Laplace Transforms and the American Call Option

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Summary. A partial Laplace transform is used to study the valuation of American call options with constant dividend yield, and to derive an integral equation for the location of the optimal exercise boundary, which is the main result of this paper.

The integral equation differs depending on whether the dividend yield is less than or exceeds the risk-free rate.

1 Introduction

One of the defining events in mathematical finance was the publication in 1973 of the Black-Scholes-Merton model for the pricing of equity options, for which the 1997 Nobel Prize in Economics was awarded. Using this model, in which the volatility σ, interest rate r, and dividend yield D were assumed constant, it was possible to obtain closed form expressions for European call and put options, which have pay-offs at expiry of $\max(S - E, 0)$ and $\max(E - S, 0)$ respectively, where S is the price of the stock upon which the option is written and E is the strike price.

Although the Black-Scholes-Merton model made it possible to obtain closed form prices for many European options, which can be exercised only at expiry, American options, which can be exercised at or before expiry at the discretion of the holder, are rather more difficult to price. American call and put options have the same pay-offs when exercised early as when held to expiry, that is $\max(S - E, 0)$ and $\max(E - S, 0)$ respectively. The right to exercise early leads to the issue of when and if an option should be exercised, which leads to an interesting free boundary problem similar to the Stefan problem which arises in melting and solidification, and it is precisely this free boundary problem which makes American options both difficult to price and mathematically interesting. To date, a closed form pricing formula for American options has remained elusive, except in a couple of special cases. One such special case is the American call with either no dividends, when exercise is never optimal so that the value of the option is the same as that of a European call, or discrete dividends [9, 10, 23, 27]. For those cases where exact solutions are not known, practitioners are of course able to price American options numerically, or use one of the approximate or series solutions which appear in the literature.

In the present study, we will consider the free boundary, or *optimal exercise boundary*, for an American call option. This boundary separates the region where it is optimal for an investor to retain an option from that where exercise is optimal, and closed form expressions for the location of the boundary have remained as elusive as a closed form pricing formula for American options, although, as with the price of the option, numerical and approximate and series solutions can be used. The location of the free boundary is key to pricing an American option. In our analysis, we will use a partial Laplace transform to arrive at an integral equation giving the location of the free boundary, which we regard as a curve in (S,t) space, denoted by $S = S_f(t)$ or the inverse relation $t = T_f(S)$.

We are of course not the first to apply integral equation methods to American options: on the contrary, it has been a very popular approach, including such studies as the early work of [18, 26], which drew on the pioneering work of [15] on Stefan problems, the studies by [4, 11, 13, 21], all of which looked at the difference between European and American prices, and the recent work of [1, 8, 14, 16, 25]. We will touch upon the differences between some of those studies and our own in the final section.

The partial Laplace transform approach used here was developed by [7] for diffusion problems, specifically the recrystallization of an infinite metal slab, and an overview of the technique can be found in [6]. The solidification problem considered in [7] was governed by the diffusion equation, and the Black-Scholes-Merton partial differential equation used in our analysis can of course be recast as that equation. [7] were able to use a partial Laplace transform, which we shall define shortly, to give an integral equation formulation of their problem, and were then able to find a series solution of that integral equation. For the problem considered here, the boundary conditions at the free boundary cause the kernel in our integral equation to be much more complicated than that in [7].

2 Analysis

Under the Black-Scholes-Merton model, in which the volatility σ, interest rate r, and dividend yield D are assumed constant, the value $V(S,t)$ of an option on an equity obeys the Black-Scholes-Merton partial differential equation or PDE [2, 19],

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D)\, S\frac{\partial V}{\partial S} - rV = 0, \tag{1}$$

where S is the price of the underlying and $t < T$ is the time, with T being the expiry when the holder will receive a pay-off of $\max(S - E, 0)$ for a call with a strike of E. To simplify the analysis, we will work in terms of the remaining life of the option, $\tau = T - t$, so that (1) is replaced by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - D)\, S \frac{\partial V}{\partial S} - rV. \tag{2}$$

For European options, (2) is valid for $\tau \geq 0$. For options where early exercise is permitted, (2) is only valid when it is optimal to hold the option, and must be solved together with the appropriate conditions at the optimal exercise boundary, whose location is unknown and must be solved for. We will label the position of the free boundary as $S = S_f(\tau)$, which we can invert to give $\tau_f(S)$ as the time at which early exercise should occur.

A review of the properties of the free boundary and the option price can be found in [20]. In our analysis, we will make use of a number of these including:
(i) The price of an American call option is given by a value function. Where it is optimal to hold the option the value function is smooth with $V(S, \tau) > \max(S - E, 0)$ and $0 \leq \partial V / \partial S < 1$ [18, 26].
(ii) There is an optimal exercise policy for American options and an optimal stopping time [12].
(iii) At the free boundary the value of the option is equal to the pay-off from immediate exercise [11, 18], $V_f(S, \tau) = S - E$.
(iv) At the free boundary the value of the option's delta, or derivative of its value with respect to the stock price, is $\partial V_f / \partial S = 1$. This high contact or smooth-pasting condition [24] has been shown to be both a necessary [19] and sufficient [3] condition for the optimality of the boundary.
(v) At expiry [18, 26], the location of the free boundary is

$$S_f(0) = S_0 = \begin{cases} Er/D > E & r > D \\ E & D \geq r, \end{cases}$$

which we can write as $\tau_f(S_0) = 0$.
(vi) As $\tau \to \infty$, from the perpetual American call [18, 19] we know $S_f(\tau) \to S_1 = \frac{E\alpha}{\alpha - 1}$ where $\alpha = \left[\frac{\sigma^2}{2} - (r - D) + \sqrt{\left(r - D + \frac{\sigma^2}{2}\right)^2 + 2D\sigma^2}\right] / \sigma^2$. We can write this as $\tau_f(S) \to \infty$ as $S \to S_1$ from below. As $D \to 0$, $S_1 \to \infty$ [19] and a perpetual call on a stock with no dividends has the same value as the stock.
(vii) The free boundary is a strictly increasing function of τ [18, 26], which enables us to define the inverse $\tau_f(S)$ mentioned above. The optimal exercise boundary will move upwards as we move away from the expiration date and will lie between the two limits, $S_0 \leq S_f(\tau) \leq S_1$, with early exercise optimal if $S \geq S_f(\tau)$ and retaining the option optimal if $0 \leq S < S_f(\tau)$.
(viii) The free boundary is a continuous, differentiable function of τ [18, 26], which enables us to take derivatives of $\tau_f(S)$.

Having formulated the problem, we shall now attempt to solve it using a Laplace transform in time. Because S_0 differs depending on whether $r > D$ or $D \geq r$, we will consider these two cases separately. Since (2) only holds where it is optimal to retain the option, we will modify the usual Laplace transform $\mathcal{L}(G)(p) = \int_0^\infty g(\tau)e^{-p\tau}d\tau$ somewhat, and define the *partial* Laplace transform for $S \leq S_1$,

$$\mathcal{V}(S,p) = \int_{\tau_f(S)}^{\infty} V(S,\tau)e^{-p\tau}d\tau, \tag{3}$$

with the lower limit of $\tau = \tau_f(S)$ rather than $\tau = 0$. As we mentioned in Section 1, the partial Laplace transform is due to [7], and has been successfully used to tackle diffusion problems in the past. This definition of the partial Laplace transform, is of course equivalent to setting $V(S,\tau) = 0$ in the region where it is optimal not to hold. Because of this definition, the price of the option $V(S,\tau)$ will obey (2) everywhere we integrate. We require the real part of p to be positive for the integral in (3) to converge. In addition, we know from the definition that $\mathcal{V}(S,p) \to 0$ as $S \to S_1$. We can also define an inverse transform

$$V(S,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{V}(S,p)e^{p\tau}dp, \tag{4}$$

which of course is only meaningful where it is optimal to retain the option. From our definition, the following transforms can be derived easily,

$$\begin{aligned}
\mathcal{L}\left[\frac{\partial V}{\partial \tau}\right] &= p\mathcal{V} - e^{-p\tau_f(S)}V_f(S,\tau_f(S)), \\
\mathcal{L}\left[\frac{\partial V}{\partial S}\right] &= \frac{d\mathcal{V}}{dS} + e^{-p\tau_f(S)}\tau_f'(S)V_f(S,\tau_f(S)), \\
\mathcal{L}\left[\frac{\partial^2 V}{\partial S^2}\right] &= \frac{d}{dS}\left(\mathcal{L}\left[\frac{\partial V}{\partial S}\right]\right) + e^{-p\tau_f(S)}\tau_f'(S)\frac{\partial V_f}{\partial S}(S,\tau_f(S)).
\end{aligned} \tag{5}$$

In the above, we have adopted the convention that $\tau_f(S)$ is the location of the free boundary for $S_0 < S < S_1$, while for $S < S_0$, we set $\tau_f = 0$ since it is optimal to hold the option to expiry. Applying this partial Laplace transform to (2), we arrive at the following (nonhomogeneous Euler) ordinary differential equation for the transform of the option price,

$$\left[\frac{\sigma^2 S^2}{2}\frac{d^2}{dS^2} + (r-D)\,S\frac{d}{dS} - (p+r)\right]\mathcal{V} + F(S) = 0, \tag{6}$$

where the nonhomogeneous term $F(S)$ takes a different value in various regions, as shown in Table 1:
with $F_0(S) = \left(1 + (r-D)\,S\tau_f'(S) + \frac{\sigma^2 S^2}{2}\left(\tau_f''(S) + p_0\tau_f'^2(S)\right)\right)(S-E) + \sigma^2 S^2\tau_f'(S)$ and $F_1(S) = -\frac{\sigma^2 S^2}{2}\,(S-E)\,\tau_f'^2(S)$ in region (c) and $p_0 = \frac{(D-r)^2}{2\sigma^2} +$

Table 1. The nonhomogeneous term $F(S)$.

Region	$V(S_f(\tau),\tau)$	$\frac{\partial V}{\partial S}(S_f(\tau),\tau)$	$\tau_f(S)$	$F(S)$
(a) $0 < S < E$	0	0	0	0
(b) $E < S < S_0$	$S-E$	1	0	$S-E$
(c) $S_0 < S < S_1$	$S-E$	1	>0	$e^{-p\tau_f(S)}\left[F_0(S)+(p+p_0)\,F_1(S)\right]$

$\frac{D+r}{2}+\frac{\sigma^2}{8}$. We would mention that in region (c), $F(S)$ is fairly complicated, being a function of $\tau_f'(S)$ and $\tau_f''(S)$, the derivatives of the inverse of the optimal exercise boundary, as are $F_0(S)$ and $F_1(S)$.

The three regions in Table 1 coexist when $0 < D < r$. When $D \geq r$, the location of the free boundary at expiry is $S_f(0) = S_0 = E$, so that the middle region (b) vanishes and we are left with only two regions, (a) and (c), with the nonhomogeneous terms in these two regions unchanged. For the moment, we will perform our analysis for the case $0 < D < r$, and explain later how the analysis differs when $D \geq r$.

2.1 $0 < D < r$

The general solution of (6) is

$$\mathcal{V} = S^{\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}}\left[C_1(p) - \int \frac{\tilde{S}^{-\frac{D-r+\lambda(p)}{\sigma^2}-\frac{3}{2}}F(\tilde{S})d\tilde{S}}{\lambda(p)}\right] + S^{\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}}\left[C_2(p) + \int \frac{\tilde{S}^{-\frac{D-r-\lambda(p)}{\sigma^2}-\frac{3}{2}}F(\tilde{S})d\tilde{S}}{\lambda(p)}\right], \tag{7}$$

where $\lambda(p) = \sqrt{2}\sigma\,[p+p_0]^{1/2}$, and C_1 and C_2 are constants of integration, which may depend on the transform variable p. Since r, D and σ are all assumed to be positive, and we assume that p has a positive real part from the definition of the Laplace transform, then the real part of the first exponent $\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}$ is assumed positive, while the real part of the second exponent, $\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}$ is assumed negative.

Applying this solution (7) to the three separate regions outlined above, we find that in region (a) we must discard the second solution in order to satisfy the boundary condition on $S = 0$ that $V(0,t) = 0$, and the corollary that $\mathcal{V}(0,p) \to 0$ as $p \to \infty$, so in this region we have

$$\mathcal{V} = C_1^{(a)}(p)\left(\frac{S}{E}\right)^{\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}}. \tag{8}$$

In region (b), we find

$$\mathcal{V} = \frac{S}{p+D} - \frac{E}{p+r} + C_1^{(b)}(p)\left(\frac{S}{E}\right)^{\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}} + C_2^{(b)}(p)\left(\frac{S}{E}\right)^{\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}}, \tag{9}$$

and in region (c), we have

$$\mathcal{V} = S^{\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}}\left[C_1(p) - \int_{S_0}^{S} \frac{\tilde{S}^{-\frac{D-r+\lambda(p)}{\sigma^2}-\frac{3}{2}}F(\tilde{S})d\tilde{S}}{\lambda(p)}\right] + S^{\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}}\left[C_2(p) + \int_{S_0}^{S} \frac{\tilde{S}^{-\frac{D-r-\lambda(p)}{\sigma^2}-\frac{3}{2}}F(\tilde{S})d\tilde{S}}{\lambda(p)}\right], \tag{10}$$

with F given in Table 1. Applying the condition that $\mathcal{V}(S,p) \to 0$ as $S \to S_1$, we require

$$C_1^{(c)}(p) = \frac{1}{\lambda(p)}\int_{S_0}^{S_1} \tilde{S}^{-\frac{D-r+\lambda(p)}{\sigma^2}-\frac{3}{2}}F(\tilde{S})d\tilde{S},$$
$$C_2^{(c)}(p) = -\frac{1}{\lambda(p)}\int_{S_0}^{S_1} \tilde{S}^{-\frac{D-r-\lambda(p)}{\sigma^2}-\frac{3}{2}}F(\tilde{S})d\tilde{S}, \tag{11}$$

so that the solution in region (c) becomes

$$\mathcal{V} = \int_{S}^{S_1} \left(\frac{\tilde{S}}{S}\right)^{-\frac{D-r}{\sigma^2}-\frac{3}{2}}\left[\left(\frac{\tilde{S}}{S}\right)^{-\frac{\lambda(p)}{\sigma^2}} - \left(\frac{\tilde{S}}{S}\right)^{\frac{\lambda(p)}{\sigma^2}}\right]\frac{F(\tilde{S})d\tilde{S}}{\lambda(p)S}. \tag{12}$$

We must now match the solutions in these three regions together. We will require that $\mathcal{V}$ and $d\mathcal{V}/dS$ are continuous across $S = E$ and $S = S_0$. Matching regions (b) and (c) together at $S = S_0$, we find we can write $C_1^{(b)}(p)$ and $C_2^{(b)}(p)$ in terms of $C_1^{(c)}(p)$ and $C_2^{(c)}(p)$, which were given in (11) above,

$$C_1^{(b)} = C_1^{(c)}E^{\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}} + \frac{1}{2\lambda(p)}\left(\frac{E}{S_0}\right)^{\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}} \times \left[\frac{\left(D-r-\frac{\sigma^2}{2}-\lambda(p)\right)S_0}{p+D} - \frac{\left(D-r+\frac{\sigma^2}{2}-\lambda(p)\right)E}{p+r}\right],$$
$$C_2^{(b)} = C_2^{(c)}E^{\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}} - \frac{1}{2\lambda(p)}\left(\frac{E}{S_0}\right)^{\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}} \times \left[\frac{\left(D-r-\frac{\sigma^2}{2}+\lambda(p)\right)S_0}{p+D} - \frac{\left(D-r+\frac{\sigma^2}{2}+\lambda(p)\right)E}{p+r}\right]. \tag{13}$$

Similarly, matching regions (a) and (b) together at $S = E$, we find we can write $C_1^{(a)}(p)$ in terms of $C_1^{(b)}(p)$ and $C_2^{(b)}(p)$, which we just found,

$$C_1^{(a)}(p) = E\left(\frac{1}{p+D} - \frac{1}{p+r}\right) + C_1^{(b)}(p) + C_2^{(b)}(p), \tag{14}$$

but we also arrive at another expression for $C_2^{(b)}(p)$,

$$C_2^{(b)}(p) = \frac{E}{2\lambda(p)}\left[\left(\frac{D-r+\frac{\sigma^2}{2}+\lambda(p)}{p+r}\right) - \left(\frac{D-r-\frac{\sigma^2}{2}+\lambda(p)}{p+D}\right)\right], \tag{15}$$

and comparing this to the earlier expression we found for $C_2^{(b)}(p)$, we arrive at the following equation

$$\begin{aligned} C_2^{(c)} &= \frac{1}{2\lambda(p)} S_0^{-\frac{D-r-\lambda(p)}{\sigma^2}-\frac{1}{2}} \\ &\quad \times \left[\frac{S_0\left(D-r-\frac{\sigma^2}{2}+\lambda(p)\right)}{p+D} - \frac{E\left(D-r+\frac{\sigma^2}{2}+\lambda(p)\right)}{p+r}\right] \\ &\quad + \frac{1}{2\lambda(p)} E^{-\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}} \\ &\quad \times \left[\frac{D-r+\frac{\sigma^2}{2}+\lambda(p)}{p+r} - \frac{D-r-\frac{\sigma^2}{2}+\lambda(p)}{p+D}\right], \end{aligned} \tag{16}$$

or using (11),

$$\begin{aligned} &\int_{S_0}^{S_1} \tilde{S}^{-\frac{D-r\lambda(p)}{\sigma^2}-\frac{3}{2}} F(\tilde{S}) d\tilde{S} \\ &= \frac{1}{2} S_0^{-\frac{D-r-\lambda(p)}{\sigma^2}-\frac{1}{2}} \\ &\quad \times \left[\frac{E\left(D-r+\frac{\sigma^2}{2}+\lambda(p)\right)}{p+r} - \frac{S_0\left(D-r-\frac{\sigma^2}{2}+\lambda(p)\right)}{p+D}\right] \\ &\quad + \frac{1}{2} E^{-\frac{D-r-\lambda(p)}{\sigma^2}+\frac{1}{2}} \\ &\quad \times \left[\frac{D-r-\frac{\sigma^2}{2}+\lambda(p)}{p+D} - \frac{D-r+\frac{\sigma^2}{2}+\lambda(p)}{p+r}\right], \end{aligned} \tag{17}$$

where again $F(S)$ is given in Table 1.

We should comment on why (13), which came from matching regions (b) and (c), appears to be more complicated than (14,15) which came from matching (a) and (b). Because we wrote S/E in the homogeneous terms in (8,9), the matching at $S = E$ was greatly simplified. If we had instead written S/S_0 in those terms, the matching at $S = S_0$ would have become much simpler, while that at $S = E$ would have become correspondingly more complex. Another reason for the complexity in (13) is of course that the nonhomogeneous terms in region (c) are fairly lengthy, while those in region (a) vanish.

2.2 $D \geq r$

The analysis when $D \geq r$ is very similar to that for $D < r$, so we will merely highlight the differences and give the main results. As mentioned earlier, when $D \geq r$, the location of the free boundary at expiry is $S_f(0) = S_0 = E$. Because of this, instead of the three regions (a)-(c) described above, the middle region (b) vanishes and we are left with only two regions, (a) and (c), with the nonhomogeneous terms in these two regions unchanged, and given in Table 1. The general solutions in these two regions are also the same, namely (8,12). However, the matching process will lead to constants that differ from those found earlier. When $D \geq r$, we have a single boundary to match across, and we require that $\mathcal{V}$ and $\frac{d\mathcal{V}}{dS}$ are continuous across $S = E$. This tells us that

$$\begin{aligned} C_1^{(a)}(p) &= C_1^{(c)}(p) E^{\frac{D-r+\lambda(p)}{\sigma^2}+\frac{1}{2}}, \\ C_2^{(c)}(p) &= 0, \end{aligned} \tag{18}$$

the latter of which gives

$$\int_E^{S_1} \tilde{S}^{-\frac{D-r-\lambda(p)}{\sigma^2}-\frac{3}{2}} F(\tilde{S}) d\tilde{S} = 0, \tag{19}$$

with $F(S)$ again given in Table 1.

3 The integral equations

The equations, (17) for $0 < D < r$ and (19) for $D \geq r$, are integral equations in transform space for the location of the free boundary $\tau_f(S)$, which appears in equations via the nonhomogeneous term $F(S)$. To be more specific, (17,19) are nonlinear Fredholm integral equations, or to be even more specific, Urysohn equations. When $r = D$, the integral equations for the two cases are the same.

Each of (17,19) is of course the Laplace transform of an integro-differential equation in physical space, and we can obtain these latter equations by applying the inverse Laplace transform (4) to (17,19). The inversion process is conceptually straightforward, but the algebra is somewhat complicated. To invert (17), we first divide by $(p + p_0)^{3/2} S_0^{-\frac{D-r}{\sigma^2}-\frac{3}{2}} (S_f(\tau))^{\lambda(p)/\sigma^2}$, and rewrite (17) as

$$\int_{S_0}^{S_1} \exp\left[-\frac{\sqrt{2(p+p_0)}}{\sigma}\ln\frac{S_f(\tau)}{\tilde{S}}\right]$$
$$\times\left(\frac{\tilde{S}}{S_0}\right)^{-\frac{D-r}{\sigma^2}-\frac{3}{2}} \frac{e^{-p\tau_f(\tilde{S})}}{\sqrt{p+p_0}}\left[\frac{F_0(\tilde{S})}{p+p_0}+F_1(\tilde{S})\right]d\tilde{S} =$$

$$\begin{aligned}
&= \frac{S_0}{2\,(p+p_0)}\exp\left[-\frac{\sqrt{2(p+p_0)}}{\sigma}\ln\frac{S_f(\tau)}{S_0}\right] \\
&\quad\times\left(\frac{E}{p+r}\left[\frac{D-r+\frac{\sigma^2}{2}}{(p+p_0)^{1/2}}+\sigma\sqrt{2}\right]-\frac{S_0}{p+D}\left[\frac{D-r-\frac{\sigma^2}{2}}{(p+p_0)^{1/2}}+\sigma\sqrt{2}\right]\right) \\
&\quad+\frac{S_0^2}{2\,(p+p_0)}\left(\frac{E}{S_0}\right)^{-\frac{D-r}{\sigma^2}+\frac{1}{2}}\exp\left[-\frac{\sqrt{2(p+p_0)}}{\sigma}\ln\frac{S_f(\tau)}{E}\right] \qquad (20)\\
&\quad\times\left(\frac{1}{p+D}\left[\frac{D-r-\frac{\sigma^2}{2}}{(p+p_0)^{1/2}}+\sigma\sqrt{2}\right]-\frac{1}{2p+r}\left[\frac{D-r+\frac{\sigma^2}{2}}{(p+p_0)^{1/2}}+\sigma\sqrt{2}\right]\right),
\end{aligned}$$

and then use the following standard inverse transforms [22],

$$\begin{aligned}
\mathcal{L}^{-1}\left[e^{-ap}G(p)\right] &= H\left(\tau-a\right)g\left(\tau-a\right), \\
\mathcal{L}^{-1}\left[G(p+p_0)\right] &= e^{-p_0\tau}g(\tau), \\
\mathcal{L}^{-1}\left[G_1(p)G_2(p)\right] &= \int_0^{\tau} g_1(\tau-z)g_2(z)dz, \\
\mathcal{L}^{-1}\left[p^{-1/2}\exp\left(-ap^{1/2}\right)\right] &= \frac{1}{\sqrt{\pi}\tau^{1/2}}\exp\left[-\frac{a^2}{4\tau}\right], \\
\mathcal{L}^{-1}\left[p^{-1}\exp\left(-ap^{1/2}\right)\right] &= \operatorname{erfc}\left[\frac{a}{2\sqrt{\tau}}\right], \\
\mathcal{L}^{-1}\left[p^{-3/2}\exp\left(-ap^{1/2}\right)\right] &= \frac{2\tau^{1/2}}{\sqrt{\pi}}\exp\left[-\frac{a^2}{4\tau}\right]-a\operatorname{erfc}\left[\frac{a}{2\sqrt{\tau}}\right], \qquad (21)
\end{aligned}$$

where $H(t)$ is the Heaviside step function, to obtain

$$\int_{S_0}^{S_f(\tau)}\sqrt{\tau-\tau_f(\tilde{S})}\left(\frac{\tilde{S}}{S_0}\right)^{-\frac{D-r}{\sigma^2}-\frac{3}{2}} e^{-p_0(\tau-\tau_f(\tilde{S}))}$$
$$\times\left[\frac{1}{\sqrt{\pi}}\left(2F_0(\tilde{S})+\frac{F_1(\tilde{S})}{\tau-\tau_f(\tilde{S})}\right)\exp\left(-\frac{\left(\ln(S_f(\tau)/\tilde{S})\right)^2}{2\sigma^2(\tau-\tau_f(\tilde{S}))}\right)\right.$$
$$\left.-\frac{F_0(\tilde{S})\sqrt{2}\ln(S_f(\tau)/\tilde{S})}{\sigma(\tau-\tau_f(\tilde{S}))}\operatorname{erfc}\left(\frac{\ln(S_f(\tau)/\tilde{S})}{\sigma\sqrt{2(\tau-\tau_f(\tilde{S}))}}\right)\right]d\tilde{S} =$$

$$
\begin{aligned}
&= S_0 \int_0^\tau \left[E\left(D - r + \frac{\sigma^2}{2}\right) e^{-r(\tau-z)} - S_0\left(D - r - \frac{\sigma^2}{2}\right) e^{-D(\tau-z)} \right] e^{-p_0 z} \\
&\times \left(\frac{z^{1/2}}{\sqrt{\pi}} \exp\left[-\frac{[\ln(S_f(z)/S_0)]^2}{2\sigma^2 z} \right] - \frac{\ln(S_f(z)/S_0)}{\sigma\sqrt{2}} \operatorname{erfc}\left[\frac{\ln(S_f(z)/S_0)}{\sigma\sqrt{2z}} \right] \right) dz \\
&+ S_0^2 \left(\frac{E}{S_0} \right)^{-\frac{D-r}{\sigma^2}+\frac{1}{2}} \\
&\times \int_0^\tau \left[\left(D - r - \frac{\sigma^2}{2}\right) e^{-D(\tau-z)} - \left(D - r + \frac{\sigma^2}{2}\right) e^{-r(\tau-z)} \right] e^{-p_0 z} \\
&\times \left(\frac{z^{1/2}}{\sqrt{\pi}} \exp\left[-\frac{[\ln(S_f(z)/E)]^2}{2\sigma^2 z} \right] - \frac{\ln(S_f(z)/E)}{\sigma\sqrt{2}} \operatorname{erfc}\left[\frac{\ln(S_f(z)/E)}{\sigma\sqrt{2z}} \right] \right) dz \\
&+ \frac{S_0 \sigma}{\sqrt{2}} \int_0^\tau \left(E e^{-r(\tau-z)} - S_0 e^{-D(\tau-z)} \right) e^{-p_0 z} \operatorname{erfc}\left[\frac{\ln(S_f(z)/S_0)}{\sigma\sqrt{2z}} \right] dz \\
&+ \frac{S_0^2 \sigma}{\sqrt{2}} \left(\frac{E}{S_0} \right)^{-\frac{D-r}{\sigma^2}+\frac{1}{2}} \\
&\times \int_0^\tau \left(e^{-D(\tau-z)} - e^{-r(\tau-z)} \right) e^{-p_0 z} \operatorname{erfc}\left[\frac{\ln(S_f(z)/E)}{\sigma\sqrt{2z}} \right] dz, \qquad (22)
\end{aligned}
$$

which is an integro-differential equation in physical space for the location of the free boundary for the call with $0 < D < r$, and is the inverse transform of the equation in transform space (17). The equation for $D \geq r$ can be obtained by setting the right-hand side of (22) to zero,

$$
\begin{aligned}
&\int_{S_0}^{S_f(\tau)} \sqrt{\tau - \tau_f(\tilde{S})} \left(\frac{\tilde{S}}{S_0} \right)^{-\frac{D-r}{\sigma^2}-\frac{3}{2}} e^{-p_0(\tau - \tau_f(\tilde{S}))} \\
&\times \left[\frac{1}{\sqrt{\pi}} \left(2F_0(\tilde{S}) + \frac{F_1(\tilde{S})}{\tau - \tau_f(\tilde{S})} \right) \exp\left(-\frac{\left(\ln(S_f(\tau)/\tilde{S})\right)^2}{2\sigma^2(\tau - \tau_f(\tilde{S}))} \right) \right. \\
&\left. - \frac{F_0(\tilde{S})\sqrt{2}\ln(S_f(\tau)/\tilde{S})}{\sigma(\tau - \tau_f(\tilde{S}))} \operatorname{erfc}\left(\frac{\ln(S_f(\tau)/\tilde{S})}{\sigma\sqrt{2(\tau - \tau_f(\tilde{S}))}} \right) \right] d\tilde{S} = 0, \qquad (23)
\end{aligned}
$$

which is an integro-differential equation in physical space for the location of the free boundary for the call with $D \geq r$, and is the inverse transform of the equation in transform space (19).

4 Discussion

The purpose of this study was to apply one of the tools of classical applied mathematics, the Laplace transform, to the pricing of American options, using the partial Laplace transform method developed by [7] for diffusion problems. The resulting integral equations for the location of the free boundary, (17,19) in transform space and their inverses (22,23) in physical space, form the main result of this paper. These equations were obtained by applying a partial Laplace transform [7] to the Black-Scholes-Merton PDE and solving the resultant ordinary differential equation in transform space. The equations when $D \geq r$ are somewhat simpler than those when $0 < D < r$. It should be recalled that the nonhomogeneous term $F(S)$ in these equations is a function of $\tau_f'(S)$ and $\tau_f''(S)$, the derivatives of the inverse of the optimal exercise boundary, so that (17,19,22,23) involve the first and second derivatives of the unknown boundary and because of this, the integral equations are more complicated than those in [4, 11, 13, 21] which involve the boundary but not the derivatives.

As we mentioned briefly in section 1, integral equation methods have been used to analyze American options before, including the studies of [4, 8, 11, 13, 14, 16, 18, 21, 25, 26]. However, those studies tackled the problem in very different ways to that used here, and ended up with equations of a somewhat different form to those found by us. For example, in their recent studies, [8, 16] used Green's functions to solve the Black-Scholes PDE for American options, and their results involved an integral equation for $S_f(\tau)$, whereas we have an integral equation for the inverse of that function, $\tau_f(S)$. As with our equation, those authors were unable to obtain exact solutions of their integral equations. [25] used a Fourier transform method, while [14] essentially took a Mellin transform with respect to the stock price, and each obtained a (different) integral equation for $S_f(\tau)$. Obviously some connection exists between our results and those other studies, since the integral equations from each study describe the same boundary. It is interesting to note that the Laplace transform, Mellin transform, and Green's function approaches all yield the same expression for the value of a European option but each yields a different integral equation for the free boundary of an American option.

Moving on to the issue of the value of the option, in (8,9,12), we have a series of expressions for $\mathcal{V}(p, S)$, the transform of the option price $V(S,t)$. The constants which appear in these expressions were also given in the previous sections. In theory, given these expressions, we could apply the inverse transform (4), and then we would arrive at the option price itself. Unfortunately, these expressions involve $\tau_f(S)$, the location of the free boundary, which we know only abstractly as the solution of the applicable integral equation; however, if $\tau_f(S)$ were known explicitly, taking the inverse Laplace transform would give the value of the option.

Although the results presented in this study were for the call, it is straightforward to apply them to the put using the well-known put-call 'symmetry' condition of [5, 17], under which the prices of the American call and put are related by

$$C\left[S, E, D, r\right] = P\left[E, S, r, D\right], \tag{24}$$

and the positions of the optimal exercise boundary for the call and put are related by

$$S_f^c\left[t, E, r, D\right] = E^2 / S_f^p\left[t, E, D, r\right]. \tag{25}$$

Of course, (24,25) can also be applied to other studies of American options, such as [4, 11, 13, 21] where the price of the option is given as a closed form expression involving the location of the free boundary.

At this point it behooves us to mention that although we have derived the integro-differential equations (22,23) for the location of the free boundary, we have not addressed either the existence or the regularity or the uniqueness of any solutions to these equations, and these issues remain open, although obviously we would expect the physical free boundary to be a solution. We would suggest that a study addressing these important issues would be a worthwhile endeavor. Indeed, the existence, regularity and uniqueness of solutions remain unresolved for many of the other integral equation formulations of the American option pricing problem mentioned in Section 1, the noticeable exception of course being [4, 11, 13, 21] for which these issues have been successfully resolved. For the integral equations in [4, 11, 13, 21], which involve the boundary but not the derivatives, the existence of solutions follows from the fixed point theorem, while uniqueness has very recently finally been resolved [21].

Finally, we would address the usefulness of the equations (22,23), which describe the location of the free boundary. Although we would not pretend to be proficient in the numerical solution of integral equations, it is reasonable to assume that (22,23) could be used to compute the optimal exercise boundary numerically, as has been done for many of the other integral equation formulations of the American option pricing problem, and we would suggest this, along with a local solution close to expiry, as possible directions for future research. Of course, with such a numerical solution, great care must be taken to verify that any solution of the integral equations corresponds to a solution of the underlying optimal stopping problem, and it would also be of interest to compare the boundary computed using (22,23) with those computed using the other integral equation formulations.

References

1. Alobaidi G, Mallier R (2006) The American straddle close to expiry. Boundary Value Problems 2006 article ID 32835.
2. Black F, Scholes M (1973) The pricing of options and corporate liabilities. Journal of Political Economy 81:637–659.
3. Brekke KA, Oksendal B (1991) The high contact principle as a sufficiency condition for optimal stopping. In: Lund D, Oksendal B (eds) Stochastic Models and Option Values: Applications to Resources, Environment and Investment Problems. North-Holland, Amsterdam
4. Carr P, Jarrow R, Myneni R (1992) Alternative characterizations of the American put. Mathematical Finance 2:87–106.

5. Chesney M, Gibson R (1993) State space symmetry and two-factor option pricing models. Advances in Futures and Options Research 8:85–112.
6. Crank J (1984) Free and moving boundary problems. Clarendon, Oxford
7. Evans GW, Isaacson E, MacDonald JKL (1950) Stefan-like problems. Quarterly of Applied Mathematics 8:312–319.
8. Evans JD, Kuske RE, Keller JB (2002) American options with dividends near expiry. Mathematical Finance 12:219–237.
9. Geske R (1979) A note on an analytical valuation formula for unprotected American call options on stocks with known dividends. Journal of Financial Economics 7:375–380.
10. Geske R (1981) Comments on Whaley's note. Journal of Financial Economics 9:213-215.
11. Jacka SD (1991) Optimal stopping and the American put. Mathematical Finance 1:1–14.
12. Karatzas I (1988) On the pricing of American options. Applied Mathematics and Optimization 17:37–60.
13. Kim IJ (1990) The analytic valuation of American options. Review of Financial Studies 3:547–572.
14. Knessl C (2001) A note on a moving boundary problem arising in the American put option. Studies in Applied Mathematics 107:157–183.
15. Kolodner II (1956) Free Boundary Problem for the Heat Equation with Applications to Problems of Change of Phase. I. General Method of Solution. Communications in Pure and Applied Mathematics 9:1-31.
16. Kuske RE, Keller JB (1998) Optimal exercise boundary for an American put option. Applied Mathematical Finance 5:107–116.
17. McDonald R, Scroder M (1998) A parity result for American options. Journal of Computational Finance 1:5–13.
18. McKean HP Jr. (1965) Appendix A: A free boundary problem for the heat equation arising from a problem in mathematical economics. Industrial Management Review 6:32–39.
19. Merton RC (1973) The theory of rational option pricing. Bell Journal of Economics and Management Science 4:141–183.
20. Myneni R (1992) The Pricing of the American Option. Annals of Applied Probability 2:1-23.
21. Peskir G (2005) On the American option problem. Mathematical Finance 15:169-181.
22. Polyanin AD, Manzhirov V (1998) Handbook of integral equations. CRC Press, New York
23. Roll R (1977) An analytical formula for unprotected American call options on stocks with known dividends. Journal of Financial Economics 5:251–258.
24. Samuelson PA (1965) Rational theory of warrant pricing. Industrial Management Review 6:13–31.
25. Stamicar R, Sevcovic D, Chadam J (1999) The early exercise boundary for the American put near expiry: numerical approximation. Canadian Applied Mathematics Quarterly 7:427–444.
26. Van Moerbeke J (1976) On optimal stopping and free boundary problems. Archive for Rational Mechanics Analysis 60:101–148.
27. Whaley RE (1981) On the valuation of American call options on stocks with known dividends. Journal of Financial Economics 9:207–211.

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