

CHAPTER 2

BOUNDARY INTEGRAL EQUATIONS

Synopsis: This chapter introduces boundary integral equations and their numerical approximations. For potential theory, *i.e.*, the Laplace equation, the boundary integral equations for surface potential and for surface flux are derived, these equations involve the Green's function $G(P, Q)$ (fundamental solution) and its first and second derivatives. These functions are divergent when $P = Q$, the singularity becoming progressively worse with higher derivatives, and thus the definition and evaluation of (highly) singular integrals is of paramount importance. The fundamental approach adopted in this book is to define all singular integrals as a 'limit to boundary'. To illustrate this concept with a concrete example, two explicit calculations are carried out. The integrals of the second derivative of $G(P, Q)$, termed *hypersingular*, arising from the two basic approximations – **collocation** and **Galerkin** – are considered. The important observation from these calculations is that the collocation hypersingular integral is *not* finite unless a differentiability condition is met, whereas the corresponding Galerkin calculation is finite under a weaker continuity requirement.

2.1 BOUNDARY POTENTIAL EQUATION

Although the boundary integral formulations for different partial differential equations will necessarily change due to the presence of different Green's functions, these equations all *look* the same. Moreover, the Green's function behavior at the singular point $P = Q$ is essentially the same for all problems (even, roughly

speaking, for the more complex Green's function arising in anisotropic elasticity, (Section 4.6). Thus, for the most part, it suffices to discuss the simplest boundary integral equation, that for the Laplace equation.

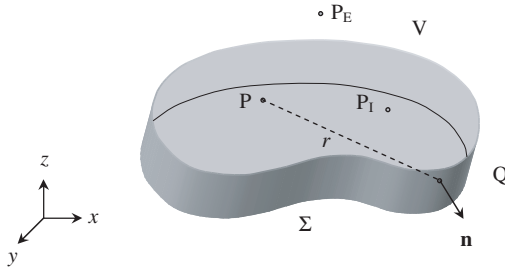


Figure 2.1 A volume V with boundary Σ , the source point P and the field point Q and the interior and exterior points P_I and P_E .

The three-dimensional (isotropic) Laplace equation for the function $\phi(x, y, z)$ is

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0, \quad (2.1)$$

where this equation is assumed to hold for $\mathbf{x} = (x, y, z)$ in an open volume V having boundary Σ (see Fig. 2.1). The function $\phi(\mathbf{x})$ is called a *potential function*. Problems governed by the Laplace equation are found in a wide range of applications e.g. steady state heat transfer, electrostatics (for example, EEG modeling), electrochemical problems [292], magnetostatics, ideal fluid flow, flow in porous media [167] and many others.

The surface flux corresponding to the potential ϕ is

$$-\kappa \frac{\partial \phi}{\partial \mathbf{n}}(Q) = -\kappa \nabla \phi \cdot \mathbf{n}(Q) \quad (2.2)$$

where $\mathbf{n}(Q)$ is the unit outward normal at the boundary point Q and the terminology for the constant κ depends upon the specific application, *e.g.*, thermal conductivity for heat transfer problems. In many instances it suffices to ignore the constant κ ; however including this parameter will be important for the discussion of multi-region problems, Chapter 7.

The derivation of the boundary integral equation that is equivalent to Eq.(2.1) begins by noting that

$$\int_V f(\mathbf{x}) \nabla^2 \phi(\mathbf{x}) dV = 0 \quad (2.3)$$

for any sufficiently well behaved function $f(\mathbf{x})$ defined in the volume. Integrating by parts, *i.e.*, applying the divergence theorem (see the Appendix A), we obtain

$$0 = \int_{\Sigma} \left[\phi(Q) \frac{\partial f}{\partial \mathbf{n}}(Q) - f(Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ - \int_V \phi(\mathbf{x}) \nabla^2 f(\mathbf{x}) dV. \quad (2.4)$$

In order to get rid of the volume integral, we wish to choose $f(\mathbf{x})$ so that it also satisfies the Laplace equation. These *fundamental solutions* can be taken as

$$G(P, Q) = \frac{1}{4\pi r} \quad (2.5)$$

where

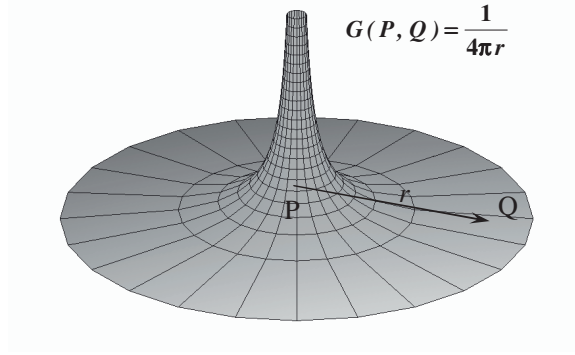


Figure 2.2 Green's function $G(P, Q)$

$$r = \|\mathbf{R}\| = \|Q - P\| = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2} . \quad (2.6)$$

is the distance between $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$. These functions are called the 'free space' Green's function (see Fig. 2.2) and physically correspond to the electrostatic potential at the *field point* Q given a point charge at P . Thus, $G(P, Q)$ is also called the 'point source potential', and for each *fixed* P , it satisfies the Laplace equation as a function for Q , for $Q \neq P$. A derivation of Eq.(2.5) is given in the end of the chapter.

The Green's function is clearly singular when $Q = P$ and, in fact, satisfies the equation

$$\nabla^2 G(P, Q) = -\delta(P, Q) , \quad (2.7)$$

where the derivative is with respect to Q and $\delta(P, Q)$ is the Dirac Delta function. As a consequence, for $P \in V \cup \Sigma$ and $f(\mathbf{x}) = G(P, \mathbf{x})$, the above use of the Divergence Theorem is not permissible. However, for $P = P_E$ exterior to V , i.e., $P_E \notin V \cup \Sigma$, substituting $G(P, \mathbf{x}) = f(\mathbf{x})$ in Eq.(2.4) yields the *exterior* integral equation

$$\int_{\Sigma} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_E, Q) - G(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0 , \quad (2.8)$$

where

$$\frac{\partial G}{\partial \mathbf{n}}(P, Q) = -\frac{1}{4\pi} \frac{\mathbf{n} \cdot \mathbf{R}}{r^3} . \quad (2.9)$$

Note that this normal derivative is with respect to the coordinates of Q , and that the singularity at $Q = P$ now behaves as r^{-2} , worse than the r^{-1} for $G(P, Q)$.

The *interior equation*, namely for $P_I \in V$,

$$\phi(P_I) + \int_{\Sigma} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_I, Q) - G(P_I, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0 \quad (2.10)$$

can, informally, be seen to arise from substituting the delta function Eq.(2.7) into the volume integral Eq.(2.4) and by the definition of δ ,

$$\int_V \phi(\mathbf{x}) \nabla^2 f(\mathbf{x}) dV - \int_V \phi(\mathbf{x}) \delta(P_I, Q) dV = -\phi(P_I) . \quad (2.11)$$

This result is more properly derived by deleting from V a sphere S_ε of radius ε centered at P_I , as shown in Fig. 2.3. The Divergence Theorem can then be legitimately applied to the punctured volume having boundary $\Sigma \setminus S_\varepsilon$, resulting in

$$\begin{aligned} \int_{\Sigma} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_I, Q) - G(P_I, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = \\ \int_{S_\varepsilon} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_I, Q) - G(P_I, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ. \end{aligned} \quad (2.12)$$

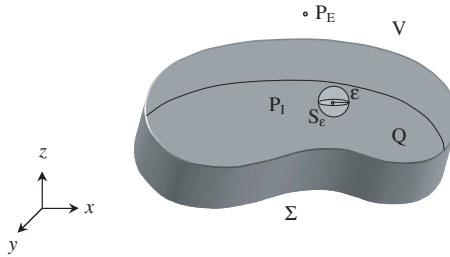


Figure 2.3 A volume V with boundary Σ , and the punctured volume S_ε .

Consider now the integral over S_ε as $\varepsilon \rightarrow 0$. Transferring to spherical coordinates, it is first seen that $G(P_I, Q)$ is integrable (weakly singular) at P_I , and thus as the surface area goes to zero, this integral must vanish in the limit $\varepsilon \rightarrow 0$. Second, for the integral with $\partial G / \partial \mathbf{n}$, employing a Taylor expansion of ϕ at the point P_I , it is easily seen that only the constant $\phi(P_I)$ term contributes in the limit. Having simplified the integral, direct evaluation of the now yields

$$\lim_{\varepsilon \rightarrow 0} \phi(P_I) \int_{S_\varepsilon} \frac{\partial G}{\partial \mathbf{n}}(P_I, Q) dQ = -\phi(P_I), \quad (2.13)$$

and once again we obtain Eq.(2.10).

The goal is now to obtain a boundary-only statement, allowing $P \in \Sigma$. As just noted, $G(P, Q)$ is only weakly singular at $P = Q$, and thus this integral exists for $P \in \Sigma$. However, the singularity for the normal derivative of the Green's function is worse, and this integral is not immediately finite. The standard practice in the literature is to invoke a 'Cauchy Principal Value', but we prefer, for reasons mentioned in the previous chapter, and discussed in detail later, to define this singular integral as a *limit to the boundary*. The *boundary* integral equations, when $P \in \Sigma$ are therefore simply defined as

$$\lim_{P_E \rightarrow P} \int_{\Sigma} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_E, Q) - G(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0 \quad (2.14)$$

$$\phi(P) + \lim_{P_I \rightarrow P} \int_{\Sigma} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_I, Q) - G(P_I, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0 \quad (2.15)$$

Of course, this simplicity is deceptive: for this approach to be useful it will be necessary to show that the limits exist and, moreover, that they can be readily

computed. The development of appropriate boundary limit algorithms for evaluating singular integrals in two and three dimensions are the primary subjects of Chapters 3 and 4.

While not immediately obvious, the interior and exterior boundary equations are indeed identical. The weakly singular integral of $G(P, Q)$ is finite for P on the boundary, and thus the interior and exterior limits are clearly the same. However, the integral involving the derivative of the Green's function is *discontinuous* crossing the boundary, and the jump in this integral balances the presence of the additional $\phi(P)$ term in the interior limit equation. We make note of this here primarily to contrast this with the behavior of the surface gradient equations, examined in detail in Chapter 5.

The interior limit form of Eq.(2.10) is clearly an expression for potential, stating that the potential at $P \in V$ can be calculated by integrating the surface potential and surface flux around the boundary. Thus, once all boundary values are known, ϕ can be computed in the interior (if needed), and the problem is effectively solved. The role of the boundary integral statements is therefore to provide equations needed to solve for the unknown boundary values of potential and flux.

Cauchy Principal Value. The Cauchy Principal Value (CPV) definition for the integral of the normal derivative of the Green's function is a standard technique in boundary integral analysis. This method will *not* be used herein, but contrasting this approach with a boundary limit evaluation can be quite instructive. For a more detailed discussion of the CPV we refer the reader to presentations found in most boundary element texts, *e.g.*, [191]; in addition, the article [127] contains an extensive analysis.

The CPV integral that arises in a two-dimensional analysis is in essence the one-dimensional integral

$$\int_{-a}^b \frac{1}{t} dt \quad a, b > 0 \quad . \quad (2.16)$$

In the CPV sense, this improper integral is defined by deleting a *symmetric neighborhood* $\{-\varepsilon, \varepsilon\}$ around the singular point $t = 0$ (see Fig. 2.4), and taking the limit ($\varepsilon > 0$)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\{ \int_{-a}^{-\varepsilon} \frac{1}{t} dt + \int_{\varepsilon}^b \frac{1}{t} dt \right\} &= \lim_{\varepsilon \rightarrow 0} \{ \log(\varepsilon/a) + \log(b/\varepsilon) \} \\ &= \log(b/a) \end{aligned} \quad (2.17)$$

Thus, CPV *starts* with a singular integral, and to obtain a finite value, it exploits the fact that the function diverges symmetrically to $\pm\infty$ on either 'side' of the singularity.

An important distinction of the boundary limit method is that the integrals are *never* improper: with P off the boundary, one is always dealing with integrals that exist. However, it must be shown that limiting value exists (and is readily computable). In the equivalent boundary limit analysis, the integral in Eq.(2.16) takes the form

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{-a}^b \frac{t}{t^2 + \varepsilon^2} dt &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \log((b^2 + \varepsilon^2)/(a^2 + \varepsilon^2)) \\ &= \log(b/a) \end{aligned} \quad (2.18)$$

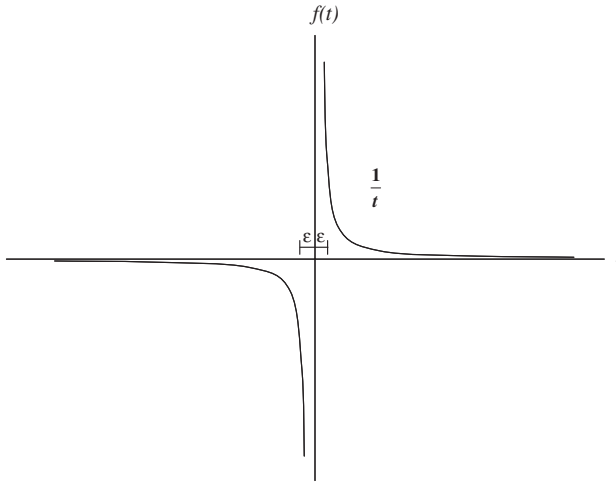


Figure 2.4 CPV: Symmetric Neighborhood around the singular point

A second observation worth noting is that the limit is taken *after* an integration. This is in fact key to the methods developed in Chapters 3 and 4, performing sufficient analytic integration that the limit process can be carried out.

Note that while CPV relies on a somewhat artificial device, the deletion of a symmetric neighborhood around the singular point, there is no such requirement in the boundary limit. The limit is based upon the physically reasonable assumption of continuity of potential at the boundary. Most importantly, the boundary limit works equally well for the boundary integral equation for surface flux: as discussed in the next section, a second derivative of the Green's function appears in this equation. The singularity in this function, termed *hypersingular*, is worse than for the first derivative, and the analogous one-dimensional integral is

$$\int_{-a}^b \frac{1}{t^2} dt \quad a, b > 0, \quad (2.19)$$

(see Section 2.5.1). In this case the integrand is strictly positive, and no CPV-type cancellation is possible, which explains why other procedures (most commonly Hadamard Finite Part) have been employed. A primary attraction of the singular integration procedures based upon the boundary limit is that they apply to *all* singular integrals. Chapters 3 and 4 will in fact primarily detail the treatment of the hypersingular integral, the implementation for the less singular integrals will then be quite straightforward. A final argument in favor of the limit approach must await the discussion of surface gradient evaluation, Chapter 5.

2.2 BOUNDARY FLUX EQUATION

The majority of problems that have a boundary integral formulation can be solved using solely the equation that corresponds to the surface potential equation for Laplace, *e.g.*, the displacement equation in elasticity. This is, however, not the case

for one important class of applications, those in which the domain volume contains a *crack* boundary. The reason for this, together with methods for solving fracture problems, will be taken up in detail in Chapter 9; for now we simply wish to point out that an effective solution of crack problems requires an additional equation, one for surface flux (in elasticity, surface traction). This equation is also quite useful for other reasons, most importantly it is an essential ingredient for obtaining a symmetric coefficient matrix in the symmetric-Galerkin approximation.

One rationale for defining the surface potential equation as a boundary limit, Eq.(2.14), is that it is a simple matter to obtain a mathematically correct statement of the corresponding boundary integral equations for surface flux. For $P \notin \Sigma$, the Green's function quantities in the integrals are well behaved, and thus Eq.(2.14) can be differentiated with respect to P by first moving the derivative underneath the integral, and then taking the limit. We thus obtain exterior and interior surface *flux* equations

$$\begin{aligned} \lim_{P_E \rightarrow P} \int_{\Sigma} \left[\phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_E, Q) - \frac{\partial G}{\partial \mathbf{N}}(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ &= 0 \\ \frac{\partial \phi}{\partial \mathbf{N}}(P) + \lim_{P_I \rightarrow P} \int_{\Sigma} \left[\phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_I, Q) - \frac{\partial G}{\partial \mathbf{N}}(P_I, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ &= 0, \end{aligned} \quad (2.20)$$

where $\mathbf{N} = \mathbf{N}(P)$ is the unit outward normal at P and $\partial \mathbf{N}$ indicates a derivative with respect to the coordinates of P . The kernel functions are

$$\frac{\partial G}{\partial \mathbf{N}}(P, Q) = \frac{1}{4\pi} \frac{\mathbf{N} \cdot \mathbf{R}}{r^3} \quad (2.21)$$

$$\frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P, Q) = \frac{1}{4\pi} \left[\frac{\mathbf{n} \cdot \mathbf{N}}{r^3} - 3 \frac{(\mathbf{n} \cdot \mathbf{R})(\mathbf{N} \cdot \mathbf{R})}{r^5} \right], \quad (2.22)$$

and it can be seen that the *hypersingular* kernel, Eq.(2.22), behaves as r^{-3} at the singular point.

It should be noted that if the potential equation is defined as a Cauchy Principal Value, one has, in effect, already evaluated the singular part of the $\partial G / \partial \mathbf{n}$ integral via a limit process. Thus, moving the derivative under the integral involves interchanging limit processes, and due to the singularity of the kernel function, this interchange is definitely not allowed. This partly explains why early in its development, hypersingular evaluation was a highly contentious issue.

As with the potential equations, the two limit equations for surface flux are identical on the boundary: the *hypersingular* integral involving two derivatives of the Green's function will turn out to be continuous crossing the boundary while, as before, the first derivative integral is not. Again, it will be necessary to demonstrate that the limits can be calculated, this task being obviously somewhat more difficult.

As the interior and exterior limits result in the same equations, and the limit process is the same in each case, either form can be employed. However, as the exterior limit lacks the 'free term' outside the integral, it is slightly more convenient to compute with this equation. Thus, until we come to consider the equation for surface gradient, we will work exclusively with the two exterior limit equations

$\mathcal{P}(P)$ for potential and $\mathcal{F}(P)$ for flux, defined as

$$\mathcal{P}(P) \equiv \lim_{P_E \rightarrow P} \int_{\Sigma} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_E, Q) - G(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0 \quad (2.23)$$

$$\mathcal{F}(P) \equiv \lim_{P_E \rightarrow P} \int_{\Sigma} \left[\phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_E, Q) - \frac{\partial G}{\partial \mathbf{N}}(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0$$

In the following, unless needed for clarity or emphasis, we will not explicitly write the boundary limit, it being understood that the singular integrals are interpreted in this sense.

Two dimensions. In two dimensions the Laplace boundary integral equations for potential and flux retain precisely the same form as above,

$$\int_{\Gamma} \left[\phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_E, Q) - G(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0 \quad (2.24)$$

$$\int_{\Gamma} \left[\phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_E, Q) - \frac{\partial G}{\partial \mathbf{N}}(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) \right] dQ = 0, \quad (2.25)$$

only now the integrals are line integrals over the boundary curve Γ . The Green's function kernels in this case are

$$\begin{aligned} G(P, Q) &= -\frac{1}{2\pi} \log(r) \\ \frac{\partial G}{\partial \mathbf{n}}(P, Q) &= -\frac{1}{2\pi} \frac{\mathbf{n} \cdot \mathbf{R}}{r^2} \\ \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P, Q) &= -\frac{1}{2\pi} \left[-\frac{\mathbf{N} \cdot \mathbf{n}}{r^2} + 2 \frac{(\mathbf{n} \cdot \mathbf{R})(\mathbf{N} \cdot \mathbf{R})}{r^4} \right], \end{aligned} \quad (2.26)$$

and the behavior at $r = 0$ is analogous to three dimensions, *e.g.*, $\log(r)$ is integrable.

2.3 ELASTICITY

One of the themes of this book is that the numerical implementation of a boundary integral equation is, for the most part, independent of the particular partial differential equation being solved. Said another way, the algorithms for singular integration – the chief task – are essentially the same irrespective of the particular fundamental solution. Some complicated Green's functions, *e.g.*, anisotropic elasticity (Chapter 4), may require some additional techniques to evaluate the singular integrals, but for most formulations, there are no significant differences from the simplest situation, the Laplace equation.

Nevertheless, elasticity problems are a primary application of boundary integral methods, and elasticity will be specifically discussed in Chapter 9 on fracture. Thus, it is necessary to introduce here the boundary integral elasticity formulation. In addition, elasticity, unlike scalar Laplace, is a vector problem and it is useful to present at least one vector formulation.

The three-dimensional isotropic elasticity equations, derived from Newton's law, state that the divergence of the stress tensor is zero [190]. Thus,

$$\nabla \cdot \sigma = \sigma_{ij,j} = 0, \quad (2.27)$$

where the summation convention is employed and a subscript after the comma denotes partial differentiation with respect to that coordinate. An isotropic solid can be defined by two material parameters, the Poisson ratio ν and the shear modulus μ , and setting $\gamma = \nu/(1 - 2\nu)$, the stress and strain tensors are defined as

$$\begin{aligned}\sigma_{ij} &= 2\mu(\epsilon_{ij} + \gamma\epsilon_{kk}\delta_{ij}) \\ \epsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) ,\end{aligned}\quad (2.28)$$

\mathbf{u} being the displacement vector. Following the procedure employed to derive the Laplace integral equation, we multiply Eq.(2.27) by functions g_i , integrate over the volume,

$$0 = \int_V g_i(\mathbf{x})\sigma_{ij,j}(\mathbf{x})dV , \quad (2.29)$$

and then integrate by parts. Summing these three equations results in

$$0 = \int_{\Sigma} [g_i(Q)\tau_i(Q) - t_i(Q)u_i(Q)] dx + \int_V S_i(\mathbf{x})u_i(\mathbf{x})dV , \quad (2.30)$$

where

$$\begin{aligned}S_1 &= \mu(\nabla^2 g_1 + (1 + 2\gamma)(g_{1,11} + g_{2,12} + g_{3,13})) \\ S_2 &= \mu(\nabla^2 g_2 + (1 + 2\gamma)(g_{1,21} + g_{2,22} + g_{3,23})) \\ S_3 &= \mu(\nabla^2 g_3 + (1 + 2\gamma)(g_{1,13} + g_{2,23} + g_{3,33}))\end{aligned}\quad (2.31)$$

and the traction vector $\tau(\xi) = \{t_i(\xi)\}$ is obtained by applying the traction operator to g_i :

$$\begin{aligned}t_1 &= \mu(n_1(2g_{1,1} + 2\gamma\nabla \cdot \mathbf{g}) + n_2(g_{1,2} + g_{2,1}) + n_3(g_{1,3} + g_{3,1})) \\ t_2 &= \mu(n_1(g_{1,2} + g_{2,1}) + n_2(2g_{2,2} + 2\gamma\nabla \cdot \mathbf{g}) + n_3(g_{2,3} + g_{3,2})) \\ t_3 &= \mu(n_1(g_{1,3} + g_{3,1}) + n_2(g_{2,3} + g_{3,2}) + n_3(2g_{3,3} + 2\gamma\nabla \cdot \mathbf{g})) .\end{aligned}\quad (2.32)$$

As before, the goal is to remove the volume integral from Eq.(2.30), and in analogy with the Laplace equation we set, for $1 \leq k \leq 3$,

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = -\delta(P, Q) \begin{pmatrix} \delta_{1k} \\ \delta_{2k} \\ \delta_{3k} \end{pmatrix} , \quad (2.33)$$

where $\delta(P, Q)$ and δ_{jk} are the Dirac and Kronecker Delta functions, respectively. The solution of these equations gives the displacement at Q given a point load at P in the direction k , and is known as the Kelvin solution.

This fundamental displacement tensor $U_{kj}(P, Q)$ is

$$U_{kj}(P, Q) = \frac{1}{16\pi\mu(1 - \nu)r} [(3 - 4\nu)\delta_{kj} + r_{,k}r_{,j}] , \quad (2.34)$$

where $r_{,i} = \partial r / \partial x_i$ (see Fig. 2.5). Differentiating with respect to ξ we obtain the traction kernel

$$\begin{aligned}T_{kj}(P, Q) &= -\frac{1}{8\pi(1 - \nu)r^2} \left[\{(1 - 2\nu)\delta_{kj} + 3r_{,k}r_{,j}\} \frac{\partial r}{\partial \mathbf{n}} - \right. \\ &\quad \left. (1 - 2\nu) \{n_j r_{,k} - n_k r_{,j}\} \right] .\end{aligned}$$

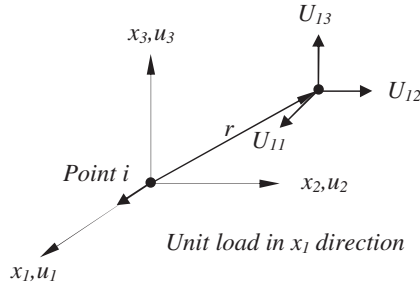


Figure 2.5 Geometric Interpretation of the components of the fundamental solution.

From Eq.(2.30), the exterior boundary integral equation for surface displacement is given by [64, 235]

$$\mathcal{U}(P) \equiv \int_{\Sigma} [T_{kj}(P, Q)u_j(Q) - U_{kj}(P, Q)\tau_j(Q)] dQ = 0 \quad (2.35)$$

Note that it is understood that for $P_k \in \Sigma$, these equations are defined as a limit to the boundary. The stress equation, from which one gets an equation for surface traction by multiplying by the appropriate normal vector, is obtained by differentiating the displacement equation with respect to P . This results in the interior

$$\begin{aligned} \mathcal{S}_I(P) &\equiv \\ \sigma_{lk}(P) + \int_{\Sigma} [S_{lkm}(P, Q)u_m(Q) - D_{lkm}(P, Q)\tau_m(Q)] dQ &= 0 \end{aligned} \quad (2.36)$$

and exterior

$$\mathcal{S}_E(P) \equiv \int_{\Sigma} [S_{lkm}(P, Q)u_m(Q) - D_{lkm}(P, Q)\tau_m(Q)] dQ = 0 \quad (2.37)$$

stress equations. The kernels $D_{lkm}(P, Q)$ (singular) and $S_{lkm}(P, Q)$ (hypersingular) for the stress equation result from differentiating U_{kj} and T_{kj} and are given by

$$\begin{aligned} D_{lkm} &= \frac{1}{8\pi(1-\nu)r^2} [(1-2\nu) \{ \delta_{lm}r_{,k} + \delta_{km}r_{,l} - \delta_{lk}r_{,m} \} + 3r_{,l}r_{,k}r_{,m}] \\ S_{lkm} &= \frac{\mu}{4\pi(1-\nu)r^3} \left[\right. \\ &\quad 3 \frac{\partial r}{\partial \mathbf{n}} \{ (1-2\nu) \delta_{lk}r_{,m} + \nu (\delta_{km}r_{,l} + \delta_{lm}r_{,k}) - 5r_{,l}r_{,k}r_{,m} \} \\ &\quad + (1-2\nu) (3n_m r_{,l}r_{,k} + n_k \delta_{lm} + n_l \delta_{km}) \\ &\quad \left. + 3\nu (n_l r_{,k}r_{,m} + n_k r_{,l}r_{,m}) - (1-4\nu)n_m \delta_{lk} \right] . \end{aligned} \quad (2.38)$$

In addition to being 3×3 matrices, the Kelvin solution and its derivatives are lengthier expressions than the corresponding formulas for the Laplace equation. Nevertheless, they are still rational algebraic expressions, and the singular integration methods developed for the Laplace equation will carry over directly to elasticity. The important point is that the order of the singularities at $r = 0$ is precisely the same for both differential equations.

Two Dimensions. The fundamental displacement tensor $U_{kj}(\xi, \mathbf{y})$ in two dimensions is

$$U_{kj}(\xi, \mathbf{y}) = \frac{1}{8\pi\mu(1-\nu)} [-(3-4\nu)\delta_{kj}\log(r) + r_{,k}r_{,j}] , \quad (2.39)$$

and the corresponding derivatives of this function are

$$\begin{aligned} T_{kj}(\xi, \mathbf{y}) &= -\frac{1}{4\pi(1-\nu)r} \left[\{(1-2\nu)\delta_{kj} + 2r_{,k}r_{,j}\} \frac{\partial r}{\partial \mathbf{n}} - \right. \\ &\quad \left. (1-2\nu) \{n_j r_{,k} - n_k r_{,j}\} \right] . \\ D_{lkm} &= \frac{1}{4\pi(1-\nu)r} [(1-2\nu) \{\delta_{lm}r_{,k} + \delta_{km}r_{,l} - \delta_{lk}r_{,m}\} + 2r_{,l}r_{,k}r_{,m}] \\ S_{lkm} &= \frac{\mu}{2\pi(1-\nu)r^2} \left[\right. \\ &\quad 2\frac{\partial r}{\partial \mathbf{n}} (\{1-2\nu\} \delta_{lk}r_{,m} + \nu(\delta_{km}r_{,l} + \delta_{lm}r_{,k}) - 4r_{,l}r_{,k}r_{,m}) \\ &\quad + (1-2\nu)(2n_m r_{,l}r_{,k} + n_k \delta_{lm} + n_l \delta_{km}) \\ &\quad \left. + 2\nu(n_l r_{,k}r_{,m} + n_k r_{,l}r_{,m}) - (1-4\nu)n_m \delta_{lk} \right] . \end{aligned} \quad (2.40)$$

2.4 NUMERICAL APPROXIMATION

Having defined the boundary integral equations, we now begin the consideration of their numerical solution. Analytic solutions of the integral equations are no easier to obtain than for the original differential equation, and thus it is necessary to reduce the continuous equations to a discrete system of linear equations that can be solved.

The main purpose of this section is to introduce the Galerkin method that will be used throughout the book. To do this we must first discuss the scheme for the two main approximations, and hence the two main sources of error in the calculation: interpolation of the boundary and the interpolation of the boundary functions. Moreover, before describing Galerkin, we first give a brief description of the simpler *collocation* method.

2.4.1 Approximations

One of the most convenient ways of accomplishing the necessary approximations is an *isoparametric* method, in which the boundary and boundary functions are represented through the same set of simple *shape functions* defined on a parameter space. The best way to describe this is through an example, and we choose the 3D linear triangular element that will be employed extensively in Chapter 4.

The boundary surface is approximated as a sum of small surface patches called elements, $\Sigma = \bigcup E_i$ (see Fig. 1.1), and each element is defined as a mapping from a fixed parameter domain in \mathcal{R}^2 . This parameter space can be selected more or less arbitrarily; however, for the Galerkin integration over the linear triangular element it will turn out to be convenient to choose an equilateral triangle as shown in

Fig. 2.6. The parametric variables will be called $\{\eta, \xi\}$, and the equilateral triangle with vertices $v_1 = (-1, 0)$, $v_2 = (1, 0)$, $v_3 = (0, \sqrt{3})$ is defined via $-1 \leq \eta \leq 1$, $0 \leq \xi \leq \sqrt{3}(1 - |\eta|)$.

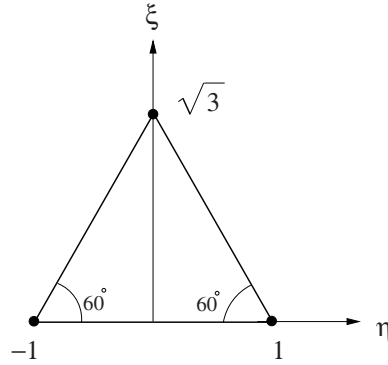


Figure 2.6 Isoparametric equilateral triangular linear element in $\{\eta, \xi\}$ space, where $-1 \leq \eta \leq 1$, $0 \leq \xi \leq \sqrt{3}(1 - |\eta|)$.

The surface and function interpolations will be constructed based upon the three linear shape functions

$$\begin{aligned}\psi_1(\eta, \xi) &= \frac{\sqrt{3}(1 - \eta) - \xi}{2\sqrt{3}} \\ \psi_2(\eta, \xi) &= \frac{\sqrt{3}(1 + \eta) - \xi}{2\sqrt{3}} \\ \psi_3(\eta, \xi) &= \frac{\xi}{\sqrt{3}},\end{aligned}\tag{2.41}$$

defined by the property that $\psi_l(v_j) = \delta_{lj}$. A linear triangular element E_i is defined by three nodal points $\{Q_j = (x_j, y_j, z_j)\}$, oriented so as to give an exterior normal, and the linear interpolation of these nodes defines the approximate boundary patch. The mapping from parameter space to E_i is then easily written as

$$\Sigma_i(\eta, \xi) = \sum_{j=1}^3 (x_j, y_j, z_j) \psi_j(\eta, \xi),\tag{2.42}$$

and the corresponding approximate surface potential and flux on this element are

$$\begin{aligned}\phi(\eta, \xi) &= \sum_{j=1}^3 \phi(Q_j) \psi_j(\eta, \xi) \\ \frac{\partial \phi}{\partial \mathbf{n}}(\eta, \xi) &= \sum_{j=1}^3 \frac{\partial \phi}{\partial \mathbf{n}}(Q_j) \psi_j(\eta, \xi).\end{aligned}\tag{2.43}$$

Note that $\phi(\eta, \xi)$ is a convenient shorthand notation: the function ϕ is in reality defined on the surface Σ , and not on the parameter space. Thus by $\phi(\eta, \xi)$ we really mean $\phi(\Sigma_i(\eta, \xi))$.

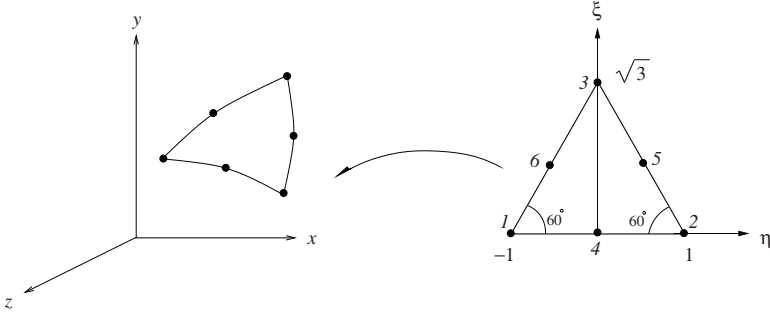


Figure 2.7 The equilateral triangle parameter space $\{\eta, \xi\}$ is mapped onto the quadratic triangular element in 3D.

A linear approximation suffices for many, if not most, applications. However, for the development of an effective approximation at a crack front, a quadratic interpolation is highly useful. For completeness, we therefore list here the quadratic shape functions, again for the equilateral triangle parameter space (see Fig. 2.7). These functions are once again constructed so that $\psi_l(v_j) = \delta_{lj}$, $1 < l, j < 6$.

$$\begin{aligned}
 \psi_1(\eta, \xi) &= \frac{(\xi + \sqrt{3}\eta - \sqrt{3})(\xi + \sqrt{3}\eta)}{6} \\
 \psi_2(\eta, \xi) &= \frac{(\xi - \sqrt{3}\eta - \sqrt{3})(\xi - \sqrt{3}\eta)}{6} \\
 \psi_3(\eta, \xi) &= \frac{(2\xi - \sqrt{3})\xi}{3} \\
 \psi_4(\eta, \xi) &= \frac{(\xi + \sqrt{3}\eta - \sqrt{3})(\xi - \sqrt{3}\eta - \sqrt{3})}{3} \\
 \psi_5(\eta, \xi) &= -2\frac{(\xi - \sqrt{3}\eta - \sqrt{3})\xi}{3} \\
 \psi_6(\eta, \xi) &= -2\frac{(\xi + \sqrt{3}\eta - \sqrt{3})\xi}{3}
 \end{aligned} \tag{2.44}$$

2.4.2 Collocation

In collocation the boundary integral equations are enforced at specified points. In its simplest form, these collocation points are chosen to be the nodes used to discretize the boundary, see Fig. 2.7. Thus, for example, if there are M nodes on the boundary, the exterior limit potential equation (we could work equally well with the flux equation) can be used to generate the M collocation equations

$$\begin{pmatrix} \mathcal{P}(P_1) \\ \mathcal{P}(P_2) \\ \vdots \\ \vdots \\ \mathcal{P}(P_M) \end{pmatrix} = 0, \tag{2.45}$$

where \mathcal{P} is defined in Eq.(2.23). If the corresponding $M \times 1$ vectors of boundary values of potential and flux are denoted by

$$\begin{bmatrix} \phi(P_j) \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi}{\partial \mathbf{n}}(P_j) \end{bmatrix}, \quad (2.46)$$

the collocation equations (after the numerical approximations discussed below) eventually become the matrix equation

$$\mathcal{H} \begin{bmatrix} \phi(P_j) \end{bmatrix} = \mathcal{G} \begin{bmatrix} \frac{\partial \phi}{\partial \mathbf{n}}(P_j) \end{bmatrix}, \quad (2.47)$$

where \mathcal{H} and \mathcal{G} are $M \times M$ matrices. Of the $2M$ surface quantities in this equation, M are given by the boundary conditions; rearranging columns, collecting unknowns as the vector x , a linear system $Ax = b$ is obtained and x can be determined.

The elements of the \mathcal{H} matrix, and a similar discussion for \mathcal{G} , arise from approximating the integrals

$$\int_{\Sigma} \phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_k, Q) dQ \quad (2.48)$$

for $1 \leq k \leq M$. Specifically, using the approximations for Σ and ϕ discussed above we can write

$$\begin{aligned} \int_{\Sigma} \phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_k, Q) dQ &\approx \sum_{E_l} \int_{E_l} \phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P_k, Q) dQ \\ &= \sum_{E_l} \sum_j \phi(Q_j) \int \psi_j(\eta, \xi) J(\eta, \xi) \frac{\partial G}{\partial \mathbf{n}}(P_k, Q_l(\eta, \xi)) d\xi d\eta \\ &= \sum_{E_l} \sum_j \phi(Q_j) \mathcal{I}_{kj}^l \\ &= \sum_j \phi(Q_j) \sum_{E_l} \mathcal{I}_{kj}^l = \sum_j \mathcal{H}_{kj} \phi(Q_j) \end{aligned} \quad (2.49)$$

where $J(\eta, \xi)$ is the jacobian of the surface mapping (constant for a linear interpolation). Thus, the matrix elements \mathcal{H}_{kj} (and similarly for \mathcal{G}_{kj}) are given by

$$\mathcal{H}_{kj} = \sum_{E_l} \mathcal{I}_{kj}^l, \quad (2.50)$$

the integrals being of the form

$$\mathcal{I}_{kj}^l = \int \psi_j(\eta, \xi) J(\eta, \xi) \frac{\partial G}{\partial \mathbf{n}}(P_k, Q_l(\eta, \xi)) d\xi d\eta, \quad (2.51)$$

and the notation $Q_l(\eta, \xi)$ indicates that this surface mapping is defined by the element E_l . Note that everything in the above integral is a known quantity and can, assuming the singular integrals have finite limiting values, be computed. Moreover, for collocation, these integrations have to be carried out for every node/element pair $\{P_k, E_l\}$.

Singular integrals therefore arise when $P_k \in E_l$. Also note that the \mathcal{H} and \mathcal{G} matrices are dense and non-symmetric, unlike the sparse systems obtained using finite element methods. In fact, these matrices, generally speaking, have no exploitable

structure. The best that can be said is that as the Green's function kernels generally die off for large r , the singular integrals tend to be the dominant terms. These terms are located near the diagonal, and this observation is potentially useful for an iterative method for solving the linear system. Finally, it is useful to observe that \mathcal{H} and \mathcal{G} depend only on geometry, not on the boundary conditions.

2.4.3 Galerkin Approximation

In contrast to collocation, the Galerkin approach does not require that the boundary integral equations be satisfied at any point. Instead, the equations are enforced in a weighted average:

$$\begin{aligned} \int_{\Sigma} \hat{\psi}_k(P) \mathcal{P}(P) dP &= 0 \\ \int_{\Sigma} \hat{\psi}_k(P) \mathcal{F}(P) dP &= 0 \end{aligned} \quad (2.52)$$

In mathematical terminology, we give up the ‘strong’ requirement that the integral equations are actually satisfied at any given point, in exchange for a ‘weak solution’ in which the equations hold in an integrated sense. This requirement has a nice geometric interpretation: the approximate Galerkin solution is the exact solution projected onto the subspace consisting of all functions which are linear combinations of the shape functions. The Galerkin solution is therefore the linear combination which is the ‘closest’ to the exact solution.

In the standard Galerkin procedure, the weight functions $\hat{\psi}_k(P)$ are composed of all shape functions that are non-zero at a node P_k .

For the double integral over the surface, two sets of parameters will be needed. We will use η, ξ for the outer P integration and η^*, ξ^* for the inner Q integral. Repeating for Galerkin the previous calculation that led to the expression for the collocation \mathcal{H} matrix elements, we obtain

$$\begin{aligned} & \int_{\Sigma} \hat{\psi}_k(P) \int_{\Sigma} \phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P, Q) dQ \\ &= \sum_{E_m} \sum_{E_l} \int_{E_m} \hat{\psi}_k(P) \int_{E_l} \phi(Q) \frac{\partial G}{\partial \mathbf{n}}(P, Q) dQ \\ &= \sum_{E_m, E_l} \sum_j \phi(Q_j) \int \hat{\psi}_k(\eta, \xi) J_p(\eta, \xi) \int \psi_j(\eta^*, \xi^*) J_q(\eta^*, \xi^*) G_{\mathbf{n}}^{m,l} d\xi^* d\eta^* d\xi d\eta \\ &= \sum_{E_m, E_l} \sum_j \phi(Q_j) \mathcal{I}_{kj}^{m,l} \\ &= \sum_j \phi(Q_j) \sum_{E_m, E_l} \mathcal{I}_{kj}^{m,l} = \sum_j \mathcal{H}_{kj} \phi(Q_j) \end{aligned} \quad (2.53)$$

where the shorthand

$$G_{\mathbf{n}}^{m,l} = \frac{\partial G}{\partial \mathbf{n}}(P_m(\eta, \xi), Q_l(\eta^*, \xi^*)) \quad (2.54)$$

has been introduced. The matrix elements for Galerkin are therefore given by

$$\mathcal{H}_{kj} = \sum_{E_m, E_l} \mathcal{I}_{kj}^{m,l}, \quad (2.55)$$

where the component integrals are of the form

$$\mathcal{I}_{kj}^{m,l} = \int \hat{\psi}_k(\eta, \xi) J_p(\eta, \xi) \int \psi_j(\eta^*, \xi^*) J_q(\eta^*, \xi^*) G_{,\mathbf{n}}^{m,l} d\xi^* d\eta^* d\xi d\eta . \quad (2.56)$$

Although more complicated than for collocation, everything in this expression is once again known, dependent solely on the problem geometry. As an integration is now required for every pair of elements $\{E_m, E_l\}$, singular integrals will occur when the elements are *coincident* $E_m = E_l$, or *adjacent*, sharing either an edge or a vertex. Thus, for three dimensional problems, there are four possible configurations for the P and Q elements that must be considered, as illustrated in Fig. 2.8:

- *Non-singular case*, when the source point P and the field point Q lie on distinct elements, that do not share a common vertex or edge.
- *Coincident case*, when the source point P and the field point Q lie in the same element;
- *Edge adjacent case*, when two elements share a common edge; and
- *Vertex adjacent case*, when a vertex is the only common node between the two elements.

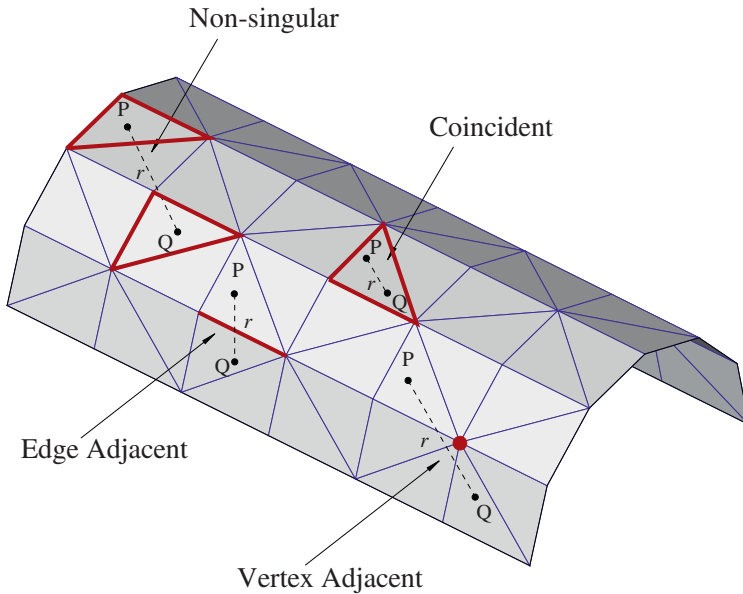


Figure 2.8 Four different cases considered for integration: (a) non-singular; (b) coincident; (c) edge adjacent; and (d) vertex adjacent.

In two dimensions there are only two singular cases, coincident and common vertex.

Although Galerkin clearly requires more computational effort, there are significant benefits. In addition to the very important concern that hypersingular integrals can be handled far more simply than collocation (see Section 2.5.1), Galerkin is generally more accurate, can lead to a symmetric coefficient matrix, and provides a more natural and effective algorithm for dealing with boundary corners and edges.

2.4.4 Symmetric-Galerkin

Unlike collocation, there is some structure to the Galerkin matrices. Of the four matrices arising from the four possible kernel functions (two for the potential equation, two for the flux equation), two matrices are symmetric. The kernel functions

$$\begin{aligned} G(P, Q) &= G(Q, P) \\ \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P, Q) &= \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(Q, P), \end{aligned} \quad (2.57)$$

are symmetric, and as P and Q are treated on an equal footing in the double boundary integration, this property is passed onto the resulting matrices. As a consequence, if the problem is purely Dirichlet (ϕ specified everywhere on Σ), use of the potential equation leads to a symmetric coefficient matrix for the unknown flux. For a Neumann problem, the flux equation similarly produces a symmetric matrix.

These facts lead to a general prescription: if the potential equation is invoked on the part of Σ having Dirichlet boundary conditions, and the negative of the flux equation is employed on the Neumann surface, the coefficient matrix for this mixed problem is symmetric. This follows from observing that the first order derivatives are related,

$$\nabla_q G(P, Q) = -\nabla_p G(P, Q) = \nabla_p G(Q, P). \quad (2.58)$$

Multiplying one of the equations by -1 is needed to account for the negative sign when columns are rearranged to bring all unknowns to one side.

2.5 HYPERSINGULAR INTEGRATION: AN EXAMPLE

Limit evaluation of the singular integrals, collocation or Galerkin, is relatively straightforward and unexciting, except for the hypersingular integral. To get a feel for the behavior of this integral, and to contrast collocation and Galerkin, it is instructive to examine a simple setting that allows for complete evaluation of the integrals.

This section will first demonstrate that collocation of the hypersingular integral

$$\lim_{P_E \rightarrow P_0} \int_{\Sigma} \phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_E, Q) dQ \quad (2.59)$$

is not a simple matter, and in fact *cannot* be accomplished for the linear interpolation discussed above. Specifically, we show that existence of the boundary limit requires that $\phi(P)$ be differentiable (\mathcal{C}^1) at the collocation point P_0 (the limit can be either interior or exterior). Second, the corresponding Galerkin integral will be shown to be finite.

The need for the differentiability condition for collocation can be understood by the following hand-waving argument. One way to attack the evaluation of the hypersingular integral would be to use integration by parts, moving the second derivative off of the Green's function and onto the function $\phi(Q)$. This would return the singularity in the Green's function quantity to the more manageable r^{-2} . However, in order to do this, $\phi(Q)$ must be differentiable. Regarding Galerkin, note that the extra outer integral with respect to P can be viewed as a counterbalance to the second derivative (with respect to $\mathbf{N}(P)$) that created the hypersingular kernel, and thus the Galerkin form should exist under the weaker continuity requirement. Standard boundary integral interpolations are continuous crossing from one element to another, but in general not differentiable. This \mathcal{C}^1 condition is therefore a serious impediment for a collocation computation, and consequently a principal argument in favor of Galerkin.

There is another good motivation for carrying out the calculations below: they illustrate, in a very concrete fashion, the general *limit to the boundary* and analytic integration procedures that will be used in Chapter 3 and Chapter 4.

2.5.1 Collocation: \mathcal{C}^1 Condition

To establish the \mathcal{C}^1 condition, evaluation of the hypersingular integral will be attempted for a boundary node P on a flat boundary segment. It will be seen that, without differentiability, the limit, and hence the integral, fails to exist. A similar calculation can be carried out in three dimensions [107].

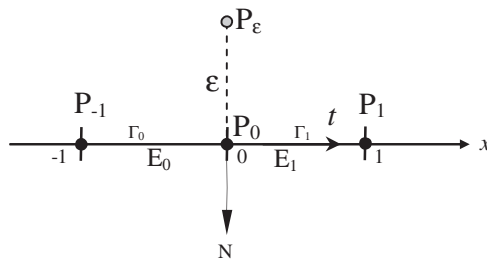


Figure 2.9 Limit to the boundary for collocating the hypersingular integral.

Consider, as shown in Fig.2.9, a flat boundary segment lying along the x -axis, consisting of the interval $[-1, 1]$. This segment is divided into the two elements $E_0 = [-1, 0]$ and $E_1 = [0, 1]$, outward normals $\mathbf{n} = \mathbf{N} = (0, -1)$, with nodes $P_{-1} = (-1, 0)$, $P_0 = (0, 0)$, and $P_1 = (1, 0)$.

We seek to collocate the hypersingular integral at the point P_0 . The interior limit to the boundary is effected by replacing P_0 with the point

$$P_\epsilon = P_0 - \epsilon \mathbf{N} , \quad (2.60)$$

where $\epsilon > 0$. With P off the boundary the integral exists, and then the limit

$$\lim_{\epsilon \rightarrow 0} -\frac{1}{2\pi} \int_{E_0 + E_1} \phi(Q) \left\{ -\frac{\mathbf{N} \cdot \mathbf{n}}{r^2} + 2 \frac{(\mathbf{n} \cdot \mathbf{R})(\mathbf{N} \cdot \mathbf{R})}{r^4} \right\} dQ \quad (2.61)$$

can be considered. On E_0 the linear interpolations for the boundary and for ϕ are

$$\begin{aligned}\Gamma_0(t) &= (-1, 0) \psi_1(t) + (0, 0) \psi_2(t) = (-1 + t, 0) \\ \phi(\Gamma_0(t)) &= \phi(P_{-1}) \psi_1(t) + \phi(P_0) \psi_2(t) = \phi(P_{-1}) + t(\phi(P_0) - \phi(P_{-1})) \\ r^2 &= (-1 + t)^2 + \varepsilon^2\end{aligned}\tag{2.62}$$

where the linear shape functions $0 < t < 1$ are

$$\psi_1(t) = 1 - t\tag{2.63}$$

$$\psi_2(t) = t.\tag{2.64}$$

Similarly for the element E_1 ,

$$\begin{aligned}\Gamma_1(t) &= (0, 0) \psi_1(t) + (1, 0) \psi_2(t) = (t, 0) \\ \phi(\Gamma_1(t)) &= \phi(P_0) \psi_1(t) + \phi(P_1) \psi_2(t) = \phi(P_0) + t(\phi(P_1) - \phi(P_0)) \\ r^2 &= t^2 + \varepsilon^2\end{aligned}\tag{2.65}$$

Integrating first the constant term from ϕ over E_0 (cf. Eq.(2.62) and Eq.(2.61)) we find, with a coefficient of $\phi(P_{-1})/2\pi$,

$$\begin{aligned}& \int_0^1 \frac{1}{(-1 + t)^2 + \varepsilon^2} dt - \int_0^1 \frac{2\varepsilon^2}{((-1 + t)^2 + \varepsilon^2)^2} dt \\ &= \frac{\tan^{-1}(1/\varepsilon)}{\varepsilon} - \left(\frac{1}{1 + \varepsilon^2} + \frac{\tan^{-1}(1/\varepsilon)}{\varepsilon} \right) \\ &= -\frac{1}{1 + \varepsilon^2} = -1\end{aligned}\tag{2.66}$$

Note that as $\tan^{-1}(1/\varepsilon) \rightarrow \pi/2$, a $1/\varepsilon$ term appears separately in each term, but they fortunately cancel by themselves. This is in fact a general feature of the hypersingular integration that will show up in the general analysis in subsequent chapters.

For the more interesting linear term, coefficient of $(\phi(P_0) - \phi(P_{-1}))/2\pi$, integrating over E_0 yields

$$\begin{aligned}& \int_0^1 t \left[\frac{1}{(-1 + t)^2 + \varepsilon^2} - \frac{2\varepsilon^2}{((-1 + t)^2 + \varepsilon^2)^2} \right] dt \\ &= \log(\varepsilon^2)/2 - \log(1 + \varepsilon^2)/2 \\ &= \log(\varepsilon^2)/2\end{aligned}\tag{2.67}$$

For E_1 the constant term is again finite, while the linear term has a coefficient of $(\phi(P_1) - \phi(P_0))/2\pi$ and resulting integral

$$\begin{aligned}& \int_0^1 t \left[\frac{1}{t^2 + \varepsilon^2} - \frac{2\varepsilon^2}{(t^2 + \varepsilon^2)^2} \right] dt \\ &= -\log(\varepsilon^2)/2 + \log(1 + \varepsilon^2)/2 - \frac{1}{1 + \varepsilon^2} \\ &= -\log(\varepsilon^2)/2 - 1\end{aligned}\tag{2.68}$$

For the hypersingular integral to have a finite limiting value, the two $\log(\epsilon^2)$ terms must cancel, and thus it follows that

$$\phi_0 - \phi_{-1} = \phi_1 - \phi_0 \quad (2.69)$$

The expressions on the left and right are in fact the tangential derivatives of $\phi(P_0)$ on E_0 and E_1 , respectively, as defined by the linear interpolations. Thus for a non- \mathcal{C}^1 interpolation at P_0 the hypersingular integral does not exist.

More precisely, without a differentiable interpolation of ϕ at P_0 , there is no way to define the integral in Eq.(2.61) so that the flux is *continuous from the interior*. Although it is possible to implement \mathcal{C}^1 interpolations (*e.g.*, Overhauser elements), they are in general difficult and computationally expensive, especially in 3D. What has generally been done in the literature is to invoke the Hadamard Finite Part (Ioakimidis [138], Hadamard [129], Ioakimidis [139], Kaya & Erdogan [144]). This definition will assign a value to this integral, essentially by ignoring the $\log(\epsilon^2)$ singularity, but lacking continuity to the boundary, the physical relevance of this definition is dubious.

Finally, if collocating the hypersingular integral is difficult at a boundary point where the surface is perfectly flat, it can be expected that at a boundary corner/edge, the situation will be even more complicated. We will return to this issue in Chapter 3 when we consider the Galerkin corner treatment.

2.5.2 Galerkin: \mathcal{C}^0

Another observation stemming from the computation in the previous section is to note that obtaining a finite value for the hypersingular integral demands integrating over *a complete neighborhood* of the singular point P . With or without differentiability, the individual integrals over E_1 and E_0 are always divergent, the linear term from the shape functions will always produce a $\log(\epsilon^2)$ contribution. The best that can be hoped for is a cancellation of the divergences when the two integrals are summed, *i.e.* the total singular integral at P_0 is computed.

As Galerkin can be viewed as collocating at points P interior to an element – namely at the Gauss points for the P integration – it might appear that this requirement of a complete neighborhood is always met. However, for a node on the edge of an element, half of its neighborhood is contained in the coincident integral, and the remaining half neighborhood executed in the corresponding adjacent edge integration. It is therefore possible that the hypersingular kernel is sufficiently singular that the Galerkin hypersingular coincident and adjacent edge integrals will be separately divergent. This is indeed the situation, with once again $\log(\epsilon^2)$ terms appearing in both integrals. Nevertheless, under the weaker condition of continuity, \mathcal{C}^0 , the Galerkin hypersingular integral is finite, the divergent terms cancel. Standard boundary integral approximations (linear, quadratic, *etc.*) are continuous crossing element boundaries, and can therefore be safely employed for hypersingular equations.

It is important to note that this presumed cancellation of divergent terms cannot be trusted to the numerics. A direct evaluation of the hypersingular Galerkin integrals (coincident and adjacent edge) must explicitly identify the divergent terms, so that they can be exactly cancelled. This can in fact be achieved in general, but will first be illustrated below by re-doing the above simple collocation calculation in Galerkin form.

Considering just the two elements E_0 and E_1 in Fig. 2.9, the Galerkin formulation gives rise to four integrals: two coincident and two adjacent (one with E_0 as the outer integral and one with E_1). We treat only the coincident (E_0, E_0) and adjacent with (E_0, E_1) (E_0 as the outer P integral), the remaining two integrals are similar. As with the collocation calculation, attention is focused on the equation written at the node P_0 . In terms of Galerkin, this means that, for E_0 , we only examine the integral for the weight function $\psi_2(P)$, the shape function that is equal to 1 at P_0 ($t = 1$).

In reference to the above discussion on complete neighborhoods, note that adding the two integrals (E_0, E_0) and (E_0, E_1) results in a complete inner Q integral that covers P_0 . Thus, we will eventually find this sum to be finite.

2.5.2.1 Coincident E_0, E_0 integration We seek to evaluate

$$\lim_{\varepsilon \rightarrow 0} \int_{E_0} \psi_2(P) \int_{E_0} \phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_\varepsilon, Q) dQ dP, \quad (2.70)$$

and unlike the collocation calculation in Section 2.5.1, the Galerkin double integral requires two parametric variables. Using $s < 1$, for the parameter for the inner Q integral, t for P , we have

$$\begin{aligned} Q(s) &= (-1 + s, 0) \\ \phi(Q(s)) &= \phi(P_{-1})\psi_1(s) + \phi(P_0)\psi_2(s) \\ P_\varepsilon(t) &= (-1 + t, \varepsilon) \\ r^2 &= (s - t)^2 + \varepsilon^2. \end{aligned} \quad (2.71)$$

As just noted, we choose to examine the equation at P_0 , and thus $\psi_2(t) = t$. Every point P has now been shifted off the boundary, and the singular point for the Q integral is $s = t$. The integrals with the two shape functions $\psi_j(s)$, $j = 1, 2$ can be evaluated analytically, resulting in

$$\begin{aligned} &-\frac{1}{4\pi} & j = 1 \\ \frac{1}{4\pi} (1 + \log(\varepsilon^2)) & j = 2 \end{aligned} \quad (2.72)$$

Thus, the coincident integral multiplying $\phi(P_0)$ is divergent, whereas the zero in the weight function at $t = 0$ is sufficient to kill the divergence at P_{-1} . Of course, had we considered the weight function $1 - t$, the opposite situation would result (and subsequent cancellation of the divergent term relying on the element to the left of E_0). It is useful to note the different origins of the collocation and Galerkin $\log(\varepsilon^2)$ terms: for collocation the divergence comes from the linear term in the expansion of the potential at P_0 , and therefore has a coefficient that involves both $\phi(P_{-1})$ and $\phi(P_0)$. For Galerkin however, the coefficient simply involves the value of ϕ at the singular point, reflecting the fact that the divergence now arises from the constant term in the shape function.

We now show that adding in the adjacent integral results in a finite quantity.

2.5.2.2 Adjacent integration For E_1 as the inner Q integral, namely

$$\lim_{\varepsilon \rightarrow 0} \int_{E_0} \psi_2(P) \int_{E_1} \phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_\varepsilon, Q) dQ dP. \quad (2.73)$$

we now have

$$\begin{aligned} Q(s) &= (s, 0) \\ \phi(Q(s)) &= \phi(P_0)\psi_1(s) + \phi(P_1)\psi_2(s) \\ r^2 &= (s - t + 1)^2 + \varepsilon^2. \end{aligned} \tag{2.74}$$

The kernel function is therefore singular only when $s = 0$ and $t = 1$, which corresponds to $Q = P = P_0$. Once again doing the complete integral analytically, we find

$$-\frac{1}{4\pi} (1 + \log(\varepsilon^2) - 4\log(2)) \quad \begin{matrix} j = 1 \\ \frac{1}{4\pi} \\ j = 2 \end{matrix} \tag{2.75}$$

Thus, as $\psi_2(s = 0) = 0$, the $j = 2$ integral is finite, whereas for $j = 1$ (again having coefficient $\phi(P_0)$) the adjacent integral is also divergent. However, the sum of coincident and adjacent is perfectly finite. The Galerkin form of the hypersingular integral therefore exists for the C^0 linear interpolation.

In general, it will not be necessary or possible – especially in three dimensions – to do the complete integral analytically. It is only necessary to do sufficient analytic integration to produce the divergent terms, and allow setting $\varepsilon = 0$ in the well-behaved remainder.

Finally, note that in three dimensions, the individual adjacent vertex integrals are also carried out over a partial neighborhood of the singular point. However in this case the singularity is limited to one point in a four dimensional parameter space, and the singularity is therefore sufficiently weak that the vertex integrals will turn out to be separately finite.

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Sutradhar, A.; Paulino, G.; Gray, L.J.

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