

1. Introduction

This chapter serves as a basic introduction to the reduction of elliptic boundary value problems to boundary integral equations. We begin with model problems for the Laplace equation. Our approach is the direct formulation based on Green's formula, in contrast to the indirect approach based on a layer ansatz. For ease of reading, we begin with the interior and exterior Dirichlet and Neumann problems of the Laplacian and their reduction to various forms of boundary integral equations, without detailed analysis. (For the classical results see e.g. Günter [113] and Kellogg [155].) The *Laplace equation*, and more generally, the *Poisson equation*,

$$-\Delta v = f \quad \text{in } \Omega \text{ or } \Omega^c$$

already models many problems in engineering, physics and other disciplines (Dautray and Lions [59] and Tychonoff and Samarski [308]). This equation appears, for instance, in conformal mapping (Gaier [88, 89]), electrostatics (Gauss [95], Martensen [199] and Stratton [298]), stationary heat conduction (Günter [113]), in plane elasticity as the membrane state and the torsion problem (Szabo [300]), in Darcy flow through porous media (Bear [12] and Liggett and Liu [188]) and in potential flow (Glauert [102], Hess and Smith [124], Jameson [147] and Lamb [181]), to mention a few.

The approach here is based on the relation between the Cauchy data of solutions via the Calderón projector. As will be seen, the corresponding boundary integral equations may have eigensolutions in spite of the uniqueness of the solutions of the original boundary value problems. By appropriate modifications of the boundary integral equations in terms of these eigensolutions, the uniqueness of the boundary integral equations can be achieved. Although these simple, classical model problems are well known, the concepts and procedures outlined here will be applied in the same manner for more general cases.

1.1 The Green Representation Formula

For the sake of simplicity, let us first consider, as a model problem, the Laplacian in two and three dimensions. As usual, we use $x = (x_1, \dots, x_n) \in$

\mathbb{R}^n ($n = 2$ or 3) to denote the Cartesian co-ordinates of the points in the Euclidean space \mathbb{R}^n . Furthermore, for $x, y \in \mathbb{R}^n$, we set

$$x \cdot y = \sum_{j=1}^n x_j y_j \quad \text{and} \quad |x| = (x \cdot x)^{\frac{1}{2}}$$

for the inner product and the Euclidean norm, respectively. We want to find the solution u satisfying the differential equation

$$-\Delta v := -\sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2} = f \text{ in } \Omega. \quad (1.1.1)$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded, simply connected domain, and its boundary Γ is sufficiently smooth, say twice continuously differentiable, i.e. $\Gamma \in C^2$. (Later this assumption will be reduced.) As is known from classical analysis, a classical solution $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ can be represented by boundary potentials via the Green representation formula and the *fundamental solution* E of (1.1.1). For the Laplacian, $E(x, y)$ is given by

$$E(x, y) = \begin{cases} -\frac{1}{2\pi} \log |x - y| & \text{for } n = 2, \\ \frac{1}{4\pi} \frac{1}{|x - y|} & \text{for } n = 3. \end{cases} \quad (1.1.2)$$

The presentation of the solution reads

$$v(x) = \int_{y \in \Gamma} E(x, y) \frac{\partial v}{\partial n}(y) ds_y - \int_{y \in \Gamma} v(y) \frac{\partial E(x, y)}{\partial n_y} ds_y + \int_{\Omega} E(x, y) f(y) dy \quad (1.1.3)$$

for $x \in \Omega$ (see Mikhlin [213, p. 220ff.]) where \mathbf{n}_y denotes the exterior normal to Γ at $y \in \Gamma$, ds_y the surface element or the arclength element for $n = 3$ or 2 , respectively, and

$$\frac{\partial v}{\partial n}(y) := \lim_{\tilde{y} \rightarrow y \in \Gamma, \tilde{y} \in \Omega} \text{grad} v(\tilde{y}) \cdot \mathbf{n}_y. \quad (1.1.4)$$

The notation $\partial/\partial n_y$ will be used if there could be misunderstanding due to more variables.

In the case when $f \not\equiv 0$ in (1.1.1), one may also use the decomposition in the following form:

$$v(x) = v_p(x) + u(x) := \int_{\mathbb{R}^n} E(x, y) f(y) dy + u(x) \quad (1.1.5)$$

where u now solves the Laplace equation

$$-\Delta u = 0 \text{ in } \Omega. \quad (1.1.6)$$

Here v_p denotes a particular solution of (1.1.1) in Ω or Ω^c and f has been extended from Ω (or Ω^c) to the entire R^n . Moreover, for the extended f we assume that the integral defined in (1.1.5) exists for all $x \in \overline{\Omega}$ (or $\overline{\Omega^c}$). Clearly, with this particular solution, the boundary conditions for u are to be modified accordingly.

Now, without loss of generality, we restrict our considerations to the solution u of the Laplacian (1.1.6) which now can be represented in the form:

$$u(x) = \int_{y \in \Gamma} E(x, y) \frac{\partial u}{\partial n}(y) - \int_{y \in \Gamma} u(y) \frac{\partial E(x, y)}{\partial n_y} ds_y. \quad (1.1.7)$$

For given boundary data $u|_\Gamma$ and $\frac{\partial u}{\partial n}|_\Gamma$, the representation formula (1.1.7) defines the solution of (1.1.6) everywhere in Ω . Therefore, the pair of boundary functions belonging to a solution u of (1.1.6) is called the *Cauchy data*, namely

$$\text{Cauchy data of } u := \left(u|_\Gamma, \frac{\partial u}{\partial n}|_\Gamma \right). \quad (1.1.8)$$

In solid mechanics, the representation formula (1.1.7) can also be derived by the principle of virtual work in terms of the so-called weighted residual formulation. The Laplacian (1.1.6) corresponds to the equation of the equilibrium state of the membrane without external body forces and vertical displacement u . Then, for fixed $x \in \Omega$, the terms

$$u(x) + \int_{y \in \Gamma} u(y) \frac{\partial E(x, y)}{\partial n_y} ds_y$$

correspond to the virtual work of the point force at x and of the resulting boundary forces $\partial E(x, y)/\partial n_y$ against the displacement field u , which are equal to the virtual work of the resulting boundary forces $\frac{\partial u}{\partial n}|_\Gamma$ acting against the displacement $E(x, y)$, i.e.

$$\int_{y \in \Gamma} E(x, y) \frac{\partial u}{\partial n}(y) ds_y.$$

This equality is known as *Betti's principle* (see e.g. Ciarlet [42], Fichera [75] and Hartmann [121, p. 159]). Corresponding formulas can also be obtained for more general elliptic partial differential equations than (1.1.6), as will be discussed in Chapter 2.

1.2 Boundary Potentials and Calderón's Projector

The representation formula (1.1.7) contains two boundary potentials, the *simple layer potential*

$$V\sigma(x) := \int_{y \in \Gamma} E(x, y)\sigma(y) ds_y, \quad x \in \Omega \cup \Omega^c, \quad (1.2.1)$$

and the *double layer potential*

$$W\varphi(x) := \int_{y \in \Gamma} \left(\frac{\partial}{\partial n_y} E(x, y) \right) \varphi(y) ds_y, \quad x \in \Omega \cup \Omega^c. \quad (1.2.2)$$

Here, σ and φ are referred to as the densities of the corresponding potentials. In (1.1.7), for the solution of (1.1.6), these are the Cauchy data which are not both given for boundary value problems. For their complete determination we consider the Cauchy data of the left- and the right-hand sides of (1.1.7) on Γ ; this requires the limits of the potentials for x approaching Γ and their normal derivatives. This leads us to the following definitions of boundary integral operators, provided the corresponding limits exist. For the potential equation (1.1.6), this is well known from classical analysis (Mikhlin [213, p. 360] and G nter [113, Chap. II]):

$$V\sigma(x) := \lim_{z \rightarrow x \in \Gamma} V\sigma(z) \quad \text{for } x \in \Gamma, \quad (1.2.3)$$

$$K\varphi(x) := \lim_{z \rightarrow x \in \Gamma, z \in \Omega} W\varphi(z) + \frac{1}{2}\varphi(x) \quad \text{for } x \in \Gamma, \quad (1.2.4)$$

$$K'\sigma(x) := \lim_{z \rightarrow x \in \Gamma, z \in \Omega} \text{grad}_z V\sigma(z) \cdot \mathbf{n}_x - \frac{1}{2}\sigma(x) \quad \text{for } x \in \Gamma, \quad (1.2.5)$$

$$D\varphi(x) := - \lim_{z \rightarrow x \in \Gamma, z \in \Omega} \text{grad}_z W\varphi(z) \cdot \mathbf{n}_x \quad \text{for } x \in \Gamma. \quad (1.2.6)$$

To be more explicit, we quote the following standard results without proof.

Lemma 1.2.1. *Let $\Gamma \in C^2$ and let σ and φ be continuous. Then the limits in (1.2.3)–(1.2.5) exist uniformly with respect to all $x \in \Gamma$ and all σ and φ with $\sup_{x \in \Gamma} |\sigma(x)| \leq 1$, $\sup_{x \in \Gamma} |\varphi(x)| \leq 1$. Furthermore, these limits can be expressed by*

$$V\sigma(x) = \int_{y \in \Gamma \setminus \{x\}} E(x, y)\sigma(y) ds_y \quad \text{for } x \in \Gamma, \quad (1.2.7)$$

$$K\varphi(x) = \int_{y \in \Gamma \setminus \{x\}} \frac{\partial E}{\partial n_y}(x, y)\varphi(y) ds_y \quad \text{for } x \in \Gamma, \quad (1.2.8)$$

$$K'\sigma(x) = \int_{y \in \Gamma \setminus \{x\}} \frac{\partial E}{\partial n_x}(x, y)\sigma(y) ds_y \quad \text{for } x \in \Gamma. \quad (1.2.9)$$

We remark that here all of the above boundary integrals are improper with weakly singular kernels in the following sense (see [213, p. 158]): The kernel $k(x, y)$ of an integral operator of the form

$$\int_{\Gamma} k(x, y)\varphi(y) ds_y$$

is called *weakly singular* if there exist constants c and $\lambda < n - 1$ such that

$$|k(x, y)| \leq c|x - y|^{-\lambda} \quad \text{for all } x, y \in \Gamma. \quad (1.2.10)$$

For the Laplacian, for $\Gamma \in C^2$ and $E(x, y)$ given by (1.1.2), one even has

$$|E(x, y)| \leq c_\lambda |x - y|^{-\lambda} \quad \text{for any } \lambda > 0 \text{ for } n = 2 \text{ and } \lambda = 1 \text{ for } n = 3, \quad (1.2.11)$$

$$\frac{\partial E}{\partial n_y}(x, y) = \frac{1}{2(n-1)\pi} \frac{(x-y) \cdot \mathbf{n}_y}{|x-y|^n}, \quad (1.2.12)$$

$$\frac{\partial E}{\partial n_x}(x, y) = \frac{1}{2(n-1)\pi} \frac{(y-x) \cdot \mathbf{n}_x}{|x-y|^n} \quad \text{for } x, y \in \Gamma. \quad (1.2.13)$$

In case $n = 2$, both kernels in (1.2.12), (1.2.13) are continuously extendable to a C^0 -function for $y \rightarrow x$ (see Mikhlin [213]), in case $n = 3$ they are weakly singular with $\lambda = 1$ (see Günter [113, Sections II.3 and II.6]). For other differential equations, as e.g. for elasticity problems, the boundary integrals in (1.2.7)–(1.2.9) are strongly singular and need to be defined in terms of Cauchy principal value integrals or even as finite part integrals in the sense of Hadamard. In the classical approach, the corresponding function spaces are the *Hölder spaces* which are defined as follows:

$$C^{m+\alpha}(\Gamma) := \{\varphi \in C^m(\Gamma) \mid \|\varphi\|_{C^{m+\alpha}(\Gamma)} < \infty\}$$

where the norm is defined by

$$\|\varphi\|_{C^{m+\alpha}(\Gamma)} := \sum_{|\beta| \leq m} \sup_{x \in \Gamma} |\partial^\beta \varphi(x)| + \sum_{|\beta|=m} \sup_{\substack{x, y \in \Gamma \\ x \neq y}} \frac{|\partial^\beta \varphi(x) - \partial^\beta \varphi(y)|}{|x - y|^\alpha}$$

for $m \in \mathbb{N}_0$ and $0 < \alpha < 1$. Here, ∂^β denotes the covariant derivatives

$$\partial^\beta := \partial_1^{\beta_1} \cdots \partial_{n-1}^{\beta_{n-1}}$$

on the $(n-1)$ -dimensional boundary surface Γ where $\beta \in \mathbb{N}_0^{n-1}$ is a multi-index and $|\beta| = \beta_1 + \cdots + \beta_{n-1}$ (see Millman and Parker [216]).

Lemma 1.2.2. *Let $\Gamma \in C^2$ and let φ be a Hölder continuously differentiable function. Then the limit in (1.2.6) exists uniformly with respect to all $x \in \Gamma$ and all φ with $\|\varphi\|_{C^{1+\alpha}} \leq 1$. Moreover, the operator D can be expressed as a composition of tangential derivatives and the simple layer potential operator V :*

$$D\varphi(x) = -\frac{d}{ds_x} V\left(\frac{d\varphi}{ds}\right)(x) \quad \text{for } n = 2 \quad (1.2.14)$$

and

$$D\varphi(x) = -(\mathbf{n}_x \times \nabla_x) \cdot V(\mathbf{n}_y \times \nabla_y \varphi)(x) \quad \text{for } n = 3. \quad (1.2.15)$$

For the classical proof see Maue [200] and Günter [113, p. 73ff].

Note that the differential operator $(\mathbf{n}_y \times \nabla_y)\varphi$ defines the tangential derivatives of $\varphi(y)$ within Γ which are Hölder-continuous functions on Γ . Often this operator is also called the *surface curl* (see Giroire and Nedelec [101, 232]). In the following, we give a brief derivation for these formulae based on classical results of potential theory with Hölder continuous densities $\frac{d\varphi}{ds}(y)$ and $(\mathbf{n}_y \times \nabla_y)\varphi(y)$, respectively. Note that $\frac{d}{ds_x}$ and $(\mathbf{n}_x \times \nabla_x)$ in (1.2.14) and (1.2.15), respectively, are *not* interchanged with integration over Γ . Later on we will discuss the connection of such an interchange with the concept of Hadamard's finite part integrals. For $n = 2$, note that, for $z \in \Omega$, $z \neq y \in \Gamma$,

$$\begin{aligned} & -\mathbf{n}_x \cdot \nabla_z \int_{\Gamma} (\mathbf{n}_y \cdot \nabla_y E(z, y)) \varphi(y) ds_y \\ &= - \int_{\Gamma} \mathbf{n}_x \cdot \nabla_z (\mathbf{n}_y \cdot \nabla_y E(z, y)) \varphi(y) ds_y, \\ &= \int_{\Gamma} \mathbf{n}_x \cdot (\nabla_y \nabla_y^\top E(z, y)) \cdot \mathbf{n}_y \varphi(y) ds. \end{aligned}$$

Here,

$$\mathbf{n}_x = \begin{pmatrix} \frac{dx_2}{ds} \\ -\frac{dx_1}{ds} \end{pmatrix},$$

hence, with

$$\mathbf{t}_x = \begin{pmatrix} \frac{dx_1}{ds} \\ \frac{dx_2}{ds} \end{pmatrix} = \mathbf{n}_x^\perp$$

where $\mathbf{a}^\perp := \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}$ is defined as counterclockwise rotation by $\frac{\pi}{2}$.

An elementary computation shows that

$$\mathbf{n}_x^\top A \mathbf{n}_y = -\mathbf{t}_x^\top A^\top \mathbf{t}_y + (\text{trace } A) \mathbf{t}_x \cdot \mathbf{t}_y$$

for any 2×2 -matrix A . Hence,

$$\begin{aligned} -\mathbf{n}_x \cdot \nabla_z W \varphi(z) &= \int_{\Gamma} \mathbf{n}_x \cdot (\nabla_y \nabla_y^\top E(z, y)) \cdot \mathbf{n}_y \varphi(y) ds_y \\ &= \int_{\Gamma} \{ \mathbf{t}_x \cdot \Delta_y E(z, y) \mathbf{t}_y - (\mathbf{t}_x \cdot \nabla_y)(\mathbf{t}_y \cdot \nabla_y E(z, y)) \} \varphi(y) ds_y. \end{aligned}$$

Since $y \neq z$ and $\Delta_y E(z, y) = 0$, the second term on the right takes the form

$$\begin{aligned} -\mathbf{n}_x \cdot \nabla_z W\varphi(z) &= \int_{\Gamma} (\mathbf{t}_x \cdot \nabla_z)(\mathbf{t}_y \cdot \nabla_y E(z, y))\varphi(y)ds_y \\ &= (\mathbf{t}_x \cdot \nabla_z) \int_{\Gamma} \left(\frac{d}{ds_y} E(z, y)\right)\varphi(y)ds_y \end{aligned}$$

and, after integration by parts,

$$\begin{aligned} &= -\mathbf{t}_x \cdot \nabla_z \int_{\Gamma} E(z, y) \frac{d\varphi}{ds_y}(y)ds_y \\ &= -\mathbf{t}_x \cdot \nabla_z \left\{ V\left(\frac{d\varphi}{ds}\right)(z) \right\}. \end{aligned}$$

First note that $\nabla_z V\left(\frac{d\varphi}{ds}\right)(z)$ is a Hölder continuous function for $z \in \Omega$ which admits a Hölder continuous extension up to Γ (Günter [113, p. 68]). The definition of derivatives at the boundary gives us

$$\nabla_x V\left(\frac{d\varphi}{ds}\right)(x) = \lim_{z \rightarrow x} \nabla_z V\left(\frac{d\varphi}{ds}\right)(z)$$

which yields

$$\frac{d}{ds_x} V\left(\frac{d\varphi}{ds}\right)(x) = \mathbf{t}_x \cdot \nabla_x V\left(\frac{d\varphi}{ds}\right)(x) = \lim_{z \rightarrow x} \mathbf{t}_x \cdot \nabla_z V\left(\frac{d\varphi}{ds}\right)(z),$$

i.e. (1.2.14).

Similarly, for $n = 3$, we see that

$$-\mathbf{n}_x \cdot \nabla_z W\varphi(z) = \int_{\Gamma} \mathbf{n}_x \cdot (\nabla_y \nabla_y^T E(z, y)) \cdot \mathbf{n}_y \varphi(y)ds_y$$

and with the formulae of vector analysis

$$\begin{aligned} &= \int_{\Gamma} \{(\mathbf{n}_y \cdot \nabla_y)(\mathbf{n}_x \cdot \nabla_y)E(z, y)\}\varphi(y)ds_y \\ &= - \int_{\Gamma} \{(\mathbf{n}_y \times \nabla_y) \cdot (\mathbf{n}_x \times \nabla_y)E(z, y)\}\varphi(y)ds_y \\ &\quad + \int_{\Gamma} \{(\mathbf{n}_x \cdot \mathbf{n}_y)\Delta_y E(z, y)\}\varphi(y)ds_y, \end{aligned}$$

where the last term vanishes since $z \notin \Gamma$. Now, with elementary vector analysis,

$$\begin{aligned}
-\mathbf{n}_x \cdot \nabla_z W\varphi(z) &= - \int_{\Gamma} \{ \mathbf{n}_y \cdot (\nabla_y \times (\mathbf{n}_x \times \nabla_y)) E(z, y) \} \varphi(y) ds_y \\
&= - \int_{\Gamma} \mathbf{n}_y \cdot \{ \nabla_y \times ((\mathbf{n}_x \times \nabla_y) E(z, y) \varphi(y)) \} ds_y \\
&\quad - \int_{\Gamma} \mathbf{n}_y \cdot \{ (\mathbf{n}_x \times \nabla_y) E(z, y) \times \nabla_y \varphi(y) \} ds_y,
\end{aligned}$$

where $\varphi(y)$ denotes any $C^{1+\alpha}$ -extension from Γ into \mathbb{R}^3 . The first term on the right-hand side vanishes due to the Stokes theorem, whereas the second term gives

$$\begin{aligned}
-\mathbf{n}_x \cdot \nabla_z W\varphi(z) &= - \int_{\Gamma} \mathbf{n}_y \cdot \{ \nabla_y \varphi(y) \times (\mathbf{n}_x \times \nabla_y) E(z, y) \} ds_y \\
&= - \int_{\Gamma} (\mathbf{n}_x \times \nabla_z E(z, y)) \cdot (\mathbf{n}_y \times \nabla_y \varphi(y)) ds_y \\
&= - (\mathbf{n}_x \times \nabla_z) \cdot \int_{\Gamma} E(z, y) (\mathbf{n}_y \times \nabla_y) \varphi(y) ds_y.
\end{aligned}$$

Since $(\mathbf{n}_y \times \nabla_y) \varphi(y)$ defines tangential derivatives of φ ,

$$\nabla_z \cdot V((\mathbf{n}_y \times \nabla_y) \varphi)(z)$$

defines a Hölder continuous function for $z \in \Omega$ which admits a Hölder continuous limit for $z \rightarrow x \in \Gamma$ due to Günter [113, p. 68] implying (1.2.15).

From (1.2.15) we see that the hypersingular integral operator (1.2.6) can be expressed in terms of a composition of differentiation and a weakly singular operator. This, in fact, is a regularization of the hypersingular distribution, which will also be useful for the variational formulation and related computational procedures.

A more elementary, but different regularization can be obtained as follows (see Giroire and Nedelec [101]). From the definition (1.2.6), we see that

$$\begin{aligned}
D\varphi(x) &= \lim_{\Omega \ni z \rightarrow x \in \Gamma} \left\{ -\mathbf{n}_x \cdot \nabla_z \int_{\Gamma} \frac{\partial E}{\partial n_y}(z, y) (\varphi(y) - \varphi(x)) ds_y \right. \\
&\quad \left. - \mathbf{n}_x \cdot \nabla_z \int_{\Gamma} \frac{\partial E}{\partial n_y}(z, y) \varphi(x) ds_y \right\}.
\end{aligned}$$

If we apply the representation formula (1.1.7) to $u \equiv 1$, then we obtain Gauss' well known formula

$$\int_{\Gamma} \frac{\partial E}{\partial n_y}(z, y) ds_y = -1 \quad \text{for all } z \in \Omega.$$

This yields

$$\nabla_z \int_{\Gamma} \frac{\partial E}{\partial n_y}(z, y) \varphi(x) ds_y = 0 \quad \text{for all } z \in \Omega,$$

hence, we find the simple regularization

$$D\varphi(x) = \lim_{\Omega \ni z \rightarrow x \in \Gamma} -\mathbf{n}_x \cdot \int_{\Gamma} \nabla_z \frac{\partial E}{\partial n_y}(z, y) (\varphi(y) - \varphi(x)) ds_y. \quad (1.2.16)$$

In fact, the limit in (1.2.16) can be expressed in terms of a Cauchy principal value integral,

$$\begin{aligned} D\varphi(x) &= - \text{p.v.} \int_{\Gamma} \left(\frac{\partial}{\partial n_x} \frac{\partial}{\partial n_y} E(x, y) \right) (\varphi(y) - \varphi(x)) ds_y \\ &:= - \lim_{\varepsilon \rightarrow 0+} \int_{y \in \Gamma \wedge |y-x| \geq \varepsilon} \left(\frac{\partial}{\partial n_x} \frac{\partial}{\partial n_y} E(x, y) \right) (\varphi(y) - \varphi(x)) ds_y \\ &= \lim_{\varepsilon \rightarrow 0+} \int_{y \in \Gamma \wedge |y-x| \geq \varepsilon} \frac{1}{2(n-1)\pi} \left\{ \frac{\mathbf{n}_x \cdot \mathbf{n}_y}{r^n} + n \frac{(y-x) \cdot \mathbf{n}_y (x-y) \cdot \mathbf{n}_x}{r^{2+n}} \right\} \\ &\quad \times (\varphi(y) - \varphi(x)) ds_y. \end{aligned} \quad (1.2.17)$$

The derivation of (1.2.17) from (1.2.16), however, requires detailed analysis (see Günter [113, Section II, 10]).

Since the boundary values for the various potentials are now characterized, we are in a position to discuss the relations between the Cauchy data on Γ by taking the limit $x \rightarrow \Gamma$ and the normal derivative of the left- and right-hand sides in the representation formula (1.1.7). For any solution of (1.1.6), this leads to the following relations between the Cauchy data:

$$u(x) = \left(\frac{1}{2}I - K \right) u(x) + V \frac{\partial u}{\partial n}(x) \quad (1.2.18)$$

and

$$\frac{\partial u}{\partial n}(x) = Du(x) + \left(\frac{1}{2}I + K' \right) \frac{\partial u}{\partial n}(x). \quad (1.2.19)$$

Consequently, for any solution of (1.1.6), the Cauchy data $(u, \frac{\partial u}{\partial n})^\top$ on Γ are reproduced by the operators on the right-hand side of (1.2.18), (1.2.19), namely by

$$\mathcal{C}_\Omega := \begin{pmatrix} \frac{1}{2}I - K, & V \\ D, & \frac{1}{2}I + K' \end{pmatrix} \quad (1.2.20)$$

This operator is called the *Calderón projector* (with respect to Ω) (Calderón [34]). The operators in \mathcal{C}_Ω have mapping properties in the classical Hölder function spaces as follows:

Theorem 1.2.3. *Let $\Gamma \in C^2$ and $0 < \alpha < 1$, a fixed constant. Then the boundary potentials V, K, K', D define continuous mappings in the following spaces,*

$$V : C^\alpha(\Gamma) \rightarrow C^{1+\alpha}(\Gamma), \quad (1.2.21)$$

$$K, K' : C^\alpha(\Gamma) \rightarrow C^\alpha(\Gamma), \quad C^{1+\alpha}(\Gamma) \rightarrow C^{1+\alpha}(\Gamma), \quad (1.2.22)$$

$$D : C^{1+\alpha}(\Gamma) \rightarrow C^\alpha(\Gamma). \quad (1.2.23)$$

For the proofs see Mikhlin and Prössdorf [215, Sections IX, 4 and 7].

Remark 1.2.1: The double layer potential operator K and its adjoint K' for the Laplacian have even stronger continuity properties than those in (1.2.22), namely K, K' map continuously $C^\alpha(\Gamma) \rightarrow C^{1+\alpha}(\Gamma)$ and $C^{1+\alpha}(\Gamma) \rightarrow C^{1+\beta}(\Gamma)$ for any $\alpha \leq \beta < 1$ (Mikhlin and Prössdorf [215, Sections IX, 4 and 7] and Colton and Kress [47, Chap. 2]). Because of the compact imbeddings $C^{1+\alpha}(\Gamma) \hookrightarrow C^\alpha(\Gamma)$ and $C^{1+\beta}(\Gamma) \hookrightarrow C^{1+\alpha}(\Gamma)$, K and K' are compact. These smoothing properties of K and K' do not hold anymore, if K and K' correspond to more general elliptic partial differential equations than (1.1.6). This is e.g. the case in linear elasticity. However, the continuity properties (1.2.22) remain valid.

With Theorem 1.2.3, we now are in a position to show that \mathcal{C}_Ω indeed is a projection. More precisely, there holds:

Lemma 1.2.4. *Let $\Gamma \in C^2$. Then \mathcal{C}_Ω maps $C^{1+\alpha}(\Gamma) \times C^\alpha(\Gamma)$ into itself continuously. Moreover,*

$$\mathcal{C}_\Omega^2 = \mathcal{C}_\Omega. \quad (1.2.24)$$

Consequently, we have the following identities:

$$VD = \frac{1}{4}I - K^2, \quad (1.2.25)$$

$$DV = \frac{1}{4}I - K'^2, \quad (1.2.26)$$

$$KV = VK', \quad (1.2.27)$$

$$DK = K'D. \quad (1.2.28)$$

These relations will show their usefulness in our variational formulation later on, and as will be seen in the next section, the Calderón projector leads in a direct manner to boundary integral equations for boundary value problems.

1.3 Boundary Integral Equations

As we have seen from (1.2.18) and (1.2.19), the Cauchy data of a *solution* of the differential equation in Ω are related to each other by these two equations. As is well known, for regular elliptic boundary value problems, only

half of the Cauchy data on Γ is given. For the remaining part, the equations (1.2.18), (1.2.19) define an overdetermined system of boundary integral equations which may be used for determining the complete Cauchy data. In general, any combination of (1.2.18) and (1.2.19) can serve as a boundary integral equation for the missing part of the Cauchy data. Hence, the boundary integral equations associated with one particular boundary condition are by no means uniquely determined. The ‘direct’ approach for formulating boundary integral equations becomes particularly simple if one considers the Dirichlet problem or the Neumann problem. In what follows, we will always prefer the direct formulation.

1.3.1 The Dirichlet Problem

In the Dirichlet problem for (1.1.6), the boundary values

$$u|_{\Gamma} = \varphi \text{ on } \Gamma \quad (1.3.1)$$

are given. Hence,

$$\sigma = \frac{\partial u}{\partial n}|_{\Gamma} \quad (1.3.2)$$

is the missing Cauchy datum required to satisfy (1.2.18) and (1.2.19) for any solution u of (1.1.6). In the direct formulation, if we take the first equation (1.2.18) of the Calderón projection then σ is to be determined by the boundary integral equation

$$V\sigma(x) = \frac{1}{2}\varphi(x) + K\varphi(x), \quad x \in \Gamma. \quad (1.3.3)$$

Explicitly, we have

$$\int_{y \in \Gamma} E(x, y)\sigma(y)ds_y = f(x), \quad x \in \Gamma \quad (1.3.4)$$

where f is given by the right-hand side of (1.3.3) and E is given by (1.1.2), a weakly singular kernel. Hence, (1.3.4) is a *Fredholm integral equation of the first kind*. In the case $n = 2$ and a boundary curve Γ with conformal radius equal to 1, the integral equation (1.3.4) has exactly one eigensolution, the so-called *natural charge* $e(y)$ (Plemelj [248]). However, the modified equation

$$\int_{y \in \Gamma} E(x, y)\sigma(y)ds_y - \omega = f(x), \quad x \in \Gamma \quad (1.3.5)$$

together with the normalizing condition

$$\int_{y \in \Gamma} \sigma ds = \Sigma \quad (1.3.6)$$

is always solvable for σ and the constant ω for given f and given constant Σ [136]. Later on we will come back to this modification.

For $\Sigma = 0$ it can be shown that $\omega = 0$. Hence, with $\Sigma = 0$, this modified formulation can also be used for solving the interior Dirichlet problem.

Alternatively to (1.3.4), if we take the second equation (1.2.19) of the Calderón projector, we arrive at

$$\frac{1}{2}\sigma(x) - K'\sigma(x) = D\varphi(x) \quad \text{for } x \in \Gamma. \quad (1.3.7)$$

In view of (1.2.9) and (1.2.11), the explicit form of (1.3.7) reads

$$\sigma(x) - 2 \int_{y \in \Gamma \setminus \{x\}} \frac{\partial E}{\partial n_x}(x, y) \sigma(y) ds_y = g(x) \quad \text{for } x \in \Gamma, \quad (1.3.8)$$

where $\frac{\partial E}{\partial n_x}(x, y)$ is weakly singular due to (1.2.13), provided Γ is smooth. $g = 2D\varphi$ is defined by the right-hand side of (1.3.7). Therefore, in contrast to (1.3.4), this is a *Fredholm integral equation of the second kind*.

This simple example shows that for the same problem we may employ different boundary integral equations. In fact, (1.3.8) is one of the celebrated integral equations of classical potential theory — the adjoint to the Neumann–Fredholm integral equation of the second kind with the double layer potential — which can be obtained by using the double layer ansatz in the indirect approach. In the classical framework, the analysis of integral equation (1.3.8) has been studied intensively for centuries, including its numerical solution. For more details and references, see, e.g., Atkinson [8], Bruhn et al. [26], Dautray and Lions [59, 60], Jeggli [149, 150], Kellogg [155], Kral et al [167, 168, 169, 170, 171], Martensen [198, 199], Maz'ya [202], Neumann [238, 239, 240], Radon [259] and [316]. In recent years, increasing efforts have also been devoted to the integral equation of the first kind (1.3.4) which — contrary to conventional belief — became a very rewarding and fundamental formulation theoretically as well as computationally. It will be seen that this equation is particularly suitable for the variational analysis.

1.3.2 The Neumann Problem

In the Neumann problem for (1.1.6), the boundary condition reads as

$$\frac{\partial u}{\partial n}|_{\Gamma} = \psi \quad \text{on } \Gamma \quad (1.3.9)$$

with given ψ . For the interior problem (1.1.6) in Ω , the normal derivative ψ needs to satisfy the necessary compatibility condition

$$\int_{\Gamma} \psi ds = 0 \quad (1.3.10)$$

for any solution of (1.1.6), (1.3.9) to exist. Here, $u|_\Gamma$ is the missing Cauchy datum required to satisfy (1.2.18) and (1.2.19) for any solution u of the Neumann problem (1.1.6), (1.3.9). If we take the first equation (1.2.18) of the Calderón projector, then $u|_\Gamma$ is determined by the solution of the boundary integral equation

$$\frac{1}{2}u(x) + Ku(x) = V\psi(x), \quad x \in \Gamma. \quad (1.3.11)$$

This is a classical Fredholm integral equation of the second kind on Γ , namely

$$u(x) + 2 \int_{y \in \Gamma \setminus \{x\}} \frac{\partial E}{\partial n_y}(x, y) u(y) ds_y = 2 \int_{y \in \Gamma \setminus \{x\}} E(x, y) \psi(y) ds_y =: f(x). \quad (1.3.12)$$

For the Laplacian we have (1.2.12), which shows that the kernel of the integral operator in (1.3.12) is continuous for $n = 2$ and weakly singular for $n = 3$. It is easily shown that $u_0 = 1$ defines an eigensolution of the homogeneous equation corresponding to (1.3.12); and that $f(x)$ in (1.3.12) satisfies the classical orthogonality condition if and only if ψ satisfies (1.3.10). Classical potential theory provides that (1.3.12) is always solvable if (1.3.10) holds and that the null-space of (1.3.12) is one-dimensional, see e.g. Mikhlin [212, Chap. 17, 11].

Alternatively, if we take the second equation (1.2.19) of the Calderón projector, we arrive at the equation

$$Du(x) = \frac{1}{2}\psi(x) - K'\psi(x) \quad \text{for } x \in \Gamma. \quad (1.3.13)$$

This is a *hypersingular boundary integral equation* of the first kind for $u|_\Gamma$ which also has the one-dimensional null-space spanned by $u_0|_\Gamma = 1$, as can easily be seen from (1.2.14) and (1.2.15) for $n = 2$ and $n = 3$, respectively. Although this integral equation (1.3.13) is not one of the standard types, we will see that, nevertheless, it has advantages for the variational formulation and corresponding numerical treatment.

1.4 Exterior Problems

In many applications such as electrostatics and potential flow, one often deals with exterior problems which we will now consider for our simple model equation.

1.4.1 The Exterior Dirichlet Problem

For boundary value problems exterior to Ω , i.e. in $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$, infinity belongs to the boundary of Ω^c and, therefore, we need additional growth or radiation conditions for u at infinity. Moreover, in electrostatic problems, for

instance, the total charge Σ on Γ will be given. This leads to the following *exterior Dirichlet problem*, defined by the differential equation

$$-\Delta u = 0 \quad \text{in } \Omega^c, \quad (1.4.1)$$

the boundary condition

$$u|_{\Gamma} = \varphi \quad \text{on } \Gamma \quad (1.4.2)$$

and the additional growth condition

$$u(x) = \frac{1}{2\pi} \Sigma \log |x| + \omega + o(1) \quad \text{as } |x| \rightarrow \infty \quad \text{for } n = 2 \quad (1.4.3)$$

or

$$u(x) = -\frac{1}{4\pi} \Sigma \frac{1}{|x|} + \omega + O(|x|^{-2}) \quad \text{as } |x| \rightarrow \infty \quad \text{for } n = 3. \quad (1.4.4)$$

The Green representation formula now reads

$$u(x) = Wu(x) - \left(V \frac{\partial u}{\partial n} \right) (x) + \omega \quad \text{for } x \in \Omega^c, \quad (1.4.5)$$

where the direction of \mathbf{n} is defined as before and the normal derivative is now defined as in (1.1.7) but with $z \in \Omega^c$. In the case $\omega = 0$, we may consider the Cauchy data from Ω^c on Γ , which leads with the boundary data of (1.4.5) on Γ to the equations

$$\begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K, & -V \\ -D, & \frac{1}{2}I - K' \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} \quad \text{on } \Gamma. \quad (1.4.6)$$

Here, the boundary integral operators V, K, K', D are related to the limits of the boundary potentials from Ω^c similar to (1.2.3)–(1.2.6), namely

$$V\sigma(x) = \lim_{z \rightarrow x} V\sigma(z) \quad x \in \Gamma, \quad (1.4.7)$$

$$K\varphi(x) = \lim_{z \rightarrow x, z \in \Omega^c} W\varphi(z) - \frac{1}{2}\varphi(x), \quad x \in \Gamma, \quad (1.4.8)$$

$$K'\sigma(x) = \lim_{z \rightarrow x, z \in \Omega^c} \text{grad}_z V(z) \cdot n_x + \frac{1}{2}\sigma(x), \quad x \in \Gamma, \quad (1.4.9)$$

$$D\varphi(x) = - \lim_{z \rightarrow x, z \in \Omega^c} \text{grad}_z W\varphi(z) \cdot n_x, \quad x \in \Gamma. \quad (1.4.10)$$

Note that in (1.4.8) and (1.4.9) the signs at $\frac{1}{2}\varphi(x)$ and $\frac{1}{2}\sigma(x)$ are different from those in (1.2.4) and (1.2.5), respectively.

For any solution u of (1.4.1) in Ω^c with $\omega = 0$, the Cauchy data on Γ are reproduced by the right-hand side of (1.4.6), which therefore defines the *Calderón projector* \mathcal{C}_{Ω^c} for the Laplacian with respect to the exterior domain Ω^c . Clearly,

$$\mathcal{C}_{\Omega^c} = \mathcal{I} - \mathcal{C}_{\Omega}, \quad (1.4.11)$$

where \mathcal{I} denotes the identity matrix operator.

For the solution of (1.4.1), (1.4.2) and (1.4.3), we obtain from (1.4.5) a modified boundary integral equation,

$$V\sigma(x) - \omega = -\frac{1}{2}\varphi(x) + K\varphi(x) \quad \text{for } x \in \Gamma. \quad (1.4.12)$$

This, again, is a *first kind integral equation* for $\sigma = \frac{\partial u}{\partial n}|_{\Gamma}$, the unknown Cauchy datum. However, in addition, the constant ω is also unknown. Hence, we need an additional constraint, which here is given by

$$\int_{\Gamma} \sigma ds = \Sigma. \quad (1.4.13)$$

This is the same modified system as (1.3.5), (1.3.6), which is always uniquely solvable for (σ, ω) .

If we take the normal derivative at Γ on both sides of (1.4.5), we arrive at the following *Fredholm integral equation of the second kind* for σ , namely

$$\frac{1}{2}\sigma(x) + K'\sigma(x) = -D\varphi(x) \quad \text{for } x \in \Gamma. \quad (1.4.14)$$

This is the classical integral equation associated with the exterior Dirichlet problem which has a one-dimensional space of eigensolutions. Here, the special right-hand side of (1.4.14) always satisfies the orthogonality condition in the classical Fredholm alternative. Hence, (1.4.14) always has a solution, which becomes unique if the additional constraint of (1.4.13) is included.

1.4.2 The Exterior Neumann Problem

Here, in addition to (1.4.1), we require the Neumann condition

$$\frac{\partial u}{\partial n}|_{\Gamma} = \psi \quad \text{on } \Gamma \quad (1.4.15)$$

where ψ is given. Moreover, we again need a condition at infinity. We choose the growth condition (1.4.3) or (1.4.4), respectively, where the constant Σ is given by

$$\Sigma = \int_{\Gamma} \psi ds$$

from (1.4.15), where ω is now an additional parameter, which can be prescribed arbitrarily according to the special situation. The representation formula (1.4.5) remains valid. Often $\omega = 0$ is chosen in (1.4.3), (1.4.4) and (1.4.5). The direct approach with $x \rightarrow \Gamma$ in (1.4.5) now leads to the boundary integral equation

$$-\frac{1}{2}u(x) + Ku(x) = V\psi(x) - \omega \quad \text{for } x \in \Gamma. \quad (1.4.16)$$

For any given ψ and ω , this is the *classical Fredholm integral equation of the second kind* which has been studied intensively (Günter [113]). (See also

Atkinson [8], Dieudonné [61], Kral [168, 169], Maz'ya [202], Mikhlin [211, 212]) (1.4.16) is uniquely solvable for $u|_F$.

If we apply the normal derivative to both sides of (1.4.5), we find the *hypersingular integral equation of the first kind*,

$$Du(x) = -\frac{1}{2}\psi(x) - K'\psi(x) \text{ for } x \in F. \quad (1.4.17)$$

This equation has the constants as an one-dimensional eigenspace. The special right-hand side in (1.4.17) satisfies an orthogonality condition in the classical Fredholm alternative, which is also valid for (1.4.17), e.g., in the space of Hölder continuous functions on F , as will be shown later. Therefore, (1.4.17) always has solutions $u|_F$. Any solution of (1.4.17) inserted into the right hand side of (1.4.5) together with any choice of ω will give the desired unique solution of the exterior Neumann problem.

For further illustration, we now consider the historical example of the two-dimensional potential flow of an inviscid incompressible fluid around an airfoil. Let \mathbf{q}_∞ denote the given traveling velocity of the profile defining a uniform velocity at infinity and let \mathbf{q} denote the velocity field. Then we have the following exterior boundary value problem for $\mathbf{q} = (q_1, q_2)^\top$:

$$(\nabla \times \mathbf{q})_3 = \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2} = 0 \text{ in } \Omega^c, \quad (1.4.18)$$

$$\operatorname{div} \mathbf{q} = \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0 \text{ in } \Omega^c, \quad (1.4.19)$$

$$\mathbf{q} \cdot \mathbf{n}|_F = 0 \text{ on } F, \quad (1.4.20)$$

$$\lim_{\Omega^c \ni x \rightarrow TE} |\mathbf{q}(x)| = |\mathbf{q}|_{TE} \text{ exists at the trailing edge } TE, \quad (1.4.21)$$

$$\mathbf{q} - \mathbf{q}_\infty = o(1) \text{ as } |x| \rightarrow \infty. \quad (1.4.22)$$

Here, the airfoil's profile F is given by a simply closed curve with one corner point at the trailing edge TE . Moreover, F has a C^∞ -parametrization $\mathbf{x}(s)$ for the arc length $0 \leq s \leq L$ with $\mathbf{x}(0) = \mathbf{x}(L) = TE$, whose periodic extension is only piecewise C^∞ . With Bernoulli's law, the condition (1.4.21) is equivalent to the Kutta–Joukowski condition, which requires bounded and equal pressure at the trailing edge. (See also Ciavaldini et al [44]). The origin 0 of the co-ordinate system is chosen within the airfoil with TE on the x_1 -axis and the line $\overline{0TE}$ within Ω , as shown in Figure 1.4.1.

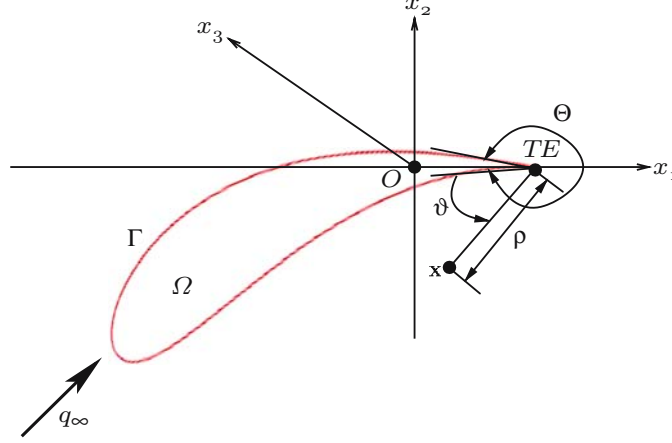


Figure 1.4.1: Airfoil in two dimensions

As before, the exterior domain is denoted by $\Omega^c := \mathbb{R}^2 \setminus \overline{\Omega}$. Since the flow is irrotational and divergence-free, \mathbf{q} has a potential which allows the reformulation of (1.4.18)–(1.4.22) as

$$\mathbf{q} = \mathbf{q}_\infty + \frac{\omega}{2\pi} \frac{1}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + \nabla u \quad (1.4.23)$$

where u is the solution of the exterior Neumann boundary value problem

$$-\Delta u = 0 \quad \text{in } \Omega^c, \quad (1.4.24)$$

$$\frac{\partial u}{\partial n}|_\Gamma = -\mathbf{q}_\infty \cdot \mathbf{n}|_\Gamma - \frac{\omega_0}{2\pi} \frac{1}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \cdot \mathbf{n} \quad \text{on } \Gamma, \quad (1.4.25)$$

$$u(x) = o(1) \quad \text{as } |x| \rightarrow \infty. \quad (1.4.26)$$

In this formulation, $u \in C^2(\Omega^c) \cap C^0(\overline{\Omega^c})$ is the unknown disturbance potential, ω_0 is the unknown circulation around Γ which will be determined by the additional Kutta–Joukowski condition, that is

$$\lim_{\Omega^c \ni x \rightarrow TE} |\nabla u(x)| = |\nabla u|_{TE} \quad \text{exists}. \quad (1.4.27)$$

We remark that condition (1.4.27) is a direct consequence of condition (1.4.21). By using conformal mapping, the solution u was constructed by Kirchhoff [157], see also Goldstein [105].

We now reduce this problem to a boundary integral equation. As in (1.4.5) and in view of (1.4.26), the solution admits the representation

$$u(x) = Wu(x) + V \left(\mathbf{q}_\infty \cdot \mathbf{n} + \frac{\omega_0}{2\pi|y|^2} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \cdot \mathbf{n} \right) (x) \quad \text{for } x \in \Omega^c. \quad (1.4.28)$$

Since

$$\int_{\Gamma} \mathbf{q}_{\infty} \cdot \mathbf{n}(y) ds_y = \int_{\Omega} (\operatorname{div} \mathbf{q}_{\infty}) dx = 0,$$

and, by Green's theorem for

$$\int_{\Gamma} \frac{1}{|y|^2} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \cdot \mathbf{n}(y) ds_y = \int_{|y|=R} \frac{1}{|y|^2} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \cdot \mathbf{n}(y) ds_y = 0,$$

it follows that every solution u represented by (1.4.28) satisfies (1.4.26) for any choice of ω_0 . We now set

$$u = u_0 + \omega_0 u_1, \quad (1.4.29)$$

where the potentials u_0 and u_1 are solutions of the exterior Neumann problems

$$\begin{aligned} -\Delta u_i &= 0 \quad \text{in } \Omega^c \quad \text{with } i = 0, 1, \quad \text{and} \\ \frac{\partial u_0}{\partial n}|_{\Gamma} &= -\mathbf{q}_{\infty} \cdot \mathbf{n}|_{\Gamma}, \quad \frac{\partial u_1}{\partial n}|_{\Gamma} = -\frac{1}{2\pi} |x|^{-2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \cdot \mathbf{n}|_{\Gamma}, \end{aligned}$$

respectively, and

$$u_i = o(1) \quad \text{as } |x| \rightarrow \infty, \quad i = 0, 1.$$

Hence, these functions admit the representations

$$u_0(x) = W u_0(x) + V(\mathbf{q}_{\infty} \cdot \mathbf{n})(x), \quad (1.4.30)$$

$$u_1(x) = W u_1(x) + V\left(\frac{1}{2\pi} \frac{1}{|y|^2} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \cdot \mathbf{n}\right)(x) \quad (1.4.31)$$

for $x \in \Omega^c$, whose boundary traces are the unique solutions of the boundary integral equations

$$\frac{1}{2} u_0(x) - K u_0(x) = V(\mathbf{q}_{\infty} \cdot \mathbf{n})(x), \quad x \in \Gamma, \quad (1.4.32)$$

$$\frac{1}{2} u_1(x) - K u_1(x) = V\left(\frac{1}{2\pi} \frac{1}{|y|^2} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \cdot \mathbf{n}\right)(x), \quad x \in \Gamma. \quad (1.4.33)$$

Note that K is defined by (1.5.2) which is valid at TE , too. The right-hand sides of (1.4.32) and (1.4.33) are both Hölder continuous functions on Γ . Due to the classical results by Carleman [37] and Radon [259], there exist unique solutions u_0 and u_1 in the class of continuous functions. A more detailed analysis shows that the derivatives of these solutions possess singularities at the trailing edge TE . More precisely, one finds (e.g. in the book by Grisvard [108, Theorem 5.1.1.4 p.255]) that the solutions admit local singular expansions of the form

$$u_i(x) = \alpha_i \varrho^{\frac{\pi}{\Theta}} \cos\left(\frac{\pi}{\Theta} \vartheta\right) + O\left(\varrho^{\frac{2\pi}{\Theta} - \varepsilon}\right), \quad i = 0, 1, \quad (1.4.34)$$

where Θ is the exterior angle of the two tangents at the trailing edge, ϱ denotes the distance from the trailing edge to x , ϑ is the angle from the lower trailing edge tangent to the vector $(\mathbf{x} - TE)$, where ε is any positive number. Consequently, the gradients are of the form

$$\nabla u_i(x) = \alpha_i \frac{\pi}{\Theta} \varrho^{\frac{\pi}{\Theta}-1} \mathbf{e}_\vartheta + O\left(\varrho^{\frac{2\pi}{\Theta}-1-\varepsilon}\right), \quad i = 0, 1, \quad (1.4.35)$$

where \mathbf{e}_ϑ is a unit vector with angle $(1 - \frac{\pi}{\Theta})\vartheta$ from the lower trailing edge tangent, for both cases, $i = 0, 1$.

Hence, from equations (1.4.27) and (1.4.29) we obtain, for $\varrho \rightarrow 0$, the condition for ω_0 ,

$$\alpha_0 + \omega_0 \alpha_1 = 0. \quad (1.4.36)$$

The solution u_1 corresponds to $\mathbf{q}_\infty = 0$, i.e. the pure circulation flow, which can easily be found by mapping Ω conformally onto the unit circle in the complex plane. The mapping has the local behavior as in (1.4.34) with $\alpha_1 \neq 0$ since TE is mapped onto a point on the unit circle (see Lehman [183]). Consequently, ω_0 is uniquely determined from (1.4.36). We remark that this choice of ω_0 shows that

$$\nabla u = O\left(\varrho^{\frac{2\pi}{\Theta}-1-\varepsilon}\right)$$

due to (1.4.35) and, hence, the singularity vanishes for $\Theta < 2\pi$ at the trailing edge TE ; which indeed is then a stagnation point for the disturbance velocity ∇u . This approach can be generalized to two-dimensional transonic flow problems (Coclici et al [46]).

1.5 Remarks

For applications in engineering, the strong smoothness assumptions for the boundary Γ need to be relaxed allowing corners and edges. Moreover, for crack and screen problems as in elasticity and acoustics, respectively, Γ is not closed but only a part of a curve or a surface. To handle these types of problems, the approach in the previous sections needs to be modified accordingly.

To be more specific, we first consider *Lyapounov boundaries*. Following Mikhlin [212, Chap. 18], a Lyapounov curve in \mathbb{R}^2 or Lyapunov surface Γ in \mathbb{R}^3 (Günter [113]) satisfies the following two conditions:

1. There exists a normal \mathbf{n}_x at any point x on Γ .
2. There exist positive constants a and $\kappa \leq 1$ such that for any two points x and ξ on Γ with corresponding vectors \mathbf{n}_x and \mathbf{n}_ξ the angle ϑ between them satisfies

$$|\vartheta| \leq ar^\kappa \quad \text{where } r = |x - \xi|.$$

In fact, it can be shown that for $0 < \kappa < 1$, a Lyapounov boundary coincides with a $C^{1,\kappa}$ boundary curve or surface [212, Chap. 18].

For a Lyapounov boundary, all results in Sections 1.1–1.4 remain valid if C^2 is replaced by $C^{1,\kappa}$ accordingly. These non-trivial generalizations can be found in the classical books on potential theory. See, e.g., Günter [113], Mikhlin [211, 212, 213] and Smirnov [284].

In applications, one often has to deal with boundary curves with corners, or with boundary surfaces with corners and edges. The simplest generalization of the previous approach can be obtained for piecewise Lyapounov curves in \mathbb{R}^2 with finitely many corners where $\Gamma = \cup_{j=1}^N \bar{\Gamma}_j$ and each Γ_j being an open arc of a particular closed Lyapounov curve. The intersections $\bar{\Gamma}_j \cap \bar{\Gamma}_{j+1}$ are the corner points where $\Gamma_{N+1} := \Gamma_1$. In this case it easily follows that there exists a constant C such that

$$\int_{\Gamma \setminus \{x\}} \left| \frac{\partial E}{\partial n_y}(x, y) \right| ds \leq C \quad \text{for all } x \in \mathbb{R}^2. \quad (1.5.1)$$

This property already ensures that for continuous φ on Γ , the operator K is well defined by (1.2.4) and that it is a continuous mapping in $C^0(\Gamma)$. However, (1.2.8) needs to be modified and becomes

$$\begin{aligned} K\varphi(x) &:= \int_{\Gamma \setminus \{x\}} \varphi(y) \frac{\partial E}{\partial n_y}(x, y) ds_y \\ &\quad - \left(\frac{1}{2} + \int_{\Gamma \setminus \{x\}} \frac{\partial E}{\partial n_y}(x, y) ds_y \right) \varphi(x), \end{aligned} \quad (1.5.2)$$

where the last expression takes care of the corner points and vanishes if x is not a corner point. Here K is not compact anymore as in the case of a Lyapounov boundary, however, it can be shown that K can be decomposed into a sum of a compact operator and a contraction, provided the corner angles are not 0 or 2π . This decomposition is sufficient for the classical Fredholm alternative to hold for (1.3.11) with continuous u , as was shown by Radon [259]. For the most general two-dimensional case we refer to Kral [168].

For the Neumann problem, one needs a generalization of the normal derivative in terms of the so-called *boundary flow*, which originally was introduced by Plemelj [248] and has been generalized by Kral [169]. It should be mentioned that in this case the adjoint operator K' to K is no longer a bounded operator on the space of continuous functions (Netuka [237]). The simple layer potential $V\sigma$ is still Hölder continuous in \mathbb{R}^2 for continuous σ . However, its normal derivative needs to be interpreted in the sense of boundary flow.

This situation is even more complicated in the three-dimensional case because of the presence of edges and corners. Here, for continuous φ it is still not clear whether the Fredholm alternative for equation (1.3.11) remains valid even for general piecewise Lyapounov surfaces with finitely many corners and

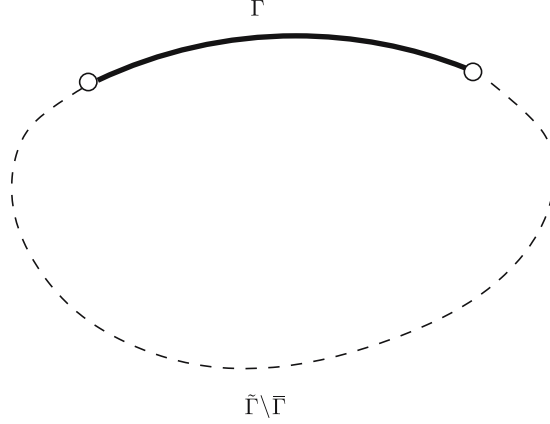


Fig. 1.5.1. Configuration of an arc Γ in \mathbb{R}^2

edges; see, e.g., Angell et al [7], Burago et al [31, 32], Kral et al [170, 171], Maz'ya [202] and [316].

On the other hand, as we will see, in the variational formulation of the boundary integral equations, many of these difficulties can be circumvented for even more general boundaries such as Lipschitz boundaries (see Section 5.6).

To conclude these remarks, we consider Γ to be an oriented, open part of a closed curve or surface $\tilde{\Gamma}$ (see Figure 1.5.1). The *Dirichlet problem* here is to find the solution u of (1.4.1) in the domain $\Omega^c = \mathbb{R}^n \setminus \bar{\Gamma}$ subject to the boundary conditions

$$u_+ = \varphi_+ \text{ on } \Gamma_+ \text{ and } u_- = \varphi_- \text{ on } \Gamma_- \quad (1.5.3)$$

where Γ_+ and Γ_- are the respective sides of Γ and u_+ and u_- the corresponding traces of u . The functions φ_+ and φ_- are given with the additional requirement that

$$\varphi_+ - \varphi_- = 0$$

at the endpoints of Γ for $n = 2$, or at the boundary edge of Γ for $n = 3$. In the latter case we require $\partial\Gamma$ to be a C^∞ -smooth curve. Similar to the regular exterior problem, we again require the growth condition (1.4.3) for $n = 2$ and (1.4.4) for $n = 3$. For a sufficiently smooth solution, the Green representation formula has the form

$$u(x) = W_\Gamma[u](x) - V_\Gamma \left[\frac{\partial u}{\partial n} \right] (x) + \omega \text{ for } x \in \Omega^c, \quad (1.5.4)$$

where W_Γ , V_Γ are the corresponding boundary potentials with integration over Γ only,

$$W_\Gamma \varphi(x) := \int_{y \in \Gamma} \left(\frac{\partial}{\partial n_y} E(x, y) \right) \varphi(y) ds_y, \quad x \notin \bar{\Gamma}; \quad (1.5.5)$$

$$V_\Gamma \sigma(x) := \int_{y \in \Gamma} E(x, y) \sigma(y) ds_y, \quad x \notin \bar{\Gamma}. \quad (1.5.6)$$

$$[u] = u_+ - u_-, \quad \sigma := \left[\frac{\partial u}{\partial n} \right] = \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \quad \text{on } \Gamma \quad (1.5.7)$$

with \mathbf{n} the normal to Γ pointing in the direction of the side Γ_+ . If we substitute the given boundary values φ_+, φ_- into (1.5.4), the missing Cauchy datum σ is now the jump of the normal derivative across Γ . Between this unknown datum and the behaviour of u at infinity viz.(1.4.3) we arrive at

$$\int_\Gamma \sigma ds = \Sigma. \quad (1.5.8)$$

By taking x to Γ_+ (or Γ_-) we obtain (in both cases) the boundary integral equation of the first kind for σ on Γ ,

$$V_\Gamma \sigma(x) - \omega = -\frac{1}{2}(\varphi_+(x) + \varphi_-(x)) + K_\Gamma(\varphi_+ - \varphi_-)(x) =: f(x) \quad (1.5.9)$$

where K_Γ is defined by

$$K_\Gamma \varphi(x) = \int_{y \in \Gamma \setminus \{x\}} \left(\frac{\partial}{\partial n_y} E(x, y) \right) \varphi(y) ds_y \quad \text{for } x \in \Gamma. \quad (1.5.10)$$

As for the previous case of a closed curve or surface Γ , respectively, the system (1.5.8), (1.5.9) admits a unique solution pair (σ, ω) for any given φ_+, φ_- and Σ . Here, however, σ will have singularities at the endpoints of Γ or the boundary edge of Γ for $n = 2$ or 3 , respectively, and our classical approach, presented here, requires a more careful justification in terms of appropriate function spaces and variational setting.

In a similar manner, one can consider the *Neumann problem*: Find u satisfying (1.4.1) in Ω^c subject to the boundary conditions

$$\frac{\partial u_+}{\partial n} = \psi_+ \quad \text{on } \Gamma_+ \quad \text{and} \quad \frac{\partial u_-}{\partial n} = \psi_- \quad \text{on } \Gamma_- \quad (1.5.11)$$

where ψ_+ and ψ_- are given smooth functions. By applying the normal derivatives $\partial/\partial n_x$ to the representation formula (1.5.4) from both sides of Γ , it is not difficult to see that the missing Cauchy datum $\varphi =: [u] = u_+ - u_-$ on Γ satisfies the hypersingular boundary integral equation of the first kind for φ ,

$$D_\Gamma \varphi = -\frac{1}{2}(\psi_+ + \psi_-) - K'_\Gamma[\psi] \quad \text{on } \Gamma, \quad (1.5.12)$$

where the operators D_Γ and K'_Γ again are given by (1.2.6) and (1.2.5) with W and V replaced by W_Γ and V_Γ , respectively. As we will see later on

in the framework of variational problems, this integral equation (1.5.12) is uniquely solvable for φ with $\varphi = 0$ at the endpoints of Γ for $n = 2$ or at the boundary edge $\partial\Gamma$ of Γ for $n = 3$, respectively. Here, Σ in (1.4.3) or (1.4.4) is already given by (1.5.8) and ω can be chosen arbitrarily. For further analysis of these problems see [146], Stephan et al [294, 297], Costabel et al [49, 52].



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