1 dx - \delta_{n2}\frac{|\Omega|^2}{2\pi}k^2\log k\Big\}. \end{aligned}$$

For $n = 2$, the above relations show that $f_R = O(k^2\log k)$, which implies that $u_R = O(k^2\log k)$, as well.

In case $n = 3$, by using the Gaussian theorem for the first two terms from (2.1.21) we obtain explicitly

$$R_k 1 = \int\limits_{\Gamma} R_k(x,y)ds_y = -\frac{1}{4\pi}k^2\int\limits_{\Omega}\frac{1}{|x-y|}dy - \frac{i}{4\pi}k^3|\Omega| + O(k^4).$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial n_x}R_k 1 &= -\frac{1}{4\pi}k^2\int\limits_{\Omega}\frac{\partial}{\partial n_x}\frac{1}{|x-y|}dy + O(k^4), \\ \int\limits_{\Omega} R_k 1 dx &= -\frac{1}{4\pi}k^2\int\limits_{\Omega}\int\limits_{\Omega}\frac{1}{|x-y|}dydx - \frac{i}{4\pi}k^3|\Omega|^2 + O(k^4). \end{aligned}$$

Together with (2.1.44) this yields

$$f_R = O(k^2)$$

which implies that u_R is of the same order as $k \to 0$.

<u>ENP:</u> Finally, let us consider the boundary integral equations of (ENP) and begin with the simplest case, i.e. the integral equation of the second kind (see (2) in Table 2.1.1),

$$\frac{1}{2}u - Ku - R_k u = -V\psi + \left\{\delta_{n2}\left(\frac{1}{2\pi}\log k + \gamma_0\right) - \delta_{n3}\frac{ik}{4\pi}\right\}\int\limits_\Gamma \psi ds - S_k\psi\,. \tag{2.1.48}$$

The operator on the left–hand side is a regular perturbation of the reduced invertible operator $\frac{1}{2}I - K$ (see Section 1.4.2).

For $n = 2$, $S_k\psi$ is of order $O(k^2 \log k)$ in view of (2.1.19). Hence, for $u_{|\Gamma}$ we use the expansion

$$u = \frac{1}{2\pi}(\log k + \gamma_0)\alpha + \widetilde{u} + u_R\,.$$

The highest order term

$$\alpha = \int\limits_\Gamma \psi ds$$

is the unique solution of the reduced equation

$$\tfrac{1}{2}\alpha - K\alpha = \int\limits_\Gamma \psi ds\,;$$

the second term $\widetilde{u}$ is the unique solution of the reduced equation

$$\tfrac{1}{2}\widetilde{u} - K\widetilde{u} = -V\psi\,. \tag{2.1.49}$$

For the remainder u_R we obtain the equation

$$\tfrac{1}{2}u_R - Ku_R - R_k u_R = \frac{\alpha}{2\pi}(\log k + \gamma_0)R_k 1 + R_k\widetilde{u} - S_k\psi\,.$$

The dominating term on the right–hand side is defined by

$$R_k 1 = \frac{1}{2\pi}|\Omega| k^2 \log k + O(k^2)\,,$$

therefore u_R is of order $O((k\log k)^2)$.

For $n = 3$, $S_k\psi$ in (2.1.48) is of order $O(k^2)$ in view of (2.1.19). Hence, now u is of the form

$$u = \widetilde{u} - \alpha\frac{ik}{4\pi} + u_R$$

where $\widetilde{u}$ is the unique solution of the reduced equation (2.1.49) and $\alpha = \int_\Gamma \psi ds$. The remainder u_R satisfies the equation

$$\tfrac{1}{2}u_R - Ku_R - R_k u_R = R_k\widetilde{u} - S_k\psi - \frac{ik}{4\pi}R_k\alpha\,,$$

and, therefore, is of the order $O(k^2)$.

The hypersingular boundary integral equation (1) of the first kind in Table 2.1.2 reads

$$Du - \frac{\partial}{\partial n_x}R_k u = -\Big(\frac{1}{2}I + K'\Big)\psi - R_k'\psi\,, \tag{2.1.50}$$

where D has the constant functions as eigensolutions.

Multiplying (2.1.48) by the natural charge q, integrating over Γ and using (2.1.36) and $V_q = c_0$, we obtain the relation

$$\begin{aligned}
&\int\limits_\Gamma uq ds - \int\limits_\Gamma q(R_k u) ds \\
&= -\left\{c_0 - \delta_{n2}\frac{1}{2\pi}(\log k + \gamma_0) + \delta_{n3}\frac{ik}{4\pi}\right\}\int\limits_\Gamma \psi ds - \int\limits_\Gamma q(S_k\psi) ds
\end{aligned} \tag{2.1.51}$$

with c_0 given by (2.1.35). Now (2.1.51) can be used as a normalizing condition by combining (2.1.50) and (2.1.51); we arrive at the modified equation

$$\begin{aligned}
Du + \int\limits_\Gamma uq ds - \int\limits_\Gamma (R_k u) q ds - \frac{\partial}{\partial n_x} R_k u & \\
= \left(-c_0 + \delta_{n2}\frac{1}{2\pi}(\log k + \gamma_0) - \delta_{n3}\frac{ik}{\pi}\right)\int\limits_\Gamma \psi ds & \\
- \left(\tfrac{1}{2}I + K'\right)\psi - R_k'\psi - \int\limits_\Gamma q(S_k\psi) ds\,. &
\end{aligned} \tag{2.1.52}$$

Again, we assume u in the form

$$u = \delta_{n2}\frac{1}{2\pi}(\log k + \gamma_0 - 2\pi c_0)\alpha - \delta_{n3}\frac{ik}{4\pi}\alpha + \widetilde{u} + u_R\,,$$

where $k \to 0$ yields

$$\alpha = \int\limits_\Gamma \psi ds\,.$$

The term $\widetilde{u}$ is the unique solution of

$$D\widetilde{u} + \int\limits_\Gamma \widetilde{u} q ds = -\delta_{n3} c_0 \int\limits_\Gamma \psi ds - (\tfrac{1}{2}I + K')\psi\,.$$

The remainder u_R satisfies the equation

$$Du_R + \int\limits_\Gamma u_R q ds - \int\limits_\Gamma q(R_k u_R) ds - \frac{\partial}{\partial n_x} R_k u_R = f_R\,,$$

where

$$\begin{aligned}
f_R \;=\; & \alpha\left\{\delta_{n2}(\log k + \gamma_0 - 2\pi c_0) - \delta_{n3}\frac{ik}{4\pi}\right\}\left\{\frac{\partial}{\partial n_x} R_k 1 + \int\limits_\Gamma q(R_k 1) ds\right\} \\
& + \frac{\partial}{\partial n_x} R_k \widetilde{u} + \int\limits_\Gamma q(R_k\widetilde{u}) ds - R_k'\psi - \int\limits_\Gamma q(S_k\psi) ds
\end{aligned}$$

is of order $O\left((k \log k)^2\right)$ for $n = 2$ and of order $O(k^2)$ for $n = 3$, which is in agreement with the result from the previous analysis of the boundary integral equation of the second kind.

With the solutions of the BIEs available, we now summarize the asymptotic behaviour of the solutions of the BVPs for small k by substituting the boundary densities into the representation formula (2.1.5). In all the cases we arrive at the following asymptotic expression

$$u(x) = \pm[V\widetilde{\sigma}(x) - W\widetilde{u}(x)] + C(k;x) + R(k;x) \tag{2.1.53}$$

where the $\pm$ sign corresponds to the interior and exterior domain and $x \in \Omega$ or Ω^c as in (2.1.5). For the Dirichlet problems, $\widetilde{u}_{|\Gamma} = \varphi$, and for the Neumann problems, $\widetilde{\sigma}_{|\Gamma} = \psi$ on Γ, are the given boundary data, respectively, whereas the missing densities are the solutions of the corresponding BIEs presented above. In Formula (2.1.53), $C(k;x)$ denotes the highest order terms of the perturbations in Ω or Ω^c; whereas $R(k;x)$ denotes the remaining boundary potentials. The behaviour of C and R for $k \to 0$ is summarized in Table 2.1.2 below.

The remainders $R(x;k)$ are of the orders as given in the table, uniformly in $x \in \Omega$ for the interior problems and in compact subsets of $\overline{\Omega^c}$ only, for the exterior problems.

Table 2.1.2. Low frequency characteristics

BVP	$C(k;x)$	$R(k;x)$	n
IDP	0	$O((k\log k)^2)$	$n=2$
		$O(k^2)$	$n=3$
EDP	$-\widetilde{\omega}$	$O((k\log k)^2)$ [1]	$n=2$
	$-k\{V\widetilde{\sigma}_1(x) + \frac{i}{4\pi}\int\limits_\Gamma \widetilde{\sigma} ds\}$	$O(k^2)$	$n=3$
INP	$-\{\frac{1}{k^2} - \frac{1}{2\pi}\int\limits_\Omega \log\lvert x-y\rvert dy\}\frac{1}{\lvert\Omega\rvert}\int\limits_\Gamma \psi ds$	$O(k^2\log k)$	$n=2$
	$-\{\frac{1}{k^2} + \frac{1}{4\pi}\int\limits_\Omega \frac{1}{\lvert x-y\rvert}dy\}\frac{1}{\lvert\Omega\rvert}\int\limits_\Gamma \psi ds$	$O(k^2)$	$n=3$
ENP	$\frac{1}{2\pi}(\log k + \gamma_0)\int\limits_\Gamma \psi ds$	$O((k\log k)^2)$	$n=2$
	$-\frac{ik}{4\pi}\int\limits_\Gamma \psi ds$	$O(k^2)$	$n=3$

[1]For this case a sharper result with $O(k^2)$ is given by MacCamy [194]

2.2 The Lamé System

In linear elasticity for isotropic materials, the governing equations are

$$-\Delta^* \boldsymbol{v} := -\mu \Delta \boldsymbol{v} - (\mu + \lambda)\,\mathrm{graddiv}\,\boldsymbol{v} = \boldsymbol{f} \quad \text{in } \Omega \text{ (or } \Omega^c) \,, \tag{2.2.1}$$

where $\boldsymbol{v}$ is the desired displacement field and $\boldsymbol{f}$ is a given body force. The parameters μ and λ are the *Lamé constants* which characterize the elastic material. (See e.g. Ciarlet [42], Fichera [75], Gurtin [115], Kupradze et al [177] and Leis [184]).

For $n = 3$, one also has the relation $\lambda = 2\mu\nu/(1-2\nu)$ with $0 \le \nu < \frac{1}{2}$, the *Poisson ratio*. The latter relation is also valid for the *plane strain* problem in two dimensions, where (2.2.1) is considered with $n = 2$. In the special case of so–called *generalized plane stress* problems one still has (2.2.1) with $n = 2$ for the first two displacement components $(v_1, v_2)^\top$ but with a modified $\overline{\lambda} = 2\mu\overline{\nu}/(1 - 2\overline{\nu})$ and a modified Poisson ratio $\overline{\nu} = \nu/(1 + \nu)$. In this case and in what follows, we will keep the same notation for λ.

For the Lamé system, the *fundamental solution* is given by

$$E(x,y) = \tfrac{\lambda+3\mu}{4\pi(n-1)\mu(\lambda+2\mu)} \Big\{ \gamma_n(x,y)\boldsymbol{I} + \tfrac{\lambda+\mu}{\lambda+3\mu} \tfrac{1}{r^n} (\boldsymbol{x} - \boldsymbol{y})(\boldsymbol{x} - \boldsymbol{y})^\top \Big\} \,, \tag{2.2.2}$$

a *matrix*–valued function, where $\boldsymbol{I}$ is the identity matrix, $r = |x - y|$ and

$$\gamma_n(x,y) = \begin{cases} -\log r & \text{for } n = 2, \\ \frac{1}{r} & \text{for } n = 3. \end{cases} \tag{2.2.3}$$

The boundary integral equations for the so–called fundamental boundary value problems are based on the Green representation formula, which in elasticity also is termed the *Betti–Somigliana representation formula*. For interior problems, we have the representation

$$\boldsymbol{v}(x) = \int\limits_\Gamma E(x,y) T\boldsymbol{v}(y) ds_y - \int\limits_\Gamma \left(T_y E(x,y)\right)^\top \boldsymbol{v}(y) ds_y + \int\limits_\Omega E(x,y) \boldsymbol{f}(y) dy \tag{2.2.4}$$

for $x \in \Omega$. Here the traction on Γ is defined by

$$T\boldsymbol{v}_{|\Gamma} = \left(\lambda(\mathrm{div}\ \boldsymbol{v})\boldsymbol{n} + 2\mu \frac{\partial \boldsymbol{v}}{\partial n} + \mu \boldsymbol{n} \times curl \boldsymbol{v} \right)\Big|_\Gamma \tag{2.2.5}$$

for $n = 3$ which reduces to the case $n = 2$ by setting $u_3 = 0$ and the third component of the normal $n_3 = 0$. The subscript y in $T_y E(x,y)$ again denotes differentiations in (2.2.4) with respect to the variable y.

The last term in the representation (2.2.4) is the *volume potential* (or *Newton potential*) due to the body force $\boldsymbol{f}$ defining a particular solution $\boldsymbol{v}_p$ of (2.2.1). As in Section 2.1, we decompose the solution in the form

$$\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{u}$$

where $\boldsymbol{u}$ now satisfies the homogeneous Equation (2.2.1) with $\boldsymbol{f} = \boldsymbol{0}$ and has a representation (2.2.4) with $\boldsymbol{f} = \boldsymbol{0}$, i.e.

$$\boldsymbol{u}(x) = V\boldsymbol{\sigma}(x) - W\boldsymbol{\varphi}(x)\,. \tag{2.2.6}$$

Here V and W are the simple and double layer potentials, now defined by

$$V\boldsymbol{\sigma}(x) := \int\limits_{\Gamma} E(x,y)\boldsymbol{\sigma}(y)ds_y\,, \tag{2.2.7}$$

$$W\boldsymbol{\varphi}(x) := \int\limits_{\Gamma} \left(T_y(x,y)E(x,y)\right)^\top \boldsymbol{\varphi}(y)ds_y\,; \tag{2.2.8}$$

and where in (2.2.6) the boundary charges are the Cauchy data $\boldsymbol{\varphi}(x) = \boldsymbol{u}(x)_{|\Gamma}$, $\boldsymbol{\sigma}(x) = T\boldsymbol{u}(x)_{|\Gamma}$ of the solution to

$$-\Delta^*\boldsymbol{u} = \boldsymbol{0} \ \text{ in } \ \Omega\,. \tag{2.2.9}$$

For linear problems, because of the above decomposition, in the following, we shall consider, without loss of generality, only the case of the homogeneous equation (2.2.9), i.e., $\boldsymbol{f} = \boldsymbol{0}$.

For the exterior problems, the representation formula for $\boldsymbol{v}$ needs to be modified by taking into account growth conditions at infinity. For $\boldsymbol{f} = \boldsymbol{0}$, the growth conditions are

$$\boldsymbol{u}(x) = -E(x,0)\boldsymbol{\Sigma} + \boldsymbol{\omega}(x) + O(|x|^{1-n}) \ \text{ as } \ |x| \to \infty\,, \tag{2.2.10}$$

where $\boldsymbol{\omega}(x)$ is a rigid motion defined by

$$\boldsymbol{\omega}(x) = \begin{cases} \boldsymbol{a} + b(-x_2, x_1)^\top & \text{for } n = 2, \\ \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} & \text{for } n = 3. \end{cases} \tag{2.2.11}$$

Here, $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{\Sigma}$ are constant vectors; the former denote translation and rotation, respectively. The representation formula for solutions of

$$-\Delta^*\boldsymbol{u} = \boldsymbol{0} \ \text{ in } \ \Omega^c \tag{2.2.12}$$

with the growth condition (2.2.10) has the form

$$\boldsymbol{u}(x) = -V\boldsymbol{\sigma}(x) + W\boldsymbol{\varphi}(x) + \boldsymbol{\omega}(x) \tag{2.2.13}$$

with the Cauchy data $\boldsymbol{\varphi} = \boldsymbol{u}_{|\Gamma}$ and $\boldsymbol{\sigma} = T\boldsymbol{u}_{|\Gamma}$ and with

$$\boldsymbol{\Sigma} = \int\limits_{\Gamma} \boldsymbol{\sigma} ds\,. \tag{2.2.14}$$

2.2.1 The Interior Displacement Problem

The Dirichlet problem for the Lamé equations (2.2.9) in Ω is called the *displacement problem* since here the boundary displacement

$$\boldsymbol{u}_{|\Gamma} = \boldsymbol{\varphi} \text{ on } \Gamma \tag{2.2.15}$$

is prescribed. The missing Cauchy datum on Γ is the boundary traction $\boldsymbol{\sigma} = T\boldsymbol{u}_{|\Gamma}$. Applying the trace and the traction operator T (2.2.5) to both sides of the representation formula (2.2.4), we obtain the overdetermined system of boundary integral equations

$$\boldsymbol{\varphi}(x) = (\tfrac{1}{2}I - K)\boldsymbol{\varphi}(x) + V\boldsymbol{\sigma}(x)\,, \tag{2.2.16}$$

$$\boldsymbol{\sigma}(x) = D\boldsymbol{\varphi} + (\tfrac{1}{2}I + K')\boldsymbol{\sigma}(x) \text{ on } \Gamma\,. \tag{2.2.17}$$

Here, the boundary integral operators are defined as in (1.2.3)–(1.2.6), however, it is understood that the fundamental solution now is given by (2.2.2) and the differentiation $\frac{\partial}{\partial n_{|\Gamma}}$ (1.2.4)–(1.2.6) is to be replaced by the traction operator $T_{|\Gamma}$ (2.2.5). Explicitly, we also have:

Lemma 2.2.1. *Let $\Gamma \in C^2$ and let $\boldsymbol{\varphi} \in C^\alpha(\Gamma), \boldsymbol{\sigma} \in C^\alpha(\Gamma)$ with $0 < \alpha < 1$. Then, for the case of elasticity, the limits* (1.2.3)–(1.2.5) *exist uniformly with respect to all $x \in \Gamma$ and all $\boldsymbol{\varphi}$ and $\boldsymbol{\sigma}$ with $||\boldsymbol{\varphi}||_{C^\alpha} \le 1$, $||\boldsymbol{\sigma}||_{C^\alpha} \le 1$. Furthermore, these limits can be expressed by*

$$V\boldsymbol{\sigma}(x) = \int\limits_{y\in\Gamma\setminus\{x\}} E(x,y)\boldsymbol{\sigma}(y)ds_y, \qquad x \in \Gamma\,, \tag{2.2.18}$$

$$K\boldsymbol{\varphi}(x) = \text{p.v.} \int\limits_{y\in\Gamma\setminus\{x\}} \left(T_y E(x,y)\right)^\top \boldsymbol{\varphi}(y)ds_y\,, \quad x \in \Gamma\,, \tag{2.2.19}$$

$$K'\boldsymbol{\sigma}(x) = \text{p.v.} \int\limits_{y\in\Gamma\setminus\{x\}} \left(T_x E(x,y)\right)\boldsymbol{\sigma}(y)ds_y\,, \quad x \in \Gamma\,. \tag{2.2.20}$$

These results are originally due to Giraud [99], see also Kupradze [176, 177] and the references therein. We remark that the integral in (2.2.18) is a weakly singular improper integral, whereas the integrals in (2.2.19) and (2.2.20) are to be defined as *Cauchy principal value integrals*, i.e.

$$\text{p.v.} \int\limits_{y\in\Gamma\setminus\{x\}} \left(T_y E(x,y)\right)^\top \boldsymbol{\varphi}(y)ds_y = \lim_{\varepsilon\to 0} \int\limits_{|y-x|\ge\varepsilon>0\wedge y\in\Gamma} \left(T_y E(x,y)\right)^\top \boldsymbol{\varphi}(y)ds_y\,,$$

since the operators K and K' have Cauchy singular kernels:

$$
\begin{aligned}
&(T_y E(x,y))^\top \\
&= \frac{\mu}{2(n-1)\pi(\lambda+2\mu)}\Big\{\Big(I + \frac{n(\lambda+\mu)}{\mu|x-y|^2}(x-y)(x-y)^\top\Big)\frac{\partial\gamma_n}{\partial n_y}(x,y) \\
&\qquad + \frac{1}{|x-y|^n}\big((x-y)\boldsymbol{n}_y^\top - \boldsymbol{n}_y(x-y)^\top\big)\Big\}^\top, \qquad (2.2.21)
\end{aligned}
$$

$$
\begin{aligned}
&(T_x E(x,y)) \\
&= \frac{\mu}{2(n-1)\pi(\lambda+2\mu)}\Big\{\Big(I + \frac{n(\lambda+\mu)}{\mu|x-y|^2}(x-y)(x-y)^\top\Big)\frac{\partial\gamma_n}{\partial n_x}(x,y) \\
&\qquad - \frac{1}{|x-y|^n}\big((x-y)\boldsymbol{n}_x^\top - \boldsymbol{n}_x(x-y)^\top\big)\Big\}. \qquad (2.2.22)
\end{aligned}
$$

We note that in case $n=2$, the last term in (2.2.21) can also be written in the form

$$
\frac{1}{|x-y|^2}\big((x-y)\boldsymbol{n}_y^\top - \boldsymbol{n}_y(x-y)^\top\big) = \begin{pmatrix} 0 & , & 1 \\ -1 & , & 0 \end{pmatrix}\frac{d}{ds_y}\log|x-y|\,. \qquad (2.2.23)
$$

The last terms in the kernels (2.2.21) and (2.2.22) are of the order $O(r^{-n})$ as $r=|x-y|\to 0$ and, in addition, satisfy the *Mikhlin condition* (see Mikhlin [213, Chap. 5] or [215, Chap. IX]) which is necessary and sufficient for the existence of the above Cauchy principal value integrals for any $x\in\Gamma$. We shall return to this condition later on. The proof of Lemma 2.2.1 can be found in [213, Chap. 45 p. 210 ff].

It turns out that there exists a close relation between the single and double layer operators of the Laplacian and the Lamé system based on the following lemma and the *Günter derivatives*

$$
m_{jk}\big(\partial_y,\boldsymbol{n}(y)\big) := n_k(y)\frac{\partial}{\partial y_j} - n_j(y)\frac{\partial}{\partial y_k} = -m_{kj}\big(\partial_y,\boldsymbol{n}(y)\big) \qquad (2.2.24)
$$

(see Kupradze et al [177, (4.7), p. 99]) which are particular tangential derivatives; in matrix form

$$
\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big) := \big(\big(m_{jk}\big(\partial_y,\boldsymbol{n}(y)\big)\big)\big)_{3\times 3}\,. \qquad (2.2.25)
$$

Lemma 2.2.2. *For the Günter derivatives there hold the identities*

$$
m_{j\ell}\big(\partial_y,\boldsymbol{n}(y)\big)\gamma_n = \frac{n_j(y_\ell-x_\ell)-n_\ell(y_j-x_j)}{|x-y|^n}\,, \qquad (2.2.26)
$$

$$
\begin{aligned}
&\sum_{\ell=1}^n m_{j\ell}\big(\partial_y,\boldsymbol{n}(y)\big)\frac{(y_\ell-x_\ell)(y_k-x_k)}{|x-y|^n} \\
&\qquad = -\Big(\delta_{jk} - \frac{n(y_j-x_j)(y_k-x_k)}{|x-y|^2}\Big)\frac{\partial}{\partial n_y}\gamma_n. \qquad (2.2.27)
\end{aligned}
$$

The proof is straight forward by direct computation.

Inserting the identities (2.2.26) and (2.2.27) into (2.2.21) we arrive at

$$\begin{aligned}\big(T_y E(x,y)\big)^\top &= \frac{1}{2(n-1)\pi}\Big\{\frac{\partial}{\partial n_y}\gamma_n(x,y) + \mathcal{M}\big(\partial_y,(\boldsymbol{n}(y)\big)\gamma_n(x,y)\Big\} \\ &\quad +2\mu\big(\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)E(x,y)\big)^\top . \end{aligned} \tag{2.2.28}$$

In the same manner we find

$$\begin{aligned}\big(T_x E(x,y)\big) &= \frac{1}{2(n-1)\pi}\Big\{\frac{\partial}{\partial n_x}\gamma_n(x,y) - \mathcal{M}\big(\partial_x,(\boldsymbol{n}(x)\big)\gamma_n(x,y)\Big\} \\ &\quad +2\mu\big(\mathcal{M}\big(\partial_x,\boldsymbol{n}(x)\big)E(x,y)\big) . \end{aligned} \tag{2.2.29}$$

As a consequence, we may write the double layer potential operator in (2.2.19) after integration by parts in the form of the Stokes theorem as

$$\begin{aligned} K\boldsymbol{\varphi}(x) &= \frac{1}{2(n-1)\pi}\Big\{\int\limits_\Gamma \Big(\frac{\partial}{\partial n_y}\gamma_n(x,y)\Big)\boldsymbol{\varphi}(y)ds_y \\ &\quad - \int\limits_\Gamma \gamma_n(x,y)\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)ds_y\Big\} \\ &\quad + 2\mu\int\limits_\Gamma E(x,y)\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)ds_y , \end{aligned} \tag{2.2.30}$$

see Kupradze et al [177, Chap. V Theorem 6.1].

Finally, the hypersingular operator D in (2.2.17) is defined by

$$\begin{aligned} D\boldsymbol{\varphi}(x) &= -T_x W\boldsymbol{\varphi}(x) \\ &:= -\lim_{z\to x\in\Gamma, z\notin\Gamma}\Big(\lambda\big(\operatorname{div}_z W\boldsymbol{\varphi}(z)\big) + 2\mu(\operatorname{grad}_z W\boldsymbol{\varphi}(z)\big)\cdot\boldsymbol{n}_x \\ &\qquad + \mu\boldsymbol{n}_x\times\operatorname{curl}_z W\boldsymbol{\varphi}(z)\Big) . \end{aligned} \tag{2.2.31}$$

Lemma 2.2.3. *Let $\Gamma\in C^2$ and let $\boldsymbol{\varphi}$ be a Hölder continuously differentiable function. Then in the case of elasticity, the limits* (1.2.6) *exist uniformly with respect to all $x\in\Gamma$ and all $\boldsymbol{\varphi}$ with $\|\boldsymbol{\varphi}\|_{C^{1+\alpha}}\le 1$. Moreover, the operator D can be expressed as a composition of tangential differential operators and the simple layer operators of the Laplacian and the Lamé system:*

For $n=2$:

$$D\boldsymbol{\varphi}(x) = -\frac{d}{ds_x}\widetilde{V}\left(\frac{d\boldsymbol{\varphi}}{ds}\right)(x) \tag{2.2.32}$$

where

$$\widetilde{V}\boldsymbol{\chi}(x) := \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\int\limits_\Gamma\left(-\log|x-y| + \frac{(x-y)(x-y)^\top}{|x-y|^2}\right)\boldsymbol{\chi}(y)ds_y . \tag{2.2.33}$$

For $n = 3$:

$$
\begin{aligned}
D\boldsymbol{\varphi}(x) &= -\frac{\mu}{4\pi}(\boldsymbol{n}_x \times \nabla_x) \cdot \int\limits_\Gamma \frac{1}{|x-y|}(\boldsymbol{n}_y \times \nabla_y)\boldsymbol{\varphi}(y)ds_y \\
&- \mathcal{M}\big(\partial_x, \boldsymbol{n}(x)\big) \int\limits_\Gamma \Big\{4\mu^2 E(x,y) - \frac{\mu}{2\pi}\ \frac{1}{|x-y|} I\Big\} \mathcal{M}\big(\partial_y, \boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)ds_y \\
&+ \frac{\mu}{4\pi}\Big(\sum_{\ell,k=1}^3 m_{\ell k}\big(\partial_x, \boldsymbol{n}(x)\big) \int\limits_\Gamma \frac{1}{|x-y|}\big(m_{kj}\big(\partial_y, \boldsymbol{n}(y)\big)\varphi_\ell\big)(y)ds_y\Big)_{j=1,2,3}.
\end{aligned}
\tag{2.2.34}
$$

Proof: The proof for $n = 2$ was given by Bonnemay [17] and Nedelec [233]; here we follow the proof given by Houde Han [118, 119] for $n = 3$. The proof is based on a different representation of the traction operator T by employing Günter's derivatives $\mathcal{M}$; more precisely we have:

$$
T\big(\partial_x, \boldsymbol{n}(x)\big)\boldsymbol{u}(x) = (\lambda + \mu)(\mathrm{div}\boldsymbol{u})\boldsymbol{n}(x) + \mu\Big(\frac{\partial \boldsymbol{u}}{\partial n} + \mathcal{M}\boldsymbol{u}\Big). \tag{2.2.35}
$$

We note

$$
\mathcal{M}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial n} - (\mathrm{div}\boldsymbol{u})\boldsymbol{n}(x) + \boldsymbol{n}(x) \times \mathrm{curl}\,\boldsymbol{u}\,. \tag{2.2.36}
$$

Now we apply T in the form (2.2.35) to the three individual terms in the double layer potential K in (2.2.30) successively, and begin with

$$
\begin{aligned}
&T\big(\partial_z, \boldsymbol{n}(x)\big)\frac{\partial}{\partial n_y}\gamma_n(z,y)\boldsymbol{I} \\
&= \mu(\boldsymbol{n}(x)\cdot\nabla_z)\frac{\partial}{\partial n_y}\gamma_n(z,y)\boldsymbol{I} + \mu\mathcal{M}\big(\partial_z, \boldsymbol{n}(x)\big)\frac{\partial}{\partial n_y}\gamma_n(z,y)\boldsymbol{I} \\
&\quad + (\lambda+\mu)\boldsymbol{n}(x)\Big(\mathrm{grad}_z \frac{\partial}{\partial n_y}\gamma_n(z,y)\Big)^\top.
\end{aligned}
$$

Next we apply T to the remaining two terms by using (2.2.29) to obtain

$$
\begin{aligned}
&T\big(\partial_z, \boldsymbol{n}(x)\big)\Big(2\mu E(z,y) - \frac{1}{2(n-1)\pi}\gamma_n(z,y)\boldsymbol{I}\Big) \\
&= \mathcal{M}\big(\partial_z, \boldsymbol{n}(x)\big)\Big\{4\mu^2 E(x,y) - \frac{2\mu}{2(n-1)\pi}\gamma_n(x,y)\boldsymbol{I}\Big\} \\
&\quad + \frac{2\mu}{2(n-1)\pi}\boldsymbol{n}(x)\cdot\nabla_z\gamma_n(z,y)\boldsymbol{I} - T\big(\partial_z, \boldsymbol{n}(x)\big)\frac{1}{4\pi}\gamma_n(z,y)\boldsymbol{I}\,.
\end{aligned}
$$

Now we apply (2.2.35) to the last term in this expression and find

$$
\begin{aligned}
& T\big(\partial_z, \boldsymbol{n}(x)\big)\frac{1}{2(n-1)\pi}\gamma_n(z,y)\boldsymbol{I} \\
&= \frac{1}{2(n-1)\pi}(\lambda+\mu)\boldsymbol{n}(x)\big(\nabla_z\gamma_n(z,y)\big)^\top + \frac{\mu}{2(n-1)\pi}\boldsymbol{n}(x)\cdot\nabla_z\gamma_n(z,y)\boldsymbol{I} \\
&\qquad\qquad + \frac{\mu}{2(n-1)\pi}\mathcal{M}\big(\partial_z, \boldsymbol{n}(x)\big)\gamma_n(z,y)\boldsymbol{I}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& T\big(\partial_z, \boldsymbol{n}(x)\big)\Big(2\mu E(z,y) - \frac{1}{2(n-1)\pi}\gamma_n(z,y)\boldsymbol{I}\Big) \\
&= \mathcal{M}\big(z, \boldsymbol{n}(x)\big)\Big\{4\mu^2 E(z,y) - \frac{3\mu}{2(n-1)\pi}\gamma_n\boldsymbol{I}\Big\} - \frac{\lambda+\mu}{2(n-1)\pi}\boldsymbol{n}(x)\big(\nabla_z\gamma_n(x,z)\big)^\top \\
&\quad + \frac{\mu}{2(n-1)\pi}\boldsymbol{n}(x)\cdot\nabla_z(z,y)\boldsymbol{I}\,.
\end{aligned}
$$

Collecting these results we obtain

$$
\begin{aligned}
& T\big(\partial_z, \boldsymbol{n}(x)\big)(K\varphi)(z) \\
&= \frac{\mu}{2(n-1)\pi}\boldsymbol{n}(x)\cdot\nabla_z\int\limits_\Gamma \frac{\partial}{\partial n_y}\gamma_n(z,y)\boldsymbol{\varphi}(y)ds_y \\
&\quad + \mathcal{M}\big(\partial_z, \boldsymbol{n}(x)\big)\int\limits_\Gamma \Big\{4\mu^2 E(z,y) - \frac{3\mu}{2(n-1)\pi}\gamma_n(z,y)\Big\}\mathcal{M}\big(\partial_y, \boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)ds_y \\
&\quad + \frac{\lambda+\mu}{2(n-1)\pi}\Big\{\boldsymbol{n}(x)\int\limits_\Gamma \Big(\nabla_z\frac{\partial}{\partial n_y}\gamma_n(z,y)\Big)\cdot\boldsymbol{\varphi}(y)ds_y \\
&\qquad\qquad - \boldsymbol{n}(x)\big(\nabla_z\int\limits_\Gamma \gamma_n(z,y)\cdot\big(\mathcal{M}\big(\partial_y, \boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)\big)ds_y\Big\} \\
&\quad + \frac{\mu}{2(n-1)\pi}\Big\{\mathcal{M}\big(\partial_z, \boldsymbol{n}(x)\big)\int\limits_\Gamma \frac{\partial}{\partial n_y}\gamma_n(z,y)\boldsymbol{\varphi}(y)ds_y \\
&\qquad\qquad + \boldsymbol{n}(x)\cdot\nabla_z\int\limits_\Gamma \gamma_n(z,y)\mathcal{M}\big(\partial_y, \boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)ds_y\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mu}{2(n-1)\pi}\boldsymbol{n}(x)\cdot\nabla_z\int\limits_\Gamma \Big(\frac{\partial}{\partial n_y}\gamma_n(z,y)\Big)\boldsymbol{\varphi}(y)ds_y \\
&\quad + \mathcal{M}\big(\partial_z, \boldsymbol{n}(x)\big)\int\limits_\Gamma \Big\{4\mu^2 E(z,y) - \frac{3\mu}{2(n-1)\pi}\gamma_n(z,y)\Big\}\mathcal{M}\big(\partial_y, \boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)ds_y \\
&\quad + \frac{(\lambda+\mu)}{2(n-1)\pi}\mathcal{J}_1\boldsymbol{\varphi}(x) + \frac{\mu}{2(n-1)\pi}\mathcal{J}_2\boldsymbol{\varphi}(x)
\end{aligned}
$$

where

$$\mathcal{J}_1\boldsymbol{\varphi}(x) := \int\limits_\Gamma \Big\{\boldsymbol{n}(x)\Big(\nabla_z \frac{\partial}{\partial n_y}\gamma_n(z,y)\cdot\boldsymbol{\varphi}(y)\Big) - \boldsymbol{n}(x)\big(\nabla_z\gamma_n(z,y)\big)\cdot\big(\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)\big)\Big\}ds_y$$

and

$$\mathcal{J}_2\boldsymbol{\varphi}(x) := \int\limits_\Gamma \Big\{\mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big)\frac{\partial}{\partial n_y}\gamma_n(z,y) + \big(\boldsymbol{n}(x)\cdot\nabla_z\gamma_n(z,y)\big)\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\Big\}\boldsymbol{\varphi}(y)ds_y\,.$$

The product rule gives

$$\begin{aligned}
&\big(\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\nabla_z\gamma_n(z,y)\big)\cdot\boldsymbol{\varphi}(y)\\
&\quad= \sum_{k,\ell=1}^n \Big(m_{k\ell}\big(\partial_y,\boldsymbol{n}(y)\big)\frac{\partial}{\partial z_\ell}\gamma_n(z,y)\Big)\varphi_k(y)\\
&\quad= \sum_{k,\ell=1}^n \Big(m_{k\ell}\big(\partial_y,\boldsymbol{n}(y)\big)\Big(\frac{\partial}{\partial z_\ell}\gamma_n(z,y)\varphi_k(y)\Big)\\
&\qquad\qquad -\big(m_{k\ell}\big(\partial_y,\boldsymbol{n}(y)\big)\varphi_k(y)\big)\frac{\partial}{\partial z_\ell}\gamma_n(z,y)\Big)\\
&\quad= \sum_{k,\ell=1}^n \Big(-m_{\ell k}\big(\partial_y,\boldsymbol{n}(y)\big)\Big(\frac{\partial}{\partial z_\ell}\gamma_n(z,y)\varphi_k(y)\Big)\\
&\qquad\qquad +\Big(\frac{\partial}{\partial z_\ell}\gamma_n(z,y)\Big)\big(m_{\ell k}\big(\partial_y,\boldsymbol{n}(y)\big)\big)\varphi_k(y)\Big)\,.
\end{aligned}\tag{2.2.37}$$

The symmetry of $\gamma_n(z,y)$ and $\Delta_y\gamma_n(z,y)=0$ for $z\neq y$ implies

$$\begin{aligned}
\sum_{k=1}^n m_{jk}(y)\frac{\partial\gamma_n(z,y)}{\partial z_k} &= \sum_{k=1}^n n_k(y)\frac{\partial^2}{\partial y_j\partial z_k}\gamma_n(z,y) - n_j(y)\sum_{k=1}^n\frac{\partial^2}{\partial y_k\partial z_k}\gamma_n(z,y)\\
&= \sum_{k=1}^n n_k(y)\frac{\partial^2}{\partial y_k\partial z_j}\gamma_n(z,y) + n_j(y)\Delta_y\gamma_n(z,y)\\
&= \frac{\partial}{\partial z_j}\;\frac{\partial\gamma_n(z,y)}{\partial n_y}\,,
\end{aligned}$$

i.e.,

$$\nabla_z\frac{\partial\gamma_n(z,y)}{\partial n_y} = \mathcal{M}\big(\partial_y(y)\big)\nabla_z\gamma_n(z,y)\,. \tag{2.2.38}$$

Using (2.2.37) and (2.2.38) for the first term of the integrand of $\mathcal{J}_1$, we find

$$
\begin{aligned}
\mathcal{J}_1\boldsymbol{\varphi}(x) &= \boldsymbol{n}(x)\int\limits_{\Gamma}\big\{\big(\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\nabla_z\gamma_n(z,y)\big)\cdot\boldsymbol{\varphi}(y)\\
&\qquad\qquad - \nabla_z\gamma_n(z,y)\cdot\big(\mathcal{M}\big(\partial_y(y)\big)\boldsymbol{\varphi}(y)\big)\big\}ds_y\\
&= -\boldsymbol{n}(x)\int\limits_{\Gamma}\sum_{\ell,k=1}^{n} m_{\ell k}\big(\partial_y,\boldsymbol{n}(y)\big)\Big(\frac{\partial}{\partial z_k}\gamma_n(z,y)\varphi_k(y)\Big)ds_y = 0
\end{aligned}
$$

for $x \notin \Gamma$ due to the Stokes theorem.

For the integral $\mathcal{J}_2$ we use

$$
\begin{aligned}
&\mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big)\frac{\partial}{\partial n_y}\gamma_n(z,y) - \mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\big(\nabla_z\cdot\boldsymbol{n}(x)\big)\gamma_n(z,y)\boldsymbol{I}\\
&= \big\{\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big) - \mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big)\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\big\}\gamma_n(z,y)\boldsymbol{I}
\end{aligned}
$$

to obtain

$$
\begin{aligned}
\mathcal{J}_2\boldsymbol{\varphi}(x) = &\int\limits_{\Gamma}\Big(\mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big)\gamma_n(z,y)\boldsymbol{I}\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)^{\top}\boldsymbol{\varphi}(y)ds_y\\
&+\mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big)\int\limits_{\Gamma}\gamma_n(z,y)\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\boldsymbol{\varphi}(y)ds_y\,.
\end{aligned}
$$

The final result is then with (2.2.36):

$$
\begin{aligned}
&-T\big(\partial_z,\boldsymbol{n}(x)\big)\int\limits_{\Gamma}K(z,y)\boldsymbol{\varphi}(y)ds_y\\
&= -\frac{\mu}{2(n-1)\pi}\big(\boldsymbol{n}(x)\times\nabla_z\big)\cdot\int\limits_{\Gamma}\gamma_n(z,y)\big(\boldsymbol{n}(y)\times\nabla_y\big)\boldsymbol{\varphi}(y)ds_y\\
&\quad - \mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big)\int\limits_{\Gamma}\Big\{4\mu^2E(z,y) - \frac{2\mu}{2(n-1)\pi}\,\gamma_n(z,y)\boldsymbol{I}\Big\}\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\boldsymbol{\varphi}ds_y\\
&\quad + \frac{\mu}{2(n-1)\pi}\int\limits_{\Gamma}\Big(\mathcal{M}\big(\partial_z,\boldsymbol{n}(x)\big)\gamma_n(z,y)\mathcal{M}\big(\partial_y,\boldsymbol{n}(y)\big)\Big)^{\top}\boldsymbol{\varphi}(y)ds_y\,,
\end{aligned}
$$

where $z \notin \Gamma$.

As can be seen, this is a combination of applications of tangential derivatives to weakly singular operators operating on tangential derivatives of $\boldsymbol{\varphi}$. Therefore, the limits $z \to x \in \Gamma$ with $z \in \Omega$ or $z \in \Omega^c$ exist, which leads to the desired result involving Cauchy singular integrals (see Hellwig [123, p. 197]). Note that for $n = 2$ we have $\big(n(x)\times\nabla_x\big)|_\Gamma = \frac{d}{ds_x}$. ∎

The singular behaviour of the hypersingular operator now can be regularized as above. This facilitates the computational algorithm for the Galerkin method (Of et al [243]).

As for the hypersingular integral operator associated with the Laplacian in Section 1.2, here we can apply a more elementary, but different regularization. Based on (2.2.31), we get

$$\begin{aligned} D\boldsymbol{\varphi}(x) &= -\lim_{\Omega\ni z\to x\in\Gamma}\{T_z\int\limits_\Gamma (T_yE(x,y)^\top(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x))ds_y \\ &\quad + T_z\int\limits_\Gamma (T_yE(x,y))^\top\boldsymbol{\varphi}(x)ds_y\}\,. \end{aligned}$$

Here, in the neighborhood of Γ, the operator T_z is defined by (2.2.22) where we identify $\boldsymbol{n}_z = \boldsymbol{n}_x$ for $z \notin \Gamma$. If we apply the representation formula (2.2.4) to any constant vector field $\boldsymbol{a}$, representing a rigid displacement, then we obtain

$$\boldsymbol{a} = -\int\limits_\Gamma (T_yE(z,y))^\top\boldsymbol{a}ds_y \text{ for } z\in\Omega,$$

which yields for $z \in \Omega$ in the neighborhood of Γ

$$T_z\int\limits_\Gamma (T_yE(z,y))^\top ds_y\boldsymbol{\varphi}(x) = \mathbf{0}\,.$$

Hence,

$$D\boldsymbol{\varphi}(x) = -\lim_{\Omega\ni z\to x\in\Gamma} T_z\int\limits_\Gamma (T_yE(z,y))^\top(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x))ds_y\,,$$

from which it can finally be shown that

$$D\boldsymbol{\varphi}(x) = -p.v.\int\limits_\Gamma T_x\left(T_yE(x,y)\right)^\top(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x))\,ds_y\,, \tag{2.2.39}$$

(see Kupradze et al [177, p.294], Schwab et al [274]).

If the boundary potential operators in (1.2.18) and (1.2.19) are replaced by the corresponding elastic potential operators, then the Calderón projector $\mathcal{C}_\Omega$ for solutions $\boldsymbol{u}$ of (2.2.9) in Ω again is given in the form of (1.2.20) with the corresponding elastic potential operators.

Also, Theorem 1.2.3. and Lemma 1.2.4. remain valid for the corresponding elastic potentials V, K, K' and D.

For the solution of the interior displacement problem we now may solve the Fredholm boundary integral equation of the first kind

$$V\boldsymbol{\sigma} = \tfrac{1}{2}\boldsymbol{\varphi} + K\boldsymbol{\varphi} \text{ on } \Gamma\,, \tag{2.2.40}$$

or the Cauchy singular integral equation of the second kind

$$\tfrac{1}{2}\boldsymbol{\sigma} - K'\boldsymbol{\varphi} = D\boldsymbol{\varphi} \quad \text{on} \quad \Gamma\,, \tag{2.2.41}$$

which both are equations for $\boldsymbol{\sigma}$.

The first kind integral equation (2.2.40) may have eigensolutions for special Γ similar to (1.3.4) for the Laplacian (see Steinbach [291]). Again we can modify (2.2.40) by including rigid motions (2.2.11). More precisely, we consider the system

$$\begin{aligned} V\boldsymbol{\sigma} - \boldsymbol{\omega}_0 &= \tfrac{1}{2}\boldsymbol{\varphi} + K\boldsymbol{\varphi} \quad \text{on} \quad \Gamma\,, \\ \int\limits_\Gamma \boldsymbol{\sigma} ds &= \mathbf{0} \end{aligned} \tag{2.2.42}$$

together with

$$\begin{aligned} \int\limits_\Gamma (-\sigma_1 x_2 + \sigma_2 x_1) ds &= 0 \qquad \text{for} \;\; n = 2 \;\; \text{or} \\ \int\limits_\Gamma (\boldsymbol{\sigma} \times \boldsymbol{x}) ds &= \mathbf{0} \qquad \text{for} \;\; n = 3\,, \end{aligned} \tag{2.2.43}$$

where $\boldsymbol{\sigma}$ and the unknown constant vector $\boldsymbol{\omega}_0$ are to be determined. As was shown in [142], the rotation $\boldsymbol{b}$ in (2.2.11) can be prescribed as $\boldsymbol{b} = \mathbf{0}$; in this case the side conditions (2.2.43) will not be needed. For $n = 3$, many more choices in $\boldsymbol{\omega}_0$ can be made (see [142] and Mikhlin et al [214]). The modified system (2.2.42) and (2.2.43) is always uniquely solvable in the Hölder space, $\boldsymbol{\sigma} \in C^\alpha(\Gamma)$ for given $\boldsymbol{\varphi} \in C^{1+\alpha}(\Gamma)$.

For the special Cauchy singular integral equation of the second kind (2.2.41), Mikhlin showed in [210] that the Fredholm alternative, originally designed for compact operators K, remains valid here. Therefore, (2.2.41) admits a unique classical solution $\boldsymbol{\sigma} \in C^\alpha(\Gamma)$ provided $\boldsymbol{\varphi} \in C^{1+\alpha}(\Gamma), 0 < \alpha < 1$. (See Kupradze et al [177, Chap. VI], Mikhlin et al [215, p. 382 ff]).

2.2.2 The Interior Traction Problem

The Neumann problem for the Lamé system (2.2.9) in Ω is called the *traction problem*, since here the boundary traction

$$T\boldsymbol{u}_{|\Gamma} = \boldsymbol{\psi} \text{ on } \Gamma \tag{2.2.44}$$

is given, whereas the missing Cauchy datum $\boldsymbol{u}_{|\Gamma}$ needs to be determined. Corresponding to (2.2.16) and (2.2.17), we have the overdetermined system

$$\left(\tfrac{1}{2}I + K\right)\boldsymbol{u} = V\boldsymbol{\psi}\,, \tag{2.2.45}$$

$$D\boldsymbol{u} = \left(\tfrac{1}{2}I - K'\right)\boldsymbol{\psi} \quad \text{on} \quad \Gamma \tag{2.2.46}$$

for the unknown boundary displacemant $\boldsymbol{u}_{|\Gamma}$.

As for the Neumann Problem for the Laplacian, here $\boldsymbol{\psi}$ needs to satisfy certain equilibrium conditions for a solution to exist. These can be obtained from the Betti formula, which for $\boldsymbol{u}$ and any rigid motion $\boldsymbol{\omega}(x)$ reads

$$\begin{aligned} 0 &= \int\limits_{\Omega} (\boldsymbol{\omega} \cdot \Delta^* \boldsymbol{u}) - (\boldsymbol{u} \cdot \Delta^* \boldsymbol{\omega}) dx \\ &= \int\limits_{\Gamma} \boldsymbol{\omega} \cdot T\boldsymbol{u} ds - \int\limits_{\Gamma} \boldsymbol{u} \cdot T\boldsymbol{\omega} ds \,. \end{aligned}$$

This implies, with $T\boldsymbol{\omega} = \mathbf{0}$, the necessary compatibility conditions for the given traction $\boldsymbol{\psi}$, namely

$$\int\limits_{\Gamma} \boldsymbol{\omega} \cdot \boldsymbol{\psi} ds = 0 \;\text{ for all rigid motions } \boldsymbol{\omega} \tag{2.2.47}$$

given by (2.2.11). This condition turns out to be also sufficient for the existence of $\boldsymbol{u}$ in the classical Hölder function spaces. If $\boldsymbol{\psi} \in C^{\alpha}(\Gamma)$ with $0 < \alpha < 1$ is given satisfying (2.2.47), then the right–hand side, $V\boldsymbol{\psi}$ in (2.2.45), automatically satisfies the orthogonality conditions from Fredholm's alternative; and the Cauchy singular integral equation (2.2.45) admits a solution$\boldsymbol{u} \in C^{1+\alpha}(\Gamma)$. The solution, however, is unique only up to all rigid motions $\boldsymbol{\omega}$, which are eigensolutions. For further details see [142].

The hypersingular integral equation of the first kind (2.2.46) also has eigensolutions which again are given by all rigid motions (2.2.11). As will be seen, the classical Fredholm alternative even holds for (2.2.46), and the right–hand side $\frac{1}{2}\boldsymbol{\psi} - K'\boldsymbol{\psi}$ satisfies the corresponding orthogonality conditions, provided, $\boldsymbol{\psi} \in C^{\alpha}(\Gamma)$ satisfies the equilibrium conditions (2.2.47). In both cases, the integral equations , together with appropriate side conditions, can be modified so that the resulting equations are uniquely solvable (see [141]).

2.2.3 Some Exterior Fundamental Problems

In this section we shall summarize the approach from [142]. For the *exterior displacement problem* for $\boldsymbol{u}$ satisfying the Lamé system (2.2.12) and the Dirichlet condition (2.2.15),

$$\boldsymbol{u}_{|\Gamma} = \boldsymbol{\varphi} \text{ on } \Gamma \,,$$

we require at infinity appropriate conditions according to (2.2.10); namely that there exist $\boldsymbol{\Sigma}$ and some rigid motion $\boldsymbol{\omega}$ such that

$$\boldsymbol{u}(x) + E(x,0)\boldsymbol{\Sigma} - \boldsymbol{\omega} = O(|x|^{1-n}) \quad \text{as} \;\; |x| \to \infty \,.$$

In general, the constants in $\boldsymbol{\Sigma}$ and $\boldsymbol{\omega}$ are related to each other. However, some of them can still be specified.

For $n = 2$, we consider the following two cases. In the first case,

$$b \text{ in } \boldsymbol{\omega} = \boldsymbol{a} + b(-x_2, x_1)^T \text{ is given}.$$

In addition, the total forces $\boldsymbol{\Sigma}$ in (2.2.14) are also given, often as $\boldsymbol{\Sigma} = \mathbf{0}$ due to equilibrium. Then $\boldsymbol{a}$ in $\boldsymbol{\omega}$ is an additional unknown vector. The representation formula (2.2.13) for the Dirichlet problem (2.2.12) with (2.2.14) yields the modified boundary integral equation of the first kind

$$V\boldsymbol{\sigma} - \boldsymbol{a} = -\tfrac{1}{2}\boldsymbol{\varphi}(x) + K\boldsymbol{\varphi}(x) + b(-x_2, +x_1)^\top \text{ on } \Gamma\,, \tag{2.2.48}$$

$$\int\limits_\Gamma \boldsymbol{\sigma} ds = \boldsymbol{\Sigma}\,. \tag{2.2.49}$$

Here, $\boldsymbol{\varphi}, \boldsymbol{\Sigma}$ and b are given and $\boldsymbol{\sigma}, \boldsymbol{a}$ are the unknowns. As we will see, these equations are always uniquely solvable. In particular, for any given $\boldsymbol{\varphi} \in C^{1+\alpha}(\Gamma), \boldsymbol{\Sigma}$ and b, one finds in the classical Hölder–spaces $\boldsymbol{\sigma} \in C^\alpha(\Gamma)$.

In the second case, in addition to the total force $\boldsymbol{\Sigma}$, the total momentum

$$\int\limits_\Gamma (-x_2\sigma_1 + x_1\sigma_2) ds_x = \Sigma_3$$

is also given, whereas b is now an additional unknown. Now the modified boundary integral equation of the first kind reads

$$V\boldsymbol{\sigma}(x) - \boldsymbol{a} - b(-x_2, +x_1)^\top = \frac{1}{2}\boldsymbol{\varphi}(x) + K\boldsymbol{\varphi}(x) \text{ on } \Gamma\,, \tag{2.2.50}$$

$$\int\limits_\Gamma \boldsymbol{\sigma} ds = \boldsymbol{\Sigma},\ \int\limits_\Gamma (-x_2\sigma_1 + x_1\sigma_2) ds_x = \Sigma_3\,, \tag{2.2.51}$$

where $\boldsymbol{\varphi} \in C^{1+\alpha}(\Gamma), \boldsymbol{\Sigma}, \Sigma_3$ are given and $\boldsymbol{\sigma}, \boldsymbol{a}$ and b are to be determined. The system (2.2.50), (2.2.51) always has a unique solution $\boldsymbol{\sigma} \in C^\alpha(\Gamma), \boldsymbol{a}, b$.

Both these problems can also be reduced to Cauchy singular integral equations by applying the traction operator to (2.2.13). This yields the singular integral equation

$$\tfrac{1}{2}\boldsymbol{\sigma}(x) + K'\boldsymbol{\sigma}(x) = -D\boldsymbol{\varphi}(x) \text{ for } x \in \Gamma\,, \tag{2.2.52}$$

with the additional equation (2.2.49) in the first case or the additional equations (2.2.51) in the second case, respectively. The operator $\frac{1}{2}I + K'$ is adjoint to $\frac{1}{2}I + K$ in (2.2.17). Due to Mikhlin [210], for these special operators, Fredholm's classical alternative is still valid in the space $C^\alpha(\Gamma)$. Since

$$\tfrac{1}{2}\boldsymbol{\omega} + K\boldsymbol{\omega} = 0 \text{ on } \Gamma$$

for all rigid motions $\boldsymbol{\omega}$, the adjoint equation (2.2.52) has an $3(n-1)$–dimensional eigenspace, as well. Moreover, $D\boldsymbol{\omega} = \mathbf{0}$ for all rigid motions;

hence, the right–hand side of (2.2.52) always satisfies the orthogonality conditions for any given $\boldsymbol{\varphi} \in C^{1+\alpha}(\Gamma)$. This implies that equation (2.2.52) always admits a solution $\boldsymbol{\sigma} \in C^{\alpha}(\Gamma)$ which is not unique. If, for $n = 2$, the total force and total momentum in addition are prescribed by (2.2.51), i.e. in the second case, then these equations determine $\boldsymbol{\sigma}(x)$ uniquely. For finding $\boldsymbol{a}$ and b we first determine three vector–valued functions $\boldsymbol{\lambda}_j(x), j = 1, 2, 3$, satisfying on Γ the equations

$$\begin{array}{rclcrl} \int\limits_\Gamma \boldsymbol{a}\cdot\boldsymbol{\lambda}_j ds &=& a_j & \text{and} & \int\limits_\Gamma (-x_2, x_1)\cdot\boldsymbol{\lambda}_j(x) ds & = 0 \ \text{ for } \ j = 1, 2\,, \\ \int\limits_\Gamma \boldsymbol{a}\cdot\boldsymbol{\lambda}_3 ds &=& 0 & \text{and} & \int\limits_\Gamma (-x_2, x_1)\cdot\boldsymbol{\lambda}_3 ds & = 1\,. \end{array} \tag{2.2.53}$$

Since $\boldsymbol{\sigma}(x)$ on Γ is already known from solving (2.2.52), equation (2.2.50) can now be used to find $\boldsymbol{a}$ and b; namely

$$\left.\begin{array}{ll} \text{for } j = 1, 2 & : a_j \\ \text{for } j = 3 & : b \end{array}\right\} = \int\limits_\Gamma V\boldsymbol{\sigma}(x)\cdot\boldsymbol{\lambda}_j(x) d_x - \frac{1}{2}\int\limits_\Gamma \boldsymbol{\varphi}\cdot\boldsymbol{\lambda}_j ds - \int\limits_\Gamma (K\boldsymbol{\varphi})\cdot\boldsymbol{\lambda}_j ds\,. \tag{2.2.54}$$

If, as in the first case, b and $\boldsymbol{\Sigma}$ are given, then the additional equations

$$\begin{array}{rcl} \int\limits_\Gamma \boldsymbol{\sigma} ds &=& \boldsymbol{\Sigma} \qquad \text{and} \\ \int\limits_\Gamma \boldsymbol{\lambda}_3(x)\cdot V\boldsymbol{\sigma}(x) ds_x &=& b - \frac{1}{2}\int\limits_\Gamma \boldsymbol{\lambda}_3\cdot\boldsymbol{\varphi} ds + \int\limits_\Gamma \boldsymbol{\lambda}_3\cdot K\boldsymbol{\varphi} ds \end{array}$$

determine $\boldsymbol{\sigma}$ uniquely; and a_1, a_2 can be found from (2.2.54), afterwards.

In the case $n = 3$, there are many more possible choices of additional conditions. To this end, we write the rigid motions (2.2.11) in the form

$$\boldsymbol{\omega}(x) = \sum_{j=1}^{3} a_j \boldsymbol{m}_j(x) + \sum_{j=4}^{6} b_{j-3}\boldsymbol{m}_j(x) =: \sum_{j=1}^{6} \omega_j \boldsymbol{m}_j(x)$$

where $\boldsymbol{m}_j(x)$ is the j–th column vector of the matrix

$$\begin{pmatrix} 1\,, & 0\,, & 0\,, & 0\,, & x_3\,, & -x_2 \\ 0\,, & 1\,, & 0\,, & x_2\,, & 0\,, & x_1 \\ 0\,, & 0\,, & 1\,, & -x_3\,, & -x_1\,, & 0\,. \end{pmatrix}. \tag{2.2.55}$$

Let $\mathcal{J} \subset \mathcal{F} := \{1, 2, 3, 4, 5, 6\}$ denote any fixed set of indices in $\mathcal{F}$. Then we may prescribe a_{j-3}, b_j for $j \in \mathcal{J}$, i.e., some of the parameters in $\boldsymbol{\omega}$ subject to the behavior of (2.2.10) at infinity. If $\mathcal{J}$ is a proper subset of $\mathcal{F}$ then we must include additional normalization conditions,

$$\int\limits_{\Gamma} \boldsymbol{m}_k(y) \cdot \boldsymbol{\sigma}(y) ds_y = \Sigma_k \quad \text{for} \quad k \in \mathcal{F} \setminus \mathcal{J} \,. \tag{2.2.56}$$

By taking the representation formula (2.2.13) on Γ, we obtain from the direct formulation the boundary integral equation of the first kind on Γ,

$$V\boldsymbol{\sigma}(x) - \sum_{k \in \mathcal{F} \setminus \mathcal{J}} \omega_k \boldsymbol{m}_k(x) = -\frac{1}{2}\boldsymbol{\varphi}(x) + K\boldsymbol{\varphi}(x) + \sum_{j \in \mathcal{J}} \omega_j \boldsymbol{m}_j(x) \,, \tag{2.2.57}$$

together with the additional equations,

$$\int\limits_{\Gamma} \boldsymbol{m}_k(y) \cdot \boldsymbol{\sigma}(y) ds_y = \Sigma_k \quad \text{for} \quad k \in \mathcal{F} \setminus \mathcal{J} \,. \tag{2.2.58}$$

In the right–hand side of (2.2.57), the ω_j are given, whereas the ω_k in the left–hand side are unknown. For given $\boldsymbol{\varphi} \in C^\alpha(\Gamma), 0 < \alpha < 1$, and given constants $\omega_j, j \in \mathcal{J}$, the unknowns are $\boldsymbol{\sigma} \in C^\alpha(\Gamma)$ together with ω_k for $k \in \mathcal{F} \setminus \mathcal{J}$.

Again, we may take the traction of (2.2.13) on Γ to obtain a Cauchy singular boundary integral equation instead of (2.2.57) and (2.2.58), namely (2.2.52) together with (2.2.58). Since (2.2.52) is always solvable for any given $\boldsymbol{\varphi} \in C^{1+\alpha}(\Gamma)$ due to the special form of the right–hand side, and since the eigenspace of $\frac{1}{2}I + K'$ is the linear space of all rigid motions, the linear equations (2.2.58) need to be completed by including (card $\mathcal{J}$) additional equations which resembles the required behavior of $\boldsymbol{u}(x)$ at infinity for those ω_j given already with $j \in \mathcal{J}$. For these constraints, we again choose the vector–valued functions, $\boldsymbol{\lambda}_\ell$ on Γ, $\ell \in \mathcal{F}$, (e.g. linear combinations of $\boldsymbol{m}_\ell|_\Gamma$) which are orthonormalized to $\boldsymbol{m}_j|_\Gamma$, i.e.,

$$\int\limits_{\Gamma} \boldsymbol{m}_j(x) \cdot \boldsymbol{\lambda}_\ell(x) ds_x = \delta_{j,\ell}, \quad j, \ell \in \mathcal{F} \,. \tag{2.2.59}$$

Now, the complete system of equations for the modified Dirichlet problem can be formulated as

$$\begin{aligned} \frac{1}{2}\boldsymbol{\sigma}(x) + K'\boldsymbol{\sigma}(x) - \sum_{\ell \in \mathcal{F}} \alpha_\ell \boldsymbol{m}_\ell(x) &= -D\boldsymbol{\varphi}(x) \quad \text{for} \quad x \in \Gamma \\ \int\limits_{\Gamma} \boldsymbol{m}_k \cdot \boldsymbol{\sigma}_k(y) ds_y &= \Sigma_k, \; k \in \mathcal{F} \setminus \mathcal{J} \,, \end{aligned} \tag{2.2.60}$$

together with

$$\int\limits_{\Gamma} \boldsymbol{\lambda}_j(x) \cdot V\boldsymbol{\sigma} ds = \int\limits_{\Gamma} \boldsymbol{\lambda}_j \cdot \left\{ \frac{1}{2}\boldsymbol{\varphi}(x) + K'\boldsymbol{\varphi}(x) \right\} ds + \omega_j \quad \text{for} \quad j \in \mathcal{J} \,.$$

The desired displacement $\boldsymbol{u}(x)_{|\Gamma}$ then is obtained from (2.2.13) with $\boldsymbol{\sigma}(x)$ determined from (2.2.60) and ω_k given by

$$\omega_k = \int\limits_\Gamma \boldsymbol{\lambda}_k \cdot V\boldsymbol{\sigma} ds + \int\limits_\Gamma \boldsymbol{\lambda}_k \cdot \left\{\frac{1}{2}\boldsymbol{\varphi} - K\boldsymbol{\varphi}\right\} ds \quad \text{for } k \in \mathcal{F} \setminus \mathcal{J}\,.$$

Note that we have included the extra unknown term $\sum \alpha_\ell \boldsymbol{m}_\ell(x)$ in (2.2.60) so that the number of unknowns and equations coincide. One can show that in fact $\alpha_\ell = 0$ for $\ell \in \mathcal{F}$. The last set of equations has been obtained from (2.2.50) with (2.2.59).

For the *exterior traction problem*, the Neumann datum is given by

$$T\boldsymbol{u}|_\Gamma = \boldsymbol{\psi} \quad \text{on } \Gamma\,,$$

and the total forces and momenta by

$$\int\limits_\Gamma \boldsymbol{\psi} \cdot \boldsymbol{m}_k ds = \Sigma_k, \quad k \in \mathcal{F}.$$

Here, the standard direct approach with $x \to \Gamma$ in (2.2.13) yields the Cauchy singular boundary integral equation for $\boldsymbol{u}_{|\Gamma}$,

$$\tfrac{1}{2}\boldsymbol{u}(x) - K\boldsymbol{u}(x) = V\boldsymbol{\psi}(x) + \boldsymbol{\omega}(x) \quad \text{for } x \in \Gamma\,. \tag{2.2.61}$$

As is well known, (2.2.61) is always uniquely solvable for any given $\boldsymbol{\psi} \in C^\alpha(\Gamma)$ and given $\boldsymbol{\omega}$ with $\boldsymbol{u} \in C^{1+\alpha}(\Gamma)$; we refer for the details to Kupradze [176, p. 118].

If we apply T to (2.2.13) then we obtain the hypersingular boundary integral equation

$$D\boldsymbol{u}(x) = -\tfrac{1}{2}\boldsymbol{\psi}(x) - K'\psi(x) \quad \text{for } x \in \Gamma\,. \tag{2.2.62}$$

It is easily seen that the rigid motions $\boldsymbol{\omega}(y)$ on Γ are eigensolutions of (2.2.62). Therefore, in order to guarantee unique solvability of the boundary integral equation , we modify (2.2.62) by including restrictions and adding unknowns, e.g.,

$$\begin{aligned} D\boldsymbol{u}_0(x) = -\frac{1}{2}\boldsymbol{\psi}(x) - K'\boldsymbol{\psi}(x) + \sum_{\ell=1}^{3(n-1)} \alpha_\ell \boldsymbol{m}_\ell(x) \quad \text{for } x \in \Gamma \text{ and} \\ \int\limits_\Gamma \boldsymbol{m}_k(y) \cdot \boldsymbol{u}_0(y) ds_y = 0\,, \quad k = 1, \cdots, 3(n-1)\,. \end{aligned} \tag{2.2.63}$$

Also here, the unknowns α_ℓ are introduced for obtaining a quadratic system. For the Neumann problem, they all vanish because of the special form of the right–hand side. As we will see, the system (2.2.63) is always uniquely solvable; for any given $\boldsymbol{\psi} \in C^{\alpha}(\Gamma)$ we find exactly one $\boldsymbol{u}_0 \in C^{1+\alpha}(\Gamma)$. In (2.2.63), the additional compatibility conditions can also be incorporated into the first equation of (2.2.63) which yields the stabilized uniquely solvable version

$$\begin{aligned} \widetilde{D}\boldsymbol{u}_0(x) &:= D\boldsymbol{u}_0(x) + \sum_{k=1}^{3(n-1)} \int\limits_{\Gamma} \boldsymbol{m}_k(y)\boldsymbol{u}_0(y)ds_y \boldsymbol{m}_k(x) \\ &= -\tfrac{1}{2}\boldsymbol{\psi}(x) - K'\boldsymbol{\psi}(x) \quad \text{for } \ x \in \Gamma\,. \end{aligned} \tag{2.2.64}$$

Once $\boldsymbol{u}_0$ is known, the actual displacement field $\boldsymbol{u}(x)$ is then given by

$$\boldsymbol{u}(x) = -V\boldsymbol{\psi}(x) + W\boldsymbol{u}_0(x) \ \text{ for } \ x \in \Omega^c\,. \tag{2.2.65}$$

Note that the actual boundary values of $\boldsymbol{u}|_\Gamma$ may differ from $\boldsymbol{u}_0$ by a rigid motion. $\boldsymbol{u}|_\Gamma$ can be expressed via (2.2.65) in the form

$$\boldsymbol{u}(x)|_\Gamma = \frac{1}{2}\boldsymbol{u}_0(x) + K\boldsymbol{u}_0(x) - V\boldsymbol{\psi}(x)\,. \tag{2.2.66}$$

In concluding this section we remark that the boundary integral equations on Hölder continuous charges are considered in most of the more classical works on this topic with applications in elasticity (e.g., Ahner et al [5], Balaš et al [11], Bonnemay [18], Kupradze [175, 176], Muskhelishvili [223] and Natroshvili [225]). However, as mentioned before, we shall come back to these equations in Chapters 5–10.

Now we extend our approach to incompressible materials.

2.2.4 The Incompressible Material

If the elastic material becomes *incompressible*, the Poisson ratio $\nu := \lambda/2(\lambda+\mu)$ tends to $1/2$ or $\lambda = 2\mu\nu/(1-2\nu) \to \infty$ for $n = 3$ and for the plane strain case where $n = 2$. However, for the plane stress case we have $\overline{\nu} \to 1/3$ and $\overline{\lambda} \to 2\mu$ if the material is incompressible; and our previous analysis remains valid without any restriction.

In order to analyze the incompressible case, we now rewrite the Lamé equation (2.2.1) in the form of a system

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= 0\,, \\ \text{div } \boldsymbol{u} &= -cp \quad \text{where } \ p = -\frac{\lambda+\mu}{\mu}\text{div}\boldsymbol{u} \end{aligned} \tag{2.2.67}$$

and $c = 1 - 2\nu = \frac{\mu}{\lambda+\mu} \to 0+$ (see Duffin and Noll [66]). This system corresponds to the *Stokes system*. In terms of the parameter c, the fundamental solution (2.2.2) now takes the form

$$E(x,y) = \frac{1+2c}{4\pi(n-1)\mu(1+c)} \Big\{ \gamma_n(x,y)\boldsymbol{I} + \tfrac{1}{(1+2c)r^n}(x-y)(x-y)^\top \Big\} \tag{2.2.68}$$

which is well defined for $c = 0$, as well. Consequently, the Betti–Somigliana representation formula (2.2.4) remains valid, where the double layer potential kernel now reads as

$$\begin{aligned}(T_y E(x,y))^\top \quad &= \quad \frac{1}{2\pi(n-1)(1+c)} \Big\{ \big(c\boldsymbol{I} + \frac{n}{r^2}(x-y)(x-y)^\top\big)\frac{\partial\gamma}{\partial n_y} \\ &\qquad + \frac{c}{r^n}\left((x-y)\boldsymbol{n}_y^\top - \boldsymbol{n}_y(x-y)^\top\right) \Big\}^\top ,\end{aligned} \tag{2.2.69}$$

which again is well defined for the limiting case $c = 0$. However, the limiting case of the differential equations (2.2.63) leads to the more complicated Stokes system involving a mixed variational formulation (see Brezzi and Fortin [25]) whereas the associated boundary integral equations remain valid. In fact, from (2.2.65) and (2.2.66) it seems that the case $c = 0$ does not play any exceptional role. However, for small $c > 0$, a more detailed analysis is required. We shall return to this point after discussing the Stokes problem in Section 2.3.

2.3 The Stokes Equations

The linearized and stationary equations of the *incompressible viscous fluid* are modeled by the Stokes system consisting of the equations in the form

$$\begin{aligned} -\mu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f}\,, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega \text{ (or } \Omega^c)\,. \end{aligned} \tag{2.3.1}$$

Here $\boldsymbol{u}$ and p are the velocity and pressure of the fluid flow, respectively, which are the unknowns; $\boldsymbol{f}$ corresponds to a given forcing term, while μ is the given *dynamic viscosity* of the fluid. We have already seen this system previously in (2.2.67) for the elastic material when it becomes incompressible, although for viscous flow with given *fluid density* ρ, one introduces

$$\nu := \frac{\mu}{\rho} \gg 1$$

which is usually referred to as the *kinematic viscosity* of the fluid, not the Poisson ratio as in the case of elasticity. The fundamental solution of the Stokes system (2.3.1) is defined by the pair of distributions $\boldsymbol{v}^k$, and q^k satisfying

$$\begin{aligned} -\{\mu\Delta_x\boldsymbol{v}^k(x,y) - \nabla_x q^k(x,y)\} &= \delta(x,y)\boldsymbol{e}^k\,, \\ \operatorname{div}_x \boldsymbol{v}^k &= 0\,, \end{aligned} \tag{2.3.2}$$

where $\boldsymbol{e}^k$ denotes the unit vector along the x_k-axis, $k = 1,.,n$ with $n = 2$ or 3 (see Ladyženskaya [179]). By using the Fourier transform, we may obtain the

fundamental solution explicitly:
For $n = 2$,

$$\begin{aligned}\boldsymbol{v}^k(x,y) &= \frac{1}{4\pi\mu}\Big\{\log\frac{1}{|x-y|}\boldsymbol{e}^k + \sum_{j=1}^{2}\frac{(x_k-y_k)(x_j-y_j)\boldsymbol{e}^j}{|x-y|^2}\Big\}\,,\\ q^k(x,y) &= \frac{\partial}{\partial x_k}\{-\frac{1}{2\pi}\log\frac{1}{|x-y|}\}\,;\end{aligned} \tag{2.3.3}$$

and for $n = 3$,

$$\begin{aligned}\boldsymbol{v}^k(x,y) &= \frac{1}{8\pi\mu}\Big\{\frac{1}{|x-y|}\boldsymbol{e}^k + \sum_{j=1}^{3}\frac{(x_k-y_k)(x_j-y_j)\boldsymbol{e}^j}{|x-y|^3}\Big\}\,,\\ q^k(x,y) &= \frac{\partial}{\partial x_k}\{-\frac{1}{4\pi}\frac{1}{|x-y|}\}.\end{aligned} \tag{2.3.4}$$

We note that from their explicit forms, $\boldsymbol{v}^k(x,y)$ and $q^k(x,y)$ also satisfy the adjoint system in the y–variables, namely

$$\begin{aligned}-\{\mu\Delta_y\boldsymbol{v}(x,y) + \nabla_y q^k(x,y)\} &= \delta(x,y)\boldsymbol{e}^k\,,\\ -\mathrm{div}_y\,\boldsymbol{v}^k &= 0\,.\end{aligned} \tag{2.3.5}$$

This means that we may use the same fundamental solution for the Stokes system and for its adjoint depending on which variables are differentiated. As will be seen, we do not need to switch the variables x and y in the representation of the solution of (2.3.1) from Green's formula (see [179]). For the flow $(\boldsymbol{u},p)$, we define the *stress operators* as in elasticity,

$$\begin{aligned}T(\boldsymbol{u}) &:= -p\boldsymbol{n} + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)\boldsymbol{n}\\ &= \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}\,,\\ T'(\boldsymbol{u}) &:= p\boldsymbol{n} + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)\boldsymbol{n}\\ &= \boldsymbol{\sigma}(\boldsymbol{u},-p)\boldsymbol{n}\,,\end{aligned} \tag{2.3.6}$$

where

$$\boldsymbol{\sigma} := -pI + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)$$

denotes the *stress tensor* in the viscous flow. We remark that it is understood that the stress operator T is always defined for the pair $(\boldsymbol{u},p)$. For smooth $(\boldsymbol{u},p)$ and $(\boldsymbol{v},q)$, we have the second Green formula

$$\begin{aligned}&\int\limits_{\Omega}\boldsymbol{u}\cdot\{-\mu\Delta\boldsymbol{v} - \nabla q)\}\,dx - \int\limits_{\Omega}\{-\mu\Delta\boldsymbol{u} + \nabla p\}\cdot\boldsymbol{v}dy\\ &\qquad = \int\limits_{\Gamma}[T(\boldsymbol{u})\cdot\boldsymbol{v} - \boldsymbol{u}\cdot T'(\boldsymbol{v})]\,ds\end{aligned} \tag{2.3.7}$$

provided

$$\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{v} = 0\,.$$

Now replacing $(\boldsymbol{v}, q)$ by $(\boldsymbol{v}^k, q^k)$, and by following standard arguments, the velocity component of the nonhomogeneous Stokes system (2.3.1) has the representation

$$u_k(x) = \int\limits_{\Gamma} \left[T_y(\boldsymbol{u}) \cdot \boldsymbol{v}^k(x,y) - \boldsymbol{u} \cdot T'_y(\boldsymbol{v}^k)(x,y)\right] ds_y + \int\limits_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}^k dy \quad \text{for} \;\; x \in \Omega\,. \tag{2.3.8}$$

To obtain the representation of the pressure, we may simply substitute the relation

$$\frac{\partial p}{\partial x_k} = \mu \Delta u_k + f_k$$

into (2.3.8). To simplify the representation, we now introduce the *fundamental velocity tensor* $E(x,y)$ and its *associated pressure vector* $Q(x,y)$, respectively, as

$$E(x,y) = [\boldsymbol{v}^1, \cdot, \boldsymbol{v}^n], \quad \text{and} \quad Q(x,y) = [q^1, \cdot, q^n]\,,$$

which satisfy

$$\begin{aligned} -\mu \Delta_x E + \nabla_x Q &= \delta(x,y) \boldsymbol{I}\,, \\ \operatorname{div}_x E &= \boldsymbol{0}^\top. \end{aligned} \tag{2.3.9}$$

As a result of (2.3.4) and (2.3.5), we have explicitly

$$\begin{aligned} E(x,y) &= \frac{1}{4(n-1)\pi\mu} \Big(\gamma_n \boldsymbol{I} + \frac{(x-y)(x-y)^\top}{|x-y|^n} \Big)\,, \\ Q(x,y) &= \frac{1}{2(n-1)\pi} (-\nabla_x \gamma_n)^\top = \frac{1}{2(n-1)\pi} (\nabla_y \gamma_n)^\top \end{aligned} \tag{2.3.10}$$

with

$$\gamma_n(x,y) = \begin{cases} -\log|x-y| & \text{for } n=2, \\ \frac{1}{|x-y|} & \text{for } n=3 \end{cases}$$

as in (2.2.2). In terms of $E(x,y)$ and $Q(x,y)$, we finally have the representation for solutions in the form:

$$\begin{aligned} \boldsymbol{u}(x) \;\; &= \int\limits_{\Gamma} E(x,y) T(\boldsymbol{u})(y) ds_y - \int\limits_{\Gamma} \big(T'_y E(x,y)\big)^\top \boldsymbol{u}(y) ds_y \\ &\quad + \int\limits_{\Omega} E(x,y) \boldsymbol{f}(y) dy \quad \text{for} \;\; x \in \Omega\,, \end{aligned} \tag{2.3.11}$$

$$\begin{aligned} p(x) \;\; &= \int\limits_{\Gamma} Q(x,y) \cdot T(\boldsymbol{u})(y) ds_y - 2\mu \int\limits_{\Gamma} (\frac{\partial Q}{\partial n_y}(x,y)) \cdot \boldsymbol{u}(y) ds_y \\ &\quad + \int\limits_{\Omega} Q(x,y) \cdot \boldsymbol{f}(y) dy \quad \text{for} \;\; x \in \Omega\,. \end{aligned} \tag{2.3.12}$$

It is understood that the representation of p is unique only up to an additive constant. Also, as was explained before,

$$T'_y E(x,y) := \boldsymbol{\sigma}\big(E(x,y), -Q(x,y)\big)\boldsymbol{n}(y)\,.$$

2.3.1 Hydrodynamic Potentials

The last terms in the representation (2.3.11) and (2.3.12) corresponding to the body force $\boldsymbol{f}$ define a particular solution $(\boldsymbol{U}, P)$ of the nonhomogeneous Stokes system (2.3.1). As in elasticity, if we decompose the solution in the form

$$\boldsymbol{u} = \boldsymbol{u}_c + \boldsymbol{U}, \quad p = p_c + P,$$

then the pair $(\boldsymbol{u}_c, p_c)$ will satisfy the corresponding homogeneous system of (2.3.1). Hence, in the following, without loss of generality, we shall confine ourselves only to the homogeneous Stokes system. The solution of the homogeneous system now has the representation from (2.3.11) and (2.3.12) with $\boldsymbol{f} = \boldsymbol{0}$, i.e.,

$$\begin{aligned} \boldsymbol{u}(x) &= V\boldsymbol{\tau}(x) - W\boldsymbol{\varphi}(x)\,, && (2.3.13)\\ p(x) &= \Phi\boldsymbol{\tau}(x) - \Pi\boldsymbol{\varphi}(x). && (2.3.14) \end{aligned}$$

(The subscript c has been suppressed.) Here the pair (V, Φ) and (W, Π) are the respective simple– and double layer hydrodynamic potentials defined by

$$\begin{aligned} V\boldsymbol{\tau}(x) &:= \int\limits_\Gamma E(x,y)\boldsymbol{\tau}(y)ds_y\,,\\ \Phi\boldsymbol{\tau}(x) &:= \int\limits_\Gamma Q(x,y)\cdot\boldsymbol{\tau}(y)ds_y\,; \end{aligned} \qquad (2.3.15)$$

$$\begin{aligned} W\boldsymbol{\varphi}(x) &:= \int\limits_\Gamma \big(T'_y(E(x,y))\big)^\top \boldsymbol{\varphi}(y)ds_y\,,\\ \Pi\boldsymbol{\varphi}(x) &:= 2\mu\int\limits_\Gamma \Big(\frac{\partial}{\partial n_y}Q(x,y)\Big)\cdot\boldsymbol{\varphi}(y)ds_y \quad \text{for } x\notin\Gamma\,. \end{aligned} \qquad (2.3.16)$$

In (2.3.13) and (2.3.14) the boundary charges are the Cauchy data $\boldsymbol{\varphi} = \boldsymbol{u}(x)|_\Gamma$ and $\boldsymbol{\tau}(x) = T\boldsymbol{u}(x)|_\Gamma$ of the solution to the Stokes equations

$$\begin{aligned} -\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{0}\,,\\ \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\,. \end{aligned} \qquad (2.3.17)$$

For the exterior problems, the representation formula for $\boldsymbol{u}$ needs to be modified by taking into account the growth conditions at infinity. Here proper growth conditions are

$$\boldsymbol{u}(x) = \begin{cases} \boldsymbol{\Sigma} \ \log|x| + O(1) & \text{for } n=2, \\ O(|x|^{-1}) & \text{for } n=3; \end{cases} \tag{2.3.18}$$

$$p(x) = O(|x|^{1-n}) \quad \text{as} \quad |x| \to \infty. \tag{2.3.19}$$

In the two–dimensional case $\boldsymbol{\Sigma}$ is a given constant vector. The representation formula for solutions of the Stokes equations (2.3.17) in Ω^c with the growth conditions (2.3.18) and (2.3.19) has the form

$$\boldsymbol{u}(x) = -V\boldsymbol{\tau}(x) + W\boldsymbol{\varphi}(x) + \boldsymbol{\omega}\,, \tag{2.3.20}$$

$$p(x) = -\Phi\boldsymbol{\tau}(x) + \Pi\boldsymbol{\varphi}(x) \tag{2.3.21}$$

with the Cauchy data $\boldsymbol{\varphi} = \boldsymbol{u}_{|\Gamma}$ and $\boldsymbol{\tau} = T(\boldsymbol{u})_{|\Gamma}$ satisfying

$$\boldsymbol{\Sigma} = \int\limits_{\Gamma} \boldsymbol{\tau} ds\,; \tag{2.3.22}$$

and $\boldsymbol{\omega}$ is an unknown constant vector which vanishes when $n = 3$.

2.3.2 The Stokes Boundary Value Problems

We consider two boundary value problems for the Stokes system (2.3.17) in Ω as well as in Ω^c. In the first problem (the Dirichlet problem), the boundary trace of the velocity

$$\boldsymbol{u}|_\Gamma = \boldsymbol{\varphi} \qquad \text{on} \ \ \Gamma \tag{2.3.23}$$

is specified, and in the second problem (the Neumann problem), the hydrodynamic boundary traction

$$T(\boldsymbol{u})|_\Gamma = \boldsymbol{\tau} \qquad \text{on} \ \ \Gamma \tag{2.3.24}$$

is given. As consequences of the incompressible flow equations and the Green formula for the interior problem, the prescribed Cauchy data need to satisfy, respectively, the *compatibility conditions*

$$\begin{aligned} &\int\limits_{\Gamma} \boldsymbol{\varphi}\cdot\boldsymbol{n}\, ds = 0\,, \\ &\int\limits_{\Gamma} \boldsymbol{\tau}\cdot(\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x})\, ds = \boldsymbol{0} \quad \text{for all} \ \ \boldsymbol{a}\in\mathbb{R}^n \quad \text{and} \quad \boldsymbol{b}\in\mathbb{R}^{1+2(n-2)}\,, \end{aligned} \tag{2.3.25}$$

with $\boldsymbol{b}\times\boldsymbol{x} := b(x_2, -x_1)^\top$ for $n = 2$.

For the exterior problem we require the decay conditions (2.3.18) and (2.3.19). We again solve these problems by using the direct method of boundary integral equations.

Since the pressure p will be completely determined once the Cauchy data for the velocity are known, in the following, it suffices to consider only the boundary integral equations for the velocity $\boldsymbol{u}$. We need, of course, the representation formula for p implicitly when we deal with the stress operator. In analogy to elasticity, we begin with the representation formula (2.3.13) for the velocity $\boldsymbol{u}$ in Ω and (2.3.20) and (2.3.21) in Ω^c. Applying the trace operator and the stress operator T to both sides of the representation formula, we obtain the overdetermined system of boundary integral equations (the Calderón projection) for the interior problem

$$\boldsymbol{\varphi}(x) = (\tfrac{1}{2}I - K)\boldsymbol{\varphi}(x) + V\boldsymbol{\tau}(x)\,, \tag{2.3.26}$$

$$\boldsymbol{\tau}(x) = D\boldsymbol{\varphi} + (\tfrac{1}{2}I + K')\boldsymbol{\tau}(x) \text{ on } \Gamma\,. \tag{2.3.27}$$

Hence, the Calderón projector for Ω can also be written in operator matrix form as

$$\mathcal{C}_\Omega = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix}\,.$$

Here V, K, K' and D are the four corresponding basic boundary integral operators of the Stokes flow. Hence, the Calderón projector $\mathcal{C}_\Omega$ for the interior domain has the same form as in (1.2.20) with the corresponding hydrodynamic potential operators.

For the exterior problem, the Calderón projector on solutions having the decay properties (2.3.18) and (2.3.19) with $\boldsymbol{\Sigma}$ given by (2.3.22) is also given by (1.4.11), i.e.,

$$\mathcal{C}_{\Omega^c} = \mathcal{I} - \mathcal{C}_\Omega = \begin{pmatrix} \frac{1}{2}I + K & -V \\ -D & \frac{1}{2}I - K' \end{pmatrix}\,. \tag{2.3.28}$$

As always, the solutions of both Dirichlet problems as well as both Neumann problems in Ω and Ω^c can be solved by using the boundary integral equations of the first as well as of the second kind by employing the relations between the Cauchy data given by the Calderón projectors.

The four basic operators appearing in the Calderón projectors for the Stokes problem are defined in the same manner as in elasticity (see Lemmata 2.3.1 and 2.2.3) but with appropriate modifications involving the pressure terms. More specifically, the double layer operator is defined as

$$\begin{aligned} K\boldsymbol{\varphi}(x) &:= \frac{1}{2}\boldsymbol{\varphi}(x) + \lim_{\Omega\ni z\to x\in\Gamma} \int\limits_\Gamma \big(T'_y E(z,y)\big)^\top \boldsymbol{\varphi}(y)ds_y \qquad (2.3.29)\\ &= \int\limits_{\Gamma\setminus\{x\}} \big(T'_y E(x,y)\big)^\top \boldsymbol{\varphi}(y)ds_y \\ &= \int\limits_{\Gamma\setminus\{x\}} \sum_{i,j,k,=1}^n \Big\{Q^k(x,y)\delta_{ij} + \mu\Big(\frac{\partial E_{ik}(x,y)}{\partial y_j} + \frac{\partial E_{jk}(x,y)}{\partial y_i}\Big)\Big\} n_j(y)\varphi_i(y)ds_y \\ &= \frac{n}{2(n-1)\pi} \int\limits_{\Gamma\setminus\{x\}} \frac{\big((x-y)\cdot\boldsymbol{n}(y)\big)\big((x-y)\cdot\boldsymbol{\varphi}(y)\big)(x-y)}{|x-y|^{n+2}} ds_y \end{aligned}$$

having a weakly singular kernel for $\Gamma \subset C^2$, and, hence, defines a continuous mapping $K : C^\alpha(\Gamma) \to C^{1+\alpha}(\Gamma)$ (see Ladyženskaya [179, p. 35] where the fundamental solution and the potentials carry the opposite sign). The hypersingular operator D is now defined by

$$\begin{aligned} D\boldsymbol{\varphi} &= -T_x W\boldsymbol{\varphi}(x) \\ &:= -\lim_{\Omega \ni z \to x \in \Gamma} T_z(\partial_z, x)\big(W\boldsymbol{\varphi}(z)\big) \\ &= \lim_{\Omega \ni z \to x \in \Gamma} \big\{\big(\Pi\boldsymbol{\varphi}(z)\big)\boldsymbol{n}(x) - \mu\big(\nabla_z W\boldsymbol{\varphi}(z) + \big(\nabla_z W\boldsymbol{\varphi}(z)\big)^\top\big)\boldsymbol{n}(x)\big\}\,. \end{aligned} \tag{2.3.30}$$

With the standard regularization this reads

$$\begin{aligned} D\boldsymbol{\varphi}(x) = -\text{p.v.} \int_\Gamma \big\{T_x\big(T'_y E(x,y)\big)^\top\big\}\big(\boldsymbol{\varphi}(y) - \boldsymbol{\varphi}(x)\big)ds_y \qquad (2.3.31) \\ = \frac{-\mu}{2(n-1)\pi}\text{p.v.} \int_\Gamma \Big\{2\frac{1}{|x-y|^n}\boldsymbol{n}(y)\cdot\big(\boldsymbol{\varphi}(y) - \boldsymbol{\varphi}(x)\big) \\ + \frac{n}{|x-y|^{n+2}}\boldsymbol{n}(y)\cdot\boldsymbol{n}(x)\big[(x-y)\cdot\big(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x)\big)\big](\boldsymbol{x}-\boldsymbol{y})\Big\}ds_y \\ + \frac{\mu}{2(n-1)\pi}\int_\Gamma \Big\{\frac{2n(n+2)}{|x-y|^{n+4}}\big[(x-y)\cdot\boldsymbol{n}(x)\big] \\ \times\big[(x-y)\cdot\boldsymbol{n}(y)\big]\big[(x-y)\cdot\big(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x)\big)\big](\boldsymbol{x}-\boldsymbol{y}) \\ - \frac{n}{|x-y|^{n+2}}\Big(\big[(x-y)\cdot\boldsymbol{n}(x)\big]\big[(x-y)\cdot\big(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x)\big)\big]\boldsymbol{n}(y) \\ + \boldsymbol{n}(x)\cdot\big(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x)\big)\big[(y-x)\cdot\boldsymbol{n}(y)\big](\boldsymbol{x}-\boldsymbol{y}) \\ + \big[(x-y)\cdot\boldsymbol{n}(x)\big]\big[(x-y)\cdot\boldsymbol{n}(y)\big]\big(\boldsymbol{\varphi}(y)-\boldsymbol{\varphi}(x)\big)\Big)\Big\}ds_y\,. \end{aligned}$$

Again, as in Lemma 2.2.3, the hypersingular operator can be reformulated.

Lemma 2.3.1. *Kohr et al* [164] *Let $\Gamma \in C^2$ and let $\boldsymbol{\varphi}$ be a Hölder continuously differentiable function. Then the operator D in* (2.3.30) *can be expressed as a composition of tangential differential operators and simple layer potentials as in* (2.2.32)–(2.2.34) *where in the case $n = 2$ set $\frac{\lambda+\mu}{\lambda+2\mu} = 1$ in* (2.2.33) *and in the case $n = 3$ take $E(x,y)$ from* (2.3.10) *in* (2.2.34).

Now let us assume that the boundary $\Gamma = \bigcup_{\ell=1}^{L} \Gamma_\ell$ consists of L separate, mutually non intersecting compact boundary components $\Gamma_1, \dots, \Gamma_L$.

Before we exemplify the details of solvability of the boundary integral equations, we first summarize some basic properties of their eigenspaces.

Theorem 2.3.2. *(See also Kohr and Pop* [163]*.) Let $n = 3$. Then we have the following relations.*

i) *The normal vector fields* $\boldsymbol{n}_\ell \in C^\alpha(\Gamma)$ *where* $\boldsymbol{n}_\ell|_{\Gamma_j} = \boldsymbol{0}$ *for* $\ell \neq j$ *generate exterior to* Ω *on* $\Gamma = \bigcup\limits_{\ell=1}^{L} \Gamma_\ell$ *the* L*–dimensional eigenspace or kernel of the simple layer operator* V *as well as of* $(\frac{1}{2}I - K')$*. Then the operator* $(\frac{1}{2}I - K)$ *also has an* L*–dimensional eigenspace generated by* $\boldsymbol{\varphi}_{0\ell} \in C^{1+\alpha}(\Gamma)$ *with* $\boldsymbol{\varphi}_{0\ell}|_{\Gamma_j} = \boldsymbol{0}$ *for* $\ell \neq j$ *satisfying the equations*

$$\boldsymbol{n}_\ell = D\boldsymbol{\varphi}_{0\ell} \quad \textit{for} \;\; \ell = \overline{1, L}\,. \tag{2.3.32}$$

Any eigenfunction $\sum\limits_{j=1}^{L} \gamma_j \boldsymbol{n}_j$ *generates a solution*

$$\boldsymbol{0} \equiv \sum_{j=1}^{L} \gamma_j V \boldsymbol{n}_j \quad \textit{and} \;\; p_0 = \gamma_1 \quad \textit{in} \;\; \Omega$$

(see Kohr and Pop [163]*, Reidinger and Steinbach* [260]*).*

ii) *On each component* Γ_ℓ *of the boundary, the boundary integral operators* $D|_{\Gamma_\ell}$ *as well as* $(\frac{1}{2}I + K)|_{\Gamma_\ell}$ *have the* 6*–dimensional eigenspace* $\boldsymbol{v}_\ell = (\boldsymbol{a}_\ell + \boldsymbol{b}_\ell \times \boldsymbol{x})|_\Gamma$ *for all* $\boldsymbol{a}_\ell \in \mathbb{R}^3$ *with* $\boldsymbol{b}_\ell \in \mathbb{R}^3$*.*

If $\boldsymbol{v}_{j,\ell}$ *with* $j = 1, \dots, 6$ *and* $\ell = 1, \dots, L$ *denotes a basis of this eigenspace then there exist* $6L$ *linearly independent eigenvectors* $\boldsymbol{\tau}_{j,\ell} \in C^\alpha(\Gamma)$ *of the adjoint operator* $(\frac{1}{2}I + K')|_{\Gamma_\ell}$*; and there holds the relation*

$$\boldsymbol{v}_{j,\ell} = V|_{\Gamma_\ell} \boldsymbol{\tau}_{j,\ell} \tag{2.3.33}$$

between these two eigenspaces.

Any of the eigenfunctions $v_{j,\ell}$ *on* Γ *generates a solution*

$$\boldsymbol{u}_{0j,\ell}(x) = -\int\limits_{\Gamma_\ell} K(x,y)\boldsymbol{v}_{j,\ell}(x) ds_x = \begin{cases} \boldsymbol{0} & \text{for } x \in \overline{\Omega} \text{ if } \ell = \overline{2, L}, \\ v_{j,1}(x) & \text{for } x \in \Omega \text{ if } \ell = 1 \end{cases}$$

$$p_{0j,\ell}(x) = \begin{cases} 0 & \text{for } x \in \overline{\Omega} \text{ if } \ell = \overline{2, L}, \\ \operatorname{div}_x \frac{\mu}{\pi(n-1)} \int\limits_{\Gamma_1} \left(\frac{\partial}{\partial n_y} \gamma_n(x,y)\right) \boldsymbol{v}_{j,1}(y) ds_y & \text{for } x \in \Omega \text{ if } \ell = 1. \end{cases}$$

In the case $n = 2$*, the operator* V *needs to be replaced by*

$$\widetilde{V}\boldsymbol{\tau} := V\boldsymbol{\tau} + \alpha\Big(\int_\Gamma \boldsymbol{\tau} ds\Big) \quad \textit{with} \;\; \alpha > 0$$

an appropriately large chosen scaling constant α *and* $\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ *replaced by* $\boldsymbol{a} + b(x_2, -x_1)^\top$ *and* 6 *by* 3 *in* **ii)***.*

Proof: Let $n = 3$ and, for brevity, $L = 1$.

i) It is shown by Ladyženskaya in [179, p.61] that $\boldsymbol{n}$ is the only eigensolution of $(\frac{1}{2}I - K')$. Therefore, due to the classical Fredholm alternative, the adjoint operator $(\frac{1}{2}I - K)$ has only one eigensolution $\boldsymbol{\varphi}_0$, as well. It remains to show that $\boldsymbol{n}$ is also the only linear independent eigensolution to V and satisfies (2.3.32).

As we will show later on, for V, the Fredholm theorems are also valid and $V : C^\alpha(\Gamma) \to C^{\alpha+1}(\Gamma)$ has the Fredholm index zero. Let $\boldsymbol{\tau}_0$ be any solution of $V\boldsymbol{\tau}_0 = \mathbf{0}$. Then the single layer potential

$$\boldsymbol{u}_0 = V\boldsymbol{\tau}_0 \quad \text{with} \;\; p_0 = \Phi\boldsymbol{\tau}_0$$

is a solution of the Stokes system in Ω as well as in Ω^c with $\boldsymbol{u}_0|_\Gamma = \mathbf{0}$. Then $\boldsymbol{u}_0 \equiv \mathbf{0}$ in $\mathbb{R}^3$ and the associated pressure is zero in Ω^c and $p_0 = \beta =$ constant in Ω. As a consequence we have from the jump relations

$$T_x(\boldsymbol{u}_{0-}, p_0) - T_x(\boldsymbol{u}_{0+}, 0) = \big[\sigma(\boldsymbol{u}_0, p_0)\boldsymbol{n}\big]|_\Gamma = -\beta\boldsymbol{n}\,,$$

therefore $\boldsymbol{\tau}_0 = -\beta\boldsymbol{n}$.

On the other hand, $V\boldsymbol{n}|_\Gamma = \mathbf{0}$ follows from the fact that $\boldsymbol{u} := V\boldsymbol{n}$ and $p := \Pi\boldsymbol{n}$ is the solution of the exterior homogeneous Neumann problem of the Stokes system since

$$T(V\boldsymbol{u})|_\Gamma = (-\tfrac{1}{2}I + K')\boldsymbol{n} = \mathbf{0}\,.$$

and therefore vanishes identically (see [179, Theorem 1 p.60]).

In order to show (2.3.32), we consider the solution of the exterior Dirichlet Stokes problem with $\boldsymbol{u}^+|_\Gamma = \boldsymbol{\varphi}_0 \neq 0$ which admits the representation

$$\boldsymbol{u}(x) = W\boldsymbol{\varphi}_0 - V\boldsymbol{\tau}\,.$$

Then it follows that the corresponding simple layer term has vanishing boundary values,

$$-V\boldsymbol{\tau}|_\Gamma = \boldsymbol{\varphi}_0 - (\tfrac{1}{2}I + K)\boldsymbol{\varphi}_0 = (\tfrac{1}{2}I - K)\boldsymbol{\varphi}_0 = \mathbf{0}\,.$$

Hence, $\boldsymbol{\tau} = \beta\boldsymbol{n}$ with some constant $\beta \in \mathbb{R}$. Application of $T_x|_\Gamma$ gives

$$\boldsymbol{\tau} = \beta\boldsymbol{n} = -D\boldsymbol{\varphi}_0 + (\tfrac{1}{2}I - K')\beta\boldsymbol{n} = -D\boldsymbol{\varphi}_0\,.$$

The case $\beta = 0$ would imply $\boldsymbol{\tau} = \mathbf{0}$ and then $\boldsymbol{u}(x)$ solved the homogeneous Neumann problem which has only the trivial solution [179, p. 60] implying $\boldsymbol{\varphi}_0 = \mathbf{0}$ which is excluded. So, $\beta \neq 0$ and scaling of $\boldsymbol{\varphi}_0$ implies (2.3.32).

ii) For the operator $(\frac{1}{2}I+K)$ having the eigenspace $(\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x})|_\Gamma$ of dimension 6 we refer to [179, p. 62]. Hence, the adjoint operator $(\frac{1}{2}I + K')$ also has a 6–dimensional eigenspace due to the classical Fredholm theory since K is a compact operator. For the operator D let us consider the potential $\boldsymbol{u}(x) = W\boldsymbol{v}(x)$ in Ω^c with $\boldsymbol{v} = \boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}|_\Gamma$. Then $\boldsymbol{u}$ is a solution of the Stokes problem and on the boundary we find

$$\boldsymbol{u}^+|_\Gamma = (\tfrac{1}{2}I + K)\boldsymbol{v} = \mathbf{0}\,.$$

Therefore $\boldsymbol{u}(x) = \boldsymbol{0}$ for all $x \in \overline{\Omega^c}$ and, hence,

$$TW\boldsymbol{v} = -D\boldsymbol{v} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{v} \in \ker D\,.$$

Conversely, if $D\boldsymbol{v} = \boldsymbol{0}$ then let $\boldsymbol{u}$ be the solution of the interior Dirichlet problem with $\boldsymbol{u}^-|_\Gamma = \boldsymbol{v}$ which has the representation

$$\boldsymbol{u}(x) = V\boldsymbol{\tau} - W\boldsymbol{\tau} \quad \text{for } \ x \in \Omega$$

with an appropriate $\boldsymbol{\tau}$. Then applying T we find

$$\boldsymbol{\tau} = (\tfrac{1}{2}I + K')\boldsymbol{\tau} + D\boldsymbol{v} = (\tfrac{1}{2}I + K')\boldsymbol{\tau}\,.$$

Therefore $\boldsymbol{\tau}$ satisfies $(\frac{1}{2}I - K')\boldsymbol{\tau} = \boldsymbol{0}$ which implies $\boldsymbol{\tau} = \beta\boldsymbol{n}$ with some $\beta \in \mathbb{R}$. Hence,

$$\boldsymbol{u}(x) = \beta V\boldsymbol{n}(x) - W\boldsymbol{v}(x) = -W\boldsymbol{v}(x)$$

and its trace yields

$$(\tfrac{1}{2}I + K)\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \ \Gamma\,.$$

Therefore $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ with some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$.

Now let $\boldsymbol{\tau}_0 \in \ker(\frac{1}{2}I + K')\,,\ \boldsymbol{\tau}_0 \neq \boldsymbol{0}$. Then $\boldsymbol{u}(x) := V\boldsymbol{\tau}_0(x)$ in Ω is a solution of the homogeneous Neumann problem in Ω since $T\boldsymbol{u}|_\Gamma = (\frac{1}{2}I + K')\boldsymbol{\tau}_0 = \boldsymbol{0}$. Therefore $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ with some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ and

$$V\boldsymbol{\tau}_0 \in \ker(\tfrac{1}{2}I + K)\,.$$

The mapping $V\ :\ \ker(\frac{1}{2}I + K') \to \ker(\frac{1}{2}I + K)$ is also injective since for $\boldsymbol{\tau}_0 \neq \boldsymbol{0}$,

$$V\boldsymbol{\tau}_0 = 0 \quad \text{would imply } \ \boldsymbol{\tau}_0 = \beta\boldsymbol{n}$$

and, hence,

$$(\tfrac{1}{2}I + K')\boldsymbol{\tau}_0 = \boldsymbol{0} = \beta(\tfrac{1}{2}I + K')\boldsymbol{n} = \beta\boldsymbol{n} - (\tfrac{1}{2}I - K')\beta\boldsymbol{n} = \beta\boldsymbol{n}\,.$$

So, $\beta = 0$, which is a contradiction. The case $L > 1$ we leave to the reader (see [143]).

For $n = 2$ the proof follows in the same manner and we omit the details. ■

In the Table 2.3.3 below we summarize the boundary integral equations of the first and second kind for solving the four fundamental boundary value problems together with the corresponding eigenspaces as well as the compatibility conditions. We emphasize that, as a consequence of Theorem 2.3.2, the orthogonality conditions for the right–hand side given data in the boundary integral equations will be automatically satisfied provided the given Cauchy data satisfy the compatibility conditions if required because of the direct approach.

In the case of $n = 2$ in Table 2.3.3, replace V by $\widetilde{V}$ and $\boldsymbol{b} \times \boldsymbol{x}$ by $(bx_2, -x_1)^\top\,,\ b \in \mathbb{R}\,,\ \boldsymbol{a} \in \mathbb{R}^2$.

Table 2.3.3. Boundary Integral Equations for the 3–D Stokes Problem

BVP	BIE	Eigenspaces of BIO $\ell, k = \overline{1, L};\ j = \overline{1, 3(n-1)}$	Compability conditions for given $\boldsymbol{\varphi}, \boldsymbol{\psi}$
IDP	$(1) V\boldsymbol{\tau} = (\frac{1}{2}I + K)\boldsymbol{\varphi}$	$\ker V = \ker(\frac{1}{2} - K') = \text{span}\ \{\boldsymbol{n}_\ell\}$	$\int\limits_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n} ds = 0$
	$(2) (\frac{1}{2}I - K')\boldsymbol{\tau} = D\boldsymbol{\varphi}$		
INP	$(1) D\boldsymbol{u} = (\frac{1}{2}I - K')\boldsymbol{\psi}$	$\ker D = \ker(\frac{1}{2}I + K) = \text{span}\ \{\boldsymbol{v}_{j,\ell}\}$	$\int\limits_\Gamma \boldsymbol{\psi} \cdot (\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}) ds = 0$
	$(2) (\frac{1}{2}I + K)\boldsymbol{u} = V\boldsymbol{\psi}$	$\boldsymbol{v}_{j,\ell}$ basis of $\{\boldsymbol{v}\vert_{\Gamma_\ell} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}\vert_{\Gamma_\ell}, \boldsymbol{v}\vert_{\Gamma_k} = \boldsymbol{0}, \ell \neq k\}$	for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$.
EDP	$(1) V\boldsymbol{\tau} = (-\frac{1}{2}I + K)\boldsymbol{\varphi}$	$\ker V = \text{span}\ \{\boldsymbol{n}_\ell\}$	None
	$(2) (\frac{1}{2}I + K')\boldsymbol{\tau} = -D\boldsymbol{\varphi}$	$\ker(\frac{1}{2}I + K') = \text{span}\ \{\boldsymbol{\tau}_{j,\ell}\}$, $V\boldsymbol{\tau}_{j,\ell} = \boldsymbol{v}_{j,\ell} \in \ker(\frac{1}{2}I + K)$	
ENP	$(1) D\boldsymbol{u} = -(\frac{1}{2} + K')\boldsymbol{\psi}$	$\ker D = \text{span}\ \{\boldsymbol{v}_{j,\ell}\}$	None
	$(2) (-\frac{1}{2}I + K)\boldsymbol{u} = V\boldsymbol{\psi}$	$\ker(\frac{1}{2}I - K) = \text{span}\ \{\boldsymbol{u}_{0\ell}\}$, $D\boldsymbol{u}_{0\ell} = \boldsymbol{n}_\ell \in \ker V$	

Table 2.3.4. Modified Integral Equations for the 3–D Stokes Problems

	Modified Equations I, $\ell=\overline{1,L}$, $j=\overline{1,3(n-1)}$	Modified Equations II
IDP	$(1) V\boldsymbol{\tau}+\sum_{\ell=1}^{L}\omega_\ell \boldsymbol{n}_\ell=(\frac{1}{2}I+K)\boldsymbol{\varphi}$ and $\int_{\Gamma_\ell}\boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds=0$	$V\boldsymbol{\tau}+\sum_{\ell=1}^{L}\int_{\Gamma}\boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds\boldsymbol{n}_\ell=(\frac{1}{2}I+K)\boldsymbol{\varphi}$
	$(2) (\frac{1}{2}I-K')\boldsymbol{\tau}+\sum_{\ell=1}^{\ell}\omega_\ell \boldsymbol{n}_\ell=D\boldsymbol{\varphi}$ and $\int_{\Gamma_\ell}\boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds=0$	$(\frac{1}{2}I-K')\boldsymbol{\tau}+\sum_{\ell=1}^{L}\int_{\Gamma_\ell}\boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds\boldsymbol{n}_\ell=D\boldsymbol{\varphi}$
INP	$(1) D\boldsymbol{u}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\omega_{j,\ell}\boldsymbol{v}_{j,\ell}=(\frac{1}{2}I-K')\boldsymbol{\psi}$ and $\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{v}_{j,\ell}ds=0$	$D\boldsymbol{u}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{v}_{j,\ell}ds\boldsymbol{v}_{j,\ell}=(\frac{1}{2}I-K')\boldsymbol{\tau}$
	$(2) (\frac{1}{2}I+K)\boldsymbol{u}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\omega_{j,\ell}\boldsymbol{v}_{j,\ell}=V\boldsymbol{\psi}$ and $\int_{\Gamma_\ell}\boldsymbol{u}\cdot V\boldsymbol{\tau}_{j,\ell}ds=\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{v}_{j,\ell}ds=0$	$(\frac{1}{2}I+K)\boldsymbol{u}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{v}_{j,\ell}ds\boldsymbol{v}_j=V\boldsymbol{\psi}$
EDP	$(1) V\boldsymbol{\tau}+\sum_{\ell=1}^{L}\omega_\ell \boldsymbol{n}_\ell=(-\frac{1}{2}I+K)\boldsymbol{\varphi}$ and $\int_{\Gamma_\ell}\boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds=0$	$V\boldsymbol{\tau}+\sum_{\ell=1}^{L}\int_{\Gamma_\ell}\boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds\boldsymbol{n}_\ell=(-\frac{1}{2}I+K)\boldsymbol{\varphi}$
	$(2) (\frac{1}{2}I-K')\boldsymbol{\tau}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\omega_{j,\ell}\boldsymbol{\tau}_{j,\ell}=-D\boldsymbol{\varphi}$ and $\int_{\Gamma_\ell}\boldsymbol{\tau}\cdot\boldsymbol{v}_{j,\ell}ds=0$	$(\frac{1}{2}I-K')\boldsymbol{\tau}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\int_{\Gamma_\ell}\boldsymbol{\tau}\cdot\boldsymbol{v}_{j,\ell}ds\boldsymbol{\tau}_{j,\ell}=-D\boldsymbol{\varphi}$
ENP	$(1) D\boldsymbol{u}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\omega_{j,\ell}\boldsymbol{v}_{j,\ell}=-(\frac{1}{2}I+K')\boldsymbol{\psi}$ and $\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{v}_{j,\ell}ds=0,$	$D\boldsymbol{u}+\sum_{\ell=1}^{L}\sum_{j=1}^{3(n-1)}\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{v}_{j,\ell}ds\boldsymbol{v}_{j,\ell}=-(\frac{1}{2}I+K')\boldsymbol{\psi}$
	$(2) (\frac{1}{2}I-K)\boldsymbol{u}+\sum_{\ell=1}^{L}\omega_\ell \boldsymbol{n}_\ell=-V\boldsymbol{\psi}$ and $\int_{\Gamma_\ell}\boldsymbol{u}\cdot D\boldsymbol{u}_{0\ell}ds=\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{n}_\ell ds=0$	$(\frac{1}{2}I-K)\boldsymbol{u}+\sum_{\ell=1}^{L}\int_{\Gamma_\ell}\boldsymbol{u}\cdot\boldsymbol{n}_\ell ds\boldsymbol{n}_\ell=-V\boldsymbol{\psi}$

Since each of the integral equations in Table 2.3.3 has a nonempty kernel, we now modify these equations in the same manner as in elasticity by incorporating eigenspaces to obtain uniquely solvable boundary integral equations. Again, in order not to be repetitious, we summarize the modified equations in Table 2.3.4.

A few comments are in order.

In the two–dimensional case, it should be understood that V should be replaced by $\widetilde{V}$ and that $\ker D = \operatorname{span}\{\boldsymbol{v}_{j,\ell}\}$ with $\boldsymbol{v}_{j,\ell}$ a basis of $\{\boldsymbol{a}+b\binom{x_2}{-x_1}\}|_{\Gamma_\ell}$ with $\boldsymbol{a} \in \mathbb{R}^2$, $b \in \mathbb{R}$. Moreover, as in elasticity in Section 2.2, one has to incorporate $\int\limits_\Gamma \boldsymbol{\sigma} ds$ appropriately, in order to take into account the decay conditions (2.3.18).

For exterior problems, special attention has to be paid to the behavior at infinity. In particular, $\boldsymbol{u}$ has the representation (2.3.20), i.e.,

$$\boldsymbol{u} = W\boldsymbol{\varphi} - V\boldsymbol{\tau} + \boldsymbol{\omega} \quad \text{in} \quad \Omega^c .$$

Then the Dirichlet condition leads on Γ to the system

$$V\boldsymbol{\tau} - \boldsymbol{\omega} = -(\tfrac{1}{2}I - K)\boldsymbol{\varphi} \quad \text{and} \quad \int\limits_\Gamma \boldsymbol{\tau} ds = \boldsymbol{\Sigma}\,, \tag{2.3.34}$$

where in the last equation $\boldsymbol{\Sigma}$ is a given constant vector determining the logarithmic behavior of $\boldsymbol{u}$ at infinity (see (2.3.18)). For uniqueness, this system is modified by adding the additional conditions

$$\int\limits_\Gamma \boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds = 0 \;\; \ell = \overline{1, L}\,.$$

Then the system (2.3.34) is equivalent to the uniquely solvable system

$$\begin{aligned} V\boldsymbol{\tau} - \boldsymbol{\omega} + \sum_{\ell=1}^{L} \omega_{3\ell}\boldsymbol{n}_\ell &= -(\tfrac{1}{2}I - K)\boldsymbol{\varphi}\,, \\ \int\limits_\Gamma \boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds = 0\,, \;\; \int\limits_\Gamma \boldsymbol{\sigma} ds &= \boldsymbol{\Sigma}\,, \;\; \overline{\ell = 1, L}\,, \quad \text{or} \end{aligned} \tag{2.3.35}$$

$$V\boldsymbol{\tau} - \boldsymbol{\omega} + \sum_{\ell=1}^{L} \int\limits_\Gamma \boldsymbol{\tau}\cdot\boldsymbol{n}_\ell ds\boldsymbol{n}_\ell = -(\tfrac{1}{2}I - K)\boldsymbol{\varphi}\,, \quad \int\limits_\Gamma \boldsymbol{\sigma} ds = \boldsymbol{\Sigma}\,. \tag{2.3.36}$$

These last two versions (2.3.35) and (2.3.36) correspond to mixed formulations and have been analyzed in detail in Fischer et al [80] and [134, 135, 137, 139].

In the same manner, appropriate modifications are to be considered for other boundary co nditions and the time harmonic unsteady problems and corresponding boundary integral equations as well [137, 139], Kohr et al [163, 162] and Varnhorn [310]. There, as in this section, the boundary integral equations are considered for charges in Hölder spaces. We shall come back to these problems in a more general setting in Chapter 5.

Note that for the interior Neumann problem, the modified integral equations will provide specific uniquely determined solutions of the integral equations, whereas the solution of the original Stokes Neumann problem still has the nullspace $\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ for $n = 3$ and $\{\boldsymbol{a} + b(x_2, -x_1)^\top\}$ for $n = 2$.

Finally, the second versions of the modified integral equations (II) in Table 2.3.4 are often referred to as *stabilized versions* in scientific computing. Clearly, the two versions are always equivalent Fischer et al [79].

2.3.3 The Incompressible Material — Revisited

With the analysis of the Stokes problems available, we now return to the interior elasticity problems in Section 2.2.4 for almost incompressible materials, i.e., for small $c \geq 0$, but restrict ourselves to the case that Γ is one connected compact manifold (see also [143], and Steinbach [289]). The case of $\Gamma = \bigcup\limits_{\ell=1}^{L} \Gamma_\ell$ as in Theorem 2.3.2 is considered in [143].

For the interior displacement problem, the unknown boundary traction $\boldsymbol{\sigma}$ satisfies the boundary integral equation (2.2.40) of the first kind,

$$V_{e\ell}\boldsymbol{\sigma} = (\tfrac{1}{2}I + K_{e\ell})\boldsymbol{\varphi} \quad \text{on } \Gamma . \tag{2.3.37}$$

where the index $e\ell$ indicates that these are the operators in elasticity where the kernel $E_{e\ell}(x, y)$ can be expressed via (2.2.68). Then with the simple layer potential operator V_{st} of the Stokes equation and its kernel given in (2.3.10) we have the relation

$$V_{e\ell} = \frac{1}{1+c} V_{st} + \frac{2c}{1+c}\,\frac{1}{\mu} V_{\Delta} I \tag{2.3.38}$$

where V_Δ denotes the simple layer potential operator (1.2.1) of the Laplacian. Inserting (2.3.38) into (2.3.37) yields the equation

$$V_{st}\boldsymbol{\sigma} = (1+c)(\tfrac{1}{2}I + K_{e\ell})\boldsymbol{\varphi} - c\frac{2}{\mu} V_\Delta \boldsymbol{\sigma} , \tag{2.3.39}$$

which corresponds to the equation (1) of the interior Stokes problem in Table 2.3.3.

As was shown in Theorem 2.3.2, the solution of (2.3.39) can be decomposed in the form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \alpha \boldsymbol{n} \quad \text{with } \int\limits_\Gamma \boldsymbol{\sigma}_0 \cdot \boldsymbol{n} ds = 0 \quad \text{and } \alpha \in \mathbb{R} . \tag{2.3.40}$$

Hence,

$$\begin{aligned} V_{st}\boldsymbol{\sigma}_0 = (1+c)(\tfrac{1}{2}I + K_{e\ell})\boldsymbol{\varphi} - c\frac{2}{\mu} V_\Delta (\boldsymbol{\sigma}_0 + \alpha \boldsymbol{n}) , \\ \int\limits_\Gamma \boldsymbol{\sigma}_0 \cdot \boldsymbol{n} ds = 0 . \end{aligned} \tag{2.3.41}$$

A necessary and sufficient condition for the solvability of this system is the orthogonality condition

$$\int_{\Gamma} \Big\{(1+c)(\tfrac{1}{2}I + K_{e\ell})\boldsymbol{\varphi} - c\frac{2}{\mu}V_{\Delta}\boldsymbol{\sigma}_0 - \alpha c\frac{2}{\mu}V_{\Delta}\boldsymbol{n}\Big\} \cdot \boldsymbol{n} ds = 0\,.$$

Now we combine (2.2.68) with (2.3.29) and obtain the relation

$$(1+c)K_{e\ell}\boldsymbol{\varphi} = K_{st}\boldsymbol{\varphi} + c(K_{\Delta}\boldsymbol{\varphi} + L_1\boldsymbol{\varphi}) \tag{2.3.42}$$

between the double layer potential operators of the Lamé and the Stokes systems where K_{Δ} is the double layer potential operator (1.2.8) of the Laplacian and L is the linear Cauchy singular integral operator defined by

$$L_1\boldsymbol{\varphi} = \frac{1}{2\pi(n-1)}\ \text{p.v.}\int_{\Gamma\setminus\{x\}} \Big(\frac{\boldsymbol{n}(y)\cdot\boldsymbol{\varphi}(y)(x-y) - (x-y)\cdot\boldsymbol{\varphi}(y)\boldsymbol{n}(y)}{|x-y|^n}\Big) ds_y\,. \tag{2.3.43}$$

Therefore the orthogonality condition becomes

$$\begin{aligned}\int_{\Gamma}\{(\tfrac{1}{2}I + K_{st})\boldsymbol{\varphi}\}\cdot\boldsymbol{n}ds + c\Big[&\int_{\Gamma}\{(\tfrac{1}{2}I + K_{\Delta} + L_1)\boldsymbol{\varphi}\}\cdot\boldsymbol{n}ds \\ &-\frac{2}{\mu}\int_{\Gamma}(V_{\Delta}\boldsymbol{\sigma}_0)\cdot\boldsymbol{n}ds - \alpha\frac{2}{\mu}\beta_{\Delta}\Big] = 0\end{aligned}$$

where $\beta_{\Delta} := \int_{\Gamma}(V_{\Delta}\boldsymbol{n})\cdot\boldsymbol{n}ds$. Since $\int_{\Gamma}\boldsymbol{n}ds = 0$, it can be shown that $\beta_{\Delta} > 0$ (see [138], [141, Theorem 3.7]). In the first integral, however, we interchange orders of integration and obtain

$$\int_{\Gamma}\big((\tfrac{1}{2}I + K_{st})\boldsymbol{\varphi}\big)\cdot\boldsymbol{n}ds = \int_{\Gamma}\boldsymbol{\varphi}\cdot\big((\tfrac{1}{2}I + K'_{st})\boldsymbol{n}\big)ds = \int_{\Gamma}\boldsymbol{\varphi}\cdot\boldsymbol{n}ds$$

from Theorem 2.3.2. Hence, the orthogonality condition implies that α must be chosen as

$$\alpha = \frac{1}{c}\ \frac{\mu}{2\beta_{\Delta}}\int_{\Gamma}\boldsymbol{\varphi}\cdot\boldsymbol{n}ds + \frac{\mu}{2\beta_{\Delta}}\int_{\Gamma}\{(\tfrac{1}{2}I + K_{\Delta} + L_1)\boldsymbol{\varphi}\}\cdot\boldsymbol{n}ds - \frac{1}{\beta_{\Delta}}\int_{\Gamma}(V_{\Delta}\boldsymbol{\sigma}_0)\cdot\boldsymbol{n}ds. \tag{2.3.44}$$

Replacing α from (2.3.44) in (2.3.41) we finally obtain the corresponding stabilized equation,

$$V_{st}\boldsymbol{\sigma}_0 + \int_{\Gamma}\boldsymbol{\sigma}_0\cdot\boldsymbol{n}ds\boldsymbol{n} + cB\boldsymbol{\sigma}_0 = \boldsymbol{f} \tag{2.3.45}$$

where the linear operator B is defined by

$$B\boldsymbol{\sigma}_0 := \frac{2}{\mu}\Big(V_{\Delta}\boldsymbol{\sigma}_0 - \frac{1}{\beta_{\Delta}}\int_{\Gamma}(V_{\Delta}\boldsymbol{\sigma}_0)\cdot\boldsymbol{n}dsV_{\Delta}\boldsymbol{n}\Big)$$

and the right–hand side $\boldsymbol{f}$ is given by

$$\boldsymbol{f} = (\tfrac{1}{2}I + K_{st})\boldsymbol{\varphi} - \frac{1}{\beta_\Delta}\int_\Gamma \varphi \cdot \boldsymbol{n} ds V_\Delta \boldsymbol{n}$$
$$+c\Big[(\tfrac{1}{2}I + K_\Delta + L_1)\boldsymbol{\varphi} - \frac{1}{\beta_\Delta}\int_\Gamma \{(\tfrac{1}{2}I + K_\Delta + L_1)\boldsymbol{\varphi}\} \cdot \boldsymbol{n} ds V_\Delta \boldsymbol{n}\Big] .$$

Since for $c = 0$ the equation (2.3.45) is uniquely solvable, the regularly perturbed equation (2.3.45) for small c but $c \neq 0$ is still uniquely solvable.

With $\boldsymbol{\sigma}_0$ available, α can be found from (2.3.44) and, finally, the boundary traction $\boldsymbol{\sigma}$ is given by (2.3.40). Then the representation formula (2.2.6) provides us with the elastic displacement field $\boldsymbol{u}$ and the solution's behavior for the elastic, but almost incompressible materials, which one may expand with respect to small $c \geq 0$, as well. In particular we see that, for the almost incompressible material

$$\boldsymbol{u}_{e\ell} = \boldsymbol{u}_{st} + \tfrac{1}{\beta_\Delta}\int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n} ds V_\Delta \boldsymbol{n} + O(c) \quad \text{as} \quad c \to 0$$

where $\boldsymbol{u}_{st}$ is the unique solution of the Stokes problem with

$$\boldsymbol{u}_{st}|_\Gamma = \boldsymbol{\varphi} - \tfrac{1}{\beta_\Delta}\int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n} ds V_\Delta \boldsymbol{n} + O(c) .$$

We also have the relation

$$\int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n} ds = \int_\Omega \operatorname{div} \boldsymbol{u} dx = -c\int_\Omega p dx .$$

This shows that only if the given datum $\int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n} ds = O(c)$ then we have

$$\boldsymbol{u}_{e\ell} = \boldsymbol{u}_{st} + O(c) .$$

Next, we consider the interior traction problem for the almost incompressible material. For simplicity, we now employ Equation (2.2.46),

$$D_{e\ell}\boldsymbol{u} = (\tfrac{1}{2}I - K'_{e\ell})\boldsymbol{\psi} \quad \text{on} \quad \Gamma \tag{2.3.46}$$

where now $\boldsymbol{\psi}$, the boundary stress, is given on Γ satisfying the compatibility conditions (2.2.47), and the boundary displacement $\boldsymbol{u}$ is the unknown.

With (2.3.38), i.e.,

$$E_{e\ell}(x,y) = \frac{1}{1+c}E_{st}(x,y) + \frac{c}{1+c}\,\frac{1}{2(n-1)\pi\mu}\gamma_n(x,y)I \tag{2.3.47}$$

and with Lemma 2.3.1 we obtain for the hypersingular operators

$$D_{e\ell}\boldsymbol{\varphi} = D_{st}\boldsymbol{\varphi} + cL_2\boldsymbol{\varphi} \tag{2.3.48}$$

where for $n = 3$

$$L_2\boldsymbol{\varphi}(x) = \mathcal{M}_x \int\limits_{\Gamma\setminus\{x\}} 4\mu^2 \frac{1}{1+c}\Big(E_{st}(x,y) - \frac{1}{2(n-1)\mu\pi}\gamma_n(x,y)I\Big)\mathcal{M}_y\boldsymbol{\varphi}(y)ds_y\,, \tag{2.3.49}$$

and for $n = 2$ the differential operators can be replaced as $\mathcal{M}_x = \frac{d}{ds_x}$ and $\mathcal{M}_y = \frac{d}{ds_y}$. Hence, (2.3.46) can be written as

$$D_{st}\boldsymbol{u} = (\tfrac{1}{2}I - K'_{e\ell})\boldsymbol{\psi} - cL_2\boldsymbol{u}\,. \tag{2.3.50}$$

In view of Theorem 2.3.2, one may decompose the solution $\boldsymbol{u}$ in the form

$$\boldsymbol{u}(x) = \boldsymbol{u}_0(x) + \sum_{j=1}^{M}\alpha_j\boldsymbol{m}_j(x) \tag{2.3.51}$$

where

$$\int\limits_{\Gamma}\boldsymbol{u}_0\cdot\boldsymbol{m}_j ds = 0 \quad \text{for} \;\; j = 1,\dots,M \quad \text{with} \;\; M := \tfrac{1}{2}n(n+1)\,,$$

and $\boldsymbol{m}_j(x)$ are the traces of the rigid motions given in (2.2.55). These vector valued functions form a basis of the kernel to $D_{e\ell}$ as well as to D_{st} which implies also that

$$L_2\boldsymbol{m}_j = 0 \quad \text{for} \;\; j = 1,\dots,M \quad \text{and} \;\; c \in \mathbb{R}\,. \tag{2.3.52}$$

Substituting (2.3.51) into (2.3.50) yields the uniquely solvable system of equations

$$\begin{aligned} D_{st}\boldsymbol{u}_0 + cL_2\boldsymbol{u}_0 &= (\tfrac{1}{2}I - K'_{e\ell})\boldsymbol{\psi}\,,\\ \int\limits_{\Gamma}\boldsymbol{u}_0\cdot\boldsymbol{m}_j ds &= 0 \quad \text{for} \;\; j = 1,\dots,M\,; \end{aligned} \tag{2.3.53}$$

or, in stabilized form

$$D_{st}\boldsymbol{u}_0 + \sum_{j=1}^{M}\int\limits_{\Gamma}\boldsymbol{u}_0\cdot\boldsymbol{m}_j ds\boldsymbol{m}_j + cL_2\boldsymbol{u}_0 = (\tfrac{1}{2}I - K'_{e\ell})\boldsymbol{\psi}\,. \tag{2.3.54}$$

The right–hand side in (2.3.53) satisfies the orthogonality conditions

$$\int\limits_{\Gamma}\big((\tfrac{1}{2}I - K'_{e\ell})\boldsymbol{\psi}\big)\cdot\boldsymbol{m}_j ds = 0 \quad \text{for} \;\; j = 1,\dots,M$$

since the given $\boldsymbol{\psi}$ satisfies the compatibility conditions

$$\int\limits_{\Gamma}\boldsymbol{\psi}\cdot\boldsymbol{m}_j ds = 0 \quad \text{for} \;\; j = 1,\dots,M$$

and the vector valued function $\boldsymbol{m}_j$ satisfies

$$(\tfrac{1}{2}I + K_{e\ell})\boldsymbol{m}_j = \boldsymbol{0} \quad \text{on} \;\; \Gamma\,.$$

The equations (2.3.53) or (2.3.54) are uniquely solvable for every $c \in [0, \infty)$ and so, the general elastic solution $\boldsymbol{u}_{e\ell}$ for almost incompressible material has the form

$$\begin{aligned}
\boldsymbol{u}_{e\ell}(x) &= V_{e\ell}\boldsymbol{\psi}(x) - W_{e\ell}\boldsymbol{u}_0(x) + \sum_{j=1}^{M} \alpha_j \boldsymbol{m}_j(x) \\
&= \frac{1}{1+c}\Big\{\boldsymbol{u}_{st} + \frac{2c}{\mu} V_{\Delta}\boldsymbol{\psi} - c(W_{\Delta}\boldsymbol{u}_0 + L_1\boldsymbol{u}_0\Big\}(x) + \sum_{j=1}^{M} \alpha_j \boldsymbol{m}_j(x)
\end{aligned}$$

for $x \in \Omega$ with arbitrary $\alpha_j \in \mathbb{R}$ and where $\boldsymbol{u}_{st}$ is the solution of the Stokes problem with given boundary tractions $\boldsymbol{\psi}$, and L_1 is defined in (2.3.43). For $c \to 0$ we see that for the elastic Neumann problem

$$\boldsymbol{u}_{e\ell} = \boldsymbol{u}_{st} + O(c)$$

up to rigid motions, i.e., a regular perturbation with respect to the Stokes solution.

2.4 The Biharmonic Equation

In both problems, plane elasticity and plane Stokes flow, the systems of partial differential equations can be reduced to a single scalar 4th–order equation,

$$\Delta^2 u = 0 \quad \text{in} \quad \Omega \text{ (or } \Omega^c) \subset \mathbb{R}^2, \tag{2.4.1}$$

kwown as the *biharmonic equation*. In the elasticity case, u is the *Airy stress function*, whereas in the Stokes flow u is the *stream function* of the flow. The Airy function $W(x)$ is defined in terms of the stress tensor $\sigma_{ij}(\boldsymbol{u})$ for the displacement field $\boldsymbol{u}$ as

$$\sigma_{11}(\boldsymbol{u}) = \frac{\partial^2 W}{\partial x_2^2}, \quad \sigma_{12}(\boldsymbol{u}) = -\frac{\partial^2 W}{\partial x_1 \partial x_2}, \quad \sigma_{22}(\boldsymbol{u}) = \frac{\partial^2 W}{\partial x_1^2},$$

which satisfies the equilibrium equation $\mathrm{div}\boldsymbol{\sigma}(\boldsymbol{u}) = 0$ automatically for any smooth function W. Then from the stress-strain relation in the form of Hooke's law, it follows that

$$\Delta W = \sigma_{11}(\boldsymbol{u}) + \sigma_{22}(\boldsymbol{u}) = 2(\lambda + \mu)\mathrm{div}\boldsymbol{u}\,;$$

and thus, W satisfies (2.4.1) since $\Delta(\mathrm{div}\ \boldsymbol{u}) = \boldsymbol{0}$ from the Lamé system. On the other hand, the stream function u is defined in terms of the velocity $\boldsymbol{u}$ in the form

$$\boldsymbol{u} = (\nabla u)^{\perp}.$$

Here $\perp$ indicates the operation of rotating a vector counter–clockwise by a right angle. From the definition, the continuity equation for the velocity is satisfied for any choice of a smooth stream function u. One can verify directly that u satisfies (2.4.1) by taking the curl of the balance of momentum equation in the Stokes system. We note that in terms of the stream function, the vorticity is equal to $\omega \boldsymbol{k} = \nabla \times \boldsymbol{u} = \Delta u \boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector perpendicular to the $(x_1, x_2)-$ plane of the flow. For the homogeneous Stokes system, the vorticity is a harmonic function, and as a consequence, u satisfies the biharmonic equation (2.4.1).

To discuss boundary value problems for the biharmonic equation (2.4.1), it is best to begin with Green's formula for the equation in Ω. As is well known, for fourth-order differential equations, the Green formula generally varies and depends on the choice of boundary operators, i.e., how to apply the integration by parts formulae. In order to include boundary conditions arising for the thin plate, we rewrite $\Delta^2 u$ in terms of the Poisson ratio ν in the form

$$\Delta^2 u = \frac{\partial^2}{\partial x_1^2}\Big(\frac{\partial^2 u}{\partial x_1^2} + \nu \frac{\partial^2 u}{\partial x_2^2}\Big) + 2(1-\nu)\frac{\partial^2}{\partial x_1 \partial x_2}\Big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big) + \frac{\partial^2}{\partial x_2^2}\Big(\frac{\partial^2 u}{\partial x_2^2} + \nu \frac{\partial^2 u}{\partial x_1^2}\Big) .$$

Now integration by parts leads to the first Green formula in the form

$$\int\limits_{\Omega} (\Delta^2 u)\, v dx = a(u,v) - \int\limits_{\Gamma} \Big\{\frac{\partial v}{\partial n} Mu + vNu\Big\} ds \,, \tag{2.4.2}$$

where the bilinear form $a(u,v)$ is defined by

$$a(u,v) := \int\limits_{\Omega} \Big\{\nu \Delta u \;\Delta v + (1-\nu) \sum_{i,j=1}^{2} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\;\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)\Big\} dx \,; \tag{2.4.3}$$

and M and N are differential operators defined by

$$Mu := \nu \Delta u + (1-\nu)\big((\boldsymbol{n}(z)\cdot \nabla_x)^2 u)|_{z=x}\big) \,, \tag{2.4.4}$$

$$Nu := -\Big\{\frac{\partial}{\partial n}\Delta u + (1-\nu)\frac{d}{ds}\big((\boldsymbol{n}(z)\cdot\nabla_x)(\boldsymbol{t}(z)\cdot\nabla_x)u(x)\big)\Big\}_{|_{z=x}} \tag{2.4.5}$$

where $\boldsymbol{t} = \boldsymbol{n}^{\perp}$ is the unit tangent vector, i.e. $t_1 = -n_2$, $t_2 = n_1$. Then

$$Mu = \nu\Delta u + (1-\nu)\Big[\frac{\partial^2 u}{\partial x_1^2} n_1^2 + 2\frac{\partial^2 u}{\partial x_1 \partial x_2} n_1 n_2 + \frac{\partial^2 u}{\partial x_2^2} n_2^2\Big] \,,$$

$$Nu = -\frac{\partial}{\partial n}\Delta u + (1-\nu)\frac{d}{ds}\Big\{\Big(\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}\Big) n_1 n_2 - \frac{\partial^2 u}{\partial x_1 \partial x_2}(n_1^2 - n_2^2)\Big\} \,.$$

For the interior boundary value problems for (2.4.1), the starting point is the representation formula

$$\begin{aligned} u(x) \;\; = \;\; & \int\limits_{\Gamma} \{E(x,y) Nu(y) + \Big(\frac{\partial E}{\partial n_y}(x,y)\Big) Mu(y)\} ds_y \\ & - \int\limits_{\Gamma} \{\big(M_y E(x,y)\big)\frac{\partial u}{\partial n_y} + \big(N_y E(x,y)\big) u(y)\} ds_y \quad \text{for} \;\; x \in \Omega \,, \end{aligned} \tag{2.4.6}$$

where $E(x,y)$ is the fundamental solution for the biharmonic equation given by

$$E(x,y) = \frac{1}{8\pi}|x-y|^2 \log|x-y| \qquad (2.4.7)$$

which satisfies

$$\Delta_x^2 E(x,y) = \delta(x-y) \quad \text{in} \quad \mathbb{R}^2\,.$$

As in case of the Laplacian, we may rewrite u in the form

$$u(x) = V(Mu, Nu) - W\Big(u, \frac{\partial u}{\partial n}\Big) \qquad (2.4.8)$$

where

$$V(Mu,Nu) := \int\limits_{\Gamma} \{E(x,y)Nu(y) + \Big(\frac{\partial E}{\partial n_y}(x,y)\Big)Mu(y)\}ds_y\,, \quad (2.4.9)$$

$$W\Big(u, \frac{\partial u}{\partial n}\Big) := \int\limits_{\Gamma} \big(M_y E(x,y)\big)\frac{\partial u}{\partial n}(y) + \big(N_y E(x,y)\big)u(y)\}ds_y \quad (2.4.10)$$

are the simple and double layer potentials, respectively, and $u|_\Gamma, \frac{\partial u}{\partial n}|_\Gamma, Mu|_\Gamma$ and $Nu|_\Gamma$ are the (modified) Cauchy data. This representation formula (2.4.6) suggests two basic types of boundary conditions:
The **Dirichlet boundary condition**, where $u|_\Gamma$ and $\frac{\partial u}{\partial n}|_\Gamma$ are prescribed on Γ, and the **Neumann boundary condition**, where $Mu|_\Gamma$ and $Nu|_\Gamma$ are prescribed on Γ. In thin plate theory, where u stands for the deflection of the middle surface of the plate, the Dirichlet condition specifies the displacement and the angle of rotation of the plate at the boundary, whereas the Neumann condition provides the bending moment and shear force at the boundary. Clearly, various linear combinations will lead to other mixed boundary conditions, which will not be discussed here.

From the bilinear form (2.4.2), we see that

$$a(u,v) = 0 \quad \text{for} \quad v \in \mathcal{R} := \{v = c_1x_1 + c_2x_2 + c_3 \mid \quad \text{for all} \;\; c_1, c_2, c_3 \in \mathbb{R}\}\,. \qquad (2.4.11)$$

This implies that the Neumann data need to satisfy the compatibility condition

$$\int\limits_{\Gamma} \Big\{\frac{\partial v}{\partial n}Mu + vNu\Big\}ds = 0 \quad \text{for all} \;\; v \in \mathcal{R}\,. \qquad (2.4.12)$$

We remark that looking at (2.4.2), one might think of choosing Δu and $-\frac{\partial}{\partial n}\Delta u$ as the Neumann boundary conditions which correspond to the Poisson ratio $\nu = 1$. This means that the compatibility condition (2.4.12) requires that it should hold for all harmonic functions v. However, the space of harmonic functions in Ω has *infinite* dimension, and this does not lead to a regular boundary value problem in the sense of Agmon [2, p. 151].

As for the exterior boundary value problems, in order to ensure the uniqueness of the solution of (2.4.1) in Ω^c, we need to augment (2.4.1) with an appropriate radiation condition (see (2.3.18)). We require that

$$u(x) = \Big(A_0 r + \frac{\boldsymbol{A}_1 \cdot \boldsymbol{x}}{|x|}\Big)\, r \,\log r + O(r) \quad \text{as} \quad r = |x| \to \infty \tag{2.4.13}$$

for given constant A_0 and constant vector $\boldsymbol{A}_1$. Under the condition (2.4.9), we then have the representation formula for the solution of (2.4.1) in Ω^c,

$$u(x) = -V(Mu, Nu) + W\Big(u, \frac{\partial u}{\partial n}\Big) + p(x), \tag{2.4.14}$$

where $p \in \mathcal{R}$ is a polynomial of degree less than or equal to one.

Before we formulate the boundary integral equations we first summarize some classical basic results.

Theorem 2.4.1. (Gakhov [90], Mikhlin [208, 209, 211] and Muskhelishvili [223]). *Let $\Gamma \in C^{2,\alpha}$, $0 < \alpha < 1$.*
i) *Let*

$$\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^\top = \Big(u|_\Gamma \,,\, \frac{\partial u}{\partial n}|_\Gamma\Big) \in C^{3+\alpha}(\Gamma) \times \Gamma^{2+\alpha}(\Gamma)$$

be given. Then there exists a unique solution $u \in C^{3+\alpha}(\overline{\Omega}) \cap C^4(\Omega)$ of the interior Dirichlet problem satisfying the Dirichlet conditions

$$u|_\Gamma = \varphi_1 \quad \textit{and} \quad \frac{\partial u}{\partial n}|_\Gamma = \varphi_2 \,. \tag{2.4.15}$$

For given $A_0 \in \mathbb{R}$ and $\boldsymbol{A}_1 \in \mathbb{R}^2$ there also exists a unique solution $u \in C^{3+\alpha}(\Omega^c \cup \Gamma) \cap C^4(\Omega^c)$ of the exterior Dirichlet problem which behaves at infinity as in (2.4.13).
ii) *For given*

$$\boldsymbol{\psi} = (\psi_1, \psi_2)^\top \in C^{1+\alpha}(\Gamma) \times C^\alpha(\Gamma)$$

satisfying the compatibility conditions (2.4.12)*, i.e.,*

$$\int\limits_\Gamma \Big\{\psi_1 \frac{\partial v}{\partial n} + \psi_2 v\Big\} ds = 0 \quad \textit{for all} \;\; v \in \mathcal{R}\,, \tag{2.4.16}$$

the interior Neumann problem consisting of (2.4.1) *and the Neumann conditions*

$$Mu|_\Gamma = \psi_1 \quad \textit{and} \;\; Nu|_\Gamma = \psi_2 \tag{2.4.17}$$

has a solution $u \in C^{3+\alpha}(\overline{\Omega}) \cap C^4(\Omega)$ which is unique up to a linear function $p \in \mathcal{R}$.

If, for the exterior Neumann problem in Ω^c, in addition to $\boldsymbol{\psi}$ the linear function $p \in \mathcal{R}$ is given, then it has a unique solution $u \in C^{3+\alpha}(\Omega^c \cup \Gamma) \cap C^4(\Omega)$ with the behaviour (2.4.13)*,* (2.4.14) *where*

$$A_0 = -\int\limits_\Gamma \psi_2 ds \quad \textit{and} \quad \boldsymbol{A}_1 = \int\limits_\Gamma (\psi_1 \boldsymbol{n} + \psi_2 \boldsymbol{x}) ds\,. \tag{2.4.18}$$

As a consequence of Theorem 2.4.1 one has the useful identity of Gaussian type,

$$-W\Big(p, \frac{\partial p}{\partial n}\Big) = \begin{cases} p & \text{for} \quad x \in \Omega\,, \\ \frac{1}{2}p & \text{for} \quad x \in \Gamma\,, \\ 0 & \text{for} \quad x \in \Omega^c\,, \end{cases} \qquad \text{for any} \;\; p \in \mathcal{R}\,. \tag{2.4.19}$$

2.4.1 Calderón's Projector

(See also [144].) In order to obtain the boundary integral operators as x approaches Γ, from the simple– and double–layer potentials in the representation formulae (2.4.8) and (2.4.14), we need explicit information concerning the kernels of the potentials. A straightforward calculation gives

$$\begin{aligned} V(Mu, Nu)(x) &= \int\limits_\Gamma E(x,y) Nu(y) ds_y + \int\limits_\Gamma \Big(\frac{\partial E}{\partial n_y}(x,y)\Big) Mu(y) ds_y \\ &= \frac{1}{8\pi}\int\limits_\Gamma |x-y|^2 \log|x-y| Nu(y) ds_y \\ &\quad + \frac{1}{8\pi}\int\limits_\Gamma \boldsymbol{n}(y)\cdot(y-x)(2\log|x-y|+1) Mu(y) ds_y \end{aligned} \tag{2.4.20}$$

$$\begin{aligned} W\Big(u, \frac{\partial u}{\partial n}\Big)(x) &= \int\limits_\Gamma \big(M_y E(x,y)\big)\frac{\partial u(y)}{\partial n} ds_y + \int\limits_\Gamma \big(N_y E(x,y)\big) u(y) ds_y \\ &= \frac{1}{8\pi}\int\limits_\Gamma \Big\{(2\log|x-y|+1) + \nu(2\log|x-y|+3) \\ &\qquad + 2(1-\nu)\frac{((y-x)\cdot\boldsymbol{n}(y))^2}{|x-y|^2}\Big\}\frac{\partial u(y)}{\partial n} ds_y \\ &\quad + \frac{1}{2\pi}\int\limits_\Gamma \Big\{\frac{\partial}{\partial n_y}\log\Big(\frac{1}{|x-y|}\Big) \\ &\quad - \frac{1}{2}(1-\nu)\frac{d}{ds_y}\Big(\frac{(x-y)\cdot\boldsymbol{t}(y)(x-y)\cdot\boldsymbol{n}(y)}{|x-y|^2}\Big)\Big\} u(y) ds_y\,. \end{aligned} \tag{2.4.21}$$

This leads to the following 16 boundary integral operators.

$$
\begin{aligned}
\mathcal{K}\begin{pmatrix}\varphi\\ \sigma\end{pmatrix} &= \\
\lim_{\Omega\ni z\to x\in\Gamma} & \begin{pmatrix}
-W(\varphi_1,0)(z)-\frac{1}{2}\varphi_1(z) & -W(0,\varphi_2)(z) & V(\sigma_1,0)(z) & V(0,\sigma_2)(z)\\
-\boldsymbol{n}_x\cdot\nabla_z W(\varphi_1,0)(z) & -\boldsymbol{n}_x\cdot\nabla_z W(0,\varphi_2)(z)-\frac{1}{2}\varphi_2(x) & \boldsymbol{n}_x\cdot\nabla_z V(\sigma_1,0)(z) & \boldsymbol{n}_x\cdot\nabla_z V(0,\sigma_2)(z)\\
-M_z W(\varphi_1,0)(z) & -M_z W(0,\varphi_2)(z) & M_z V(\sigma_1,0)(z)-\frac{1}{2}\sigma_1(x) & M_z V(0,\sigma_2)(z)\\
-N_z W(\varphi_1,0)(z) & -N_z W(0,\varphi_2)(z) & N_z V(\sigma_1,0)(z) & N_z V(0,\sigma_2)(z)-\frac{1}{2}\sigma_2(x)
\end{pmatrix}\\
&= \begin{pmatrix}
-K_{11} & V_{12} & V_{13} & V_{14}\\
D_{21} & K_{22} & V_{23} & V_{24}\\
D_{31} & D_{32} & -K_{33} & V_{34}\\
D_{41} & D_{42} & D_{43} & K_{44}
\end{pmatrix}
\end{aligned}
\tag{2.4.22}
$$

where we write

$$
\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\sigma}\end{pmatrix} = (\varphi_1,\varphi_2,\sigma_1,\sigma_2)^\top = \Big(u, \frac{\partial u}{\partial n}, Mu, Nu\Big)^\top|_\Gamma\,.
$$

Then the Calderon projector associated with the bi–Laplacian is defined by

$$
\mathcal{C}_\Omega := \frac{1}{2}\mathcal{I}+\mathcal{K} = \begin{pmatrix}
\frac{1}{2}I-K_{11} & V_{12} & V_{13} & V_{14}\\
D_{21} & \frac{1}{2}I+K_{22} & V_{23} & V_{24}\\
D_{31} & D_{32} & \frac{1}{2}I-K_{33} & V_{34}\\
D_{41} & D_{42} & D_{43} & \frac{1}{2}I+K_{44}
\end{pmatrix}
\tag{2.4.23}
$$

Some more explanations are needed here. In order to maintain consistency with our notations for the Laplacian, we have adopted the notations V_{ij}, K_{ij} and D_{ij} for the weakly and hypersingular boundary integral operators according to our terminology. These boundary integral operators are obtained by taking limits of the operations $\nabla_z(\bullet)\cdot n_x, M_z, N_z$, respectively on the corresponding potentials V and W as $\Omega \ni z \to x \in \Gamma$. As in the case of the Laplacian, for any solution of (2.4.1), the Cauchy data $(u, \frac{\partial u}{\partial n}, Mu, Nu)_\Gamma$ on Γ are reproduced by the matrix operators in (2.4.23), and $\mathcal{C}_\Omega$ is the Calderón projector corresponding to the bi–Laplacian. In the classical Hölder function spaces, we have the following lemma.

Lemma 2.4.2. *Let $\Gamma \in C^{2,\alpha}$, $0<\alpha<1$. Then $\mathcal{C}_\Omega$ maps $\prod_{k=0}^{3} C^{3+\alpha-k}(\Gamma)$ into itself continuously. Moreover,*

$$\mathcal{C}_\Omega^2 = \mathcal{C}_\Omega\,. \tag{2.4.24}$$

As a consequence of this lemma, one finds the following specific identities:

$$\begin{aligned}
V_{12}D_{21}+V_{13}D_{31}+V_{14}D_{41} &= (\tfrac{1}{4}I-K_{11}^2),\\
D_{21}V_{12}+V_{23}D_{32}+V_{24}D_{42} &= (\tfrac{1}{4}I-K_{22}^2),\\
D_{31}V_{13}+D_{32}V_{23}+V_{34}D_{43} &= (\tfrac{1}{4}I-K_{33}^2),\\
D_{41}V_{14}+D_{42}V_{24}+D_{43}V_{34} &= (\tfrac{1}{4}I-K_{44}^2).
\end{aligned}$$

Clearly, from (2.4.24) one finds 12 more identities between these operators.

In the same manner as in the case for the Laplacian, for any solution u of (2.4.1) in Ω^c with $p=0$, we may introduce the Calderón projection C_{Ω^c} for the exterior domain for the biharmonic equation. Then clearly, we have

$$\mathcal{C}_{\Omega^c} = \mathcal{I} - \mathcal{C}_\Omega\,,$$

where $\mathcal{I}$ denotes the identity matrix operator. This relation then provides the corresponding boundary integral equations for exterior boundary value problems. As will be seen, the boundary integral operators in $\mathcal{C}_\Omega$ are pseudodifferential operators on Γ and their orders are summarized systematically in the following:

$$Ord(\mathcal{C}_\Omega) := \begin{pmatrix} 0 & -1 & -3 & -3\\ +1 & 0 & -1 & -3\\ +1 & +1 & 0 & -1\\ +3 & +1 & +1 & 0 \end{pmatrix} \tag{2.4.25}$$

The orders of these operators can be calculated from their symbols and provide the mapping properties in the Sobolev spaces to be discussed in Chapter 10.

2.4.2 Boundary Value Problems and Boundary Integral Equations

We begin with the boundary integral equations for the Dirichlet problems. For the integral equations of the first kind we employ the second and the first row of $\mathcal{C}_\Omega$ which leads to the following system for the *interior Dirichlet problem*,

$$\boldsymbol{V}\boldsymbol{\sigma} := \begin{pmatrix} V_{23} & V_{24} \\ V_{13} & V_{14} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -D_{21} & \frac{1}{2}I - K_{22} \\ \frac{1}{2}I + K_{11} & -V_{12} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} =: \boldsymbol{f}_i \,. \tag{2.4.26}$$

The solution of the interior Dirichlet problem has associated Cauchy data $\boldsymbol{\sigma}$ which satisfy the three compatibility conditions:

$$\int_\Gamma (\sigma_1 \boldsymbol{n} + \sigma_2 \boldsymbol{x}) ds_x = \boldsymbol{0}\,, \; -\int_\Gamma \sigma_2 ds = 0\,. \tag{2.4.27}$$

As we shall see in Chapter 10 , $\boldsymbol{V}$ is known as a strongly elliptic operator for which the classical Fredholm alternative holds. Hence uniqueness will imply the existence of exactly one solution $\boldsymbol{\sigma} \in C^{1+\alpha}(\Gamma) \times C^\alpha(\Gamma)$.

For the exterior Dirichlet problem, by using $\mathcal{C}_{\Omega^c}$ and the representation (2.4.13) we obtain the system with integral equations of the first kind,

$$\begin{aligned} \boldsymbol{V}\boldsymbol{\sigma} + R\boldsymbol{\omega} &= -\begin{pmatrix} D_{21} & \frac{1}{2}I + K_{22} \\ \frac{1}{2}I - K_{11} & V_{12} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} =: \boldsymbol{f}_e \,, \\ \int_\Gamma (\sigma_1 \boldsymbol{n} + \sigma_2 \boldsymbol{x}) &= \boldsymbol{A}_1\,, \; -\int_\Gamma \sigma_2 ds = A_0 \end{aligned} \tag{2.4.28}$$

where

$$R(x) = -\begin{pmatrix} 0 & n_1 & n_2 \\ 1 & x_1 & x_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2)^\top \in \mathbb{R}^3 \,.$$

Lemma 2.4.3. *The homogeneous system corresponding to* (2.4.28) *has only the trivial solution in* $C^{1+\alpha}(\Gamma) \times \mathbb{C}^\alpha(\Gamma) \times \mathbb{R}^3$.

Proof: Let $\boldsymbol{\sigma}_0, \boldsymbol{\omega}_0$ be any solution of

$$\boldsymbol{V}\boldsymbol{\sigma}_0 + R\boldsymbol{\omega}_0 = 0 \quad \text{on} \;\; \Gamma\,, \; \int_\Gamma \Big(\frac{\partial v}{\partial n}, v\Big)\boldsymbol{\sigma}_0 ds = 0 \quad \text{for all} \;\; v \in \Re \tag{2.4.29}$$

and consider the solution of (2.4.1),

$$u_0(x) := V\boldsymbol{\sigma}_0(x) + p_0(x) \quad \text{with} \;\; p_0(x) = \mathring{\omega}_0 + \mathring{\omega}_1 x_1 + \mathring{\omega}_2 x_2 \quad \text{for} \;\; x \in \Omega^c \,.$$

Then $A_0 = 0\,,\; \boldsymbol{A}_1 = \boldsymbol{0}$ because of (2.4.29) and (2.4.18), hence $u_0 = O(|x|)$at infinity due to (2.4.13) which implies $u_0(x) = p_0(x)$ for all $x \in \Omega^c \cup \Gamma$. On the other hand, $u_0(x)$ is also a solution of (2.4.1) in Ω and is continuous across Γ where $u_0|_\Gamma = 0$. Hence, due to Theorem 2.4.1, $u_0(x) = 0$ for all x in Ω. Consequently, $Mu_0^\pm|_\Gamma = 0$ and $Nu_0^\pm|_\Gamma = 0$. Then the jump relations corresponding to $\mathcal{C}_{\Omega^c} - \mathcal{C}_\Omega$ imply $\boldsymbol{\sigma}_0 = ([Mu]|_\Gamma\,,\; [Nu]|_\Gamma)^\top = \boldsymbol{0}$ on Γ and $0 = u_0^-|_\Gamma = u_0^+|_\Gamma = p_0$ implies $p_0(x) = 0$ for all x, i.e., $\mathring{\boldsymbol{\omega}} = 0$. ∎

As a consequence, both, interior and exterior Dirichlet problems lead to the same uniquely solvable system (2.4.28) where only the right–hand sides are different and, for the interior Dirichlet problem, $\boldsymbol{\omega} = 0$.

Clearly, the solution of the Dirichlet problems can also be treated by using the boundary integral equations of the second kind. To illustrate the idea we consider again the interior Dirichlet problem where $u|_\Gamma = \varphi_1$ and $\frac{\partial u}{\partial n}|_\Gamma = \varphi_2$. From the representation formula (2.4.6) we obtain the following system for the unknown $\boldsymbol{\sigma} = (Mu, Nu)^\top$ on Γ:

$$\begin{pmatrix} \frac{1}{2}I + K_{33} & -V_{34} \\ -D_{43} & \frac{1}{2}I - K_{44} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} D_{31} & D_{32} \\ D_{41} & D_{42} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} =: \boldsymbol{D\varphi}\,. \tag{2.4.30}$$

This system (2.4.30) of integral equations has a unique solution. As we shall see in Chapter 10, for $0 \le \nu < 1$ the Fredholm alternative is still valid for these integral equations and $\boldsymbol{\sigma} \in C^{1+\alpha}(\Gamma) \times C^{\alpha}(\Gamma)$. So, uniqueness implies existence.

Lemma 2.4.4. *Let* $\overset{\circ}{\boldsymbol{\sigma}} \in C^{\alpha}(\Gamma) \times C^{1+\alpha}(\Gamma)$ *be the solution of the homogeneous system*

$$\begin{aligned} (\tfrac{1}{2}I + K_{33})\overset{\circ}{\sigma}_1 - V_{34}\overset{\circ}{\sigma}_2 = 0 \\ -D_{34}\overset{\circ}{\sigma}_1 + (\tfrac{1}{2}I - K_{44})\overset{\circ}{\sigma}_2 = 0 \quad on \ \ \Gamma\,. \end{aligned} \tag{2.4.31}$$

Then $\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{0}$.

Proof: For the proof we consider the simple layer potential

$$u_0(x) = V\overset{\circ}{\boldsymbol{\sigma}}$$

which is a solution of (2.4.1) for $x \notin \Gamma$. Then for $x \in \Omega$ we obtain with (2.4.31):

$$\begin{aligned} Mu_0^-|_\Gamma &= (\tfrac{1}{2}I - K_{33})\overset{\circ}{\sigma}_1 + V_{34}\overset{\circ}{\sigma}_2 &= \overset{\circ}{\sigma}_1\,, \\ Nu_0^-|_\Gamma &= (\tfrac{1}{2}I + K_{44})\overset{\circ}{\sigma}_2 + D_{43}\overset{\circ}{\sigma}_1 &= \overset{\circ}{\sigma}_2\,. \end{aligned}$$

Then the Green formula (2.4.2) implies

$$\int\limits_\Gamma \Big(\overset{\circ}{\sigma}_1 \frac{\partial v}{\partial u} + \overset{\circ}{\sigma}_2 v\Big) ds = 0 \quad \text{for all} \ \ v \in \Re\,. \tag{2.4.32}$$

For $x \in \Omega^c$, we find $Mu_0^+|_\Gamma = 0$ and $Nu_0^+|_\Gamma = 0$ due to (2.4.31). Then Theorem 2.4.1 implies with (2.4.32) that

$$u_0(x) = p(x) \quad \text{for} \ \ x \in \Omega^c \cup \Gamma \quad \text{with some} \ \ p \in \Re\,.$$

But $u_0(x)$ is continuously differentiable across Γ and satisfies (2.4.1) in Ω with boundary conditions $u_0^-|_\Gamma = p|_\Gamma$ and $\frac{\partial u_0^-}{\partial n}|_\Gamma = \frac{\partial p}{\partial n}|_\Gamma$. Hence, with Theorem 2.4.1 we find

$$V\overset{\circ}{\boldsymbol{\sigma}}(x) = u_0(x) = p(x) \quad \text{for} \;\; x \in \mathbb{R}^2\,.$$

Then

$$\overset{\circ}{\sigma}_1 = [Mu_0]|_\Gamma = 0 \quad \text{and} \;\; \overset{\circ}{\sigma}_2 = [Nu_0]|_\Gamma = 0\,.$$

■

We now conclude this section by summarizing the boundary integral equations associated with the two boundary value problems of the biharmonic equation considered here in the following Tables 2.4.5 and 2.4.6. However, the missing details will not be pursued here. We shall return to these equations in later chapters.

We remark that in Table 2.4.5 we did not include orthogonality conditions for the right–hand sides in the equations INP (1) and (2), EDP (2) and ENP (1) since due to the direct approach it is known that the right–hand sides always lie in the range of the operators. Hence, we know that the solutions exist due to the basic results in Theorem 2.4.1, and, moreover, the classical Fredholm alternative holds for the systems in Table 2.4.5. From this table we now consider the modified systems so that the latter will always be uniquely solvable. The main idea here is to incorporate additional side conditions as well as eigensolutions. These modifications are collected in Table 2.4.6. In particular, we have augmented the systems by including additional unknowns $\boldsymbol{\omega} \in \mathbb{R}^3$ in the same manner as in Section 2.2 for the Lamé system. Note that in Table 2.4.6 the matrix valued function S is defined by

$$S(x) := \big(\overset{\circ}{\boldsymbol{\sigma}}{}^1(x), \overset{\circ}{\boldsymbol{\sigma}}{}^2(x), \overset{\circ}{\boldsymbol{\sigma}}{}^3(x)\big)$$

where the columns of S are the three linearly independent eigensolutions of the operator on the left–hand side of EDP (2) in Table 2.4.5. If we solve the exterior Neumann problem with the system ENP (1) in Table 2.4.6, then we obtain a particular solution with $p(x) = 0$ in Ω^c, and for given $p(x) \neq 0$, the latter is to be added to the representation formula (2.4.14). For the interior Neumann problem, the modified boundary integral equation INP (1) and (2) provide a particular solution which presents the general solution only up to linear polynomials.

Note that here we needed $\Gamma \in C^{2,\alpha}$ and even jumps of the curvature are excluded. For piecewise $\Gamma \in C^{2,\alpha}$–boundary, Green's formula, the representation formula as well as the boundary integral equations need to be modified appropriately by including certain functionals at the discontinuity points (Knöpke [160]).

Table 2.4.5. Boundary Integral Equations for the Biharmonic Equation

	Systems of BIEs	Side conditions	Eigenspaces
IDP	(1) $\begin{pmatrix} V_{23} & V_{24} \\ V_{13} & V_{14} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -D_{21} & \frac{1}{2}I - K_{22} \\ \frac{1}{2}I + K_{11} & -V_{12} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$	$\int_\Gamma \left(\sigma_1 \frac{\partial v}{\partial n} + \sigma_2 v\right) ds = 0$ * for all $v \in \Re$	none for $\overset{\circ}{\boldsymbol{\sigma}}$ satisfying *
	(2) $\begin{pmatrix} \frac{1}{2}I + K_{33} & -V_{34} \\ -D_{43} & \frac{1}{2}I - K_{44} \end{pmatrix} = \begin{pmatrix} D_{31} & D_{32} \\ D_{41} & D_{42} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$	none	none
INP	(1) $\begin{pmatrix} D_{41} & D_{42} \\ D_{31} & D_{32} \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \begin{pmatrix} -D_{43} & \frac{1}{2}I - K_{44} \\ \frac{1}{2}I + K_{33} & -V_{34} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$	$\int_\Gamma \left(\psi_1 \frac{\partial v}{\partial n} + \psi_2 v\right) ds = 0$ for all $v \in \Re$	$\begin{pmatrix} p \\ \frac{\partial p}{\partial n} \end{pmatrix}$, $p \in \Re$
	(2) $\begin{pmatrix} \frac{1}{2}I + K_{11} & -V_{12} \\ -D_{21} & \frac{1}{2}I - K_{22} \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \begin{pmatrix} V_{13} & V_{14} \\ V_{23} & V_{24} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$		
EDP	(1) $\begin{pmatrix} V_{23} & V_{24} \\ V_{13} & V_{14} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} - \begin{pmatrix} \frac{\partial}{\partial n} p \\ p \end{pmatrix} = -\begin{pmatrix} D_{21} & \frac{1}{2}I + K_{22} \\ \frac{1}{2}I - K_{11} & V_{12} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$	$-\int_\Gamma \sigma_2 ds = A_0$ $\int_\Gamma (\sigma_1 \boldsymbol{n} + \sigma_2 \boldsymbol{x}) ds = \boldsymbol{A}_1$.	none for $\overset{\circ}{\boldsymbol{\sigma}}$ satisfying *
	(2) $\begin{pmatrix} \frac{1}{2}I - K_{33} & V_{34} \\ D_{43} & \frac{1}{2}I + K_{44} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = -\begin{pmatrix} D_{31} & D_{32} \\ D_{41} & D_{42} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$	$A_0, \boldsymbol{A}_1$ given	$\overset{\circ}{\boldsymbol{\sigma}}{}^1, \overset{\circ}{\boldsymbol{\sigma}}{}^2, \overset{\circ}{\boldsymbol{\sigma}}{}^3$
ENP	(1) $\begin{pmatrix} D_{41} & D_{42} \\ D_{31} & D_{32} \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = -\begin{pmatrix} D_{43} & \frac{1}{2}I + K_{44} \\ \frac{1}{2}I - K_{33} & V_{34} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$	none, $p(x)$ given	$\begin{pmatrix} p \\ \frac{\partial p}{\partial n} \end{pmatrix}$, $p \in \Re$
	(2) $\begin{pmatrix} \frac{1}{2}I - K_{11} & V_{12} \\ D_{21} & \frac{1}{2}I + K_{22} \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = -\begin{pmatrix} V_{13} & V_{14} \\ V_{23} & V_{24} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} p(x) \\ \frac{\partial}{\partial n} p(x) \end{pmatrix}$		none

Table 2.4.6. Modified Systems for the Biharmonic Equation

BVP	
IDP	(1) $\begin{pmatrix}V_{23} & V_{24}\\ V_{13} & V_{14}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} + R\boldsymbol{\omega} = \begin{pmatrix}-D_{21} & \frac{1}{2}I - K_{22}\\ \frac{1}{2}I + K_{11} & -V_{12}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ $\int\limits_\Gamma \sigma_2 ds = 0\,,\ \int\limits_\Gamma (\sigma_1\boldsymbol{n} + \sigma_2\boldsymbol{x})ds_x = 0$
	(2) $\begin{pmatrix}\frac{1}{2}I + K_{33} & -V_{34}\\ -D_{43} & \frac{1}{2}I - K_{44}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}D_{31} & D_{32}\\ D_{41} & D_{42}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$
INP	(1) $\begin{pmatrix}D_{41} & D_{42}\\ D_{31} & D_{32}\end{pmatrix}\begin{pmatrix}u\\ \frac{\partial u}{\partial n}\end{pmatrix} + R\boldsymbol{\omega} = \begin{pmatrix}-D_{43} & \frac{1}{2}I - K_{44}\\ \frac{1}{2}I + K_{33} & -V_{34}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ $\int\limits_\Gamma uds = 0\,,\ \int\limits_\Gamma \left(ux_1 + n_1\frac{\partial u}{\partial n}\right)ds = 0\,,\ \int\limits_\Gamma \left(ux_2 + n_2\frac{\partial u}{\partial n}\right)ds = 0$
	(2) $\begin{pmatrix}\frac{1}{2}I + K_{11} & -V_{12}\\ -D_{21} & \frac{1}{2}I - K_{22}\end{pmatrix}\begin{pmatrix}u\\ \frac{\partial u}{\partial n}\end{pmatrix} + S\boldsymbol{\omega} = \begin{pmatrix}V_{13} & V_{14}\\ V_{23} & V_{24}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ $\int\limits_\Gamma uds = 0\,,\ \int\limits_\Gamma \left(ux_1 + n_1\frac{\partial u}{\partial n}\right)ds = 0\,,\ \int\limits_\Gamma \left(ux_2 + n_2\frac{\partial u}{\partial n}\right)ds = 0$
EDP	(1) $\begin{pmatrix}V_{23} & V_{24}\\ V_{13} & V_{14}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} + R\boldsymbol{\omega} = -\begin{pmatrix}D_{21} & \frac{1}{2}I + K_{22}\\ \frac{1}{2}I - K_{11} & V_{12}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ $-\int\limits_\Gamma \sigma_2 ds = A_0\,,\ \int\limits_\Gamma (\sigma_1\boldsymbol{n} + \sigma_2\boldsymbol{x})ds = \boldsymbol{A}_1$
	(2) $\begin{pmatrix}\frac{1}{2}I - K_{33} & V_{34}\\ D_{43} & \frac{1}{2}I + K_{44}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} + \begin{pmatrix}1 & x_1 & x_2\\ 0 & n_1 & n_2\end{pmatrix}\boldsymbol{\omega}$ $= -\begin{pmatrix}D_{31} & D_{32}\\ D_{41} & D42\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\ -\int\limits_\Gamma \sigma_2 ds = A_0\,,\ \int\limits_\Gamma (\sigma_1\boldsymbol{n} + \sigma_2\boldsymbol{x})ds = \boldsymbol{A}_1$
ENP	(1) $\begin{pmatrix}D_{41} & D_{42}\\ D_{31} & D_{32}\end{pmatrix}\begin{pmatrix}u_p\\ \frac{\partial u_p}{\partial n}\end{pmatrix} + R\boldsymbol{\omega} = -\begin{pmatrix}D_{43} & \frac{1}{2}I + K_{44}\\ \frac{1}{2}I - K_{33} & V_{34}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ $\int\limits_\Gamma u_p ds = 0\,,\ \int\limits_\Gamma \left(u_p n_1 + \frac{\partial u_p}{\partial n}x_1\right)ds = 0\,,\ \int\limits_\Gamma \left(u_p n_2 + \frac{\partial u_p}{\partial n}x_2\right)ds = 0$
	(2) $\begin{pmatrix}\frac{1}{2}I - K_{11} & V_{12}\\ D_{21} & \frac{1}{2}I + K_{22}\end{pmatrix}\begin{pmatrix}u\\ \frac{\partial u}{\partial n}\end{pmatrix} = -\begin{pmatrix}V_{13} & V_{14}\\ V_{23} & V_{24}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + \begin{pmatrix}p\\ \frac{\partial}{\partial n}p\end{pmatrix}$

2.5 Remarks

Very often in applications, on different parts of the boundary, different boundary conditions are required or, as in classical crack mechanics (Cruse [58]), the boundaries are given as transmission conditions on some bounded manifold, the crack surface, in the interior of the domain. A similar situation can be found for screen problems (see also Costabel and Dauge [52] and Stephan [293]).

As an example of *mixed boundary conditions* let us consider the Lamé system with given Dirichlet data on $\Gamma_D \subset \Gamma$ and given Neumann data on $\Gamma_N \subset \Gamma$ where $\Gamma = \Gamma_D \cup \Gamma_N \cup \gamma$ with the set of collision points γ of the two boundary conditions (which might also be empty if Γ_D and Γ_N are separated components of Γ) (see e.g. Fichera [76], [145], Kohr et al [164], Maz'ya [202], Stephan [295]) where meas $(\Gamma_D) > 0$.

For $n = 2$, where Γ is a closed curve, we assume that either $\gamma = \emptyset$ or consists of finitely many points; for $n = 3$, the set γ is either empty or a closed curve and as smooth as Γ. For the Lamé system (2.2.1) the classical mixed boundary value problem reads:

Find $u \in C^2(\Omega) \cap C^\alpha(\overline{\Omega})\,,\ 0 < \alpha < 1$, *satisfying*

$$\begin{aligned} -\Delta^* \boldsymbol{u} = \boldsymbol{f} \qquad & \textit{in } \ \Omega \qquad \textit{with} \\ \gamma_0 \boldsymbol{u} = \varphi_D \quad \textit{on } \ \Gamma_D \quad & \textit{and } \ Tu = \boldsymbol{\psi}_N \quad \textit{on } \ \Gamma_N\,. \end{aligned} \tag{2.5.1}$$

For reformulating this problem with boundary integral equations we first extend $\boldsymbol{\varphi}_D$ from Γ_D and $\boldsymbol{\psi}_N$ from Γ_N onto the complete boundary Γ such that

$$\boldsymbol{\varphi}_D = \boldsymbol{\varphi}|_{\Gamma_D} \quad \text{and} \quad \boldsymbol{\psi}_N = \boldsymbol{\psi}|_{\Gamma_N} \tag{2.5.2}$$

with $\boldsymbol{\varphi} \in C^\alpha(\Gamma)\,,\ 0 < \alpha < 1$ and appropriate $\boldsymbol{\psi}$. Then

$$\gamma_0 \boldsymbol{u} = \boldsymbol{\varphi} + \widetilde{\boldsymbol{\varphi}}\,,\ T\boldsymbol{u} = \boldsymbol{\psi} + \widetilde{\boldsymbol{\psi}} \tag{2.5.3}$$

where now

$$\widetilde{\boldsymbol{\varphi}} \in C_0^\alpha(\Gamma_N) = \{\boldsymbol{\varphi} \in C^\alpha(\Gamma) \,|\, \operatorname{supp} \boldsymbol{\varphi} \subset \Gamma_N\} \tag{2.5.4}$$

and $\widetilde{\boldsymbol{\psi}}$ with supp $\widetilde{\psi} \subset \overline{\Gamma}_D$ are the yet unknown Cauchy data to be determined. With (2.5.3), the representation formula (2.2.4) reads

$$\begin{aligned} \boldsymbol{v}(x) = & \int\limits_\Gamma E(x,y)\boldsymbol{\psi}(y)ds_y - \int\limits_\Gamma \big(T_y E(x,y)\big)^\top \boldsymbol{\varphi}(y)ds_y \\ & + \int\limits_\Gamma E(x,y)\widetilde{\boldsymbol{\psi}}(y)ds_y - \int\limits_\Gamma \big(T_y E(x,y)\big)^\top \widetilde{\boldsymbol{\varphi}}(y)ds_y \\ & + \int\limits_\Omega E(x,y)\boldsymbol{f}(y)dy \quad \text{for } \ x \in \Omega\,. \end{aligned} \tag{2.5.5}$$

As is well known, even if $\boldsymbol{\psi} \in C^\alpha(\Gamma)$ then $\widetilde{\boldsymbol{\psi}}$ will have singularities at γ which need to be taken into account either by $\{\text{dist }(x,\gamma)\}^{-\frac{1}{2}}\widetilde{\boldsymbol{\psi}}_1$ with $\widetilde{\boldsymbol{\psi}}_1 \in C^\alpha(\Gamma_D)$ or by adding singular functions at γ.

Taking the trace and the traction of (2.5.5) on Γ leads with (2.5.3) to the system of boundary integral equations

$$\begin{aligned} V\widetilde{\boldsymbol{\psi}}(x) - K\widetilde{\boldsymbol{\varphi}}(x) &= \tfrac{1}{2}\boldsymbol{\varphi}(x) + K\boldsymbol{\varphi}(x) - V\boldsymbol{\psi}(x) - N\boldsymbol{f}(x) \qquad \text{for } x \in \Gamma_D\,, \\ K'\widetilde{\boldsymbol{\psi}}(x) + D\widetilde{\boldsymbol{\varphi}}(x) &= \tfrac{1}{2}\boldsymbol{\psi}(x) - K'\boldsymbol{\psi}(x) - D\boldsymbol{\varphi}(x) - T_x N\boldsymbol{f}(x) \quad \text{for } x \in \Gamma_N\,. \end{aligned} \tag{2.5.6}$$

As will be seen in Chapter 5, the system (2.5.6) is uniquely solvable for $\widetilde{\varphi} \in C_0^\alpha(\Gamma_N)$ and $\widetilde{\psi}$ either in the space with the weight $\{\text{dist }(x,\gamma)\}^{-\frac{1}{2}}$ or in an augmented space according to the asymptotic behaviour of the solution and involving the stress intensity factors (Stephan et al [297] provided $\text{meas}(\Gamma_D) > 0$.

In a similar manner one might also use the system of integral equations of the second kind

$$\begin{aligned} \tfrac{1}{2}\widetilde{\boldsymbol{\varphi}}(x) + K\widetilde{\boldsymbol{\varphi}}(x) - V\widetilde{\boldsymbol{\psi}}(x) &= V\boldsymbol{\psi}(x) - \tfrac{1}{2}\boldsymbol{\varphi}(x) - K\boldsymbol{\varphi}(x) + N\boldsymbol{f}(x) \\ &\qquad \text{for } x \in \Gamma_N\,, \\ \tfrac{1}{2}\widetilde{\boldsymbol{\psi}}(x) - K'\widetilde{\boldsymbol{\psi}}(x) - D\widetilde{\boldsymbol{\varphi}}(x) &= -\tfrac{1}{2}\boldsymbol{\psi}(x) + K'\boldsymbol{\psi}(x) + D\boldsymbol{\varphi}(x) + T_x N\boldsymbol{f}(x) \\ &\qquad \text{for } x \in \Gamma_D\,. \end{aligned} \tag{2.5.7}$$

For the Laplacian and the Helmholtz equation and mixed boundary value problems as well as for the Stokes system one may proceed in the same manner. As will be seen in Chapter 5, the variational formulation for the mixed boundary conditions provides us with the right analytical tools for showing the well–posedness of the formulation (2.5.6) (see e.g., Kohr et al [164], Sauter and Schwab [266] and Steinbach [290]). In the engineering literature, usually the system (2.5.7) is used for discretization and then the equations corresponding to (2.5.7) are obtained by assembling the discrete given and unknown Cauchy data appropriately (see e.g., Bonnet [18], Brebbia et al [23, 24] and Gaul et al [94]).

For crack and insertion problems let us again consider just the example of classical linear theory without volume forces. Let us consider a bounded open domain $\Omega \subset \mathbb{R}^n$ with $n = 2$ or 3 enclosing a given bounded crack or insertion surface as an oriented piece of a curve $\Gamma_c \in C^\alpha$, if $n = 2$ or, if $n = 3$, as an open piece of an oriented surface $\Gamma_c \in C^\alpha$, with a simple, closed boundary curve $\partial\Gamma_c = \gamma \in C^\alpha$. Further the crack should not reach the boundary $\partial\Omega = \Gamma$ of Ω, i.e., $\overline{\Gamma}_c \subset \Omega$. The annulus $\Omega_c := \Omega \setminus F_c$ is not a Lipschitz domain anymore but if we distinguish the two sides of Γ_c assigning with $+$ the points near Γ_c on the side of the normal vector $\boldsymbol{n}_c$ given due to the orientation of Γ_c and the points of the opposite side with $-$, the traces from either side are still defined. For the crack or insertion problem, an elastic field $\boldsymbol{u} \in C^2(\Omega_c)$ is sought which satisfies the homogeneous Lamé system

$$-\Delta^* \boldsymbol{u} = \boldsymbol{0} \quad \text{in} \quad \Omega_c\,, \tag{2.5.8}$$

in $C^\alpha(\Omega_c \cup \Gamma)$ and up to Γ_c from either side with possibly different traces at Γ_c,

$$\gamma_0^\pm \boldsymbol{u}|_{\Gamma_c} = \boldsymbol{\varphi}^\pm \quad \text{and} \quad T_c^\pm \boldsymbol{u} = \boldsymbol{\psi}^\pm \tag{2.5.9}$$

where we have the transmission properties

$$[\gamma_0 \boldsymbol{u}]|_{\Gamma_c} := (\gamma_0^+ \boldsymbol{u} - \gamma_0^- \boldsymbol{u})|_{\Gamma_c} = [\boldsymbol{\varphi}]|_{\Gamma_c} = (\boldsymbol{\varphi}^+ - \boldsymbol{\varphi}^-)|_{\Gamma_c} \in C_0^\alpha(\Gamma_c) \tag{2.5.10}$$

and

$$[T_c \boldsymbol{u}]|_{\Gamma_c} := (T_c^+ \boldsymbol{u} - T_c^- \boldsymbol{u})|_{\Gamma_c} = [\boldsymbol{\psi}]|_{\Gamma_c} := (\boldsymbol{\psi}^+ - \boldsymbol{\psi}^-)|_{\Gamma_c} \in C_1^\alpha(\Gamma_c) \tag{2.5.11}$$

with

$$C_0^\alpha(\Gamma_c) := \{\boldsymbol{v} \in C^\alpha(\overline{\Gamma}_c) \,|\, (\gamma_0^+ \boldsymbol{v} - \gamma_0^- \boldsymbol{v})|_\gamma = \boldsymbol{0}\}\,, \tag{2.5.12}$$

and

$$C_1^\alpha(\Gamma_c) := \{\boldsymbol{\psi} = \{\text{dist } (x-\gamma)\}^{-\frac{1}{2}} \boldsymbol{\psi}_1(x) \,|\, \boldsymbol{\psi}_1 \in C^\alpha(\overline{\Gamma}_c)\}\,. \tag{2.5.13}$$

For the classical *insertion problem* with Dirichlet conditions $\gamma_0 \boldsymbol{u} = \boldsymbol{\varphi} \in C^\alpha(\Gamma)$ on Γ the functions $\boldsymbol{\varphi}^\pm \in C_0^\alpha(\Gamma_c)$ are given. The unknown field $\boldsymbol{u}$ then has to satisfy the boundary conditions

$$\begin{aligned} \gamma_0 \boldsymbol{u}|_\Gamma = \boldsymbol{\varphi} \quad \text{on} \quad \Gamma \quad \text{and with} \quad (\boldsymbol{\varphi}^+ - \boldsymbol{\varphi}^-)|_\gamma = \boldsymbol{0}\,, \\ \gamma_0^+ \boldsymbol{u}|_{\Gamma_c} = \boldsymbol{\varphi}^+ \quad \text{and} \quad \gamma_0^- \boldsymbol{u}|_{\Gamma_c} = \boldsymbol{\varphi}^- \quad \text{on} \quad \Gamma_c \end{aligned} \tag{2.5.14}$$

as well as the transmission conditions (2.5.10) and (2.5.11).

By extending Γ_c up to the boundary Γ ficticiously and applying the Green formula to the two ficticiously separated subdomains of Ω one finds the representation formula

$$\begin{aligned} \boldsymbol{u}(x) = & \int_\Gamma E(x,y)\boldsymbol{\psi}(y)ds_y - \int_\Gamma \big(T_y E(x,y)\big)^\top \boldsymbol{\varphi}(y)ds_y \\ & - \int_{y\in\Gamma_c} E(x,y)[\boldsymbol{\psi}]|_{\Gamma_c}(y)ds_y + \int_{\Gamma_c} \big(T_y^c E(x,y)\big)^\top [\boldsymbol{\varphi}]|_{\Gamma_c}(y)ds_y \end{aligned} \tag{2.5.15}$$

for $x \in \Omega_c$.

By taking the traces of (2.5.15) at Γ and at Γ_c one obtains the following system of equations on Γ and on Γ_c:

$$
\begin{aligned}
&\int\limits_{y\in\Gamma} E(x,y)\boldsymbol{\psi}(y)ds_y - \int\limits_{y\in\Gamma_c} E(x,y)[\boldsymbol{\psi}](y)ds_y \\
&\quad = \tfrac{1}{2}\boldsymbol{\varphi}(x) + K\boldsymbol{\varphi}(x) - \int\limits_{y\in\Gamma_c} \big(T_y^c E(x,y)\big)^\top [\boldsymbol{\varphi}]ds_y \quad \text{for } \ x\in\Gamma\,, \\
&\quad -\int\limits_{y\in\Gamma} E(x,y)\boldsymbol{\psi}(y)ds_y + \int\limits_{\Gamma_c} E(x,y)[\boldsymbol{\psi}](y)ds_y \\
&\quad = -\tfrac{1}{2}\big(\boldsymbol{\varphi}^+(x) + \boldsymbol{\varphi}^-(x)\big) + \int\limits_{\Gamma_c} \big(T_y^c E(x,y)\big)^\top [\boldsymbol{\psi}]ds_y \quad \text{for } \ x\in\Gamma_c\,.
\end{aligned}
\tag{2.5.16}
$$

This is a coupled system for $\boldsymbol{\psi}\in C^\alpha(\Gamma)$ on Γ and $[\boldsymbol{\psi}]\in C_1^\alpha(\Gamma_c)$ on Γ_c, which, in fact, is uniquely solvable for any given triple $(\boldsymbol{\varphi},\boldsymbol{\varphi}^+,\boldsymbol{\varphi}^-)$ with the required properties.

For the classical *crack problem* with Dirichlet conditions on Γ, e.g. as $\boldsymbol{\psi}^+ \in C^\alpha(\overline{\Gamma}_c)$ and $\boldsymbol{\psi}^- \in C^\alpha(\overline{\Gamma}_c)$ are given with $(\boldsymbol{\psi}^+ - \boldsymbol{\psi}^-)|_\gamma = \mathbf{0}$; the desired fields $\boldsymbol{u}$ has to satisfy (2.5.8) and the boundary conditions

$$
\gamma_0\boldsymbol{u}|_\Gamma = \boldsymbol{\varphi} \quad \text{on } \ \Gamma \quad \text{and } \ T_c^+\boldsymbol{u}|_{\Gamma_c} = \boldsymbol{\psi}^+\,,\ T_c^-\boldsymbol{u}|_{\Gamma_c} = \boldsymbol{\psi}^- \quad \text{on } \ \Gamma_c \tag{2.5.17}
$$

as well as the transmission conditions (2.5.10), (2.5.11).

Again from the representation formula (2.5.15) we now obtain the coupled system

$$
\begin{aligned}
&\int\limits_{y\in\Gamma} E(x,y)\boldsymbol{\psi}(y)ds_y + \int\limits_{y\in\Gamma_c} \big(T_y^c E(x,y)\big)^\top [\boldsymbol{\varphi}](y)ds_y \\
&\qquad = \tfrac{1}{2}\boldsymbol{\varphi}(x) + K\boldsymbol{\varphi}(x) + \int\limits_{y\in\Gamma_c} E(x,y)[\boldsymbol{\psi}]|_{\Gamma_c}(y)ds_y \quad \text{for } \ x\in\Gamma\,, \\
&D_c[\boldsymbol{\varphi}](x) - \int\limits_{y\in\Gamma} T_x^c E(x,y)\boldsymbol{\psi}(y)ds_y \\
&\qquad = \tfrac{1}{2}\big(\boldsymbol{\psi}^+(x) + \boldsymbol{\psi}^-(x)\big) - K^c\,|\,([\boldsymbol{\psi}]|_{\Gamma_c})(x) \\
&\qquad\qquad - \int\limits_{y\in\Gamma} T_x^c\big(T_y E(x.y)\big)^\top \boldsymbol{\varphi}(y)ds_y \quad \text{for } \ x\in\Gamma_c
\end{aligned}
\tag{2.5.18}
$$

for the unknowns $\boldsymbol{\psi}\in C^\alpha(\Gamma)$ and $[\boldsymbol{\varphi}]\in C_0^\alpha(\Gamma_c)$. As it turns out, this system always has a unique solution for any given triple $(\boldsymbol{\varphi},\boldsymbol{\psi}^+,\boldsymbol{\psi}^-)$ with the required properties.

The desired displacement field in Ω_c is in both cases given by (2.5.15).

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