

Preliminaries

The scheduling theory uses notions and methods from different disciplines of mathematics. Therefore, any systematic presentation of an arbitrary branch of the theory needs some mathematical background. The first part of the book introduces this background.

The part is composed of five chapters. In Chap. 1, we present the mathematical notation, the basic definitions and the results. The essential concepts related to decision problems and algorithms are recalled in Chap. 2. The definitions and the most important results of the theory of \mathcal{NP} -completeness are presented in Chap. 3. The basics of the scheduling theory and time-dependent scheduling are given in Chap. 4 and Chap. 5, respectively.

Chapter 1 is composed of three sections. In Sect. 1.1, we introduce the notation and terminology used in this book. In Sect. 1.2, we give some mathematical preliminaries used in subsequent chapters. The chapter is completed with bibliographic notes in Sect. 1.3.

1.1 Mathematical notation

We assume that the reader is familiar with basic mathematical notions. Therefore, we explain here only the notation that will be used throughout this book.

1.1.1 Sets and vectors

We will write $a \in A$ ($a \notin A$) if a is (is not) an element of a set A . If $a_1 \in A$, $a_2 \in A, \dots, a_n \in A$, we will simply write $a_1, a_2, \dots, a_n \in A$.

If an element a is (is not) equal to an element b , we will write $a = b$ ($a \neq b$). If $a = b$ by definition, we will write $a := b$. In a similar way, we will denote the equality (inequality) of numbers, sets, sequences, etc.

The set composed only of elements a_1, a_2, \dots, a_n will be denoted by $\{a_1, a_2, \dots, a_n\}$. The maximal (minimal) element in set $\{a_1, a_2, \dots, a_n\}$ will be denoted by $\max\{a_1, a_2, \dots, a_n\}$ ($\min\{a_1, a_2, \dots, a_n\}$).

If set A is a subset of set B , i.e., every element of set A is an element of set B , we will write $A \subseteq B$. If A is a strict subset of B , i.e., $A \subseteq B$ and $A \neq B$, we will write $A \subset B$. The empty set will be denoted by \emptyset .

The number of elements of set A will be denoted by $|A|$. The power set of set A , i.e., the set of all subsets of A , will be denoted by 2^A .

For any sets A and B , the union, intersection and difference of A and B will be denoted by $A \cup B$, $A \cap B$ and $A \setminus B$, respectively.

The Cartesian product of sets A and B will be denoted by $A \times B$. The Cartesian product of $n \geq 2$ copies of a set A will be denoted by A^n .

A partial order, i.e., a reflexive, antisymmetric and transitive binary relation, will be denoted by \prec . If $x \prec y$ or $x = y$, we will write $x \preceq y$.

The set-theoretic sum (product) of all elements of a set A will be denoted by $\bigcup_{a_i \in A} a_i$ ($\bigcap_{a_i \in A} a_i$). The union (intersection) of a family of sets A_k , $k \in K$, will be denoted by $\bigcup_{k \in K} A_k$ ($\bigcap_{k \in K} A_k$).

The sets of all natural, integer, rational and real numbers will be denoted by \mathbb{N} , \mathbb{Z} , \mathbb{Q} and \mathbb{R} , respectively. The subsets of positive elements of sets \mathbb{Z} , \mathbb{Q} and \mathbb{R} will be denoted by \mathbb{Z}_+ , \mathbb{Q}_+ and \mathbb{R}_+ , respectively. The subset of \mathbb{N} composed of the numbers that are not greater than a fixed $n \in \mathbb{N}$ will be denoted by $\{1, 2, \dots, n\}$ or I_n .

Given a set A and a property \mathfrak{P} , we will write $B = \{a \in A : \mathfrak{P} \text{ holds for } a\}$ to denote that B is the set of all elements of set A for which property \mathfrak{P} holds. For example, a closed interval $\langle a, b \rangle$ for $a, b \in \mathbb{R}$, $a \leq b$, can be defined as the set $\langle a, b \rangle = \{x \in \mathbb{R} : a \leq x \leq b\}$.

A $(n \geq 1)$ -dimensional vector space over \mathbb{R} , its positive orthant and the interior of the orthant will be denoted by \mathbb{R}^n , \mathbb{R}_+^n and $\text{int}\mathbb{R}_+^n$, respectively.

A row (column) vector $x \in \mathbb{R}^n$ composed of numbers x_1, x_2, \dots, x_n will be denoted by $x = [x_1, x_2, \dots, x_n]$ ($x = [x_1, x_2, \dots, x_n]^\top$). A norm (the l_p norm) of vector x will be denoted by $\|x\|$ ($\|x\|_p$). The scalar product of vectors x and y will be denoted by $x \circ y$.

The set of all Pareto (weakly Pareto) optimal solutions from a set of all feasible solutions X will be denoted by X_{Par} ($X_{\text{w-Par}}$).

1.1.2 Sequences

A sequence composed of numbers x_1, x_2, \dots, x_n will be denoted by $(x_j)_{j=1}^n$ or (x_1, x_2, \dots, x_n) . If the range of indices of elements of sequence $(x_j)_{j=1}^n$ is fixed, the sequence will be denoted by (x_j) . In a similar way, we will denote sequences of sequences, e.g., the sequence of pairs $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ will be denoted by $((x_j, y_j))_{j=1}^n$, $((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n))$ or $((x_j, y_j))$.

A sequence (z_k) that is a concatenation of sequences (x_i) and (y_j) will be denoted by $(x_i | y_j)$. If A and B are sets of numbers, the sequence composed of elements of A followed by elements of B will be denoted by $(A|B)$.

A sequence (x_j) in which elements are arranged in the non-decreasing (non-increasing) order will be denoted by $(x_j \nearrow)$ ($(x_j \searrow)$). An empty sequence will be denoted by (ϕ) .

The algebraic sum (product) of numbers x_k, x_{k+1}, \dots, x_m for $k, m \in \mathbb{N}$, will be denoted by $\sum_{i=k}^m x_i$ ($\prod_{i=k}^m x_i$). If the indices of components of the sum (product) belong to a set J , then the sum (product) will be denoted by $\sum_{j \in J} x_j$ ($\prod_{j \in J} x_j$). If $k > m$ or $J = \emptyset$, then $\sum_{i=k}^m x_i = \sum_{j \in J} x_j := 0$ and $\prod_{i=k}^m x_i = \prod_{j \in J} x_j := 1$.

1.1.3 Functions

A function f from a set X to a set Y will be denoted by $f : X \rightarrow Y$. The value of function $f : X \rightarrow Y$ for some $x \in X$ will be denoted by $f(x)$.

If a function f is a monotonically increasing (decreasing) function, we will write $f \nearrow$ ($f \searrow$).

For a given set X , the function f such that $f(x) = 1$ if $x \in X$ and $f(x) = 0$ if $x \notin X$ will be denoted $\mathbf{1}_X$.

The absolute value, the binary logarithm and the natural logarithm of $x \in \mathbb{R}$ will be denoted by $|x|$, $\log x$ and $\ln x$, respectively. The largest (smallest) integer number not greater (less) than $x \in \mathbb{R}$ will be denoted by $\lfloor x \rfloor$ ($\lceil x \rceil$).

Given two functions, $f : \mathbb{N} \rightarrow \mathbb{R}_+$ and $g : \mathbb{N} \rightarrow \mathbb{R}_+$, we will say that function $f(n)$ is of order $O(g(n))$, in short $f(n) = O(g(n))$, if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, there holds the inequality $f(n) \leq cg(n)$.

Permutations of elements of set I_n , i.e., bijective functions from set I_n onto itself, will be denoted by small Greek characters. For example, permutation σ with components $\sigma_1, \sigma_2, \dots, \sigma_n$, where $\sigma_i \in I_n$ for $1 \leq i \leq n$ and $\sigma_i \neq \sigma_j$ for $i \neq j$, will be denoted by $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$. In some cases, permutations will also be denoted by small Greek characters with a superscript. For example, σ' and σ'' will refer to two distinct permutations of elements of set I_n . Partial permutations defined on I_n , i.e., bijective functions between two subsets of set I_n , will be denoted by small Greek characters with a superscript in brackets. For example, $\sigma^{(a)} = (\sigma_1^{(a)}, \sigma_2^{(a)}, \dots, \sigma_k^{(a)})$ is a partial permutation of elements of set I_n . The set of all permutations (partial permutations) of set I_n will be denoted by \mathfrak{S}_n ($\hat{\mathfrak{S}}_n$).

The sequence $(x_j)_{j=1}^n$ composed of numbers x_1, x_2, \dots, x_n , ordered according to permutation $\sigma \in \mathfrak{S}_n$, will be denoted by $x_\sigma = (x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n})$.

Because of the nature of the problems considered in this book, we will assume, unless stated otherwise, that all objects (e.g., sets, sequences, etc.) are finite.

1.1.4 Logical notation

In this book, we will use the following logical notation. A negation, conjunction and disjunction will be denoted by \neg , \wedge and \vee , respectively. The implication of formulae p and q will be denoted by $p \Rightarrow q$. The equivalence of formulae p and q will be denoted by $p \Leftrightarrow q$ or $p \equiv q$. The existential and general quantifiers will be denoted by \exists and \forall , respectively.

In the proofs presented in this book, we will use a few proof techniques. The most often applied proof technique is the *pairwise job (element) interchange argument*: we consider two schedules (sequences) that differ only in the order of two jobs (elements) and we show which schedule (sequence) is the better one. A number of proofs are made *by contradiction*: we assume that a schedule (a sequence) is optimal and we show that this assumption leads to a contradiction. Finally, some proofs are made by the *mathematical induction*.

The use of the rules of inference applied in the proofs will be limited mainly to *De Morgan's rules* ($\neg(p \wedge q) \equiv (\neg p \vee \neg q)$, $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$), *material equivalence* ($(p \Leftrightarrow q) \equiv ((p \Rightarrow q) \wedge (q \Rightarrow p))$) and *transposition* ($((p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p))$) rules.

1.1.5 Other notation

Lemmas, theorems and properties will be numbered consecutively in each chapter. In a similar way, we will number definitions, examples, figures and tables. Examples will be ended by the symbol ‘◆’.

Most results will be followed either by a full proof or by the sketch of a proof. In a few cases, no proof (sketch) will be given and the reader will be referred to the literature. The proofs, sketches and references to the sources of proofs will be ended by symbols ‘■’, ‘□’ and ‘◇’, respectively.

1.2 Basic definitions and results

In this section, we include the definitions and results that are used in proofs presented in this book.

Lemma 1.1. (Elementary inequalities)

- (a) If $y_1, y_2, \dots, y_n \in \mathbb{R}$, then $\max\{y_1, y_2, \dots, y_n\} \geq \frac{1}{n} \sum_{j=1}^n y_j$.
- (b) If $y_1, y_2, \dots, y_n \in \mathbb{R}$, then $\frac{1}{n} \sum_{j=1}^n y_j \geq \sqrt[n]{\prod_{j=1}^n y_j}$.
- (c) If $a, x \in \mathbb{R}$, $x \geq -1$, $x \neq 0$ and $0 < a < 1$, then $(1+x)^a < 1+ax$.

Proof. (a) This is the *arithmetic-mean inequality*; see Bullen et al. [37, Chap. 2, Sect. 1, Theorem 2].

(b) This is a special case of the *geometric-arithmetic mean inequality*; see Bullen et al. [37, Chap. 2, Sect. 2, Theorem 1].

(c) This is *Bernoulli's inequality*; see Bullen et al. [37, Chap. 1, Sect. 3, Theorem 1]. ◇

Lemma 1.2. (Minimizing or maximizing a sum of products)

- (a) If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$, then the sum $\sum_{i=1}^n x_{\sigma_i} \prod_{j=i+1}^n y_{\sigma_j}$ is minimized (maximized) when it is calculated over the permutation $\sigma \in \mathfrak{S}_n$ in which indices are ordered by non-decreasing (non-increasing) values of the $\frac{x_i}{y_i-1}$ ratios.

(b) If (x_1, x_2, \dots, x_n) and (y_1, y_2, \dots, y_n) are two sequences of real numbers, then the sum $\sum_{j=1}^n x_j y_j$ is minimized if the sequence (x_1, x_2, \dots, x_n) is ordered non-decreasingly and the sequence (y_1, y_2, \dots, y_n) is ordered non-increasingly or vice versa, and it is maximized, if the sequences are ordered in the same way.

Proof. (a) By pairwise element interchange argument; see Kelly [164, Theorems 1-2], Rau [242, Theorem 1].

(b) By pairwise element interchange argument; see Hardy et al. [131, p. 269]. \diamond

Definition 1.3. (V-shaped and Λ -shaped sequences)

(a) A sequence (x_1, x_2, \dots, x_n) is said to be V-shaped (has a V-shape) if there exists an index k , $1 \leq k \leq n$, such that for $1 \leq j \leq k$ the sequence is non-increasing and for $k \leq j \leq n$ the sequence is non-decreasing.

(b) A sequence (x_1, x_2, \dots, x_n) is said to be Λ -shaped (has a Λ -shape), if the sequence $(-x_1, -x_2, \dots, -x_n)$ is V-shaped.

In other words, sequence $(x_j)_{j=1}^n$ is V-shaped (Λ -shaped) if the elements which are placed before the smallest (largest) x_j , $1 \leq j \leq n$, are arranged in the non-increasing (non-decreasing) order, and those which are placed after the smallest (largest) x_j are in the non-decreasing (non-increasing) order.

The V-shaped and Λ -shaped sequences will also be called *V-sequences* and *Λ -sequences*, respectively. Moreover, if index k of the minimal (maximal) element in a V-sequence (Λ -sequence) satisfies the inequality $1 < k < n$, we will say that this sequence is *strongly V-shaped* (*Λ -shaped*).

Definition 1.4. (The partial order relation \prec)

Let $(u, v), (r, s) \in \mathbb{R}^2$. The partial order relation \prec is defined as follows:

$$(u, v) \prec (r, s), \text{ if } (u, v) \leq (r, s) \text{ coordinatewise and } (u, v) \neq (r, s). \quad (1.1)$$

Lemma 1.5. The relation $(u, v) \prec (0, 0)$ does not hold when either $(u > 0 \text{ or } v > 0)$ or $(u = 0 \text{ and } v = 0)$.

Proof. By Definition 1.4, $(u, v) \prec (0, 0)$ if $(u, v) \leq (0, 0)$ coordinatewise and $(u, v) \neq (0, 0)$. By negation of the conjunction, the result follows. \blacksquare

Definition 1.6. (A graph and a digraph)

(a) A graph (undirected graph) is an ordered pair $G = (N, E)$, where $N \neq \emptyset$ is a finite set of nodes and $E \subseteq \{\{n_1, n_2\} \in 2^N : n_1 \neq n_2\}$ is a set of edges.

(b) A digraph (directed graph) is an ordered pair $G = (V, A)$, where $V \neq \emptyset$ is a finite set of vertices and $A \subseteq \{(v_1, v_2) \in V^2 : v_1 \neq v_2\}$ is a set of arcs.

Example 1.7. Consider graph G_1 and digraph G_2 given in Fig. 1.1.

In the graph $G_1 = (N, E)$, presented in Fig. 1.1a, the set of nodes $N = \{1, 2, 3, 4\}$ and the set of edges $E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$.

In the digraph $G_2 = (V, A)$, presented in Fig. 1.1b, the set of vertices $V = \{1, 2, 3, 4\}$ and the set of arcs $A = \{(1, 2), (1, 3), (2, 4), (3, 4)\}$. \blacklozenge

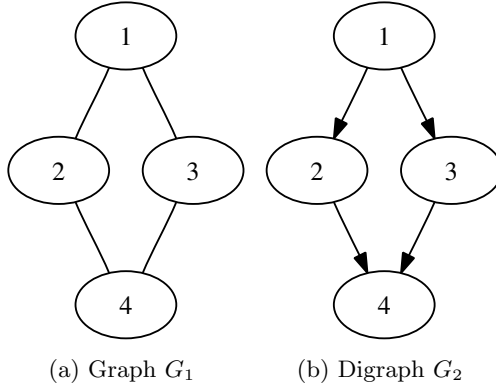


Fig. 1.1: Graph vs. digraph

In this book, we will consider mainly directed graphs. Therefore, all further definitions and remarks will refer to digraphs, unless stated otherwise.

Definition 1.8. (Basic definitions concerning digraphs)

- (a) A digraph $G' = (V', A')$ is called a *subdigraph* of a digraph $G = (V, A)$ if $V' \subseteq V$ and $(x, y) \in A'$ implies $(x, y) \in A$.
- (b) A *directed path* in a digraph $G = (V, A)$ is a sequence (v_1, v_2, \dots, v_m) of distinct vertices from V such that $(v_k, v_{k+1}) \in A$ for each $k = 1, 2, \dots, m-1$. The number m is called the *length of the path*.
- (c) A vertex $x \in V$ is called a *predecessor* (successor) of a vertex $y \in V$ if in a digraph $G = (V, A)$ there is a directed path from x to y (from y to x). If the path has unit length, then x is called a *direct predecessor* (successor) of y .
- (d) A vertex $x \in V$ that has no direct predecessor (successor) is called an *initial* (a *terminal*) vertex in a digraph $G = (V, A)$. A vertex $x \in V$ that is neither initial nor terminal is called an *internal vertex* in the digraph.
- (e) A digraph $G = (V, A)$ is *connected* if for every $x, y \in V$ there exists in G a directed path starting with x and ending with y ; otherwise, it is *disconnected*.

For a given graph (digraph) G and $v \in N$ ($v \in V$), the set of all predecessors and successors of v will be denoted by $Pred(v)$ and $Succ(v)$, respectively.

Definition 1.9. (Parallel and series composition of digraphs)

Let $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ be two digraphs such that $V_1 \cap V_2 = \emptyset$ and let $Term(G_1) \subseteq V_1$ and $Init(G_2) \subseteq V_2$ denote the set of terminal vertices of G_1 and the set of initial vertices of G_2 , respectively. Then

- (a) digraph G_P is said to be a *parallel composition* of digraphs G_1 and G_2 , if $G_P = (V_1 \cup V_2, A_1 \cup A_2)$;
- (b) digraph G_S is said to be a *series composition* of digraphs G_1 and G_2 , if $G_S = (V_1 \cup V_2, A_1 \cup A_2 \cup (Term(G_1) \times Init(G_2)))$.

In other words, digraph G_P is a disjoint union of digraphs G_1 and G_2 , while digraph G_S is a composition of digraphs G_1 and G_2 in which the arcs from all terminal vertices in G_1 are connected to all initial vertices in G_2 .

Definition 1.10. (Special classes of digraphs)

- (a) A chain (v_1, v_2, \dots, v_k) is a digraph $G = (V, A)$ with $V = \{v_i : 1 \leq i \leq k\}$ and $A = \{(v_i, v_{i+1}) : 1 \leq i \leq k-1\}$.
- (b) An in-tree (out-tree) is a digraph which is connected, has a single terminal (initial) vertex called the root of this in-tree (out-tree) and in which any other vertex has exactly one direct successor (predecessor). The initial (terminal) vertices of an in-tree (out-tree) are called leaves.
- (c) A digraph $G = (V, A)$ is a series-parallel digraph (sp-digraph, in short) if either $|V| = 1$ or G is obtained by application of parallel or series composition to two series-parallel digraphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, $V_1 \cap V_2 = \emptyset$.

Remark 1.11. A special type of a tree is a 2-3 tree, i.e., a balanced tree in which each internal node (vertex) has 2 or 3 successors. In 2-3 trees the operations of insertion (deletion) of a node (vertex) and the operation of searching through the tree can be implemented in $O(\log k)$ time, where k is the number of nodes (vertices) in the tree (see, e.g., Aho et al. [2, Chap. 2]).

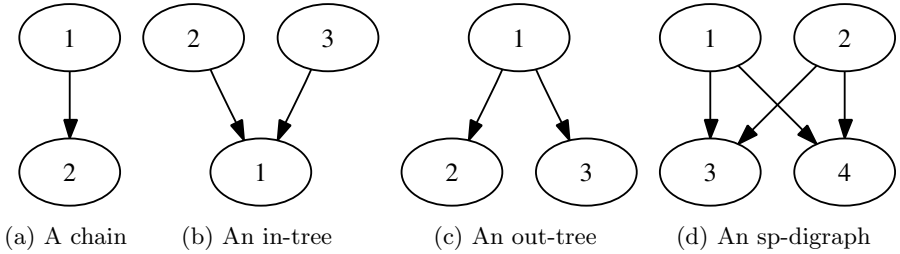


Fig. 1.2: Examples of digraphs from Definition 1.10

Example 1.12. Consider the four digraphs depicted in Fig. 1.2. The chain given in Fig. 1.2a is a digraph $G_1 = (V_1, A_1)$ in which $V_1 = \{1, 2\}$ and $A_1 = \{(1, 2)\}$. The in-tree given in Fig. 1.2b is a digraph $G_2 = (V_2, A_2)$ in which $V_2 = \{1, 2, 3\}$ and $A_2 = \{(2, 1), (3, 1)\}$. The out-tree given in Fig. 1.2c is a digraph $G_3 = (V_3, A_3)$ in which $V_3 = \{1, 2, 3\}$ and $A_3 = \{(1, 2), (1, 3)\}$.

The sp-digraph depicted in Fig. 1.2d is a digraph $G_4 = (V_4, A_4)$ in which $V_4 = \{1, 2, 3, 4\}$ and $A_4 = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. The sp-digraph G_4 is a series composition of sp-digraphs $G_5 = (V_5, A_5)$ and $G_6 = (V_6, A_6)$, where $V_5 = \{1, 2\}$, $V_6 = \{3, 4\}$ and $A_5 = A_6 = \emptyset$. Notice that the sp-digraph G_5 ,

in turn, is a parallel composition of single-vertex sp-digraphs $G'_5 = (V'_5, A'_5)$ and $G''_5 = (V''_5, A''_5)$, where $V'_5 = \{1\}$, $V''_5 = \{2\}$ and $A'_5 = A''_5 = \emptyset$. Similarly, the sp-digraph G_6 is a parallel composition of single-vertex sp-digraphs $G'_6 = (V'_6, A'_6)$ and $G''_6 = (V''_6, A''_6)$, where $V'_6 = \{3\}$, $V''_6 = \{4\}$ and $A'_6 = A''_6 = \emptyset$. ♦

Remark 1.13. From Definition 1.9 it follows that every series-parallel digraph $G = (V, A)$ can be represented in a natural way by a binary *decomposition tree* $T(G)$. Each leaf of the tree represents a vertex in G and each internal node is a series (parallel) composition of its successors. Hence we can construct G , starting from the root of the decomposition tree $T(G)$, by successive compositions of the nodes of the tree. For a given series-parallel digraph, its decomposition tree can be constructed in $O(|V| + |A|)$ time (see Valdes et al. [271]). The decomposition tree of the sp-digraph from Fig. 1.2d is given in Fig. 1.3.

Remark 1.14. Throughout the book, the internal nodes of a decomposition tree that correspond to the parallel composition and series composition will be labelled by P and S , respectively.

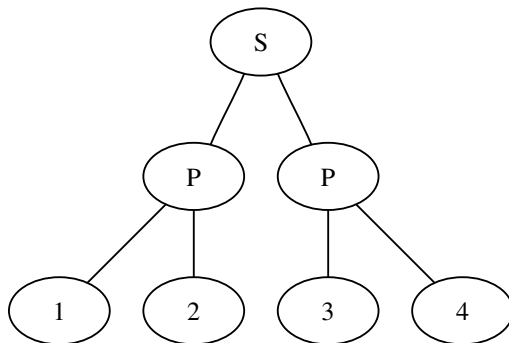


Fig. 1.3: The decomposition tree of the sp-digraph from Fig. 1.2d

Theorem 1.15. (Mean value theorems)

- (a) If functions $f : \langle a, b \rangle \rightarrow \mathbb{R}$ and $g : \langle a, b \rangle \rightarrow \mathbb{R}$ are differentiable on the interval (a, b) and continuous on the interval $\langle a, b \rangle$, then there exists at least one point $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
- (b) If function $f : \langle a, b \rangle \rightarrow \mathbb{R}$ is differentiable on the interval (a, b) and continuous on the interval $\langle a, b \rangle$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. (a) This is the *generalized mean-value theorem*; see Rudin [248, Chap. 5, Theorem 5.9]. ♦

(b) This is the *mean-value theorem*. Applying Theorem 1.15 (a) for $g(x) = x$, we obtain the result. \blacksquare

Remark 1.16. The counterparts of mean-value theorems for functions defined in vector spaces are given, e.g., by Maurin [205, Chap. VII]. \diamond

Definition 1.17. (A norm)

A norm on a vector space X is the function $\|\cdot\| : X \rightarrow \mathbb{R}$ such that for all $x, y \in X$ and any $a \in \mathbb{R}$ the following conditions are satisfied:

- (a) $\|x + y\| \leq \|x\| + \|y\|$,
- (b) $\|ax\| = |a|\|x\|$,
- (c) $\|x\| = 0 \Leftrightarrow x = 0$.

The value $\|x\|$ is called a norm of vector $x \in X$.

Definition 1.18. (Hölder's vector norm l_p)

Given an arbitrary $p \geq 1$, the l_p norm of vector $x \in \mathbb{R}^n$ is defined as follows:

$$\|x\|_p := \begin{cases} (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \max_{1 \leq i \leq n} \{|x_i|\}, & p = +\infty. \end{cases}$$

Definition 1.19. (A priority-generating function)

Let $\pi' = (\pi^{(1)}, \pi^{(a)}, \pi^{(b)}, \pi^{(2)})$, $\pi'' = (\pi^{(1)}, \pi^{(b)}, \pi^{(a)}, \pi^{(2)}) \in \mathfrak{S}_n$, where $\pi^{(1)}, \pi^{(a)}, \pi^{(b)}, \pi^{(2)} \in \hat{\mathfrak{S}}_n$.

(a) A function $\mathcal{F} : \mathfrak{S}_n \rightarrow \mathbb{R}$ is called a priority-generating function, if there exists a function $\omega : \hat{\mathfrak{S}}_n \rightarrow \mathbb{R}$ (called priority function) such that there holds either the implication $\omega(\pi^{(a)}) > \omega(\pi^{(b)}) \Rightarrow \mathcal{F}(\pi') \leq \mathcal{F}(\pi'')$ or the implication $\omega(\pi^{(a)}) = \omega(\pi^{(b)}) \Rightarrow \mathcal{F}(\pi') = \mathcal{F}(\pi'')$.

(b) If $\pi^{(a)}, \pi^{(b)} \in \hat{\mathfrak{S}}_1$, then a priority-generating function is called 1-priority-generating function.

Remark 1.20. Notice that by Definition 1.19, every priority-generating function is a 1-priority-generating function (but not vice versa).

Remark 1.21. Definition 1.19 concerns the priority-generating function of a single variable (see Tanaev et al. [264, Chap. 3]).

Remark 1.22. Priority-generating functions of a single variable are also considered by Gordon et al. [117]. The authors identify several cases in which an objective function for a scheduling problem with some time-dependent job processing times is priority generating (see [117, Sect. 7–9]). They also explore the relationship between the existence of priority functions for different criterion functions for such problems (see [117, Theorems 1–2]).

Remark 1.23. Priority-generating functions of many variables are considered by Janiak et al. [149].

Theorem 1.24. (Tanaev et al. [264]) *If $\mathcal{F} : \mathfrak{S}_n \rightarrow \mathbb{R}$ is a 1-priority generating function and $\omega : \mathfrak{S}_n \rightarrow \mathbb{R}$ is a priority function corresponding to \mathcal{F} , then the permutation in which the elements are arranged in the non-increasing order of their priorities minimizes \mathcal{F} over \mathfrak{S}_n .*

Proof. See Tanaev et al. [264, Chap. 3, Theorem 7.1]. \diamond

Let X denote the set of feasible solutions of a bicriterion optimization problem and let $f : X \rightarrow \mathbb{R}^2$, $f = (f_1, f_2)$, be the minimized criterion function, where $f_i : X \rightarrow \mathbb{R}$ are single-valued criteria for $i = 1, 2$.

Definition 1.25. (Pareto optimal solutions)

- (a) *A solution $x^* \in X$ is said to be Pareto optimal, $x^* \in X_{\text{Par}}$ in short, if there is no $x \in X$ such that $f(x) \prec f(x^*)$.*
- (b) *A solution $x^* \in X$ is said to be weakly Pareto optimal, $x^* \in X_{\text{w-Par}}$ in short, if there is no $x \in X$ such that $f_i(x) < f_i(x^*)$ for $i = 1, 2$.*

The images of sets X_{Par} and $X_{\text{w-Par}}$ under the function $f = (f_1, f_2)$, $f(X_{\text{Par}})$ and $f(X_{\text{w-Par}})$, will be denoted by Y_{eff} and $Y_{\text{w-eff}}$, respectively. (Notice that $X_{\text{Par}} \subset X_{\text{w-Par}}$ and $Y_{\text{eff}} \subset Y_{\text{w-eff}}$.)

Example 1.26. (Ehrgott [76]) Consider a set X and a function $f = (f_1, f_2)$, where $X := \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1 \wedge 0 \leq x_2 \leq 1\}$, $f_1 := x_1$ and $f_2 := x_2$. Then $Y_{\text{eff}} = \emptyset$ and $Y_{\text{w-eff}} = \{(x_1, x_2) \in X : 0 < x_1 < 1, x_2 = 0\}$.

If we define $X := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_i \leq 1 \text{ for } i = 1, 2\}$, then for the function f as above we have $Y_{\text{eff}} = \{(0, 0)\}$ and $Y_{\text{w-eff}} = \{(x_1, x_2) \in X : x_1 = 0 \vee x_2 = 0\}$. \blacklozenge

Lemma 1.27. (Scalar optimality vs. Pareto optimality)

If x^ is an optimal solution with respect to the scalar criterion $\omega \circ f$ for a certain $f = (f_1, f_2)$ and $\omega = (\omega_1, \omega_2)$, then*

- (a) *if $\omega \in \mathbb{R}^2$, then $x^* \in X_{\text{w-Par}}$,*
- (b) *if $\omega \in \text{int}\mathbb{R}^2$, then $x^* \in X_{\text{Par}}$.*

Proof. (a), (b) See Ehrgott [76, Proposition 3.7]. \diamond

Definition 1.28. (A convex function)

A function f is convex on the interval $\langle a, b \rangle$ if for any $x_1, x_2 \in \langle a, b \rangle$ and any $\lambda \in (0, 1)$ there holds the inequality $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$.

In other words, a convex function is such a continuous function that the value at any point within every interval in its domain does not exceed the value of a convex combination of its values at the ends of the interval. (A convex combination of elements x_1 and x_2 is the element $y := \lambda x_1 + (1 - \lambda)x_2$, where $\lambda \in \langle 0, 1 \rangle$ is a given number.)

Remark 1.29. If the symbol ‘ \leq ’ is replaced by ‘ $<$ ’ in Definition 1.28, then the function f is *strictly* convex.

Remark 1.30. A function f is (strictly) *concave*, if $-f$ is (strictly) *convex*.

Definition 1.31. (A convex set)

A set $X \subseteq \mathbb{R}^n$ is *convex* if for any $x_1, x_2 \in X$ and any $\lambda \in \langle 0, 1 \rangle$ the convex combination $\lambda x_1 + (1 - \lambda)x_2 \in X$.

In other words, a set X is convex if the line segment joining any pair of points of X lies entirely in X .

Remark 1.32. Basic facts concerning convex functions and convex sets are given, e.g., by Walk [278, Chap. 1]. \diamond

Lemma 1.33. (Pareto optimality vs. scalar optimality)

If X is a convex set and f_1, f_2 are convex functions, then if $x^* \in X_{\text{w-Par}}$, there exists $\omega \in \text{int}\mathbb{R}^2$ such that x^* is an optimal solution with respect to the scalar criterion $\omega \circ f$.

Proof. See Ehrgott [76, Proposition 3.8]. \diamond

With this lemma, we end the presentation of notation, definitions and auxiliary results used throughout the book. In subsequent chapters, we will introduce basic definitions and results concerning algorithms (Chap. 2), \mathcal{NP} -complete problems (Chap. 3), the scheduling theory (Chap. 4) and time-dependent scheduling (Chap. 5).

1.3 Bibliographic notes

A comprehensive presentation of basic mathematical notions and mathematical notation may be found in Rasiowa [241]. Inference rules and proof techniques are discussed in Copi [65].

Bullen et al. [37], Hardy et al. [131] and Mitrinović et al. [212] give a wide range of various inequalities.

Berge [22], Harary [130] and Wilson [294] present the graph theory from different perspectives. Brandstädt et al. [32] study the properties of different classes of graphs. Applications of graphs in computer science and engineering are discussed by Deo [70].

Maurin [205] and Rudin [248] give a concise presentation of calculus and mathematical analysis.

Priority-generating functions are discussed by Tanaev et al. [264, Chap. 3] and by Gordon et al. [117]. The extension of these functions to the multiple criteria case is presented by Janiak et al. [149].

The properties of sp-(di)graphs and applications of these (di)graphs in the scheduling theory are discussed, e.g., by Gordon [116], Gordon et al. [120], Lawler [185], Möhring [221] and Valdes et al. [271].

Ehrgott [76] presents a comprehensive introduction to Pareto optimality.



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