Chapter 2
Lebesgue Spaces of Matrix Functions

In this Chapter, we introduce the notations and define the spaces $L^p(G,M_n)$ of matrix L^p functions on locally compact groups G as a setting for later developments. We recall some basic definitions and derive some results for convolution operators in the scalar case. We discuss differentiability of the norm in $L^p(G,M_n)$ which is needed later, and compute the Gateaux derivative of the norm when the matrix space M_n is equipped with the Hilbert-Schmidt norm.

2.1 Preliminaries

We denote by G throughout a locally compact group with identity e and a *right* invariant Haar measure λ. To avoid the inconvenience of additional measure-theoretic technicalities, we assume throughout that λ is σ-finite. If G is compact, λ is normalized to $\lambda(G) = 1$.

Let $1 \leq p < \infty$. Given a complex Banach space E, we denote by $L^p(G,E)$ the Banach space of (equivalence classes of) E-valued Bochner integrable functions f on G satisfying

$$\|f\|_p = \left(\int_G \|f(x)\|^p d\lambda(x) \right)^{\frac{1}{p}} < \infty$$

(cf. [22, p.97]). We write $L^p(G)$ for $L^p(G,E)$ if $\dim E = 1$. In the sequel, E is usually the C^*-algebra M_n of $n \times n$ complex matrices in which case, a function $f : G \longrightarrow M_n$ is an $n \times n$ matrix (f_{ij}) of complex functions f_{ij} on G.

We denote by $\mathcal{B}(E)$ the Banach algebra of bonded linear self-maps on a Banach space E.

Let $\mathrm{Tr} : M_n \to \mathbb{C}$ be the canonical trace of M_n. Every continuous linear functional $\varphi : M_n \to \mathbb{C}$ is of the form $\varphi(\cdot) = \mathrm{Tr}(\cdot A_\varphi)$ where the matrix $A_\varphi \in M_n$ is unique and $\|\varphi\| = \mathrm{Tr}(|A_\varphi|) = \mathrm{Tr}((A_\varphi^* A_\varphi)^{1/2})$ which is the trace-norm $\|A_\varphi\|_{tr}$ of A_φ. We will identify the dual M_n^*, via the map $\varphi \in M_n^* \mapsto A_\varphi \in M_n$, with the vector space M_n equipped with the trace-norm $\|\cdot\|_{tr}$. If we equip M_n with the Hilbert-Schmidt norm

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$\|A\|_{hs} = \mathrm{Tr}(A^*A)^{1/2}$, then M_n is a Hilbert space with inner product $\langle A, B\rangle = \mathrm{Tr}(B^*A)$. We note that the C*-norm, the trace-norm and the Hilbert-Schmidt norm on M_n are related by

$$\|\cdot\| \leq \|\cdot\|_{tr} \leq \sqrt{n}\|\cdot\|_{hs} \leq n\|\cdot\|$$

and norm convergence is equivalent to entry-wise convergence in M_n.

If M_n is equipped with the Hilbert-Schmidt norm, then $L^2(G, (M_n, \|\cdot\|_{hs}))$ is a Hilbert space, with inner product

$$\langle f, g\rangle_2 = \int_G \mathrm{Tr}(f(x)g(x)^*)d\lambda(x).$$

Since $\|f(x)g(x)^*\|_{hs} \leq \|f(x)\|_{hs}\|g(x)\|_{hs}$ for $f, g \in L^2(G, (M_n, \|\cdot\|_{hs}))$, the Bochner integral

$$\langle\langle f, g\rangle\rangle = \int_G f(x)g(x)^* d\lambda(x)$$

exists in M_n and defines an M_n-valued inner product, turning $L^2(G, (M_n, \|\cdot\|_{hs}))$ into an inner product (left) M_n-module.

We denote by $L^\infty(G, M_n)$ the complex Banach space of M_n-valued essentially bounded (locally) λ-measurable functions on G, where M_n is equipped with the C*-norm. It is a von Neumann algebra, with predual $L^1(G, M_n^*)$, under the pointwise product and involution:

$$(fg)(x) = f(x)g(x), \quad f^*(x) = f(x)^* \qquad (f, g \in L^\infty(G, M_n),\ x \in G).$$

We will study convolution operators on $L^p(G, M_n)$ defined by matrix-valued measures. In this section, we first recall some basic definitions and derive some results for convolution operators on $L^p(G)$, for later reference. One important difference in the matrix setting is the presence of non-commutative and non-associative algebraic structures.

We equip the vector space $C(G)$ of complex continuous functions on G with the topology of uniform convergence on compact sets in G, and denote by $C_c(G)$ the subspace of functions with compact support. The Banach space of bounded complex continuous functions on G is denoted by $C_b(G)$. Let $C_0(G)$ be the Banach space of complex continuous functions on G vanishing at infinity. The dual $C_0(G)^*$ identifies with the space $M(G)$ of complex regular Borel measures on G. Each $\mu \in M(G)$ has finite total variation $|\mu|$ and $M(G)$ is a unital Banach algebra in the total variation norm and the convolution product:

$$\|\mu\| = |\mu|(G), \quad \langle f, \mu * \nu\rangle = \int_G\int_G f(xy)d\mu(x)d\nu(y) \qquad (f \in C_0(G), \mu, \nu \in M(G))$$

where we always denote the duality of a dual pair of Banach spaces E and F by

$$\langle\cdot,\cdot\rangle : E \times F \longrightarrow \mathbb{C}.$$

We also write $\mu(f)$ for $\langle f,\mu\rangle = \int_G f d\mu$. The unit mass at a point $a \in G$ is denoted by δ_a where δ_e is the identity in $M(G)$. A measure $\mu \in M(G)$ is called *absolutely continuous* if its total variation $|\mu|$ is absolutely continuous with respect to the Haar measure λ.

Given $\sigma \in M(G)$, the *support* of σ is defined to be the support of its total variation $|\sigma|$ and is denoted by $\operatorname{supp}\sigma$. We denote by G_σ the closed subgroup of G generated by the support of $|\sigma|$. A measure $\sigma \in M(G)$ is called *adapted* if $G_\sigma = G$. A measure $\sigma \in M(G)$ is said to be *non-degenerate* if $\operatorname{supp}|\sigma|$ generates a dense semigroup in G. Evidently, every non-degenerate measure is adapted. An absolutely continuous (non-zero) measure on a *connected* group must be adapted.

By a (*complex*) *measure* μ on G, we will mean a measure $\mu \in M(G)\backslash\{0\}$.

The convolutions for Borel functions f and g on G, when exit, are defined by

$$\begin{aligned}
(f*g)(x) &= \int_G f(xy^{-1})g(y)d\lambda(y);\\
(f*\mu)(x) &= \int_G f(xy^{-1})d\mu(y);\\
(\mu*f)(x) &= \int_G f(y^{-1}x)\triangle_G(y^{-1})d\mu(y)
\end{aligned}$$

where $\triangle_G$ is the modular function satisfying $d\lambda(xy) = \triangle_G(x)d\lambda(y)$ and $d\lambda(x^{-1}) = \triangle_G(x^{-1})d\lambda(x)$.

We denote by ℓ_x and r_x, respectively, the left and right translations by an element $x \in G$:

$$\ell_x f(y) = f(x^{-1}y)\,, \qquad r_x f(y) = f(yx) \qquad (y \in G)$$

for any function f on G. A complex function f on G is left uniformly continuous if $\|r_x f - f\|_\infty \longrightarrow 0$ as $x \to e$. It is right uniformly continuous if $\|\ell_x f - f\|_\infty \longrightarrow 0$ as $x \to e$. We also write ${}_x f = \ell_{x^{-1}} f$ and f_x for $r_x f$.

We note that each $f \in C_c(G)$ is both left and right uniformly continuous, and for any $\mu \in M(G)$, we have $f*\mu \in C_b(G)$ since $|f*\mu(x) - f*\mu(y)| \le \|\ell_{xy^{-1}} f - f\|\|\mu\|$. We also have

$$\langle f, \mu*\nu\rangle = \langle \widetilde{f}, \widetilde{\nu}*\widetilde{\mu}\rangle \tag{2.1}$$

where $\nu \in M(G)$ and we define $\widetilde{f}(x) = f(x^{-1})$ and $d\widetilde{\mu}(x) = d\mu(x^{-1})$. Note that

$$\widetilde{\mu}(f) = \mu(\widetilde{f}) = (f*\mu)(e) \quad \text{and} \quad \widetilde{\mu*\nu} = \widetilde{\nu}*\widetilde{\mu}$$

for $f \in C_c(G)$.

Let $\sigma \in M(G)$. For $1 \le p \le \infty$, we define the convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$ by

$$T_\sigma(f) = f*\sigma \qquad (f \in L^p(G)).$$

To avoid triviality, σ is always non-zero for T_σ. The definition of T_σ depends on its domain $L^p(G)$ although we often omit referring to it if there is no ambiguity. When regarded as an operator on $L^p(G)$, the operator T_σ is easily seen to be bounded and we denote its norm by $\|T_\sigma\|_p$, or simply $\|T_\sigma\|$ in obvious context. We have $\|T_\sigma\|_p \le \|\sigma\|$.

A convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$ commutes with left translations:

$$\ell_x T_\sigma = T_\sigma \ell_x \qquad (x \in G).$$

Conversely, for abelian groups G, every translation invariant operator $T : L^1(G) \longrightarrow L^1(G)$ is a convolution operator T_σ for some $\sigma \in M(G)$ [55, 3.8.4]. However, this result does not hold for $1 < p \leq \infty$, even if G is compact and abelian [44, p.85]. We will characterise the more general matrix convolution operators in Chapter 3. In particular, the above L^1 result is generalized to the matrix-valued case, for all locally compact groups.

For $1 \leq p \leq \infty$, we denote by q its conjugate exponent throughout, that is, $\frac{1}{p} + \frac{1}{q} = 1$, and for the dual pairing $\langle \cdot, \cdot \rangle$ between $L^p(G)$ and $L^q(G)$, we have

$$\langle f * \sigma, h \rangle = \langle f, h * \widetilde{\sigma} \rangle \tag{2.2}$$

for $f \in L^p(G)$ and $h \in L^q(G)$. This implies that T_σ is weakly continuous on $L^p(G)$ for $1 \leq p < \infty$, and is weak* continuous on $L^\infty(G)$. In particular, T_σ is a weakly compact operator on $L^p(G)$ for $1 < p < \infty$. For $p = 1, \infty$, we will discuss presently weak compactness of $T_\sigma : L^p(G) \longrightarrow L^p(G)$, but we note the following two lemmas first.

Lemma 2.1.1. *Let $\sigma \in M(G)$ and $p < \infty$. Let $T_\sigma^* : L^q(G) \longrightarrow L^q(G)$ be the dual map of the convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$. Then $T_\sigma^* = T_{\widetilde{\sigma}}$. The operator $T_\sigma : L^2(G) \longrightarrow L^2(G)$ is self-adjoint if $\widetilde{\sigma} = \sigma$ is a real measure. The weak* continuous operator $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ has predual $T_{\widetilde{\sigma}} : L^1(G) \longrightarrow L^1(G)$.*

Proof. By (2.2), we have $\langle f, T_\sigma^* h \rangle = \langle f, T_{\widetilde{\sigma}} h \rangle$ for $f \in L^p(G)$ and $h \in L^q(G)$. The adjoint of T_σ in $\mathcal{B}(L^2(G))$ is $T_{\widetilde{\overline{\sigma}}}$ where $\overline{\sigma}$ is the complex conjugate of σ. □

Lemma 2.1.2. *Let $\sigma \in M(G)$ and let T_σ be the convolution operator on $L^p(G)$ for $p = 1, \infty$. We have $\|T_\sigma\|_1 = \|T_\sigma\|_\infty = \|\sigma\|$.*

Proof. We have $\|\sigma\| = \sup\{|\int_G f d\sigma| : f \in C_c(G) \text{ and } \|f\| \leq 1\}$ in which

$$\left| \int_G f d\sigma \right| = |\widetilde{f} * \sigma(e)| \leq \|\widetilde{f} * \sigma\|_\infty \leq \|T_\sigma\|_\infty$$

where $\widetilde{f} * \sigma \in C_b(G)$. Next, we have $\|T_\sigma\|_1 = \|T_\sigma^*\|_\infty = \|T_{\widetilde{\sigma}}\|_\infty = \|\widetilde{\sigma}\| = \|\sigma\|$. □

Remark 2.1.3. We note that $\|T_\sigma\|_p$ need not equal $\|\sigma\|$ if $1 < p < \infty$. Indeed, if σ is an adapted probability measure whose support contains the identity e and if $\|T_\sigma\|_p = 1$ for some $1 < p < \infty$, then G is amenable (see, for example, [4, Theorem 1]). On the other hand, if G is amenable and σ is a probability measure, then $\|T_\sigma\|_p = 1$ for all p (cf. [33, p.48]).

By Lemma 2.1.2, the spectral radius of $T_\sigma \in \mathcal{B}(L^p(G))$, for $p = 1, \infty$, is $\lim_n \|T_\sigma^n\|^{\frac{1}{n}} = \lim_n \|T_{\sigma^n}\|^{\frac{1}{n}} = \lim_n \|\sigma^n\|^{\frac{1}{n}}$ where σ^n is the n-fold convolution of σ with itself.

Lemma 2.1.4. *Let G be a compact group and let $\sigma \in M(G)$ be absolutely continuous. Then the convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$ is compact for every $p \in [1, \infty]$.*

Proof. Let $\sigma = h \cdot \lambda$ for some $h \in L^1(G)$. Consider first $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$. By absolute continuity of σ, we have $T_\sigma(L^\infty(G)) \subset C(G)$. Hence, by Arzela-Ascoli theorem, we need only show that the set

$$\{T_\sigma(f) : \|f\|_\infty \le 1\}$$

is equicontinuous in $C(G)$. Let $\varepsilon > 0$. Pick $\varphi \in C_c(G)$ with support K and $\|\varphi - h\|_1 < \frac{\varepsilon}{4}$. Let W be a compact neighbourhood of the identity $e \in G$. By uniform continuity, we can choose a compact neighbourhood $V \subset W$ of e such that

$$|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2\lambda(KW)}$$

whenever $x^{-1}y \in V$. Then

$$\begin{aligned}\|\varphi_x - \varphi_y\|_1 &= \int_G |\varphi(zx) - \varphi(zy)| d\lambda(z) \\ &= \int_{KW} |\varphi(z) - \varphi(zx^{-1}y)| d\lambda(z) < \frac{\varepsilon}{2}.\end{aligned}$$

It follows that, for $x^{-1}y \in V$ and $\|f\|_\infty \le 1$, we have

$$\begin{aligned}|T_\sigma(f)(x) - T_\sigma(f)(y)| &= \left| \int_G f(xz^{-1})h(z)d\lambda(z) - \int_G f(yz^{-1})h(z)d\lambda(z) \right| \\ &\le \int_G |f(z^{-1})h(zx) - f(z^{-1})h(zy)| d\lambda(z) \\ &\le \|f\|_\infty \|h_x - h_y\|_1 \\ &\le \|f\|_\infty (\|h_x - \varphi_x\|_1 + \|\varphi_x - \varphi_y\|_1 + \|h_y - \varphi_y\|_1) < \varepsilon\end{aligned}$$

which proves equicontinuity and hence, compactness of $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$.

Likewise $T_{\tilde{\sigma}} : L^\infty(G) \longrightarrow L^\infty(G)$ is compact and therefore $T_\sigma : L^1(G) \longrightarrow L^1(G)$ is compact.

Let $1 < p < \infty$. Let (h_n) be a sequence in $C(G)$ such that $\|h_n - h\|_1 \longrightarrow 0$. Then $T_\sigma = \lim_{n\to\infty} T_{\sigma_n}$ in $\mathcal{B}(L^p(G))$, where $\sigma_n = h_n \cdot \lambda$. Hence it suffices to show compactness of T_σ on $L^p(G)$ for the case $h \in C(G)$.

Let (f_n) be a sequence in the unit ball of $L^p(G)$. Then $\|f_n\|_1 \le 1$ for all n and compactness of $T_\sigma : L^1(G) \longrightarrow L^1(G)$ implies that the sequence $(f_n * \sigma)$ contains a subsequence L^1-converging to some $f \in L^1(G)$, and hence a subsequence $(f_k * \sigma)$ converging pointwise to f λ-almost everywhere. Since $h \in C(G)$, we have $\|f_k * \sigma\|_\infty \le \|f_k\|_p \|h\|_q \le \|h\|_q$ for all k, and $f \in L^\infty(G)$. It follows that

$$\|f_k * \sigma - f\|_p^p \le \|f_k * \sigma - f\|_1 \|f_k * \sigma - f\|_\infty^{p-1} \longrightarrow 0 \qquad \text{as } k \to \infty.$$

This proves compactness of $T_\sigma : L^p(G) \longrightarrow L^p(G)$. $\square$

Remark 2.1.5. The above result is clearly false if σ is not absolute continuous, for instance, T_σ is the identity operator if $\sigma = \delta_e$.

A compactness criterion has been given in [48] for a class of convolution operators of the form $f \in L^1(G) \mapsto f * F \in C(G)$ where $F \in L^\infty(G)$ and G is compact abelian. Compactness of the composition of a convolution operator with a multiplier has also been considered in [59, 60]. Fredholmness of convolution operators on locally compact groups has been studied in [54, 59, 61].

Proposition 2.1.6. *Let σ be a positive measure on a group G such that $\sigma^2 * \widetilde{\sigma}^2$ is adapted. Let T_σ be the associated convolution operator. The following conditions are equivalent.*

(i) $T_\sigma : L^1(G) \longrightarrow L^1(G)$ *is weakly compact.*
(ii) $T_\sigma : L^1(G) \longrightarrow L^1(G)$ *is compact.*
(iii) $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ *is weakly compact.*
(iv) $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ *is compact.*
(v) $T_\sigma : L^p(G) \longrightarrow L^p(G)$ *is compact for all $p \in [1, \infty]$.*
(vi) *G is compact and σ is absolutely continuous.*

Proof. (i) $\Longrightarrow$ (vi). We first prove compactness of G. Note that $L^1(G)$ has the Dunford-Pettis property and in particular, every weakly compact operator on $L^1(G)$ sends weakly compact subsets to norm compact sets [22, p.154]. Hence weak compactness of T_σ implies that the operator $T_\sigma^2 : L^1(G) \longrightarrow L^1(G)$ is compact, and so is the operator $T_{\sigma*\sigma*\widetilde{\sigma}*\widetilde{\sigma}} = T_{\widetilde{\sigma}}^2 T_\sigma^2$. Since $\sigma^2 * \widetilde{\sigma}^2$ is a positive measure, the spectral radius of $T_{\sigma^2*\widetilde{\sigma}^2}$ is $\sigma(G)^4$, by a remark before Lemma 2.1.4. On the Hilbert space $L^2(G)$, the operator $T_{\sigma^2*\widetilde{\sigma}^2} = T_{\sigma^2}^* T_{\sigma^2}$ is a positive operator and therefore has only non-negative eigenvalues. The eigenvalues of $T_{\sigma^2*\widetilde{\sigma}^2} \in \mathcal{B}(L^1(G))$ are also eigenvalues of $T_{\sigma^2*\widetilde{\sigma}^2} \in \mathcal{B}(L^2(G))$ and therefore non-negative. It follows that $\sigma(G)^4$ is an eigenvalue of the compact operator $T_{\sigma^2*\widetilde{\sigma}^2} \in \mathcal{B}(L^1(G))$, that is, there is a non-zero function $f \in L^1(G)$ satisfying $f * \sigma^2 * \widetilde{\sigma}^2 = \sigma(G)^4 f$. Note that the measure $\sigma(G)^{-4}\sigma^2 * \widetilde{\sigma}^2$ is an adapted probability measure on G. Now, by [10, Theorem 3.12], f is constant which implies that G must be compact.

Next, we show that σ is absolutely continuous. By the Dunford-Pettis-Phillips Theorem [22, p.75], there is an essentially bounded function $g : G \longrightarrow L^1(G)$ such that

$$T_\sigma(f) = \int_G fg\,d\lambda \qquad (f \in L^1(G)).$$

Now the arguments in [22, p.91] can still be applied without commutativity of G. Let $a \in G$. For each $f \in L^1(G)$, we have, for λ-a.e. y,

$$\begin{aligned}\int_G f(x)g(x)(y)d\lambda(x) &= T_\sigma f(y) = \ell_{a^{-1}} T_\sigma(\ell_a f)(y)\\ &= \int_G (\ell_a f)(x)g(x)(ay)d\lambda(x)\\ &= \int_G f(a^{-1}x)g(x)(ay)d\lambda(x)\\ &= \int_G f(x)g(ax)(ay)d\lambda(x).\end{aligned}$$

It follows that

$$g(ax)(ay) = g(x)(y)$$

for λ-a.e. x and y. This implies that, for each $f \in C(G)$, the function

$$F(y) = \int_G f(x)g(yx^{-1})(y)d\lambda(x) \qquad (y \in G)$$

is invariant under the left translations ℓ_a for all $a \in G$. Using compactness of G, one can show that F is constant λ-almost every on G, as in [22, p.91], and hence we have,

$$F(y) = \int_G F(z)d\lambda(z) = \int_G \int_G f(x)g(zx^{-1})(x)d\lambda(x)d\lambda(z) \tag{2.3}$$

for λ-a.e. y. Let $h \in L^1(G)$ be defined by

$$h(x) = \int_G g(yx^{-1})(y)d\lambda(y).$$

We show that $f * \sigma = f * h$ for each $f \in C(G) \subset L^1(G)$ which then yields absolutely continuity of σ. Indeed, for each $k \in L^\infty(G)$, we have

$$\begin{aligned}\langle k, f*h\rangle &= \int_G k(y)\int_G f(yx^{-1})h(x)d\lambda(x)d\lambda(y)\\ &= \int_G k(y)\int_G\int_G f(yx^{-1})g(zx^{-1})(z)d\lambda(x)d\lambda(z)d\lambda(y)\\ &= \int_G k(y)\int_G f(yx^{-1})g(yx^{-1})(y)d\lambda(x)d\lambda(y) \qquad \text{(by (2.3))}\\ &= \int_G k(y)f*\sigma(y)d\lambda(y) = \langle k, f*\sigma\rangle\end{aligned}$$

which concludes the proof.

(vi) $\Longrightarrow$ (v). By Lemma 2.1.4.

(v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii). Trivial.

(iii) $\Longrightarrow$ (ii). The given condition implies that $T_{\widetilde{\sigma}} : L^1 G) \longrightarrow L^1(G)$ is weakly compact. Repeating (i) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) for $\widetilde{\sigma}$, we see that $T_{\widetilde{\sigma}} : L^\infty(G) \longrightarrow L^\infty(G)$ is compact, and hence $T_\sigma : L^1(G) \longrightarrow L^1(G)$ is compact.

(ii) $\Longrightarrow$ (i). Trivial. $\square$

Remark 2.1.7. In (i) $\Longrightarrow$ (vi) above, the proof of absolute continuity of σ from weak compactness of $T_\sigma \in B(L^1(G))$ is valid for *any* measure σ on a compact group G, without adaptedness of $\sigma^2 * \widetilde{\sigma}^2$.

Corollary 2.1.8. *Given a positive absolutely continuous measure σ on a connected group G, the following conditions are equivalent.*

(i) $T_\sigma : L^1(G) \longrightarrow L^1(G)$ *is weakly compact.*
(ii) $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ *is weakly compact.*
(iii) $T_\sigma : L^p(G) \longrightarrow L^p(G)$ *is compact for all* $p \in [1,\infty]$.
(iv) G *is compact.*

Proof. This is because absolutely continuous measures on a connected group are adapted. □

Definition 2.1.9. The spectrum of an element a in a unital Banach algebra $\mathcal{A}$ is denoted by $\mathrm{Spec}_{\mathcal{A}}\, a$ which is often shortened to $\mathrm{Spec}\, a$ if the Banach algebra $\mathcal{A}$ is understood. For $1 \le p \le \infty$, we write $\mathrm{Spec}(T_\sigma, L^p(G))$, or simply, $\mathrm{Spec}(T_\sigma, L^p)$, for the spectrum $\mathrm{Spec}\, T_\sigma$, when regarding $T_\sigma \in \mathcal{B}(L^p(G))$. We denote by $\Lambda(T_\sigma, L^p(G))$, or simply, $\Lambda(T_\sigma, L^p)$, the set of eigenvalues of $T_\sigma : L^p(G) \longrightarrow L^p(G)$.

Given any Banach algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$, we define, as usual, the *quasi-spectrum* of a, denoted by $\mathrm{Spec}'_{\mathcal{A}}\, a$, to be the spectrum $\mathrm{Spec}_{\mathcal{A}_1}\, a$ of a in the unit extension $\mathcal{A}_1$ of $\mathcal{A}$. We always have $0 \in \mathrm{Spec}'_{\mathcal{A}}\, a$. If $\mathcal{A}$ has an identity, then we have

$$\mathrm{Spec}'_{\mathcal{A}}\, a = \mathrm{Spec}_{\mathcal{A}}\, a \cup \{0\}.$$

We recall that

$$\mathrm{Spec}(T_\sigma, L^p) = \Lambda(T_\sigma, L^p) \cup \mathrm{Spec}^r(T_\sigma, L^p) \cup \mathrm{Spec}^c(T_\sigma, L^p)$$

where $\mathrm{Spec}^r(T_\sigma, L^p)$ denotes the *residue spectrum* of T_σ, consisting of $\alpha \in \mathrm{Spec}(T_\sigma, L^p) \backslash \Lambda(T_\sigma, L^p)$ satisfying

$$\overline{(T_\sigma - \alpha I)(L^p(G))} \neq L^p(G)$$

and $\mathrm{Spec}^c(T_\sigma, L^p)$ denotes the *continuous spectrum* of T_σ, consisting of $\alpha \in \mathrm{Spec}(T_\sigma, L^p) \backslash \Lambda(T_\sigma, L^p)$ such that

$$\overline{(T_\sigma - \alpha I)(L^p(G))} = L^p(G).$$

Since $T_\sigma^* = T_{\widetilde{\sigma}}$ for $p < \infty$, we have

$$\mathrm{Spec}(T_\sigma, L^p) = \mathrm{Spec}(T_{\widetilde{\sigma}}, L^q)$$

for $1 \le p < \infty$, and also $\mathrm{Spec}(T_\sigma, L^\infty) = \mathrm{Spec}(T_{\widetilde{\sigma}}, L^1)$.

We denote by $\mathrm{Spec}\, \sigma$ the spectrum of σ in the measure algebra $M(G)$. Note that $\mathrm{Spec}\, \sigma = \mathrm{Spec}\, \widetilde{\sigma}$ since $\widetilde{\sigma} * \widetilde{\mu} = \widetilde{\mu * \sigma}$ for each $\mu \in M(G)$.

Given a locally compact group G, we let $\widehat{G}$ be the dual space consisting of (the equivalence classes of) continuous unitary irreducible representations $\pi : G \longrightarrow \mathcal{B}(H_\pi)$, where H_π is a Hilbert space. Let $\iota \in \widehat{G}$ be the one-dimensional identity representation. For $\pi \in \widehat{G}$, $\sigma \in M(G)$ and $f \in L^1(G)$, we define the *Fourier transforms*:

$$\widehat{\sigma}(\pi) = \int_G \pi(x^{-1}) d\sigma(x) \in \mathcal{B}(H_\pi),$$
$$\widehat{f}(\pi) = \int_G f(x)\pi(x^{-1}) d\lambda(x) \in \mathcal{B}(H_\pi).$$

We have $\widehat{f * \sigma}(\pi) = \widehat{\sigma}(\pi)\widehat{f}(\pi)$ and $\widehat{\mu * \sigma}(\pi) = \widehat{\sigma}(\pi)\widehat{\mu}(\pi)$ for $\mu \in M(G)$.

The spectrum $\mathrm{Spec}_{\mathcal{B}(H_\pi)} \widehat{\sigma}(\pi)$ of $\widehat{\sigma}(\pi) \in \mathcal{B}(H_\pi)$ will be written as $\mathrm{Spec}\, \widehat{\sigma}(\pi)$ if no confusion is likely.

If G is abelian, $\widehat{G}$ is the group of characters and we often use the letter χ to denote an element in $\widehat{G}$. For $1 < p < 2$ and $f \in L^p(G)$, we define the Fourier transform $\widehat{f} \in L^q(\widehat{G})$ via Riesz-Thorin interpolation.

A continuous homomorphism χ from an abelian group G to the multiplicative group $\mathbb{C}\backslash\{0\}$ is called a *generalized character*. For such a character χ with $|\chi(\cdot)| \leq 1$, one can still define $\widehat{\sigma}(\chi)$ as above. The spectrum $\Omega(G)$ of the Banach algebra $M(G)$, i.e., the non-zero multiplicative functionals on $M(G)$, identifies with the generalized characters χ of G with $|\chi(\cdot)| \leq 1$, and by Gelfand theory, we have $\mathrm{Spec}\, \sigma = \widehat{\sigma}(\Omega(G))$ which contains $\widehat{\sigma}(\widehat{G})$. The spectrum of $L^1(G)$ identifies with the dual group $\widehat{G}$ and if G is discrete, then $M(G) = \ell^1(G)$ and $\mathrm{Spec}\, \sigma = \mathrm{Spec}_{\ell^1(\mathrm{G})} \sigma = \widehat{\sigma}(\widehat{\mathrm{G}})$. For arbitrary groups, we have the following result.

Lemma 2.1.10. *Let σ be a complex measure on a group G. Then*

$$\Lambda(T_\sigma, L^1) \subset \bigcup_{\pi \in \widehat{G}} \mathrm{Spec}\, \widehat{\sigma}(\pi) \subset \mathrm{Spec}\, \sigma.$$

The inclusions are strict in general.

Proof. Similar inclusions hold in the more general matrix setting for which a simple proof will be given in Proposition 3.3.8. If σ is an adapted probability measure and G is non-compact, then by [10, Theorem 3.12], $1 \notin \Lambda(T_\sigma, L^1)$ while $1 \in \mathrm{Spec}\, \iota(\sigma)$ where $\iota \in \widehat{G}$ is the identity representation.

If G is abelian, then $\widehat{\sigma}(\widehat{G}) = \bigcup_{\pi \in \widehat{G}} \mathrm{Spec}\, \widehat{\sigma}(\pi)$ and Example 3.3.4 shows that the last inclusion can be strict. In fact, even the closure $\overline{\widehat{\sigma}(\widehat{G})}$ may not equal $\mathrm{Spec}\, \sigma$ by Remark 3.3.24. $\square$

It has been shown in [10, Lemma 3.11] that $1 \notin \bigcup_{\pi \in \widehat{G} \backslash \{\iota\}} \mathrm{Spec}\, \widehat{\sigma}(\pi)$ if σ is an adapted probability measure on a locally compact group G. In general, there seem to be few definitive results concerning the spectrum of T_σ for non-abelian groups. We will consider this case in Chapter 3 and prove various results there.

We will make use of a version of the Wiener-Levy theorem, stated below, which has been proved in [55, Theorem 6.2.4] and will be generalized to the matrix setting in Chapter 3.

Lemma 2.1.11. *Let Ω be an open set in $\mathbb{C}$ and let $F : \Omega \longrightarrow \mathbb{C}$ be a real analytic function satisfying $F(0) = 0$ if $0 \in \Omega$. Given an abelian group G and a function $f \in L^1(G)$ such that $\overline{\widehat{f}(\widehat{G})} \subset \Omega$, then $F(\widehat{f})$ is the Fourier transform of an $L^1(G)$-function.*

Example 2.1.12. For the Cauchy distribution

$$d\sigma_t(x) = \frac{t}{\pi(t^2+x^2)}dx \qquad (t > 0)$$

on $\mathbb{R}$, we have $\widehat{\sigma}_t(\widehat{\mathbb{R}}) = \{\exp(-t|x|) : x \in \mathbb{R}\} = (0,1] = \mathrm{Spec}(T_\sigma, L^p)\backslash\{0\} = \Lambda(T_{\sigma_t}, L^\infty)$.

Example 2.1.13. Let G be any locally compact group and let $\sigma = \delta_a$ be the unit mass at $a \in G$. Then T_σ is a translation on $L^p(G)$ and we have

$$\mathrm{Spec}(T_\sigma, L^\infty) \subset \{\alpha : |\alpha| = 1\}.$$

If $G = \mathbb{T}$ and $a = i$, then $L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$ and $\mathrm{Spec}(T_\sigma, L^\infty) = \mathrm{Spec}(T_\sigma, L^2) = \widehat{\sigma}(\mathbb{Z}) = \{\exp(-in\pi/2) : n \in \mathbb{Z}\} = \{\pm 1, \pm i\} \neq \{\alpha : |\alpha| = 1\}$.

If $G = \mathbb{Z}$ and $a = 1$, then $\mathrm{Spec}(T_\sigma, \ell^2) = \{\alpha : |\alpha| = 1\} = \mathrm{Spec}(T_\sigma, \ell^\infty)$.

If $G = \mathbb{R}$ and $a \neq 0$, then $\mathrm{Spec}(T_\sigma, L^p) = \{\alpha : |\alpha| = 1\} = \Lambda(T_\sigma, L^\infty)$ as $\widehat{\delta}_a(\widehat{\mathbb{R}}) = \{\exp(-ia\theta) : \theta \in \mathbb{R}\}$.

Next consider the measure $\mu = \frac{1}{2}(\delta_0 + \delta_1)$ on $\mathbb{R}$. Its n-fold convolution

$$\mu^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\delta_k$$

is a convex sum of discrete measures and we have

$$\mathrm{Spec}(T_{\mu^n}, L^\infty) = \Lambda(T_{\mu^n}, L^\infty) = \left\{\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\exp(-ik\theta) : \theta \in \mathbb{R}\right\}$$

where, for example, $\mathrm{Spec}(T_\mu, L^\infty)$ is the circle containing 0 and internally tangent to the unit circle at 1, with $\sin \pi x$ as a 0-eigenfunction.

2.2 Differentiability of Norm in $L^p(G, M_n)$

We will be working with the complex Lebesgue spaces $L^p(G, M_n)$ where, for convenience and consistency with previous and related works elsewhere, we equip M_n with the C*-norm unless otherwise stated. Some remarks are in order here. First, there is no essential difference if one chooses to equip M_n with the trace norm since it amounts to considering the space $L^p(G,M_n^*)$ which is, for $p>1$, the dual of $L^q(G,M_n)$. Also, the Lebesgue spaces $L^p(G,M_n)$ defined in terms of the C*, trace and Hilbert-Schmidt norms on M_n are all isomorphic and most results for these three cases are identical. There is, however, a difference among the three cases if one considers the differentiability of the norm of $L^p(G,M_n)$ which will be needed later.

Let us first consider the differentiability of the C*-norm $\|\cdot\|$, the trace norm $\|\cdot\|_{tr}$ and the Hilbert-Schmidt norm $\|\cdot\|_{hs}$ on M_n, regarded as a real Banach space.

We recall that the norm $\|\cdot\|$ of a real Banach space E is said to be *Gateaux differentiable* at a point $u\in E$ if the following limit exists

$$\partial\|u\|(x)=\lim_{t\to 0}\frac{\|u+tx\|-\|u\|}{t}$$

for each $x\in E$, in which case, the limit is called the *Gateaux derivative* of the norm at u, *in the direction of* x. We note that the *right directional derivative*

$$\partial^+\|u\|(x)=\lim_{t\downarrow 0}\frac{\|u+tx\|-\|u\|}{t}$$

always exists. In fact, it is equal to

$$\sup\{\psi(x):\psi \text{ is a subdifferential at } u\}$$

where a linear functional ψ in the dual E^* is called a *subdifferential* at u if

$$\psi(x-u)\le\|x\|-\|u\|$$

for each $x\in E$. The norm is Gateaux differentiable at u if, and only if, there is a unique subdifferential at u, in which case, the subdifferential is the Gateaux derivative (cf. [53, Proposition 1.8]).

The Hilbert-Schmidt norm $\|\cdot\|_{hs}$ on M_n is Gateaux differentiable at every $A\in M_n\backslash\{0\}$. Indeed, we have

$$\begin{aligned}\lim_{t\to 0}\frac{\|A+tX\|_{hs}-\|A\|_{hs}}{t}&=\lim_{t\to 0}\frac{\mathrm{Tr}((A+tX)^*(A+tX))-\mathrm{Tr}(A^*A)}{t(\|A+tX\|_{hs}+\|A\|_{hs})}\\&=\frac{\mathrm{Tr}(A^*X+X^*A)}{2\|A\|_{hs}}\\&=\frac{1}{\|A\|_{hs}}\mathrm{Re}\,\mathrm{Tr}(A^*X).\end{aligned}$$

Although the norm of a separable Banach space is Gateaux differentiable on a dense G_δ set, it is easy to see that the C*-norm and the trace norm need not be Gateaux differentiable at every non-zero $A\in M_n$.

Lemma 2.2.1. *Let $A\in M_n\backslash\{0\}$. The C*-norm on M_n is Gateaux differentiable at A if, and only if, given any unit vectors $\xi,\eta\in\mathbb{C}^n$ with $\|A\xi\|=\|A\eta\|=\|A\|$, we have*

$$\langle A\xi, X\xi\rangle = \langle A\eta, X\eta\rangle \qquad (X \in M_n).$$

In the above case, the Gateaux derivative at A is given by

$$\partial\|A\|(X) = \frac{1}{\|A\|}\mathrm{Re}\,\langle A\xi, X\xi\rangle \qquad (X \in M_n)$$

where $\xi \in \mathbb{C}^n$ is a unit vector satisfying $\|A\xi\| = \|A\|$.

Proof. Suppose the norm is Gateaux differentiable at A. Let $\xi \in \mathbb{C}^n$ be a unit vector such that $\|A\| = \|A\xi\|$. Define a real continuous linear functional $\psi_\xi : M_n \longrightarrow \mathbb{R}$ by

$$\psi_\xi(X) = \frac{1}{\|A\|}\mathrm{Re}\,\langle A\xi, X\xi\rangle \qquad (X \in M_n).$$

Then for each $X \in M_n$, we have

$$\begin{aligned}\psi_\xi(X-A) &= \frac{1}{\|A\|}\mathrm{Re}\,\langle A\xi, X-A\xi\rangle \\ &= \frac{1}{\|A\|}\mathrm{Re}\,(\langle A\xi, X\xi\rangle - \langle A\xi, A\xi\rangle) \leq \|X\| - \|A\|.\end{aligned}$$

Hence ψ_ξ is a subdifferential at A. If η is a unit vector in $\mathbb{C}^n$ such that $\|A\eta\| = \|A\|$, then we must have $\psi_\eta = \psi_\xi$, by uniqueness of the subdifferential, which gives $\langle A\xi, X\xi\rangle = \langle A\eta, X\eta\rangle$ for every $X \in M_n$.

To show the converse, we note that (cf. [5, Proposition 4.12]),

$$\lim_{t\downarrow 0} \frac{\|A+tX\| - \|A\|}{t} = \sup\left\{\lim_{t\downarrow 0} \frac{\|(A+tX)\xi\| - \|A\xi\|}{t} : \|\xi\| = 1, \|A\xi\| = \|A\|\right\}$$

where

$$\begin{aligned}\lim_{t\downarrow 0} \frac{\|(A+tX)\xi\| - \|A\xi\|}{t} &= \lim_{t\downarrow 0} \frac{\langle (A+tX)\xi, (A+tX)\xi\rangle - \langle A\xi, A\xi\rangle}{t(\|(A+tX)\xi\| + \|A\xi\|)} \\ &= \frac{\langle A\xi, X\xi\rangle + \langle X\xi, A\xi\rangle}{2\|A\|}.\end{aligned}$$

Hence the necessary condition implies that the above set on the right reduces to a singleton which gives the right directional derivative. We also have

$$\begin{aligned}\lim_{t\uparrow 0} \frac{\|(A+tX)\xi\| - \|A\xi\|}{t} &= -\lim_{t\downarrow 0} \frac{\|(A-tX)\xi\| - \|A\xi\|}{t} \\ &= -\frac{\langle A\xi, -X\xi\rangle + \langle -X\xi, A\xi\rangle}{2\|A\|} \\ &= \lim_{t\downarrow 0} \frac{\|(A+tX)\xi\| - \|A\xi\|}{t}.\end{aligned}$$

This proves Gateaux differentiability at A. The last assertion is clear from the above computation. □

Example 2.2.2. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2$. Then the unit vectors in $\mathbb{C}^2$ where A achieves its norm are of the form $(\alpha, 0)$ with $|\alpha| = 1$. For any matrix $X = (x_{ij})$ in M_2, we have $\langle A(\alpha,0)^T, X(\alpha,0)^T\rangle = \overline{x_{12}} + \overline{x_{21}}$ which is independent of α, and the C*-norm is Gateaux differentiable at A with derivative

$$\partial\|A\|(X) = \mathrm{Re}\,\langle A(1,0)^T, X(1,0)^T\rangle = \mathrm{Re}\,x_{11}.$$

The matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ achieves its norm at $(\sqrt{2},0)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$; but

$$\left\langle B(\sqrt{2},0)^T, X(\sqrt{2},0)^T \right\rangle \neq \left\langle B\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T, X\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T \right\rangle$$

if X is the identity matrix, say. Hence the C*-norm is not Gateaux differentiable at B, however, we have the *right* I-directional derivative

$$\begin{aligned}\partial^+\|B\|(I) &= \lim_{t\downarrow 0} \frac{\|B+tI\| - \|B\|}{t} \\ &= \lim_{t\downarrow 0} \frac{\sqrt{1+t+t^2+\sqrt{1+2t+2t^2}} - \sqrt{2}}{t} = \frac{\sqrt{2}}{2}.\end{aligned}$$

On the other hand, the trace norm $\|\cdot\|_{tr}$ is not Gateaux differentiable at A since

$$\frac{\|A+tX\|_{tr} - \|A\|_{tr}}{t} = \frac{|t|}{t}$$

for $X = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, say.

Lemma 2.2.3. *Let $A \in M_n \backslash \{0\}$ with polar decomposition $A = u|A|$. If the trace norm $\|\cdot\|_{tr}$ on M_n is Gateaux differentiable at A, then the Gateaux derivative is given by*

$$\partial\|A\|_{tr}(X) = \mathrm{Re}\,\mathrm{Tr}(u^*X) \qquad (X \in M_n).$$

Proof. We only need to show that $\psi(X) = \mathrm{Re}\,Tr(u^*X)$ is a subdifferential. Indeed, we have $|A| = u^*A$ and

$$\begin{aligned}\psi(X - A) &= \mathrm{Re}\,\mathrm{Tr}(u^*X) - \mathrm{Re}\,\mathrm{Tr}(u^*A) \\ &\leq \|u^*\|\|X\|_{tr} - \|A\|_{tr} \\ &= \|X\|_{tr} - \|A\|_{tr}.\end{aligned}$$

□

Example 2.2.4. In Example 2.2.2 above, we have $u = A$ in the polar decomposition of A and $\operatorname{Re} Tr(u^*X) = 0$ for $X = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, while the *right* X-directional derivative is given by

$$\lim_{t\downarrow 0} \frac{\|A+tX\|_{tr} - \|A\|_{tr}}{t} = \lim_{t\downarrow 0} \frac{|t|}{t} = 1.$$

Due to the non-smoothness of the C*-norm and trace norm on M_n, we will consider the Lebesgue spaces $L^p(G,(M_n,\|\cdot\|_{hs}))$ with M_n equipped with the Hilbert-Schmidt norm when we need to make use of norm differentiability later. We compute below the Gateaux derivatives for $L^p(G,(M_n,\|\cdot\|_{hs}))$.

Since the function $u \in E \mapsto \|u\|^p$ is convex on any Banach space E, we have, for $0 < t < 1$ and $u, v \in E$,

$$\|u+tv\|^p \le (1-t)\|u\|^p + t\|u+v\|^p$$

and

$$\|u\|^p \le \frac{t}{1+t}\|u-v\|^p + \frac{1}{1+t}\|u+tv\|^p$$

which gives

$$\|u\|^p - \|u-v\|^p \le \frac{1}{t}(\|u+tv\|^p - \|u\|^p) \le \|u+v\|^p - \|u\|^p. \tag{2.4}$$

Proposition 2.2.5. *Let $1 < p < \infty$. The norm of $L^p(G,(M_n,\|\cdot\|_{hs}))$ is Gateaux differentiable at each non-zero f with Gateaux derivative*

$$\partial\|f\|_p(g) = \operatorname{Re}\|f\|_p^{1-p}\int_{\{x:f(x)\neq 0\}} \|f(x)\|_{hs}^{p-2}\operatorname{Tr}(f(x)^*g(x))d\lambda(x)$$

for $g \in L^p(G,(M_n,\|\cdot\|_{hs}))$.

Proof. Given $A \in M_n\backslash\{0\}$, we have, by the chain rule,

$$\left.\frac{d}{dt}\right|_{t=0}\|A+tX\|_{hs}^p = p\|A\|_{hs}^{p-1}\left.\frac{d}{dt}\right|_{t=0}\|A+tX\|_{hs} = p\|A\|_{hs}^{p-1}\operatorname{Re}\operatorname{Tr}(A^*X)$$

for $X \in M_n$.

Fix a non-zero f in $L^p(G,(M_n,\|\cdot\|_{hs}))$. Given $p > 1$ and $g \in L^p(G,(M_n,\|\cdot\|_{hs}))$, we have

$$\left.\frac{d}{dt}\right|_{t=0}\|tg(x)\|_{hs}^p = 0.$$

By (2.4) and the dominated convergence theorem, we have

$$
\begin{aligned}
p\|f\|_p^{p-1} \left.\frac{d}{dt}\right|_{t=0} \|f+tg\|_p &= \left.\frac{d}{dt}\right|_{t=0} \|f+tg\|_p^p \\
&= \int_G \left.\frac{d}{dt}\right|_{t=0} \|f(x)+tg(x)\|_{hs}^p d\lambda(x) \\
&= \int_{\{x:f(x)\neq 0\}} \left.\frac{d}{dt}\right|_{t=0} \|f(x)+tg(x)\|_{hs}^p d\lambda(x) \\
&= \int_{\{x:f(x)\neq 0\}} p\|f(x)\|_{hs}^{p-2} \mathrm{Re}\,\mathrm{Tr}(f(x)^* g(x)) d\lambda(x)
\end{aligned}
$$

which gives the formula for the Gateaux derivative at f. $\square$

Corollary 2.2.6. *For $1 < p < \infty$, the Lebesgue space $L^p(G,(M_n,\|\cdot\|_{hs}))$ is strictly convex, that is, the extreme points of its closed unit ball are exactly the functions of unit norm.*

Proof. This follows from the fact that a Banach space E is strictly convex if, and only if, the norm of its dual E^* is Gateaux differentiable on the unit sphere. $\square$

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