Chapter 2
The option pricing framework

The option markets based on swap rates or the LIBOR have become the largest fixed income markets, and caps (floors) and swaptions are the most important derivatives within these markets. Thereby, a cap (floor) can be interpreted as a portfolio of options on zero bonds. Hence, pricing a cap (floor) is very easy, if we have found an exact solution for the arbitrage-free price of a caplet (floorlet) (see e.g. Briys, Crouhy and Schöbel [11]. On the other hand, a swaption may be interpreted as an option on a portfolio of zero bonds[1]. Therefore, even in the simplest case of lognormal-distributed bond prices, the portfolio of the bonds would be described by the distribution of a sum of lognormal-distributed random variables. Unfortunately, there exists no analytic density function for such a sum of lognormal-distributed random variables. Hence, using a multi-factor model with Brownian motions or Random Fields[2] as the sources of uncertainty, it seems unlikely that exact closed-form solutions can be found for the pricing of swaptions. The characteristic function of the random variable $\bar{X}(T_{0,}\{T_i\}) = \log\sum_{i=1}^{u} c_i P(T_0,T_i)$ with the coupon payments c_i at the fixed dates $T_i \in \{T_{1,} ..., T_u\}$ cannot be computed in closed-form. Otherwise, we are able to find a closed-form solution for the moments of the underlying random variable $V(T_{0,}\{T_i\}) = \sum_{i=1}^{u} c_i P(T_0,T_i)$ at the exercise date T_0 of the swaption. Hence, using the analytic solution of the moments within our Integrated Edgeworth Expansion (IEE) enables us to compute the T_i-forward measure exercise probabilities $\Pi_t^{T_i}[K] = E_t^{T_i}\left[\mathbf{1}_{V(T_0,\{T_i\})>K}\right]$ (section (5.3.3)). Reasonable carefulness has to be paid for the fact that the characteristic function of a lognormal-distributed

[1] The owner of a swaption with strike price K maturing at time T_0, has the right to enter at time T_0 the underlying forward swap settled in arreas. A swaption may also be seen as an option on a coupon bearing bond (see e.g. Musiela and Rutkowski [61]).

[2] Eberlein and Kluge [29] find a closed-form solution for swaptions using a Lévy term structure model. A solution for bond options assuming a one-factor model has been derived by Jamishidian [42].

random variable cannot be approximated asymptotically by an infinite Taylor series expansion of the moments (Leipnik [53]). As a result of the Leipnik-effect we truncate the Taylor series before the expansion of the characteristic function tends to diverge.

In contrary to the computation of options on coupon bearing bonds via an IEE, we can apply standard Fourier inversion techniques for the derivation zero bond option prices. Applying e.g. the Fractional Fourier Transform (FRFT) technique of Bailey and Swarztrauber [4] is a very efficient method to compute option prices for a wide range of strike prices. This can either be done, by directly computing the option price via an Fourier inversion of the transformed payoff function or by separately computing the exercise probabilities $\Pi_t^{T_i}[k]$. Running the first approach has the advantage that we only have to compute one integral for the computation of the option prices. On the other hand, sometimes we are additionally interested in the computation of single exercise probabilities[3]. Therefore, we prefer the latter as the option price can be easily computed by summing over the single probabilities[4].

2.1 Zero-coupon bond options

In the following, we derive a theoretical pricing framework for the computation of options on bond applying standard Fourier inversion techniques. Starting with a plain vanilla European option on a zero-coupon bond with the strike price K, maturity T_1 of the underlying bond and exercise date T_0 of the option, we have

$$\begin{aligned} ZBO_{\mathrm{w}}(t,T_0,T_1) &= \mathrm{w}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\left(P\left(T_0,T_1\right)-K\right)\mathbf{1}_{\mathrm{w}X(T_0,T_1)>\mathrm{w}k}\right] \qquad (2.1) \\ &= \mathrm{w}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds+X(T_0,T_1)}\mathbf{1}_{\mathrm{w}X(T_0,T_1)>\mathrm{w}k}\right] \\ &\quad -\mathrm{w}KE_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\mathbf{1}_{\mathrm{w}X(T_0,T_1)>\mathrm{w}k}\right], \end{aligned}$$

with $\mathrm{w}=1$ for a European call option and $\mathrm{w}=-1$ for a European put option[5]. We define the probability $\Pi_{t,a}^{Q}[k]$ given by

[3] Note that the FRFT approach is very efficient. Hence, the computation of single exercise probabilities runs nearly without any additional computational costs and without getting an significant increase in the approximation error (see e.g. figure (5.1)).

[4] Furthermore, we want to be consistent with our IEE approach, where the price of the coupon-bond options can only be computed by summing over the single exercise probabilities $\Pi_t^{T_i}[K]$

[5] In this thesis, we mainly focus on the derivation of call options (w = 1), keeping in mind that it is always easy to compute the appropriate probabilities for w = -1 via $E_t^Q\left[e^{-\int_t^{T_0} r(s)ds+aX(T_0,T_1)}\mathbf{1}_{X(T_0,T_1)<k}\right]=1-\Pi_{t,a}^{Q}[k]$.

$$\Pi_{t,a}^{Q}[k] \equiv E_t^Q\left[e^{-\int_t^{T_0} r(s)ds + aX(T_0,T_1)}\mathbf{1}_{X(T_0,T_1)>k}\right] \tag{2.2}$$

for $a = \{0,1\}$, with $X(T_0,T_1) = \log P(T_0,T_1)$ and the (log) strike price $k = \log K$. Armed with this, we are able to compute the price of a call option via

$$ZBO_1(t,T_0,T_1) = \Pi_{t,1}^{Q}[k] - K\cdot\Pi_{t,0}^{Q}[k]$$

and accordingly the price of a put option via

$$ZBO_{-1}(t,T_0,T_1) = K\cdot\left(1-\Pi_{t,0}^{Q}[k]\right) - \left(1-\Pi_{t,1}^{Q}[k]\right).$$

Finally, defining the transform

$$\Theta_t(z) \equiv E_t^Q\left[e^{-\int_t^{T_0} r(s)ds + zX(T_0,T_1)}\right], \tag{2.3}$$

for $z \in \mathbb{C}$ we obtain the risk-neutral probabilities by performing a Fourier inversion[6]

$$\Pi_{t,a}^{Q}[k] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{\Theta_t(a+i\phi)e^{-i\phi k}}{i\phi}\right]d\phi.$$

Note that we obtain a "Black and Scholes"-like option pricing formula if the Fourier inversion can be derived in closed-form (see e.g. section (5.2.1)). Assuming more advanced models, like a multi-factor HJM-framework combined with unspanned stochastic volatility (USV), the option price often can be derived by performing a FRFT (see e.g. section (7.2)). Then, given the exercise probabilities $\Pi_{t,a}^{Q}[k]$ we easily obtain the price of the single caplets (floorlets). Finally, we get the price of the interest rate cap (floor), by summing over the single caplets (floorlets) for all payment dates $\{T_i\} = \{T_{1,\ldots,}T_N\}$. The final payoff of a caplet (floorlet) settled in arreas with the maturity T_1 and a face value of one is defined by

$$let_{\mathrm{w}}(T_1) \equiv \Delta \max\{w(L(T_0,T_1) - CR), 0\},$$

with $\Delta = T_1 - T_0$, the cap rate CR and the LIBOR $L(T_0,T_1)$ in T_0. Hence, we obtain the payoff

$$\begin{aligned} let_{\mathrm{w}}(T_0) &= \frac{\Delta}{1+\Delta L(T_0,T_1)}\max\{\mathrm{w}(L(T_0,T_1)-CR),0\} \\ &= \max\left\{\mathrm{w}\left(1-\frac{1+\Delta CR}{1+\Delta L(T_0,T_1)}\right),0\right\}, \end{aligned}$$

at the exercise date T_0, where the last term equals a zero-coupon bond paying the face value $1+\Delta CR$ at time T_1. At last, the payoff is given by

[6] See for example Duffie, Pan and Singleton [28].

$$let_{\mathrm{w}}(T_0) = \max\left\{\mathrm{w}\left(1 - P(T_0, T_1)\right), 0\right\},$$

together with the zero-coupon bond

$$P(T_0, T_1) = \frac{1 + \Delta CR}{1 + \Delta L(T_0, T_1)}.$$

This implies that the payoff of a caplet $clet(t, T_0, T_1) = let_1(T_0)$ is equivalent to a put option on a zero-coupon bond $P(t, T)$ with face value $N = 1 + \Delta CR$ and a strike price $K = 1$. Therefore, we obtain the date-t price of a caplet

$$\begin{aligned} clet(t, T_0, T_1) &= ZBO_{-1}(t, T_0, T_i) \\ &= K \cdot \left(1 - \Pi^{Q}_{t,0}[k]\right) - \left(1 - \Pi^{Q}_{t,1}[k]\right) \end{aligned}$$

and accordingly the price of a floorlet

$$\begin{aligned} flet(t, T_0, T_1) &= ZBO_1(t, T_0, T_1) \\ &= \Pi^{Q}_{t,1}[k] - K\Pi^{Q}_{t,0}[k]. \end{aligned}$$

As such, we can easily compute the price of a European cap

$$Cap(t, T_0, \{T_i\}) = \sum_{i=1}^{N} ZBO_{-1}(t, T_0, T_i)$$

and the price of the equivalent floor

$$Floor(t, T_0, \{T_i\}) = \sum_{i=1}^{N} ZBO_1(t, T_0, T_i),$$

by summing over all caplets (floorlets) for all payment dates T_i for $i = 1, \ldots, N$.

2.2 Coupon bond options

Now, applying the same approach as in section (2.1) we derive the theoretical option pricing formula for the price of a swaption based on the Fourier inversion of the new transform

$$\Xi_t(z) \equiv E^{Q}_{t}\left[e^{-\int_t^{T_0} r(s)ds + z \log V(T_0, \{T_i\})}\right].$$

Starting from the payoff function of a European option on a coupon bearing bond we can write the option price at the exercise date T_0 as follows

$$
\begin{aligned}
CBO_{\mathrm{w}}(t,T_0,\{T_i\}) &= \mathrm{w}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\left(V\left(T_0,\{T_i\}\right)-K\right)\mathbf{1}_{\mathrm{w}V(T_0,\{T_i\})>\mathrm{w}K}\right] \\
&= \mathrm{w}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds+\bar{X}(T_0,\{T_i\})}\mathbf{1}_{\mathrm{w}\bar{X}(T_0,\{T_i\})>\mathrm{w}k}\right] \\
&\quad -\mathrm{w}KE_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\mathbf{1}_{\mathrm{w}\bar{X}(T_0,\{T_i\})>\mathrm{w}k}\right]. \qquad (2.4)
\end{aligned}
$$

Together with

$$
V(T_0,\{T_i\}) = \sum_{i=1}^{u} c_i P(T_0,T_i)
$$

and

$$
\begin{aligned}
\bar{X}(T_0,\{T_i\}) &= \log V(T_0,\{T_i\}) \\
&= \log\left(\sum_{i=1}^{u} c_i P(T_0,T_i)\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
CBO_{\mathrm{w}}(t,T_0,\{T_i\}) &= \mathrm{w}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds+\bar{X}(T_0,\{T_i\})}\mathbf{1}_{\mathrm{w}\bar{X}(T_0,\{T_i\})>\mathrm{w}k}\right] \\
&\quad -\mathrm{w}KE_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\mathbf{1}_{\mathrm{w}\bar{X}(T_0,\{T_i\})>\mathrm{w}k}\right],
\end{aligned}
$$

for all payment dates $\{T_1,...,T_u\}$. By defining the probability

$$
\Pi_{t,a}^Q[k] \equiv E_t^Q\left[e^{-\int_t^{T_0} r(s)ds+a\bar{X}(T_0,\{T_i\})}\mathbf{1}_{\bar{X}(T_0,\{T_i\})>k}\right],
$$

we directly obtain the price of a zero-coupon bond call option

$$
CBO_1(t,T_0,\{T_i\}) = \Pi_{t,1}^Q[k] - K\Pi_{t,0}^Q[k]
$$

and respectively the price of the put option

$$
CBO_{-1}(t,T_0,\{T_i\}) = K\left(1-\Pi_{t,0}^Q[k]\right) - \left(1-\Pi_{t,1}^Q[k]\right).
$$

Note that the payoff function of a swaption[7] with exercise date T_0 and equidistant payment dates T_i for $i=1,...,u$ is given by[8]

$$
S_{\mathrm{w}}(T_0,\{T_i\}) = \max\left\{\Delta\cdot\mathrm{w}\sum_{i=1}^{u}(SR-L(T_0,T_{i-1},T_i))P(T_0,T_i),0\right\}, \qquad (2.5)
$$

[7] The owner of a payer (receiver) swaption maturing at time T_0, has the right to enter at time T_0 the underlying forward payer (receiver) swap settled in arreas (see e.g. Musiela and Rutkowski [61])

[8] The payoff function can be defined easily for non equidistant payment dates Δ_i.

with $\Delta = T_i - T_{i-1}$ for $i = 2,...,u$. Again, w = 1 equals a receiver swaption and w = -1 a payer swaption. Now, plugging the swap rate

$$SR = \frac{1 - P(t,T_u)}{\Delta \sum_{i=1}^{u} P(t,T_i)},$$

together with the forward rate

$$L(T_0,T_{i-1},T_i) \equiv \frac{1}{\Delta}\left(\frac{P(T_0,T_{i-1})}{P(T_0,T_i)} - 1\right)$$

in equation (2.5) finally leads to

$$S_{\mathrm{w}}(T_0,\{T_i\}) = \max\left\{\mathrm{w}\left(\Delta SR \sum_{i=1}^{u} P(T_0,T_i) + P(T_0,T_u) - 1\right), 0\right\}$$

or more easily

$$S_{\mathrm{w}}(T_0,\{T_i\}) = \max\left\{\mathrm{w}\left(\sum_{i=1}^{u} c_i P(T_0,T_i) - 1\right), 0\right\}, \tag{2.6}$$

where the coupon payments for $i = 1,...,u-1$ are given by

$$c_i = \Delta SR,$$

together with the final payment

$$c_u = 1 + \Delta SR.$$

Now, we directly see that a swaption[9] in general can be seen as an option on a coupon bond with strike $K = 1$ and exercise date T_0 paying the coupons c_i at the payment dates $\{T_i\} = \{T_{1,...,}T_u\}$. Armed with this, we obtain the price of a receiver swaption

$$\begin{aligned} S_1(t,T_0,\{T_i\}) &= CBO_1(t,T_0,\{T_i\}) \\ &= \Pi_{t,1}^{Q}[0] - \Pi_{t,0}^{Q}[0] \end{aligned} \tag{2.7}$$

and respectively the price of a payer swaption

$$\begin{aligned} S_{-1}(t,T_0,\{T_i\}) &= CBO_{-1}(t,T_0,\{T_i\}) \\ &= \left(1 - \Pi_{t,0}^{Q}[0]\right) - \left(1 - \Pi_{t,1}^{Q}[0]\right), \end{aligned}$$

[9] In the following we use the term swaption and option on a coupon bond option interchangeably. Nevertheless, keeping in mind that a swaption is only one special case of an option on a coupon bond.

given the (log) strike price $k = 0$. Now, together with the transform

$$\Xi_t(z) \equiv E_t^Q \left[e^{-\int_t^{T_0} r(s)ds + z\tilde{X}(T_0,\{T_i\})} \right], \tag{2.8}$$

with $z \in \mathbb{C}$ we theoretically could compute the risk-neutral probabilities $\Pi_{t,a}^Q[k]$ by performing a Fourier inversion via

$$\Pi_{t,a}^Q[k] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left[\frac{\Xi_t(a+i\phi)e^{-i\phi k}}{i\phi} \right] d\phi.$$

Unfortunately, there exists no closed-form solution for the transform $\Xi_t(a + i\phi)$. This directly implies that we need a new method for the approximation of the single exercise probabilities $\Pi_{t,a}^Q[k]$ assuming a multi-factor model with more than one payment date. On the other hand, the transform $\Xi_t(n)$ can be solved analytically for nonnegative integer numbers n. This special solutions of $\Xi_t(z)$ can be used to compute the n-th moments of the underlying random variable $V(T_{0,}\{T_i\})$ under the T_i forward measure. Then, by plugging these moments in the IEE scheme we are able to obtain an excellent approximation of the single exercise probabilities (see e.g. section (5.3.3) and (5.3.4)).

Recapitulating, we have derived theoretically a unified setup for the computation of bond option prices in a generalized multi-factor framework. In general, the option price can be computed by the use of exponential affine solutions of the transforms $\Theta_t(z)$, for $z \in \mathbb{C}$ applying a FRFT and $\Xi_t(n)$, for $n \in \mathbb{N}$ performing an IEE.

The transforms $\Theta_t(z)$ and $\Xi_t(z)$, by itself can be seen as a modified characteristic function. Unfortunately, there exists no closed-form of the transform $\Xi_t(z)$, meaning that the standard Fourier inversion techniques can be applied only for the computation of options on discount bonds. On the other hand, the transform $\Xi_t(n)$ can be used to compute the n-th moments of the underlying random variable $V(T_0,\{T_i\})$. Then, by plugging the moments (cumulants) in the IEE scheme the price of an option on coupon bearing bond can be computed, even in a multi-factor framework.

http://www.springer.com/978-3-540-70721-9

Pricing of Bond Options
Unspanned Stochastic Volatility and Random Field Models
Repplinger, D.
2008, X, 138 p. 23 illus., Softcover
ISBN: 978-3-540-70721-9