Chapter 2
Method of decomposition (the first approach)

In this chapter, we present the method of decomposition for the control design in nonlinear dynamical systems. We suppose the system is described by Lagrangian ordinary differential equations and subjected to controls, the number of independent control forces being equal to the number of degrees of freedom. The material of the chapter is based on papers [27, 28, 29, 34, 98, 102, 42, 43, 44].

2.1 Problem statement and game approach

2.1.1 Controlled mechanical system

Consider a nonlinear control system whose dynamics is described by Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = U_i + Q_i, \quad i = 1, \ldots, n. \tag{2.1.1}$$

Here, $q = (q_1, \ldots, q_n)$ denotes the generalized coordinates of the system, n is the number of its degrees of freedom, and a dot over a letter denotes the derivative with respect to the time t. The generalized forces consist of the control forces U_i to be determined and terms Q_i representing all other external and internal forces, including the uncontrolled perturbations.

The kinetic energy of the system T is given by a quadratic form

$$T(q, \dot{q}) = \frac{1}{2} \sum_{j,k=1}^{n} a_{jk}(q) \dot{q}_j \dot{q}_k, \tag{2.1.2}$$

where a_{jk} are elements of a symmetric positive-definite matrix $A(q)$ of order $n \times n$. Substituting (2.1.2) into (2.1.1), we write the equations of motion in the form

$$A(q)\ddot{q} = U + S(q, \dot{q}, t). \tag{2.1.3}$$

Here, $U = (U_1, \ldots, U_n)$ is the vector of the control forces and $S = (S_1, \ldots, S_n)$ is the vector function

$$S(q, \dot{q}, t) = Q(q, \dot{q}, t) - \sum_{j,k=1}^{n} \Gamma_{jk} \dot{q}_j \dot{q}_k, \tag{2.1.4}$$

where $\Gamma_{jk} = (\Gamma_{1jk}, \ldots, \Gamma_{njk})$ are n-dimensional vectors with components

$$\Gamma_{ijk} = \frac{\partial a_{ij}}{\partial q_k} - \frac{1}{2} \frac{\partial a_{jk}}{\partial q_i}. \tag{2.1.5}$$

We impose the constraints

$$|U_i| \le U_i^0, \quad i = 1, \ldots, n, \tag{2.1.6}$$

on the control forces, where $U_i^0 > 0$ are given positive constants.

The initial conditions for system (2.1.3)

$$q(t_0) = q^0, \qquad \dot{q}(t_0) = \dot{q}^0 \tag{2.1.7}$$

lie in a given domain Ω in $2n$-dimensional phase space: $\{q, \dot{q}\} \in \Omega$.

Let us formulate the control problem:

Problem 2.1. Find a feedback control $U = U(q, \dot{q})$ satisfying constraint (2.1.6) and bringing system (2.1.3) from an arbitrary initial state (2.1.7) in domain Ω to a given state with zero velocities

$$q(t_*) = q^*, \qquad \dot{q}(t_*) = 0 \tag{2.1.8}$$

in finite time (instant $t_* > t_0$ is not fixed).

2.1.2 Simplifying assumptions

Problem 2.1 will be solved under certain simplifying assumptions (conditions), which are formulated below.

We represent the matrix $A(q)$ in the form

$$\begin{gathered} A(q) = B(q) A_*, \\ B(q) = E + [A(q) - A_*] A_*^{-1} \equiv A(q) A_*^{-1}, \end{gathered} \tag{2.1.9}$$

where A_* is some constant symmetric positive-definite $n \times n$ matrix and E is the $n \times n$ identity matrix. Matrix $B(q)$ is nonsingular; hence, the inverse matrix $B^{-1}(q)$ exists. We multiply both sides of (2.1.3) by $B^{-1}(q)$ and, using the relationships (2.1.9), transform (2.1.3) to the form

$$A_* \ddot{q} = U + V(q, \dot{q}, t, U). \tag{2.1.10}$$

Here, we use the notation

$$V = V' + V'', \quad V' = B^{-1}(q)S(q,\dot{q},t),$$
$$V'' = [B^{-1}(q) - E]U. \tag{2.1.11}$$

By virtue of this notation, (2.1.10) is equivalent to the original equation (2.1.3).

Let us suppose that the conditions

$$V_i' = -\lambda_i (A_* \dot{q})_i + V_i^*,$$
$$|V_i^* + V_i''| \le V_i^0 < U_i^0, \quad i = 1,\dots,n, \tag{2.1.12}$$

hold for all $t \ge t_0$, all $\{q, \dot{q}\} \in \Omega$, and all U satisfying (2.1.6). Here, $V_i^0 > 0$ and $\lambda_i > 0$ are constants. If all λ_i are equal to zero, conditions (2.1.12) become more simple

$$|V_i| \le V_i^0 < U_i^0, \quad i = 1,\dots,n, \tag{2.1.13}$$

for all $t \ge t_0$, all $\{q, \dot{q}\} \in \Omega$, and all U satisfying (2.1.6).

The following lemma allows to check whether condition (2.1.13) is satisfied.

Lemma 2.1. *Suppose that, for any n-dimensional vector z, the following conditions are satisfied for all $t \ge t_0$ and all $\{q, \dot{q}\} \in \Omega$:*

$$|A_* z| \ge \mu_* |z|, \quad |[A(q) - A_*]z| \le \mu |z|,$$
$$|S_i(q,\dot{q},t)| \le \vartheta U_i^0, \quad i = 1,\dots,n, \tag{2.1.14}$$
$$0 < \mu < \mu_*, \quad \vartheta > 0,$$

where μ_*, μ, and ϑ are constants. Then, for all $t \ge t_0$, all $\{q, \dot{q}\} \in \Omega$, and all U meeting (2.1.6), the components of vector V in (2.1.11) satisfy the estimates

$$|V_i| \le \vartheta U_i^0 + \mu(\mu_* - \mu)^{-1}(1+\vartheta)|U^0|, \qquad i = 1,\dots,n,$$
$$U^0 = (U_1^0,\dots,U_n^0). \tag{2.1.15}$$

We note that, since A_* is a positive-definite matrix, we can take as μ_* any positive number not exceeding its smallest eigenvalue.

Proof. From the first of inequalities (2.1.14), we have

$$|A_*^{-1} z| \le \mu_*^{-1} |z|. \tag{2.1.16}$$

Here and below, z denotes any n-dimensional vector. We define

$$L = [A(q) - A_*]A_*^{-1}. \tag{2.1.17}$$

It follows from (2.1.16) and the second of inequalities (2.1.14) that

$$|Lz| \le \mu\mu_*^{-1}|z|. \tag{2.1.18}$$

By virtue of (2.1.17), we can rewrite relationship (2.1.9) for B in the form

$$Bz = z + Lz. \tag{2.1.19}$$

With the aid of (2.1.19) and (2.1.18), we obtain the estimate

$$|Bz| \ge |z| - |Lz| \ge (1 - \mu\mu_*^{-1})|z|. \tag{2.1.20}$$

It follows from condition (2.1.14) of the lemma that $(1 - \mu\mu_*^{-1}) > 0$. Setting $z = B^{-1}z'$ in (2.1.20), we obtain

$$|B^{-1}z'| \le (1 - \mu\mu_*^{-1})^{-1}|z'|. \tag{2.1.21}$$

Inequalities (2.1.18) and (2.1.21) yield

$$|LB^{-1}z| \le \mu(\mu_* - \mu)^{-1}|z|. \tag{2.1.22}$$

Let us set $z = B^{-1}z'$ in (2.1.19):

$$B^{-1}z' = z' - LB^{-1}z'. \tag{2.1.23}$$

Using (2.1.11) for V' and (2.1.23) with $z' = S$, we represent the components V_i of the vector V' in the form

$$V_i' = (B^{-1}S)_i = S_i - (LB^{-1}S)_i, \quad i = 1, \ldots, n. \tag{2.1.24}$$

The subscripts denote the components of the vectors. By virtue of the third of conditions (2.1.14) and inequality (2.1.22), we obtain from (2.1.24)

$$\begin{aligned} |V_i'| &\le |S_i| + |(LB^{-1}S)_i| \le \vartheta U_i^0 + \mu(\mu_* - \mu)^{-1}|S| \\ &\le \vartheta U_i^0 + \mu(\mu_* - \mu)^{-1}\vartheta|U^0|, \qquad i = 1, \ldots, n. \end{aligned} \tag{2.1.25}$$

Here, we used notation (2.1.15) for U^0. We substitute (2.1.23) with $z' = U$ into (2.1.11) for vector V'':

$$V_i'' = (B^{-1}U - U)_i = -(LB^{-1}U)_i, \quad i = 1, \ldots, n. \tag{2.1.26}$$

From this equation, using inequalities (2.1.22) and (2.1.6), we obtain

$$\begin{aligned} |V_i''| &\le |(LB^{-1}U)_i| \le |LB^{-1}U| \le \mu(\mu_* - \mu)^{-1}|U| \\ &\le \mu(\mu_* - \mu)^{-1}|U^0|, \qquad i = 1, \ldots, n. \end{aligned} \tag{2.1.27}$$

This and inequality (2.1.25) imply (2.1.15). This completes the proof of the lemma. □

Corollary 2.1. *If, under the conditions of Lemma 2.1, $\vartheta < 1$ and μ is sufficiently small, then condition (2.1.13) is satisfied.*

Remark 2.1. We should take for matrix A_* some "average" value of the matrix $A(q)$ for domain Ω. In particular, we can choose for A_* the matrix $A(q^*)$ for some value of vector q^*, for example, $A(q^*)$, $A(q^0)$, or $A((q^0+q^*)/2)$. Then, if the domain Ω is sufficiently small, the matrix $A(q)$ will differ only slightly from A_* for all the motions considered, and the number μ will, under conditions (2.1.14) of Lemma 2.1, be sufficiently small. Thus, by virtue of Corollary 2.1, condition (2.1.13) can be ensured for a given nonlinear system (2.1.3) if, first, the possibilities of control are increased [that is, if the constants U_i^0 in (2.1.6) are increased so that the condition $\vartheta < 1$ holds] and, second, the domain Ω is decreased, so that $A(q)$ is close to A_* (that is, the number μ is decreased).

We shall show in Sect. 2.5 that formulation of Problem 2.1 and condition (2.1.12) are natural often and are satisfied for manipulation robots with electromechanical drives.

2.1.3 Decomposition

Let us turn to the solution of Problem 2.1 with condition (2.1.12) satisfied. We assume that all motions of system (2.1.3) considered lie in the domain Ω.

If condition (2.1.12) is satisfied, system (2.1.3) can, by virtue of (2.1.10)–(2.1.12), be represented in the form

$$(A_*\ddot{q})_i + \lambda_i(A_*\dot{q})_i = U_i + \tilde{V}_i, \quad \tilde{V}_i = V_i^* + V_i'', \quad i = 1,\ldots,n. \tag{2.1.28}$$

In system (2.1.28), we make the change of variables

$$A_*(q - q^*) = y, \tag{2.1.29}$$

where q^* was introduced in (2.1.8). We obtain

$$\ddot{y}_i + \lambda_i \dot{y}_i = U_i + \tilde{V}_i, \quad i = 1,\ldots,n. \tag{2.1.30}$$

For the terms on the right-hand sides of (2.1.30), we have, by virtue of (2.1.6), (2.1.28), and (2.1.12), the constraints

$$|U_i| \le U_i^0, \quad |V_i| \le V_i^0 < U_i^0, \quad i = 1,\ldots,n. \tag{2.1.31}$$

After the change of variables (2.1.29), the initial conditions (2.1.7) and the boundary conditions (2.1.8) take the forms

$$y(t_0) = A_*(q^0 - q^*), \quad \dot{y}(t_0) = A_*\dot{q}^0, \tag{2.1.32}$$

$$y(t_*) = \dot{y}(t_*) = 0. \tag{2.1.33}$$

Thus, Problem 2.1 reduces to the construction of a control $U(y,\dot{y})$ that brings system (2.1.30) from an arbitrary initial state (2.1.32) to state (2.1.33) under constraint (2.1.31). System (2.1.30) consists of n subsystems, each with a single degree of freedom. Each of the subsystems has its own scalar control U_i that satisfies constraint (2.1.31). In this subsystem, we treat the function V_i as a perturbation subject to constraint (2.1.31), but otherwise arbitrary. Then the result obtained can be summed up in the form of the following assertion.

Theorem 2.1. *Suppose that condition (2.1.12) is satisfied and that all the motions of system (2.1.3) that are being considered lie in the domain Ω. Then, to solve Problem 2.1, it is sufficient to solve n control problems for the linear subsystems (2.1.30) with a single degree of freedom. In each of these problems, it is necessary to construct a scalar control $U_i(y_i,\dot{y}_i)$ satisfying constraint (2.1.31) and taking the ith subsystem (2.1.30) from an arbitrary initial state (2.1.32) to the coordinate origin (2.1.33) in finite time for arbitrary admissible perturbations V_i satisfying constraint (2.1.31).*

The described approach to the control decomposition has been first suggested in [27] for the case $\lambda = 0$ and in [29] for the general case $\lambda \geq 0$.

2.1.4 Game problem

Let us consider the ith subsystem (2.1.30) and, in it, let us set

$$y_i = U_i^0 x, \quad U_i = U_i^0 u, \quad \tilde{V}_i = U_i^0 v. \tag{2.1.34}$$

Then, this system combined with constraints (2.1.31) takes the form

$$\ddot{x} + \lambda\dot{x} = u + v, \quad |u| \leq 1, \quad |v| \leq \rho < 1, \tag{2.1.35}$$

and the boundary conditions (2.1.32) and (2.1.33) become

$$x(0) = \xi, \quad \dot{x}(0) = \eta, \quad x(\tau) = \dot{x}(\tau) = 0. \tag{2.1.36}$$

In (2.1.35) and (2.1.36), we used the notation

$$\begin{gathered} \rho = \frac{V_i^0}{U_i^0} < 1, \quad \xi = (U_i^0)^{-1} y_i(t_0) = (U_i^0)^{-1}[A_*(q^0 - q^*)]_i, \\ \eta = (U_i^0)^{-1}\dot{y}_i(t_0) = (U_i^0)^{-1}(A_*\dot{q}^0)_i, \quad \lambda = \lambda_i, \\ \tau = t_* - t_0, \quad i = 1,\dots,n. \end{gathered} \tag{2.1.37}$$

Without loss of generality, the initial instant of time is taken equal to zero.

Let us consider the problem of bringing system (2.1.35) to the coordinate origin in the shortest time, that is, for minimum τ in (2.1.36). We treat this problem as a differential game, in which one of the players (the controlling side) chooses a control u, and the second player (the opponent) chooses a perturbation v. We will use the approach of the theory of differential games [79] and construct a feedback control $u(x,\dot{x})$ that brings system (2.1.35) to the coordinate origin in the shortest guaranteed time τ for an arbitrary admissible perturbation v. We note that this differential game (2.1.35) and (2.1.36) is a linear differential game of similar objects.

Its solution reduces [79] to the solution of the time-optimal control problem for the system

$$\ddot{x}+\lambda\dot{x}=(1-\rho)u, \quad |u|\leq 1, \quad \tau\to\min, \tag{2.1.38}$$

with the boundary conditions (2.1.36). The sought control $u(x,\dot{x})$ and the minimum guaranteed time τ in the game problem (2.1.35) and (2.1.36) coincide with the synthesis of the optimal control and optimal time for problem (2.1.38) and (2.1.36). We note that system (2.1.38) is obtained from (2.1.36) for a perturbation equal to $v=-\rho u$ that is the optimal control of the opponent choosing the perturbation v. In other words, the worst perturbation in this problem can be taken in the form $v=-\rho u$.

Thus, as a result of the decomposition, the solution of Problem 2.1 is reduced to the construction of the time-optimal feedback control for system (2.1.38) and (2.1.36).

2.2 Control of the subsystem and feedback control design

2.2.1 Optimal control for the subsystem

Let us rewrite the time-optimal control problem (2.1.38) and (2.1.36) in the form

$$\dot{x}_1=x_2, \quad \dot{x}_2=-\lambda x_2+w, \quad w=(1-\rho)u, \quad |u|\leq 1, \tag{2.2.1}$$

$$x_1(0)=\xi, \quad x_2(0)=\eta, \quad x_1(\tau)=x_2(\tau)=0, \tag{2.2.2}$$

$$0\leq\rho<1, \quad \lambda\geq 0, \quad \tau\to\min, \quad (x_1=x, \quad x_2=\dot{x}).$$

This problem is easily solved by means of the maximum principle, see Sect. 1.2. Here, we present the necessary relationships.

Hamilton's function for system (2.2.1) is equal to

$$H=p_1x_2+p_2[(1-\rho)u-\lambda x_2], \quad |u|\leq 1,$$

where p_1 and p_2 are conjugate variables. From this we obtain, by virtue of the maximum principle,

$$u=\operatorname{sign}p_2=\pm 1. \tag{2.2.3}$$

The adjoint system has the form

$$\dot{p}_1 = 0, \quad \dot{p}_2 = -p_1 + \lambda p_2.$$

Integrating it, we obtain

$$p_2 = C_1 + C_2 e^{\lambda t} \quad \text{for} \quad \lambda > 0,$$

$$p_2 = C_1 + C_2 t \quad \text{for} \quad \lambda = 0,$$

where C_1 and C_2 are arbitrary constants. It follows that $p_2(t)$ is a monotone function for $\lambda \geq 0$. Therefore, control (2.2.3) has no more than one switching point.

For a constant $w = \text{const}$, the general solution of system (2.2.1) has the form

$$\begin{aligned} x_1 &= B_1 + \lambda^{-1} w(t-\tau) - \lambda^{-1}(B_2 - \lambda^{-1} w)[e^{-\lambda(t-\tau)} - 1], \\ x_2 &= \lambda^{-1} w + (B_2 - \lambda^{-1} w) e^{-\lambda(t-\tau)} \qquad \text{for} \qquad \lambda > 0, \end{aligned} \tag{2.2.4}$$

$$\begin{aligned} x_1 &= B_1 + B_2(t-\tau) + \frac{1}{2} w(t-\tau)^2, \\ x_2 &= B_2 + w(t-\tau) \qquad \text{for} \qquad \lambda = 0. \end{aligned} \tag{2.2.5}$$

Here and in what follows, all the relationships are given separately for the cases $\lambda > 0$ and $\lambda = 0$. We note that the case $\lambda = 0$ can be obtained by taking the limit as $\lambda \to +0$. The arbitrary constants B_1 and B_2 in (2.2.4) and (2.2.5) are chosen in such a way that, for $B_1 = B_2 = 0$, the zero boundary conditions (2.2.2) hold for $t = \tau$. Eliminating $t - \tau$ from (2.2.4) and (2.2.5), we obtain the equations for the phase trajectories

$$x_1 = B' - \lambda^{-1} x_2 - \lambda^{-2} w \log|1 - \lambda w^{-1} x_2| \qquad \text{for} \qquad \lambda > 0, \tag{2.2.6}$$

$$x_1 = B' + (2w)^{-1} x_2^2 \qquad \text{for} \qquad \lambda = 0. \tag{2.2.7}$$

Here, B' is a new constant expressed in terms of B_1 and B_2. In deriving (2.2.6), we assume that $\lambda B_2 \neq w$. If $\lambda B_2 = w$, we obtain from (2.2.4) the equation for the phase trajectory in the form

$$x_2 = \lambda^{-1} w. \tag{2.2.8}$$

The phase trajectories (2.2.7) for $\lambda = 0$ are parabolas that are symmetric about the x_1-axis. They can be obtained successively, one from another, by a parallel translation along the x_1-axis.

Let us consider trajectory (2.2.6) for $\lambda > 0$, $u = 1$, and $B' = 0$. Using (2.2.1) with $w = 1 - \rho$, we obtain the following properties of the curve $x_1(x_2)$:

- As x_2 increases from $-\infty$ to 0, x_1 decreases from ∞ to 0 and attains a zero minimum at $x_2 = 0$;
- In the interval $x_2 \in (0, \lambda^{-1}(1-\rho))$, the value of x_1 increases from 0 to ∞;
- In the interval $(\lambda^{-1}(1-\rho), \infty)$, the value of x_1 decreases from ∞ to $-\infty$.

Thus, the curve $x_1(x_2)$ consists of two branches which approach the asymptote $x_2 = \lambda^{-1}(1-\rho)$. By (2.2.8), this asymptote is itself also a phase trajectory for $u = 1$. Dependence $x_1(x_2)$ is shown for $u = 1$ and $B' = 0$ in Fig. 2.1, where the arrows indicate the direction of increasing t. The phase trajectories corresponding to $u = 1$ and arbitrary B' in (2.2.6) are obtained from the curve described above by a translation along the x_1 axis.

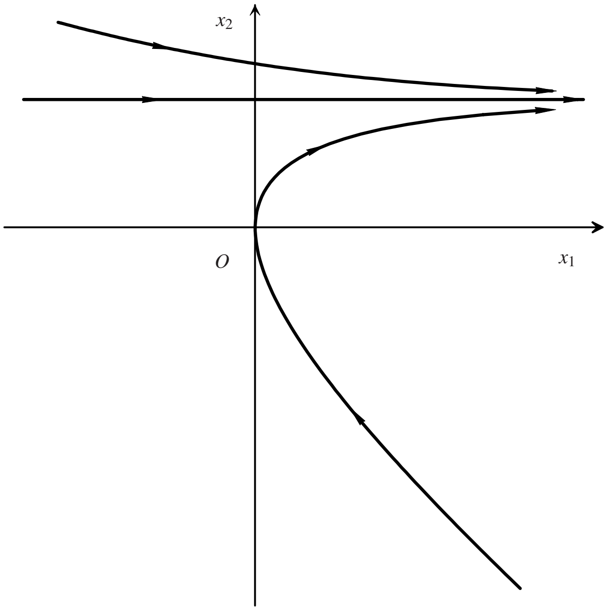

Fig. 2.1 Phase trajectories for $w = \text{const}$ and $\lambda > 0$

If we simultaneously change the signs of x_1, x_2, u, B_1, B_2, and B' in (2.2.4)–(2.2.8), the resulting relationships remain valid. Consequently, the phase trajectories corresponding to $u = -1$ are obtained by means of the central symmetry from the trajectories described above and corresponding to $u = 1$.

The only phase trajectories that reach the coordinate origin as t increases are curves (2.2.6) and (2.2.7) with $B' = 0$ and $u = \pm 1$. Motions along these curves are described by (2.2.4) and (2.2.5) with $B_1 = B_2 = 0$ and $u = \pm 1$. These two semitrajectories [(2.2.4) for $\lambda > 0$ and (2.2.5) for $\lambda = 0$] constitute the switching curve of the optimal control: the only possible change of sign of the control u along each trajectory takes place on this curve. As a result, we arrive at the field of optimal phase trajectories that is shown in Fig. 2.2 for $\lambda > 0$. Here, the bold curves represent the switching lines and the arrows indicate the direction of increasing t. For the field of optimal trajectories for $\lambda = 0$, see Fig. 1.2, Sect. 1.4.

The feedback optimal control can be represented in the form [see (1.4.13)]

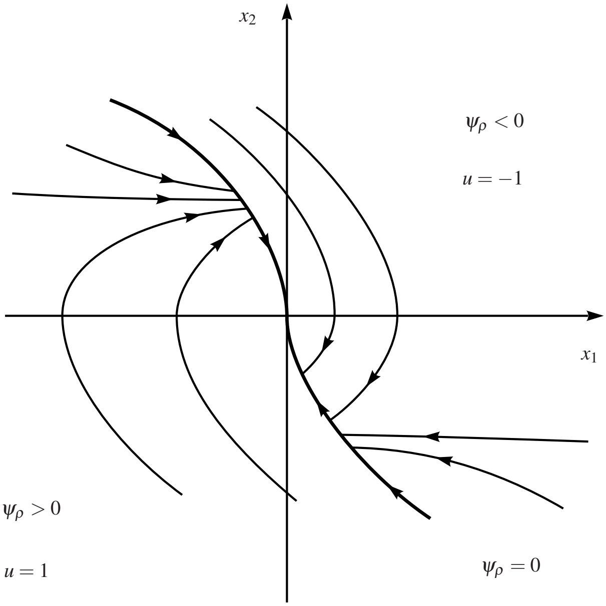

Fig. 2.2 Optimal phase trajectories

$$
\begin{aligned}
u(x_1, x_2) &= \operatorname{sign} \psi_\rho(x_1, x_2) \qquad \text{for} \qquad \psi_\rho \neq 0, \\
u(x_1, x_2) &= \operatorname{sign} x_1 = -\operatorname{sign} x_2 \qquad \text{for} \qquad \psi_\rho = 0,
\end{aligned}
\tag{2.2.9}
$$

where $\psi_\rho(x_1, x_2)$ is the switching function. It is equal to

$$
\begin{aligned}
\psi_\rho(x_1, x_2) &= -x_1 - \lambda^{-1} x_2 \\
&+ \lambda^{-2}(1-\rho)\log[1 + \lambda(1-\rho)^{-1}|x_2|] \operatorname{sign} x_2 \qquad \text{for} \qquad \lambda > 0, \\
\psi_\rho(x_1, x_2) &= -x_1 - x_2|x_2|[2(1-\rho)]^{-1} \qquad \text{for} \qquad \lambda = 0.
\end{aligned}
\tag{2.2.10}
$$

Let us also determine the time necessary to reach the coordinate origin for the optimal trajectory starting from arbitrary initial conditions (2.2.2). To be definite, suppose that the initial point lies in the region $\psi_\rho \leq 0$ and that the only possible switching takes place at an instant $s \in [0, \tau]$. The point $(x_1(s), x_2(s))$ lies, on the one hand, on the phase trajectory corresponding to $u = -1$ that passes through the initial point and, on the other hand, on the switching curve for $u = 1$. Equating the corresponding expressions (2.2.5) (we have $B_1 = B_2 = 0$ on the switching curve), we obtain, for $\lambda = 0$,

$$x_1(s) = B_1 - B_2\theta - \frac{1}{2}(1-\rho)\theta^2 = \frac{1}{2}(1-\rho)\theta^2,$$
$$x_2(s) = B_2 + (1-\rho)\theta = -(1-\rho)\theta, \quad \theta = \tau - s > 0. \tag{2.2.11}$$

Let us also write down condition (2.2.2) determining that the phase trajectory (2.2.5) with $u = -1$ passes through the initial point:

$$\xi = B_1 - B_2\tau - \frac{1}{2}(1-\rho)\tau^2,$$
$$\eta = B_2 + (1-\rho)\tau. \tag{2.2.12}$$

Eliminating constants B_1 and B_2 from (2.2.11) and (2.2.12), we obtain two equations for θ and τ. Solving them, we find

$$\tau(\xi,\eta) = \frac{1}{1-\rho}\left\{2\left[\frac{1}{2}\eta^2 - (1-\rho)\xi\gamma\right]^{1/2} - \eta\gamma\right\},$$
$$\gamma = \operatorname{sign}\psi_\rho, \quad (\lambda = 0). \tag{2.2.13}$$

Here, we take into account the symmetry of the phase trajectories. Function ψ_ρ is defined in (2.2.10). On the switching curve, that is, for $\psi_\rho = 0$, we can take for γ in (2.2.13) any of the numbers $\gamma = \pm 1$: the value of $\tau(\xi,\eta)$ is the same in both cases.

The optimal time for $\lambda > 0$ is obtained in an analogous manner. Here, instead of (2.2.5), we use formulas (2.2.4). We have finally (see [2])

$$\tau(\xi,\eta) = 2\lambda^{-1}\log\{M^{1/2} + [M - 1 + \lambda\eta\gamma(1-\rho)^{-1}]^{1/2}\},$$
$$M = \exp[-(\lambda\eta + \lambda^2\xi)\gamma(1-\rho)^{-1}], \tag{2.2.14}$$
$$\gamma = \operatorname{sign}\psi_\rho, \quad (\lambda > 0),$$

where ψ_ρ is given by (2.2.10) for $\lambda > 0$.

Equations (2.2.9), (2.2.10) and (2.2.13), (2.2.14) determine the synthesis of the optimal control and the minimum guaranteed time τ in the game problem (2.1.35) and (2.1.36). If the perturbation v is different from the worst one ($v \neq -\rho u$), then the phase trajectories will differ from the optimal ones. However, the time needed to get the system to the coordinate origin will not exceed τ given by (2.2.13) and (2.2.14). We note that, when the motion has arrived on the switching curve, it will, for arbitrary admissible perturbations, proceed along that curve until it gets to the coordinate origin. If $v \neq -\rho u$, a sliding regime of motion along the switching curves is realized. Thus, if $v = 0$ on the switching curve, the control assumes the values $u = \pm 1$ with infinitely many changes of sign, so that we have "on the average" $u = 1-\rho$ or $u = -(1-\rho)$ on the corresponding branches of the switching curve.

2.2.2 Simplified control for the subsystem

With the method of control proposed in Sect. 2.2.1, we did not assume the perturbation, that is, the function v in system (2.1.35), to be known. However, we did assume its maximum possible value [ρ with constraint (2.1.35)] to be known, and the control synthesis given by (2.2.9) and (2.2.10) depends on this maximum value.

There is another possible approach to determine the control in a system with perturbations. In it, the perturbations are completely ignored at the stage of the control design and are taken into account only in the modelling and processing of the control. This approach, which is completely natural in the case of small perturbations, we shall call the simplified approach.

Below, we compare the two approaches, and determine to what extent ignoring the perturbations in the control design is justified.

By Theorem 2.1, if condition (2.1.12) holds, the system in question in the form (2.1.3), (2.1.10), or (2.1.30) is broken down into n subsystems of the form (2.1.35). Therefore, a comparison of the two approaches needs to be made only for system (2.1.35).

If we neglect perturbation v in system (2.1.35), it takes the form

$$\ddot{x} + \lambda \dot{x} = u, \quad |u| \leq 1. \tag{2.2.15}$$

Let us write down the time-optimal control for system (2.2.15) with the boundary conditions (2.1.36). Since system (2.2.15) coincides with system (2.1.38) at $\rho = 0$, the desired control is determined by (2.2.9) and (2.2.10) in which we set $\rho = 0$. We obtain

$$\begin{gathered} u(x_1, x_2) = \operatorname{sign} \psi_0(x_1, x_2) \quad \text{for} \quad \psi_0 \neq 0, \\ u(x_1, x_2) = \operatorname{sign} x_1 = -\operatorname{sign} x_2 \quad \text{for} \quad \psi_0 = 0, \\ \psi_0(x_1, x_2) = -x_1 - \lambda^{-1} x_2 + \lambda^{-2} \log[1 + \lambda |x_2|] \operatorname{sign} x_2, \quad (\lambda > 0), \\ \psi_0(x_1, x_2) = -x_1 - \frac{1}{2} x_2 |x_2|, \quad (\lambda = 0). \end{gathered} \tag{2.2.16}$$

The switching curve $\psi_0 = 0$ for the feedback control (2.2.16) is the dashed curve in Fig. 2.3. For comparison, the solid curve in Fig. 2.3 shows the switching curve $\psi_\rho = 0$ for control (2.2.9) and (2.2.10) with $0 < \rho < 1$. Both these curves are symmetric about the coordinate origin. The equation of the curve $\psi_0 = 0$ can be represented in the form

$$x_1 = \phi(x_2), \tag{2.2.17}$$

where $\phi(x_2)$ is a monotonically decreasing odd function of its argument.

In the case of the control law (2.2.16), system (2.1.35) takes the form

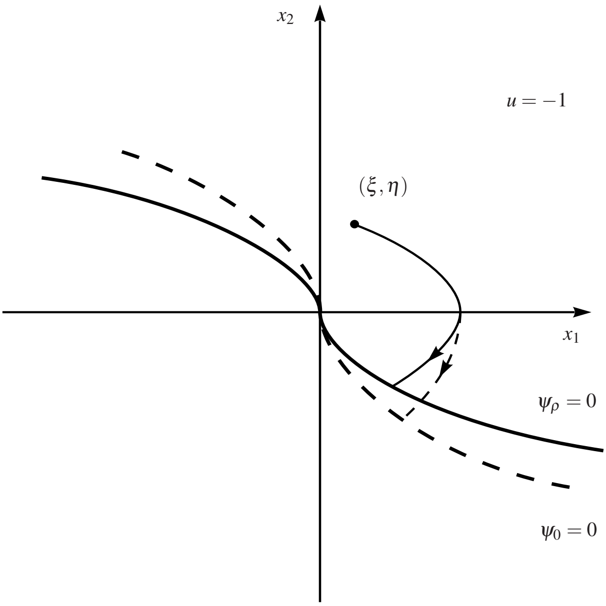

Fig. 2.3 Switching curves for $\rho > 0$ and $\rho = 0$

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\lambda\dot{x}_2 + u(x_1,x_2) + v, \quad \lambda \geq 0,$$
$$|v| \leq \rho < 1, \quad (x_1 = x, \quad x_2 = \dot{x}). \tag{2.2.18}$$

To estimate the possible influence of the perturbations on the motion of system (2.2.18), we pose the problem of finding the "worst" perturbation.

Problem 2.2. Find the optimal feedback control $v(x_1,x_2)$ for system (2.2.18) satisfying the constraint $|v| \leq \rho$ and having the property that the phase trajectory of that system first intersects the switching curve [$\psi_0 = 0$ or $x_1 = \phi(x_2)$, see (2.2.16) and (2.2.17)] as far as possible from the coordinate origin, that is, for as large $|x_1|$ as possible or, what amounts to the same thing, for as large $|x_2|$ as possible.

To be definite, we assume that the initial point (ξ,η) lies in the region $\psi_0 < 0$. Then, by (2.2.16), we have $u = -1$ on the entire trajectory in question. Then, the phase trajectory of system (2.2.18) first intersects that branch of the switching curve on which $x_1 > 0$ and $x_2 < 0$, see Fig. 2.3. As a result, Problem 2.2 is described by the following relationships:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\lambda x_2 - 1 + v, \quad |v| \leq \rho < 1,$$
$$\lambda \geq 0, \quad 0 \leq t \leq \tau, \quad x_1(0) = \xi, \quad x_2(0) = \eta, \tag{2.2.19}$$
$$x_1(\tau) = \phi(x_2(\tau)), \quad x_1(\tau) > 0, \quad x_2(\tau) < 0, \quad x_1(\tau) \to \max.$$

Here, τ is the terminal instant of the process that is not fixed. Function $\phi(x_2)$ in (2.2.17) and (2.2.19) is obtained from the equation $\psi_0 = 0$ [see (2.2.16)] with $\rho = 0$ and $x_2 < 0$. We obtain

$$\phi(x_2) = -\lambda^{-1}x_2 - \lambda^{-2}\log(1-\lambda x_2), \quad (\lambda > 0),$$

$$\phi(x_2) = \frac{1}{2}x_2^2, \quad (\lambda = 0).$$

It follows from these relationships that

$$\phi(x_2) \geq 0, \quad \phi'(x_2) = x_2(1-\lambda x_2)^{-1} < 0, \quad (x_2 < 0). \tag{2.2.20}$$

We note that maximization of $x_1(\tau)$ in (2.2.19) is equivalent to minimization of the following integral functional:

$$\int_0^\tau (-x_2)dt \to \min. \tag{2.2.21}$$

Let us apply the maximum principle, see Sect. 1.2, to Problem 2.2. The Hamiltonian function for problem (2.2.19) and (2.2.21) has the form

$$H = p_1x_2 + p_2(v - \lambda x_2 - 1) + x_2, \quad |v| \leq \rho, \tag{2.2.22}$$

where p_1 and p_2 are conjugate variables. They satisfy the following adjoint equations

$$\dot{p}_1 = 0, \quad \dot{p}_2 = \lambda p_2 - p_1 - 1 \tag{2.2.23}$$

and the transversality conditions

$$p_1(\tau)\phi'(x_2(\tau)) + p_2(\tau) = 0, \quad H(\tau) = 0. \tag{2.2.24}$$

We find $p_1(\tau)$ from the first of conditions (2.2.24) and substitute it into the second one, using expression (2.2.22) for the Hamiltonian H. We obtain

$$p_2[(v - \lambda x_2 - 1)\phi'(x_2) - x_2] + x_2\phi'(x_2) = 0, \quad (t = \tau). \tag{2.2.25}$$

Substituting $\phi'(x_2)$ given by (2.2.20) into (2.2.25), we get, after some simplifications,

$$x_2[p_2(v-2) + x_2] = 0, \quad (t = \tau). \tag{2.2.26}$$

Since, by (2.2.19), we have $|v| \leq \rho < 1$ and $x_2(\tau) < 0$, it follows from (2.2.26) that

$$p_2(\tau) < 0. \tag{2.2.27}$$

It follows from the maximum principle and (2.2.22) that the optimal control is expressed in the form

$$v(t) = \rho \operatorname{sign} p_2(t). \tag{2.2.28}$$

Integrating system (2.2.23), we obtain

$$p_1 = C_1, \quad p_2 = \lambda^{-1}(C_1+1) + C_2 e^{\lambda t}, \quad (\lambda > 0),$$
$$p_1 = C_1, \quad p_2 = C_2 - (C_1+1)t, \quad (\lambda = 0), \tag{2.2.29}$$

where C_1 and C_2 are constants. It follows from (2.2.29) that $p_2(t)$ is a monotone function. Consequently, the optimal control (2.2.28) has no more than one switching point.

Since system (2.2.19) is autonomous, its Hamiltonian H is constant along the optimal trajectory and, by virtue of (2.2.24), equal to zero. Then, by (2.2.22), we obtain

$$H(t) = (p_1+1)x_2 + p_2(v - \lambda x_2 - 1) \equiv 0. \tag{2.2.30}$$

At the instant of switching, we have, by (2.2.28), $p_2 = 0$. Then it follows from (2.2.30) that at that instant either $p_1 = -1$ or $x_2 = 0$.

Let us consider the first possibility. It follows from (2.2.29) that, if $p_1 = -1$, $p_2(t)$ does not change sign along the trajectory and hence a switching cannot take place for $p_1 = -1$.

The second possibility $x_2 = 0$ means that the control is switched when the trajectory crosses the line $x_2 = 0$. Since $p_2(\tau)$ is, by (2.2.27), negative, the optimal control (2.2.28) is negative for $x_2 < 0$ and positive for $x_2 > 0$. Thus, the feedback optimal control has the form

$$v(x_1, x_2) = \rho \operatorname{sign} x_2. \tag{2.2.31}$$

The optimal control in the region $\psi_0 < 0$ is constructed. We note that system (2.2.19) is, along with relationships (2.2.16), invariant with respect to the transformation $x_1 \to -x_1$, $x_2 \to -x_2$, and $v \to -v$. Consequently, the optimal feedback control $v(x_1, x_2)$ possesses the property of central symmetry, and synthesis (2.2.31) satisfies that condition. Thus, (2.2.31) provides the solution of Problem 2.2 formulated above in the whole phase plane (x_1, x_2).

2.2.3 Comparative analysis of the results

Let us use solution (2.2.31) of Problem 2.2 obtained above to analyze possible motions of system (2.2.18) for the case of the simplified control (2.2.16). We first assume that perturbation v is given by (2.2.31). For u given by (2.2.16) and v given by (2.2.31), all trajectories of system (2.2.18) consist of arcs of curves corresponding to constant $u = \pm 1$ and $v = \pm\rho$. The equations of these curves are determined by (2.2.4)–(2.2.7), in which, according to (2.2.16) and (2.2.31), we set

$$w = u + v = \operatorname{sign} \psi_0 + \rho \operatorname{sign} x_2. \tag{2.2.32}$$

One of the trajectories for the control law (2.2.16) in the case of perturbation (2.2.31) is shown in Fig. 2.3 by the thin dashed curve. The solid curve represents the optimal trajectory for control (2.2.9) and $v = -\rho u$. The arrows indicate the direction of increasing time. We note that the arcs of the optimal trajectories for the two control laws coincide in regions where the three functions ψ_0, ψ_ρ, and $(-x_2)$ have the same signs. The heavy solid and the dashed curves in Fig. 2.3 represent the switching curves $\psi_\rho = 0$ and $\psi_0 = 0$ for controls (2.2.16) and (2.2.9), respectively.

To be definite, we construct the optimal phase trajectory for control (2.2.16) that begins at the point (ξ, η) on the switching curve $\psi_0 = 0$ for $\eta \geq 0$ and ends at the point (ξ^*, η^*) on the other branch of the switching curve, that is, for $\xi^* > 0$ and $\eta^* < 0$. This trajectory lies in the region $\psi_0 < 0$ and consists of two sections that meet at $x_2 = 0$. On the first section, where $x_2 > 0$, we have, by (2.2.16) and (2.2.31), $u = -1$ and $v = \rho$. On the second section, $x_2 < 0$. Therefore, by (2.2.16) and (2.2.31), we have on it $u = -1$ and $v = -\rho$.

The first section of the trajectory passes through the initial point (ξ, η) and, on this section, we have, by (2.2.32), $w = u + v = -1 + \rho$. Consequently, its equation is, by (2.2.6) and (2.2.7), represented in the form

$$
\begin{gathered}
x_1 = B_1' - \lambda^{-1} x_2 + \lambda^{-2}(1-\rho)\log[1 + \lambda(1-\rho)^{-1} x_2], \\
B_1' = \xi + \lambda^{-1}\eta - \lambda^{-2}(1-\rho)\log[1 + \lambda(1-\rho)^{-1}\eta] \quad \text{for} \quad \lambda > 0; \\
x_1 = B_1' - [2(1-\rho)]^{-1} x_2^2, \\
B_1' = \xi + [2(1-\rho)]^{-1}\eta^2 \quad \text{for} \quad \lambda = 0; \\
0 \leq x_2 \leq \eta.
\end{gathered}
\tag{2.2.33}
$$

The second section of the trajectory passes through the final point (ξ^*, η^*), and, on it, $w = u + v = -1 - \rho$. Therefore, for the second section, we obtain from (2.2.6) and (2.2.7)

$$
\begin{gathered}
x_1 = B_2' - \lambda^{-1} x_2 + \lambda^{-2}(1+\rho)\log[1 + \lambda(1+\rho)^{-1} x_2], \\
B_2' = \xi^* + \lambda^{-1}\eta^* - \lambda^{-2}(1+\rho)\log[1 + \lambda(1+\rho)^{-1}\eta^*] \quad \text{for} \quad \lambda > 0; \\
x_1 = B_2' - [2(1+\rho)]^{-1} x_2^2, \\
B_2' = \xi^* + [2(1+\rho)]^{-1}(\eta^*)^2 \quad \text{for} \quad \lambda = 0; \\
\eta^* \leq x_2 \leq 0.
\end{gathered}
\tag{2.2.34}
$$

At the joining point of the sections, we have $x_2 = 0$, and the values of x_1 for the two sections coincide. We then obtain from (2.2.33) and (2.2.34)

$$
B_1' = B_2'. \tag{2.2.35}
$$

The points (ξ,η) and (ξ^*,η^*) belong to two branches of the switching curve $\psi_0=0$; also, $\eta>0$ and $\eta^*<0$. Consequently, on the basis of formulas (2.2.16), we obtain

$$\begin{gathered}\xi=-\lambda^{-1}\eta+\lambda^{-2}\log(1+\lambda\eta),\\ \xi^*=-\lambda^{-1}\eta^*-\lambda^{-2}\log(1-\lambda\eta^*)\quad\text{for}\quad\lambda>0;\\ \xi=-\frac{1}{2}\eta^2,\quad \xi^*=\frac{1}{2}(\eta^*)^2\quad\text{for}\quad\lambda=0.\end{gathered}\tag{2.2.36}$$

We substitute into (2.2.35) expressions (2.2.33) and (2.2.34) for B_1' and B_2' and also formulas (2.2.36) expressing ξ and ξ^* in terms of η and η^*. As a result, after some simplification, we obtain the relationships

$$\begin{gathered}[1+(1-\rho)^{-1}\lambda\eta]^{1-\rho}(1+\lambda\eta)^{-1}\\ =[1+(1+\rho)^{-1}\lambda\eta^*]^{1+\rho}(1-\lambda\eta^*)\quad\text{for}\quad\lambda>0,\\ \rho(1-\rho)^{-1}\eta^2=(2+\rho)(1+\rho)^{-1}(\eta^*)^2\quad\text{for}\quad\lambda=0,\end{gathered}\tag{2.2.37}$$

where $\eta>0$ and $\eta^*<0$.

Equations (2.2.37) connect the values of η^* and η. Let us first consider the case $\lambda=0$. Here, the relationship (2.2.37) takes the form

$$\begin{gathered}\left|\frac{\eta^*}{\eta}\right|=\kappa=\left[\frac{\rho(1+\rho)}{(1-\rho)(2+\rho)}\right]^{1/2},\\ 0\le\rho<1.\end{gathered}\tag{2.2.38}$$

One can easily see that κ increases monotonically from 0 to ∞ as ρ increases from 0 to 1. In particular, $\kappa=1$ for ρ equal to

$$\rho^*=\frac{1}{2}(\sqrt{5}-1)\approx 0.618.\tag{2.2.39}$$

The number ρ^* is the "golden-section" ratio. Thus, if $\lambda=0$, then, for $\rho<\rho^*$, we have, on the basis of (2.2.38), $|\eta^*/\eta|<1$; for $\rho=\rho^*$, we have $|\eta^*/\eta|=1$; and for $\rho>\rho^*$, we have $|\eta^*/\eta|>1$.

In the case $\lambda>0$, relationship (2.2.37) defines an implicit dependence of η^* on η. To investigate this connection, we set

$$\lambda\eta=X>0,\quad -\lambda\eta^*=Y>0\tag{2.2.40}$$

and represent dependence (2.2.37) in the form

$$\Phi_\rho(X) = \Psi_\rho(Y), \quad X > 0, \quad Y > 0, \quad 0 < \rho < 1,$$

$$\Phi_\rho(X) = [1 + (1-\rho)^{-1}X]^{1-\rho}(1+X)^{-1}, \tag{2.2.41}$$

$$\Psi_\rho(Y) = [1 - (1+\rho)^{-1}Y]^{1+\rho}(1+Y).$$

We note certain properties of the functions Φ_ρ and Ψ_ρ in (2.2.41). The function Φ_ρ is defined for all $X \geq 0$ and approaches zero as $X \to \infty$; the function Ψ_ρ is defined in the interval $[0, 1+\rho]$ and vanishes at $Y = 1+\rho$. Both functions are equal to unity for $X = Y = 0$. By direct differentiation of functions (2.2.41), we see that $\Phi_\rho'(X) < 0$ and $\Psi_\rho'(Y) < 0$, so that both Φ_ρ and Ψ_ρ are monotone decreasing functions. Let us also calculate the following derivative:

$$\left[\frac{\Psi_\rho(X)}{\Phi_\rho(X)}\right]' = 2[1 + (1-\rho)^{-1}X]^{\rho-2}[1 - (1+\rho)^{-1}X]^{\rho}$$
$$\times (1+X)(1-\rho^2)^{-1}X[\rho - 1 + \rho^2 - (1+\rho)X]. \tag{2.2.42}$$

We note that the expression $\rho - 1 + \rho^2$ in (2.2.42) is nonpositive for $\rho \leq \rho^*$ and positive for $\rho > \rho^*$. Consequently, for $\rho \leq \rho^*$, the ratio Ψ_ρ/Φ_ρ decreases monotonically in the interval $[0, 1+\rho]$. Therefore, $\Psi_\rho(X) < \Phi_\rho(X)$ for $0 < X \leq 1+\rho$. On the other hand, if $\rho > \rho^*$, we have $\Psi_\rho(X) > \Phi_\rho(X)$ in some interval $0 < X \leq X^* < 1+\rho$. However, $\Psi_\rho(X) < \Phi_\rho(X)$ close to $X = 1+\rho$ since

$$\Psi_\rho(1+\rho) = 0 < \Phi_\rho(1+\rho).$$

Figures 2.4 and 2.5 show graphs of the functions $\Phi_\rho(X)$ and $\Psi_\rho(Y)$ for the cases $\rho \leq \rho^*$ and $\rho > \rho^*$, respectively. These figures illustrate graphically the relationship between X and Y that is established by (2.2.41). These equations and the properties mentioned for the functions Φ_ρ and Ψ_ρ lead to the following conclusions for the case $\lambda > 0$:

- If $\rho \leq \rho^*$, we always have $Y < X$ and, by (2.2.40), $|\eta^*/\eta| \leq 1$.
- If $\rho > \rho^*$, then, for sufficiently small X, we have $Y > X$ (that is, $|\eta^*/\eta| > 1$), whereas, for sufficiently large X, we have $Y < X$ (that is, $|\eta^*/\eta| < 1$). Here, we always have $Y < 1+\rho$; that is, $|\eta^*| < (1+\rho)\lambda^{-1}$.

The trajectory that begins at an arbitrary point (ξ, η) in the phase plane can be continued indefinitely even after its intersection with the switching curve $\psi_0 = 0$ at the point (ξ^*, η^*). For this, we need to take the point (ξ^*, η^*) as the initial point and continue the motion on the basis of system (2.2.18), substituting into it control u defined by (2.2.16) and the optimal perturbation v defined by (2.2.31). The trajectory obtained in this way intersects both branches of the switching curve infinitely many times. The values of ordinates x_2 at two successive points of intersection of the switching curve $\psi_0 = 0$ are in the ratio $|\eta^*/\eta|$ that is given by formula (2.2.38) for $\lambda = 0$, and is given by (2.2.40) and (2.2.41) for $\lambda > 0$.

The nature of the motion is quite dependent on the parameters ρ and λ.

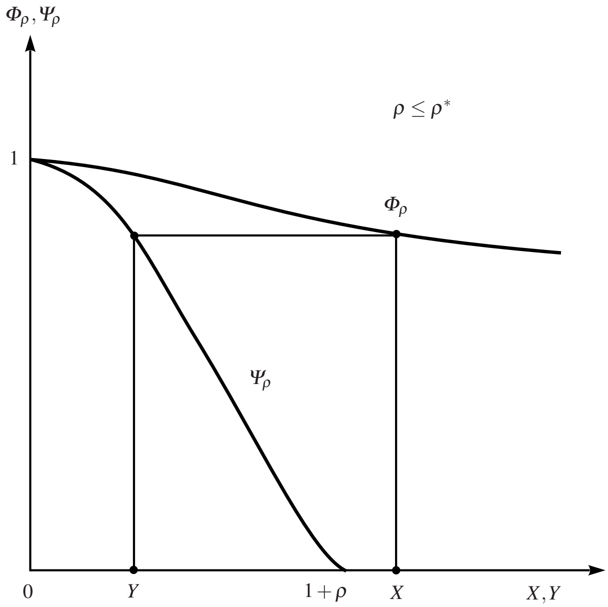

Fig. 2.4 Functions $\Phi_\rho(X)$ and $\Psi_\rho(Y)$ for the case $\rho \le \rho^*$

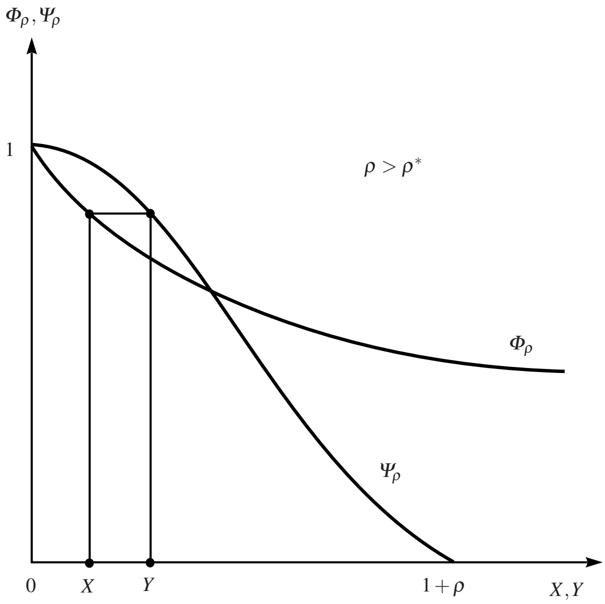

Fig. 2.5 Functions $\Phi_\rho(X)$ and $\Psi_\rho(Y)$ for the case $\rho > \rho^*$

Let us first set $\lambda = 0$. If $\rho < \rho^*$, where ρ^* is defined by (2.2.39), then $\kappa < 1$ in (2.2.38). Here, the values of $|x_2|$ at the instants of intersection of the switching curve $\psi_0 = 0$ by the trajectory decrease in a geometric progression with the denominator $\kappa < 1$. Therefore, the phase trajectory approaches the coordinate origin and reaches it in finite time, though after an infinite number of switchings.

If $\rho = \rho^*$, then $\kappa = 1$ in (2.2.38) and the phase trajectory is periodic. At equal intervals of time, it passes through the same points in the phase plane. In this case, the trajectory remains in a bounded region though it does not approach the coordinate origin.

If $\rho > \rho^*$, then $\kappa > 1$ in (2.2.38). Here, the phase trajectory moves to infinity along a spiral path.

The behavior of the phase trajectories is shown in Figs. 2.6, 2.7, and 2.8 for the cases $\rho < \rho^*$, $\rho = \rho^*$, and $\rho > \rho^*$, respectively.

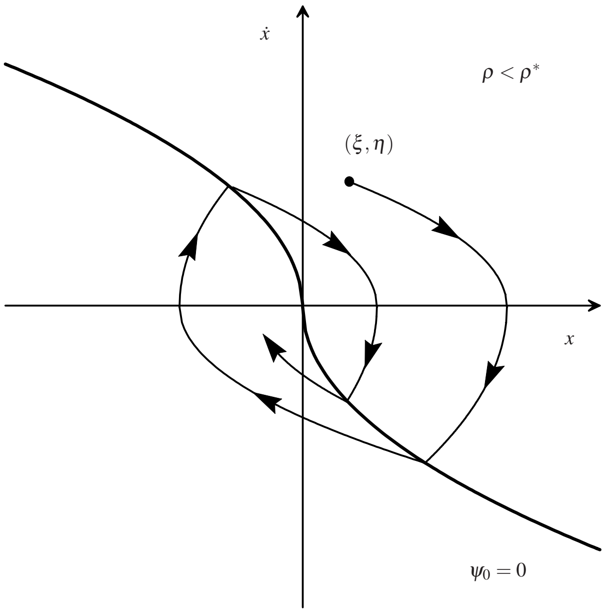

Fig. 2.6 Phase trajectory for $\rho < \rho^*$

Let us turn to the case $\lambda > 0$. If in addition $\rho \leq \rho^*$, then, by the analysis that we made, $|\eta^*/\eta| < 1$. In this case, the phase trajectory approaches the coordinate origin and reaches it in a finite time for $\rho < \rho^*$. One can show that, for $\rho = \rho^*$, the phase point approaches the coordinate origin as $t \to \infty$.

For $\rho > \rho^*$, the phase trajectory does not approach the coordinate origin but remains in a bounded region. From some instant on, we have $|x_2| \leq \lambda^{-1}(1+\rho)$ (as a consequence of the inequality $Y < 1+\rho$).

Let us now characterize the possible motions of system (2.2.18) for the control law (2.2.16) and an arbitrary perturbation $|v| \leq \rho$.

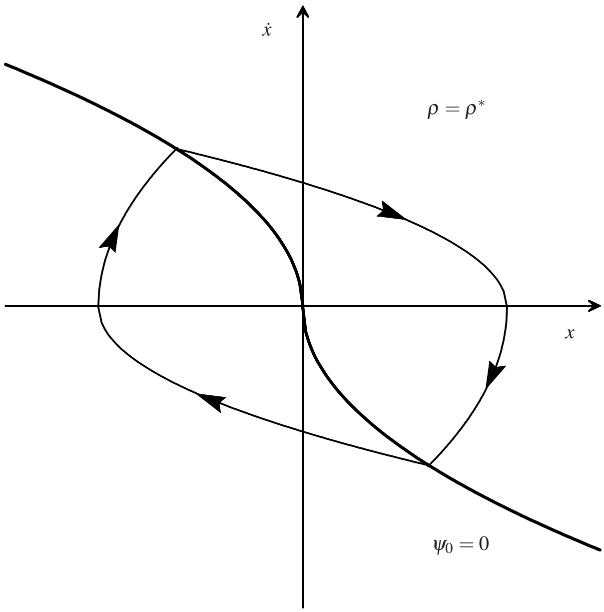

Fig. 2.7 Phase trajectory for $\rho = \rho^*$

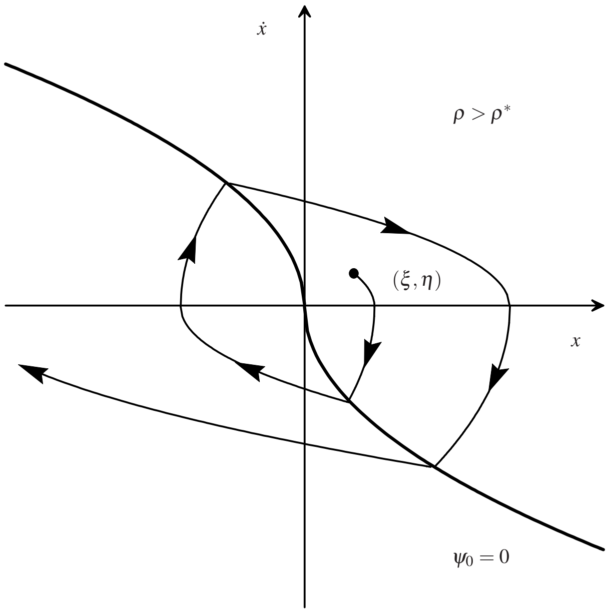

Fig. 2.8 Phase trajectory for $\rho > \rho^*$

If the ratio of the maximum possible value of the perturbation to the maximum possible value of the control is less than the "golden-section" ratio (2.2.39), namely, if $\rho < \rho^*$, then, for any admissible control and any nonnegative λ, the control law (2.2.16) ensures that system (2.2.18) will move to the coordinate origin in finite time. This follows from the fact that the origin is reached even for the "worst" perturbation (2.2.31) that tries to takes the system as far as possible from the coordinate origin.

If $\rho = \rho^*$, then, for $\lambda = 0$, the control law (2.2.16) ensures that the system will be kept in a bounded region and, for $\lambda > 0$, it ensures that it will approach the coordinate origin as $t \to \infty$.

On the other hand, if the ratio of the maximum possible perturbation to the maximum possible control exceeds the "golden-section" ratio ($\rho > \rho^*$), then there exist perturbations for which it is impossible to move the system to the coordinate origin. In the case $\lambda > 0$, perturbation (2.2.31) takes the system arbitrarily far from the coordinate origin, whereas in the case $\lambda = 0$ it takes the system out of some neighborhood of the coordinate origin though the system remains in a bounded region.

Thus, the simplified control law (2.2.16), which does not take into account the presence of perturbations, achieves the control purpose and brings the system to the coordinate origin, only if the perturbation level is sufficiently low. Specifically, the ratio of this level to the maximum level of control must not exceed the "golden-section" ratio ($\rho < \rho^*$).

In other words, one can ignore the presence of perturbations in constructing the control, only if the ratio of the maximum level of the perturbation to that of the control does not exceed the "golden-section" ratio $\rho^* \approx 0.618$.

We recall that the optimal control law (2.2.9) and (2.2.10) is based on the game approach and guarantees that system (2.1.35) will move to the coordinate origin in a finite time for any admissible perturbation, if $\rho < 1$. Thus, in comparison with the simplified control of Sect. 2.2.2, the control law described in Sect. 2.2.1 has a wider field of applicability. Furthermore, the game approach ensures the minimum guaranteed time for bringing the system to the coordinate origin because it is based on the time-optimal feedback control. However, the game approach, unlike the simplified approach, needs the knowledge of the maximal level of possible perturbations, i.e. the parameter ρ. In their structure, the two methods are similar, of a bang-bang type, and differ only in the switching curves, see Fig. 2.3.

2.2.4 Control for the initial system

Let us turn to solution of the original Problem 2.1. We obtain the feedback control in this problem on the basis of (2.1.34) and (2.1.29) in the following form:

$$U_i(q,\dot{q}) = U_i^0 u(x_1,x_2),$$

$$x_1 = x = (U_i^0)^{-1} y_i = (U_i^0)^{-1} [A_*(q - q^*)]_i, \tag{2.2.43}$$

$$x_2 = \dot{x} = (U_i^0)^{-1}\dot{y}_i = (U_i^0)^{-1}(A_*\dot{q})_i, \quad i,\dots,n.$$

Here, in the case of the optimal control of Sect. 2.2.1, function $u(x_1,x_2)$ is defined by (2.2.9), in which ψ_ρ is given by (2.2.10). Parameters λ and ρ in formulas (2.2.10) are expressed by (2.1.37); that is

$$\lambda = \lambda_i, \quad \rho = \frac{V_i^0}{U_i^0} < 1, \quad i = 1,\dots,n. \tag{2.2.44}$$

Control (2.2.43) is a bang-bang control, and it assumes the limiting admissible values: $U_i = \pm U_i^0$, $i = 1,\dots,n$. Let us describe the nature of the motion in the case of this control. We first suppose that perturbations V_i in system (2.1.28) or (2.1.30) assume at every instant the optimal ("worst") values. In terms of system (2.1.35), this means that $v = -\rho u$. In terms of system (2.1.28), we have, by virtue of (2.1.34), (2.2.43), and (2.2.44),

$$V_i = -\rho U_i^0 u = -V_i^0 (U_i^0)^{-1} U_i(q,\dot{q}), \quad i = 1,\dots,n. \tag{2.2.45}$$

In the case of perturbation (2.2.45), the motion of system (2.1.30), for the coordinate y_i, takes place along the time-optimal trajectories of system (2.1.38) or (2.2.1), that is, along the trajectories of Fig. 2.2 for $\lambda_i > 0$ or Fig. 1.2 for $\lambda = 0$. The relationship between the original coordinates q, the variables y_i, and the variables x_1 and x_2 is given by (2.1.29) and (2.2.43).

Now, if the perturbations V_i differ from the worst ones (2.2.45), as is usually the case, then the phase trajectories for each ith degree of freedom in the plane (x_1,x_2) deviate from the optimal ones. Here, the motion along the switching curves takes place in a sliding regime.

Time t_* needed to bring system (2.1.3) [or (2.1.10), (2.1.28), (2.1.30)] to the given state (2.1.8) does not exceed the maximum of the optimal times for each of subsystems (2.1.30) [or (2.1.35), (2.1.38), (2.2.1)]. We have

$$\begin{aligned} &t_* \le t_0 + \max_{1\le i\le n} \tau(\xi_i,\eta_i), \quad \xi_i = (U_i^0)^{-1}[A_*(q^0 - q^*)]_i, \\ &\eta_i = (U_i^0)^{-1}(A_*\dot{q}^0)_i, \quad i = 1,\dots,n. \end{aligned} \tag{2.2.46}$$

Here, we used (2.1.29) and (2.1.37) for ξ and η. The function $\tau(\xi,\eta)$ is defined by (2.2.13) for those i for which $\lambda_i = 0$ and by (2.2.14) for those i for which $\lambda_i > 0$.

We summarize the results in the form of a theorem.

Theorem 2.2. *Suppose that conditions (2.1.12) are satisfied, and that all the trajectories considered lie in domain Ω. Then, the feedback control $U(q,\dot{q})$ that solves Problem 2.1 is given by (2.2.43), where the function $u(x_1,x_2)$ is defined by (2.2.9) and (2.2.10). This control brings system (2.1.3) to the terminal state (2.1.8) no later than at time t_* defined by (2.2.46), (2.2.13), and (2.2.14). Parameters λ and ρ in these formulas are given for each degree of freedom by (2.2.44).*

The control constructed can be called suboptimal, since it is close to the time-optimal one and becomes optimal in the case of the "worst" perturbations.

Using the simplified approach described in Sect. 2.2.2, one should replace the function $u(x_1,x_2)$ in relations (2.2.43) by its expression from (2.2.16). Otherwise, the procedure for the control design for the original system remains the same as for the game approach.

2.3 Weak coupling between degrees of freedom

The first method of decomposition presented in Sect. 2.1.3 leads to the control that solves Problem 2.1. This control is given in explicit form in Sect. 2.2.4. The main assumption that made it possible to carry out the decomposition was the existence of a domain Ω in the $2n$-dimensional space $(q,\dot{q})$ in which all the motions being considered lie and in which inequalities (2.1.12) are satisfied.

By virtue of (2.1.11) for V, inequalities (2.1.12) and (2.1.13) impose constraints on the uncontrolled forces Q and the inertial terms S. We can see from (2.1.4) that the inertial terms depend quadratically on the generalized velocities $\dot{q}$. Therefore, inequalities (2.1.12) and (2.1.13) limit somehow the domain Ω with respect to $\dot{q}$, whereas the obtained control can carry the system to the region with large $|\dot{q}|$.

It is clear, on the one hand, that to solve Problem 2.1 it is necessary to impose constraints on the uncontrolled forces Q, otherwise the bounded controls U will not be able to overcome these forces Q. On the other hand, it follows from what was said above that it is expedient to impose constraints on quantities S_i during the control process. These considerations served as the basis for the modification of the first method of decomposition suggested in [98, 102].

2.3.1 Modification of the decomposition method

Let us turn again to the system described by relations (2.1.1)–(2.1.8).

The domain D in the n-dimensional q-space, where the motions of the system being considered can occur, is specified in the form of independent constraints on the coordinates q_i

$$D = \{q:\ q_i^- \le q_i \le q_i^+\}. \tag{2.3.1}$$

We will make certain simplifying assumptions concerning the kinetic energy and the generalized forces Q_i. Suppose that matrix $A(q)$ from (2.1.3) can be represented in the form

$$A(q) = J + \tilde{A}(q), \quad J = \mathrm{diag}(J_1,\dots,J_n), \quad J_i = \mathrm{const} > 0, \tag{2.3.2}$$

where $\tilde{A}(q)$ is a symmetric matrix such that, for any n-dimensional vector z, the inequality

$$|\tilde{A}(q)z| \leq \mu|z|, \qquad \mu > 0, \qquad \forall q \in D \tag{2.3.3}$$

is satisfied. Here, μ is a sufficiently small parameter, its possible values are specified below.

Furthermore, we assume that

$$\left|\frac{\partial a_{jk}}{\partial q_i}\right| \leq C, \qquad C = \text{const} > 0, \qquad i,j,k = 1,\ldots,n, \tag{2.3.4}$$

and that the generalized forces Q_i can be represented in the form

$$Q_i = G_i + F_i \tag{2.3.5}$$

where $G_i(q,\dot{q},t)$ are restricted forces satisfying constraints

$$|G_i| \leq G_i^0, \qquad i = 1,\ldots,n. \tag{2.3.6}$$

The magnitudes of constants G_i^0 do not exceed constants U_i^0 in constraints (2.1.6) for the permissible values of the control forces, that is

$$G_i^0 < U_i^0, \qquad i = 1,\ldots,n. \tag{2.3.7}$$

Note that, if the inequality $G_i^0 > U_i^0$ that is the inverse of (2.3.7) holds for certain i, then the system can be uncontrollable.

The forces denoted by F_i in (2.3.5) are sufficiently small at low velocities and satisfy the constraints

$$F_i = F_i(q,\dot{q},t), \qquad |F_i| \leq F^0\left(|\dot{q}|\right), \qquad i = 1,\ldots,n. \tag{2.3.8}$$

Here, $F^0(\vartheta)$ is a monotonically increasing continuous function defined for $\vartheta \geq 0$ and such that $F^0(0) = 0$. The exact form of functions $G_i(q,\dot{q},t)$ and $F_i(q,\dot{q},t)$ in (2.3.5) may be unknown.

We multiply both sides of (2.1.3) by JA^{-1} [matrix J has been introduced in (2.3.2)] and obtain

$$J_i\ddot{q}_i = U_i + V_i, \tag{2.3.9}$$

$$V_i = S_i - [\tilde{A}A^{-1}(U+S)]_i. \tag{2.3.10}$$

System (2.3.9) and (2.3.10) is equivalent to the initial equation (2.1.3). Relations (2.1.4), (2.3.4)–(2.3.6), and (2.3.8) yield the constraint on the components of vector S

$$|S_i(q,\dot{q},t)| \leq G_i^0 + \tilde{S}^0(\dot{q}), \qquad \tilde{S}^0(\dot{q}) = F^0\left(|\dot{q}|\right) + \frac{3}{2}C\left(\sum_{j=1}^{n}|\dot{q}_j|\right)^2. \tag{2.3.11}$$

We assume that the inequalities

$$|V_i| \leq \rho_i U_i^0, \qquad \rho_i < 1 \tag{2.3.12}$$

hold, where ρ_i are constants to be specified below. We shall treat the functions V_i in (2.3.9) as independent restricted perturbations not exceeding the permissible values of the controls. In this case, the initial non-linear system is decomposed into n linear subsystems [the ith subsystem is described by the ith equation (2.3.9)] subjected to perturbations, and each subsystem has a single degree of freedom. To solve Problem 2.1, it is therefore sufficient to solve n simpler control problems for the second-order subsystems.

2.3.2 Analysis of the controlled motions

As has been done previously, we shall specify the scalar control U_i which transfers the ith subsystem (2.3.9) in finite time from an arbitrary initial state $(q_i^0, \dot{q}_i^0)$ to the terminal state $(q_i^*, 0)$ for any permissible perturbation V_i satisfying condition (2.3.12). We will again define the feedback control by (2.2.43), (2.2.9), and (2.2.10). Here, one should substitute matrix J instead of matrix A_* in (2.2.43) and set $\lambda = 0$ in (2.2.10). After these transformations, we obtain

$$\begin{gathered} U_i = -U_i^0 \operatorname{sign}(\dot{q}_i - \psi_i^*) \quad \text{for} \quad \dot{q}_i \neq \psi_i^*, \\ U_i = -U_i^0 \operatorname{sign} \dot{q}_i \quad \text{for} \quad \dot{q}_i = \psi_i^*, \\ \psi_i^*(q_i) = -\left(2X_i|q_i - q_i^*|\right)^{1/2} \operatorname{sign}(q_i - q_i^*). \end{gathered} \tag{2.3.13}$$

Here, X_i is a positive control parameter connected with constant ρ_i from (2.3.12) by the relation

$$X_i = \frac{U_i^0(1-\rho_i)}{J_i}. \tag{2.3.14}$$

In inequalities (2.3.12), we express ρ_i in terms of the control parameters X_i using (2.3.14). We obtain

$$|V_i| \leq U_i^0 - J_i X_i, \qquad i = 1, \dots, n. \tag{2.3.15}$$

In order to specify the control law (2.3.13), it is necessary to choose the values of parameters $X_i > 0$, so that inequalities (2.3.15) are satisfied.

Remind that the above-mentioned control was obtained as the time-optimal control in a game problem in which U_i and V_i are considered as the controls of two players [79]. This control is a bang-bang control and takes its limiting permissible values:

$$U_i = \pm U_i^0.$$

The switching curve

$$\dot{q}_i = \psi_i^*(q_i)$$

consists of two parabolic branches which are symmetric about the point $(q_i^*, 0)$.

We will now specify the set Ω_i in the two-dimensional phase space of the ith subsystem (Fig. 2.9):

$$\Omega_i = \{(q_i, \dot{q}_i) : \; q_i^- \le q_i \le q_i^+, \quad \psi_i^- \le \dot{q}_i \le \psi_i^+\},$$
$$\psi_i^-(q_i) = \psi_i^*(q_i + q_i^* - q_i^-), \qquad \psi_i^+(q_i) = \psi_i^*(q_i + q_i^* - q_i^+). \tag{2.3.16}$$

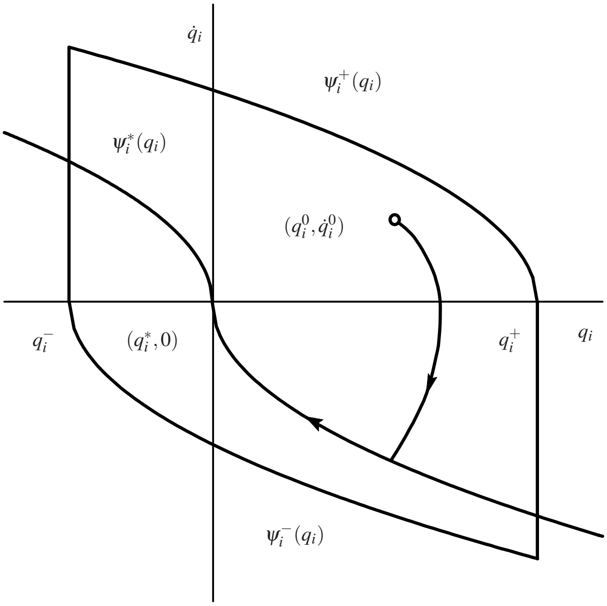

Fig. 2.9 Set Ω_i and the switching curve

We will describe the motion of subsystem (2.3.9) in the case where the control U_i is specified by (2.3.13), the disturbance V_i satisfies constraint (2.3.15), and the initial point $(q_i^0, \dot{q}_i^0)$ lies in Ω_i:

$$(q_i^0, \dot{q}_i^0) \in \Omega_i. \tag{2.3.17}$$

The control process is divided into two main stages. In the first stage, the motion is performed with a constant control until the phase point of the subsystem reaches the switching curve. To fix our ideas, we assume that $\dot{q}_i > \psi_i^*(q_i)$; then, according to (2.3.13), we have $U_i = -U_i^0$. In this case, from (2.3.9) and (2.3.15), it follows that

$$\ddot{q}_i \le -X_i. \tag{2.3.18}$$

Note that, by virtue of (2.3.16) and (2.3.13), the following equality holds

$$\frac{d\psi_i^+}{dq_i} = -\frac{X_i}{\psi_i^+}. \tag{2.3.19}$$

Along the trajectory of the subsystem in the domain Ω_i, we have, according to (2.3.16), $\dot{q}_i \leq \psi_i^+$. Therefore, taking into account (2.3.18) and (2.3.19), we obtain for $\dot{q}_i > 0$

$$\frac{d\dot{q}_i}{dq_i} = \frac{\ddot{q}_i}{\dot{q}_i} \leq -\frac{X_i}{\psi_i^+} = \frac{d\psi_i^+}{dq_i}, \qquad \dot{q}_i > 0. \tag{2.3.20}$$

For $\dot{q}_i < 0$, we have, in accordance with (2.3.18)

$$\frac{d\dot{q}_i}{dq_i} = \frac{\ddot{q}_i}{\dot{q}_i} > 0, \qquad \dot{q}_i < 0. \tag{2.3.21}$$

From inequalities (2.3.20) and (2.3.21) it follows that, for any perturbations, the phase trajectory of the subsystem under consideration does not intersect the curve $\dot{q}_i = \psi_i^+(q_i)$ and, due to (2.3.18), reaches the switching curve in finite time without going out beyond the limits of the domain Ω_i. This fact is proved in a similar way also for $\dot{q}_i < \psi_i^*$.

On reaching the switching curve, the phase point continues to move along it to the terminal state. The parabolic branches of the switching curve coincide with the phase trajectories of subsystem (2.3.9) in the case of the control U_i chosen in accordance with (2.3.13) and (2.3.14) and for $V_i = -\rho_i U_i$. If, however, $V_i \neq -\rho_i U_i$, then the motion occurs along a parabolic arc in a sliding mode. In this case, control U_i takes the values $\pm U_i^0$ with infinitely frequent changes of sign so that "on average" $\ddot{q}_i = X_i$ or $\ddot{q}_i = -X_i$ for the corresponding branches of the switching curve.

Hence, if conditions (2.3.16) and (2.3.17) are satisfied for all subsystems (2.3.9) at the initial instant of time, then their phase trajectories as a whole lie in the corresponding domains Ω_i. In this case, constraints (2.3.1) are satisfied, and the inequalities

$$|\dot{q}_i| \leq -\psi_i^-(q_i^+) = \psi_i^+(q_i^-) = \psi_i^*(q_i^- - q_i^+) \tag{2.3.22}$$

following from (2.3.16) and (2.3.13) also hold. Introducing new notation

$$Y_i = \psi_i^*(-d_i), \qquad d_i = q_i^+ - q_i^-, \tag{2.3.23}$$

we rewrite inequality (2.3.22) in the form

$$|\dot{q}_i| \leq Y_i. \tag{2.3.24}$$

Using expression (2.3.13) for ψ_i^*, we obtain

$$Y_i = (2X_i d_i)^{1/2}. \tag{2.3.25}$$

A possible phase trajectory of subsystem (2.3.9) is shown in Fig. 2.9. The direction of motion is indicated by the arrows.

It has been shown in Sect. 2.2.1 that the time for the motion of the ith subsystem (2.3.9) is maximal in the case of the "worst" perturbation $V_i = -\rho_i U_i$ and, on the strength of notation (2.3.14), is equal to [see (2.2.13)]

$$\tau_i^*(q_i^0, \dot{q}_i^0) = X_i^{-1}\left\{2\left[\frac{1}{2}(\dot{q}_i^0)^2 - X_i(q_i^0 - q_i^*)\gamma_i\right]^{1/2} - \dot{q}_i^0\gamma_i\right\}, \tag{2.3.26}$$

$$\gamma_i = -\operatorname{sign}[\dot{q}_i^0 - \psi_i^*(q_i^0)], \quad \dot{q}_i^0 \neq \psi_i^*; \qquad \gamma_i = \pm 1, \quad \dot{q}_i^0 = \psi_i^*.$$

Since time τ required to bring the original system (2.1.1) to the terminal state (2.1.8) is defined as the greatest of the control times for each of subsystems (2.3.9), we obtain the estimate

$$\tau \leq \tau^* = \max_i(\tau_i^*), \qquad i = 1, \ldots, n. \tag{2.3.27}$$

2.3.3 Determination of the parameters

Control (2.3.13) can only be used when inequalities (2.3.15) [or (2.3.12)] are satisfied throughout the whole control process. We shall now find those control parameters X_i for which the above-mentioned relations are, in fact, satisfied.

We will first estimate the moduli of the quantities V_i from (2.3.10). Using relations (2.3.3), (2.3.11), and (2.3.24), we obtain

$$|V_i| \leq G_i^0 + \tilde{S}^0(Y) + \mu|A^{-1}(U+S)|, \qquad Y = (Y_1, \ldots, Y_n). \tag{2.3.28}$$

From (2.3.2) and (2.3.3), we have for any n-dimensional vector z

$$Az = Jz + \tilde{A}z, \quad |Az| \geq J_{\min}|z| - \mu|z| = (J_{\min} - \mu)|z|. \tag{2.3.29}$$

Here, $J_{\min}$ is the least of the quantities J_i. Let us set $z = A^{-1}z'$ in inequality (2.3.29). Then for $\mu < J_{\min}$ we obtain

$$|A^{-1}z'| \leq \frac{|z'|}{J_{\min} - \mu}. \tag{2.3.30}$$

Using (2.1.6), (2.3.11), and (2.3.24), we obtain the relations

$$|(U+S)_i| \leq |U_i| + |S_i| \leq U_i^0 + G_i^0 + \tilde{S}^0(Y) = (U^0 + G^0)_i + \tilde{S}^0(Y),$$
$$U^0 = (U_1^0, \ldots, U_n^0), \qquad G^0 = (G_1^0, \ldots, G_n^0). \tag{2.3.31}$$

Combining (2.3.28), (2.3.30), and (2.3.31), we find the final estimate for the disturbances:

$$
\begin{aligned}
|V_i| &\leq G_i^0 + \tilde{S}^0(Y) + \frac{\mu}{J_{\min} - \mu} |U + S| \\
&\leq G_i^0 + \left(1 + \frac{\mu n^{1/2}}{J_{\min} - \mu}\right) \tilde{S}^0(Y) + \frac{\mu}{J_{\min} - \mu} |U^0 + G^0|.
\end{aligned}
\tag{2.3.32}
$$

In inequalities (2.3.15), instead of the quantities $|V_i|$, we substitute their estimates from (2.3.32). After transformations, we obtain

$$
J_i X_i + \left(1 + \frac{\mu n^{1/2}}{J_{\min} - \mu}\right) \tilde{S}^0(Y) \leq U_i^0 - G_i^0 - \frac{\mu}{J_{\min} - \mu} |U^0 + G^0|. \tag{2.3.33}
$$

System of inequalities (2.3.33) determines the permissible parameters X_i. It is a nonlinear system because the values Y_i are connected with X_i by (2.3.25).

If parameter μ is sufficiently small, such that the condition

$$
\mu < \frac{\min_i (U_i^0 - G_i^0) J_{\min}}{\min_i (U_i^0 - G_i^0) + |U^0 + G^0|} \tag{2.3.34}
$$

is satisfied, the expressions on the right-hand sides of inequalities (2.3.33) are positive. Since $\tilde{S}^0(Y) \to 0$ as $X_i \to 0$ due to (2.3.11), positive values of X_i can always be found, for which inequalities (2.3.33) and, consequently, inequalities (2.3.12) are satisfied.

We will sum up the results obtained in the form of a theorem.

Theorem 2.3. *Suppose that condition (2.3.34) is satisfied. Then the feedback control $U_i(q_i, \dot{q}_i)$ that solves Problem 2.1 in domain (2.3.16) is specified by relations (2.3.13) in which parameters X_i must be chosen in such a way that inequalities (2.3.33) are satisfied. This control transfers system (2.1.1) from the initial state (2.1.7) to the specified terminal state (2.1.8), if, at the initial instant of time, constraints (2.3.17) are satisfied. In this case, motion $q(t)$ of the system lies in domain D from (2.3.1), and the time of the control process τ does not exceed τ^* determined by expressions (2.3.26) and (2.3.27).*

We will now describe a method of selecting the permissible values of X_i. We shall seek these values in the form

$$
X_i = Z^2 d_i, \tag{2.3.35}
$$

where d_i is defined in (2.3.23), and the magnitude of Z is still unknown. We substitute (2.3.35) into inequalities (2.3.33) taking into account (2.3.25) and (2.3.11). Selecting the maximum value of Z

$$
Z_0 = \max Z \tag{2.3.36}
$$

that satisfies the inequality obtained, we calculate the control parameters X_i, using formulas (2.3.35). In this case, at least one of inequalities (2.3.33) is transformed into the equality.

Suppose, for example, constraint (2.3.8) has the form

$$|F_i| \leq F^0(|\dot{q}|) = a|\dot{q}| + b|\dot{q}|^2,$$

where a and b are positive constants. Then inequalities (2.3.33) can be reduced to the form

$$Z^2 + 2g_i Z \leq h_i, \tag{2.3.37}$$

where g_i and h_i are positive coefficients that can be found from (2.3.25), (2.3.11), and (2.3.33). The solution of the system of inequalities (2.3.37) can be written in the form

$$Z \leq Z_0 = \min_i [(g_i^2 + h_i)^{1/2} - g_i], \quad i = 1, \ldots, n. \tag{2.3.38}$$

Conditions (2.3.33) for determining the set of permissible values X_i together with constraint (2.3.34) imposed on parameters of system (2.1.1) are sufficient conditions that differ from the necessary ones. For specific systems, it is possible to obtain more precise estimates than (2.3.32). Substituting such estimates into inequality (2.3.15) instead of V_i, one can obtain a more broad set of permissible control parameters. As a result, increasing of X_i allows one to widen domains Ω_i from (2.3.16) that bound initial velocities for the subsystems [see (2.3.17)] and, thus, significantly shorten the time of motion τ. In some instances, this allows one to relax restrictions imposed upon parameters of the system.

The control method proposed is quite simple and does not require an exact knowledge of the nonlinear terms and perturbing forces in the equations of motion. The method is not overly sensitive to slight variations in the system parameters or to additional perturbations: to take such factors into consideration, one needs only to decrease parameters X_i, leaving a sufficient "safety margin" for the controls of the corresponding subsystems.

2.3.4 Case of zero initial velocities

While solving Problem 2.1, we assumed that the initial state of the system is an arbitrary point in the domain Ω_i, see (2.3.17). Consider a special but important case of the zero initial velocities $\dot{q}^0 = 0$.

Under control (2.3.13), the coordinates q_i of all subsystems are bounded by the inequalities $\min(q_i^0, q_i^*) \leq q_i(t) \leq \max(q_i^0, q_i^*)$. Therefore, it is possible to restrict the domain of the possible motions, setting

$$q_i^- = \min(q_i^0, q_i^*), \qquad q_i^+ = \max(q_i^0, q_i^*) \tag{2.3.39}$$

in (2.3.1) for all $i = 1, \ldots, n$. For such presetting of the domain D, the magnitudes $d_i = q_i^+ - q_i^-$ are minimal, and hence the estimates obtained in (2.3.24) for the generalized velocities and in (2.3.32) for the disturbances are of the maximum accuracy. We suppose that the boundaries q_i^- and q_i^+ of the domain of motion are given in the form (2.3.39). Estimates (2.3.26) and (2.3.27) in this case take the form

$$\tau \le \tau^* = \max_i(\tau_i^*), \qquad \tau_i^* = 2\sqrt{\frac{d_i}{X_i}}, \qquad i = 1, \dots, n. \tag{2.3.40}$$

In accordance with (2.3.35), (2.3.36), and (2.3.40), we have the equal estimates on times needed for steering each of subsystems (2.3.9) to the terminal state:

$$\tau^* = \tau_i^* = \tau_0^*, \qquad \tau_0^* = 2Z_0^{-1}, \tag{2.3.41}$$

where Z_0 is defined by (2.3.36). Let us demonstrate that for control (2.3.13) with any other permissible parameters X_i, satisfying (2.3.33), but not related to each other by equalities (2.3.35) and (2.3.36), the estimate for the motion time τ^*, calculated by using (2.3.40), will be greater than τ_0^*.

Really, in order to reduce τ^*, it is required, according to (2.3.40) and (2.3.41), to increase X_i for all $i = 1, \dots, n$. Then, due to the strict monotony of the left-hand sides of inequalities (2.3.33) with respect to X_i, all the left-hand sides will grow up, and at least one of inequalities (2.3.33), which was transformed into the equality while choosing $Z = Z_0$ according to (2.3.36), will fail. Thus, the magnitude $\tau^* = \tau_0^*$, obtained in (2.3.41), is minimal for $\dot{q}^0 = 0$ and control (2.3.13).

Modified control law

In the case $\dot{q}^0 = 0$, it is possible to modify the control law (2.3.13) so that the new (corresponding to the modified control law) estimate of the motion time will be less than the estimate obtained in (2.3.41). To this end, let us re-define functions ψ_i^* in (2.3.13) so that the switching curve $\dot{q}_i = \psi_i^*(q_i)$ (see Fig. 2.10) will consist of the parabolic arc (for $|q_i - q_i^*| \le d_i^*$) and straight-line section (for $d_i^* < |q_i - q_i^*| \le d_i$). Here, d_i^* is some positive constant that will be determined. We impose the only constraint upon this constant

$$d_i^* \le \frac{1}{2} d_i, \tag{2.3.42}$$

where d_i is defined by (2.3.23) and (2.3.39). Thus, we define functions ψ_i^* in the form:

$$\begin{aligned} \psi_i^*(q_i) &= -(2X_i|q_i - q_i^*|)^{1/2}\,\mathrm{sign}(q_i - q_i^*) \quad \text{for} \quad |q_i - q_i^*| \le d_i^*, \\ \psi_i^*(q_i) &= Y_i\,\mathrm{sign}(q_i - q_i^*) \quad \text{for} \quad d_i^* < |q_i - q_i^*| \le d_i, \end{aligned} \tag{2.3.43}$$

Here, X_i and Y_i are positive parameters for the new control law; they are not already related to each other by (2.3.25) but subjected to the additional contraints

$$Y_i \le (X_i d_i)^{1/2}. \tag{2.3.44}$$

Parameters d_i^* are expressed through X_i and Y_i by the following formula

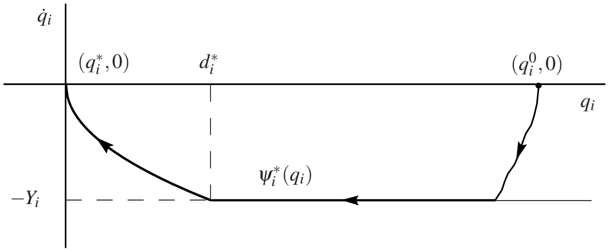

Fig. 2.10 Modified switching curve and the trajectory

$$d_i^* = \frac{Y_i^2}{2X_i}. \tag{2.3.45}$$

Due to (2.3.44) and (2.3.45), constraint (2.3.42) is satisfied.

If conditions (2.3.15) [or (2.3.12)] hold during the motion, then the control defined by (2.3.13) and (2.3.43) surely steers the system to the terminal state. Herein, the velocities of the subsystems are bounded and the inequalities $|\dot{q}_i| \le Y_i$ hold. Hence, the estimate (2.3.32) for the maximal absolute values of the disturbances $|V_i|$ is also true. Substituting (2.3.32) into (2.3.15), we obtain inequalities that completely coincide with (2.3.33). Hence, if X_i and Y_i obey inequality (2.3.33), then the control defined by (2.3.13) and (2.3.43) solves Problem 2.1 for $\dot{q}^0 = 0$.

The following estimates of the motion time are true:

$$\tau \le \tau^* = \max_i(\tau_i^*), \quad \tau_i^* = \frac{d_i}{Y_i} + \frac{Y_i}{X_i}, \qquad i = 1, \dots, n, \tag{2.3.46}$$

where τ_i^* is the estimate of the motion time for the ith subsystem. Let us prove these relations.

Proof. The motion time for the ith subsystem is maximal under the worst disturbance $V_i = -\rho_i U_i$. The motion consists of three stages, see Fig. 2.10. First, according to (2.3.9), (2.3.13), (2.3.14), and (2.3.43), the motion has the constant acceleration

$$\ddot{q}_i = -X_i \operatorname{sign}(q_i^0 - q_i^*)$$

until the phase point reaches the switching curve $\dot{q}_i = \psi_i^*(q_i)$. Further, the phase point moves along the straight-line section of the switching curve with the constant velocity

$$\dot{q}_i = -Y_i \operatorname{sign}(q_i^0 - q_i^*),$$

and then, along the parabolic section of the switching curve, with the constant acceleration

$$\ddot{q}_i = X_i \operatorname{sign}(q_i^0 - q_i^*).$$

The durations of the first and the final stages are the same and equal to Y_i/X_i. The duration of the second stage is equal to $(d_i - 2d_i^*)/Y_i$. Summing up the durations of all three stages and taking into account (2.3.45), we come to the estimate (2.3.46) for τ. □

Let us demonstrate that the modified control law (2.3.43) allows one to reduce the estimate of the motion time (2.3.41).

Proof. To this end, consider the concrete choice of the parameters X_i and Y_i in (2.3.43), supposing that each Y_i depends on the corresponding X_i so that constraints (2.3.44) are trasformed into equalities

$$Y_i = (X_i d_i)^{1/2}. \tag{2.3.47}$$

Note, that (2.3.47) differs from (2.3.25) used earlier. From (2.3.45) and (2.3.47), we have

$$d_i^* = \frac{1}{2} d_i.$$

In this case, under the worst disturbances, the strait-line stage of motion is missing and relations (2.3.46) are transformed into (2.3.40). We will seek for the parameters X_i in the form of (2.3.35), as it has been done earlier, and express the parameters Y_i according to (2.3.47). Choose the maximal value of Z:

$$Z_0' = \max Z \tag{2.3.48}$$

satisfying (2.3.33). Due to the monotone dependence of the left-hand side (2.3.33) on parameters Y_i and to (2.3.47), value Z_0' is greater than Z_0 from (2.3.36). Therefore, we obtain a new upper estimate of the time of motion that is lower than the estimate given by (2.3.40) and (2.3.41):

$$\tau^* = \tau_i^* = \tau_0'^*, \qquad \tau_0'^* = 2(Z_0')^{-1}. \tag{2.3.49}$$

Thus, the modified control law really allows one to reduce the motion time estimate. □

Optimization of the control parameters

For the modified control law defined by (2.3.13) and (2.3.43), we suggest below a numerical procedure for finding the optimal permissible parameters X_i and Y_i satisfying inequalities (2.3.33) and (2.3.44) [but not connected by relations (2.3.47) and (2.3.35)], for which the motion time estimate τ^* given by (2.3.46) is minimal.

If parameters X_i and Y_i are optimal, then the quantities τ_i^* in (2.3.46) are equal, i.e.,

$$\tau^* = \tau_i^*, \qquad i = 1, \ldots, n, \tag{2.3.50}$$

and inequalities (2.3.33) are transformed into the exact equalities

$$X_i + K_i \tilde{S}^0(Y) = \Delta_i, \qquad i = 1, \ldots, n, \tag{2.3.51}$$

where

$$K_i = \left(1 + \frac{\mu n^{1/2}}{J_{\min} - \mu}\right) J_i^{-1},$$

$$\Delta_i = \left(U_i^0 - G_i^0 - \frac{\mu}{J_{\min} - \mu}|U^0 + G^0|\right) J_i^{-1}.$$

This fact is proved by the reasoning analogous to that have been used earlier and given after (2.3.41). We assume that inequality (2.3.34) holds, so that Δ_i are positive.

The procedure for finding the optimal parameters X_i and Y_i of the modified control law is as follows. In accordance with (2.3.46) and (2.3.50), we set

$$X_i = Y_i^2 (Y_i \tau^* - d_i)^{-1} \tag{2.3.52}$$

in system (2.3.33). We obtain

$$\frac{Y_i^2}{Y_i \tau^* - d_i} + K_i \tilde{S}^0(Y) \leq \Delta_i. \tag{2.3.53}$$

Let us choose some initial value τ^* [for example, $\tau^* = \tau_0'^*$ from (2.3.49)] and find numerically some values of parameters Y_i satisfying (2.3.53). The set $[Y_i^-, Y_i^+]$, $i = 1, \dots, n$, in which it is possible to make this search, can be easily obtained by setting $\tilde{S}^0 = 0$ in inequalities (2.3.53). As a result, we obtain

$$Y_i^{\pm} = \frac{1}{2}\tau^* \Delta_i \pm \left[\left(\frac{1}{2}\tau^* \Delta_i\right)^2 - d_i \Delta_i\right]^{1/2}.$$

If any solution Y_i of inequalities (2.3.53) is found, then we decrease the value τ^* in (2.3.53) by some increment $\delta\tau^*$ and repeat the search of the permissible parameters Y_i corresponding to the new value of τ^*. The minimal value of τ^* for which inequalities (2.3.53) have the solution $Y_i > 0$ for all $i = 1, \dots, n$, defines together with (2.3.52) the optimal parameters X_i and Y_i.

Let us show also that the motion time estimate $\tau_0'^*$ [see (2.3.49)] obtained analytically earlier for the modified control law defined by (2.3.13) and (2.3.43) is not minimal and can be improved by using the suggested numerical procedure.

Proof. It is sufficient to show that, if $\tau_0'^*$ is chosen as the initial estimate τ^*, then, for the sufficiently low step size $\delta\tau^*$, the suggested algorithm surely finds, during the second iteration, the values of the parameters X_i and Y_i providing even less the motion time estimate

$$\tau^* = \tau_0'^* - \delta\tau^*.$$

Suppose the contrary. Let $\tau^* = \tau_0'^*$ from (2.3.49) is the minimal motion time estimate. Then, as it has been pointed above, (2.3.50) and (2.3.51) should be satisfied. Let us choose some value i ($1 \leq i \leq n$) and, by using (2.3.46), find the derivative $\partial\tau_i^*/\partial Y_i$, assuming that parameter X_i is connected with Y_i by the ith equality in (2.3.51):

$$\frac{\partial \tau_i^*}{\partial Y_i} = -\frac{d_i}{Y_i^2} + \frac{1}{X_i} - \frac{Y_i}{X_i^2}\frac{\partial X_i}{\partial Y_i}. \tag{2.3.54}$$

For the given i, due to the monotony of $\tilde{S}^0(Y)$ with respect to Y_i, see (2.3.11), we have from (2.3.51)

$$\frac{\partial X_i}{\partial Y_i} = -K_i \frac{\partial \tilde{S}^0(Y)}{\partial Y_i} < 0. \tag{2.3.55}$$

Values X_i and Y_i providing the estimate $\tau_i^* = \tau_0'^*$ are connected by (2.3.47). Therefore, the sign of the derivative (2.3.54) is positive

$$\frac{\partial \tau_i^*}{\partial Y_i} = -\frac{Y_i}{X_i^2}\frac{\partial X_i}{\partial Y_i} > 0. \tag{2.3.56}$$

Let us decrease the parameter Y_i for the given i by a sufficiently small value δY_i (all Y_j, $j \neq i$, are fixed) and, at the same time, increase parameters X_i, $i = 1, \ldots, n$, for all subsystems, without violating equations (2.3.51). Obviously, for such variations, the new values of the parameters will satisfy constraints (2.3.44). Herein, due to (2.3.46) and (2.3.56), the motion time estimates for all subsystems decrease. Thus, we come to the contradiction, and $\tau_0'^*$ cannot be the minimal estimate for τ^*. $\square$

Note that control (2.3.13) with the modified switching curve (2.3.43) can be used also in the case where $\dot{q}^0 \neq 0$. The corresponding domain Ω_i that contains possible initial states for the ith subsystem is described by relations (2.3.16) and depicted in Fig. 2.11.

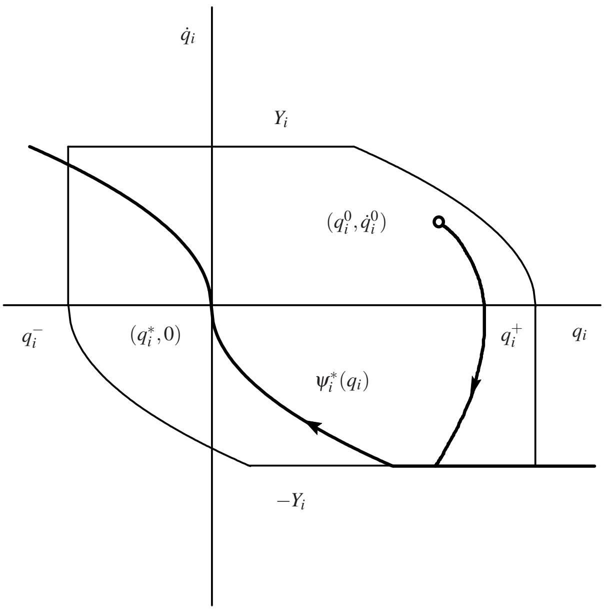

Fig. 2.11 Domain Ω_i for the modified control law

2.4 Nonlinear damping

The method of solving Problem 2.1 that we have presented in Sects. 2.1.3–2.2.4 consists of two stages: 1) the decomposition of the original nonlinear system (2.1.3) into subsystems (2.1.30) with one degree of freedom each [see (2.1.35)]; 2) the use of the game approach to construct a control for the subsystems.

Certain modifications of the suggested approach are possible at both stages. At the first stage, a system can be reduced to a collection of other subsystems, either more simple or more complicated than (2.1.35). System (2.1.35) with $\lambda = 0$ is evidently the simplest subsystem with one degree of freedom. In the case $\lambda = 0$, the imposed constraint (2.1.12) is replaced by a more simple condition (2.1.13) that can be immediately verified by means of Lemma 2.1 from Sect. 2.1.2.

At the second stage, it is not necessary to use the game approach in order to construct a control for the subsystems (see Sect. 2.2.2).

Let us consider in greater detail the modification of the described approach, in which a system with a nonlinear resistance is considered as a subsystem with one degree of freedom [28].

2.4.1 Subsystem with nonlinear damping

Let the dynamics of a system with one degree of freedom be described by the equation

$$m\ddot{q} = R(\dot{q}) + U + V(q,\dot{q},t). \tag{2.4.1}$$

Here, q is the generalized coordinate of the system, $m > 0$ is a constant inertial coefficient (the mass), $R(\dot{q})$ is the resistance, U is the control, and $V(q,\dot{q},t)$ is the disturbance.

We will assume that the resistance $R(\dot{q})$ is directed opposite to the velocity, and its magnitude increases with the velocity; it is zero in the state of rest. Also, $R(\dot{q})$ is a smooth function. Hence, we have

$$\dot{q}R(\dot{q}) < 0, \quad \frac{dR(\dot{q})}{d\dot{q}} < 0 \quad (\dot{q} \neq 0), \quad R(0) = 0. \tag{2.4.2}$$

The control and the disturbance are assumed to be bounded by geometric constraints, and the maximum disturbance is strictly less than the maximum control. We have

$$|U| \leq U_0, \quad |V(q,\dot{q},t)| \leq \rho U_0, \quad \rho < 1, \tag{2.4.3}$$

where $U_0 > 0$ and $\rho < 1$ are constants. In all other respects, the disturbance $V(q,\dot{q},t)$ may be an arbitrary function of its arguments.

It is required to construct a feedback control $U(q,\dot{q})$ that takes system (2.4.1) from an arbitrary initial state

$$q(t_0) = q^0, \quad \dot{q}(t_0) = \dot{q}^0 \tag{2.4.4}$$

to the prescribed terminal state with zero velocity

$$q(t_*) = q^*, \quad \dot{q}(t_*) = 0 \tag{2.4.5}$$

in finite time. Here, t_0, q^0, $\dot{q}^0$, and q^* are some given values, time t_* is not fixed.

Let $l > 0$ be a quantity of the same dimension as coordinate q. We introduce the dimensionless variables

$$\begin{gathered} x = \frac{q - q^*}{l}, \quad t' = \frac{t - t_0}{\tau_0}, \quad u = \frac{U}{U_0}, \quad f = -\frac{R}{U_0}, \\ v = \frac{V}{U_0}, \quad \tau_0 = \left(\frac{ml}{U_0}\right)^{1/2}. \end{gathered} \tag{2.4.6}$$

Making the change of variables (2.4.6) in (2.4.1), we obtain

$$\ddot{x} + f(\dot{x}) = u + v(x, \dot{x}, t). \tag{2.4.7}$$

Here and in what follows, dots denote derivatives with respect to the dimensionless time t'; in (2.4.7) and below t' is replaced by t. According to (2.4.2) and (2.4.6), the smooth function $f(z)$ has the following properties:

$$zf(z) > 0, \quad f'(z) > 0 \quad (z \neq 0), \quad f(0) = 0. \tag{2.4.8}$$

The variables u and v in (2.4.7) are constrained by [see (2.4.3) and (2.4.6)]

$$|u| \leq 1, \quad |v| \leq \rho, \quad \rho < 1. \tag{2.4.9}$$

After the change of variables (2.4.6), the initial conditions (2.4.4) and the final conditions (2.4.5) take the form

$$x(0) = \xi, \quad \dot{x}(0) = \eta, \tag{2.4.10}$$

$$x(\tau) = 0, \quad \dot{x}(\tau) = 0. \tag{2.4.11}$$

Here,

$$\xi = \frac{q^0 - q^*}{l}, \quad \eta = \frac{\dot{q}^0 \tau_0}{l}, \quad \tau = \frac{t_* - t_0}{\tau_0}.$$

Our control problem can now be stated in the following form.

Problem 2.3. Construct a feedback control $u(x, \dot{x})$ that satisfies constraint (2.4.9) and takes system (2.4.7) with an arbitrary disturbance v constrained by (2.4.9) from any initial state (2.4.10) to the prescribed terminal state (2.4.11) in finite time.

The formulation of the problem and the approach applied below for its solution are analogous to those described in Sects. 2.1.3–2.2.3 and are their generalization.

2.4.2 Control for the nonlinear subsystem

The game-theoretical approach

Let us consider (2.4.7) from the point of view of the differential games theory, assuming that u and v are the controls of two opponents constrained by (2.4.9). We will seek a feedback control $u(x,\dot{x})$ that takes system (2.4.7) from state (2.4.10) to state (2.4.11) in the shortest guaranteed time τ for any admissible disturbance v. Control $u(x,\dot{x})$ obtained by solving the differential game produces, as is easily seen, a solution of Problem 2.3. On the other hand, the solution of the differential game reduces [79, 80] to the solution of the time-optimal control problem for the system

$$\ddot{x}+f(\dot{x})=(1-\rho)u; \quad |u|\le 1,\ 0\le\rho<1,\ \tau\to\min \tag{2.4.12}$$

with the boundary conditions (2.4.10) and (2.4.11). Equation (2.4.12) is obtained from (2.4.7) for $v=-\rho u$ that corresponds to the worst (for u) opponent control: the optimal controls of the players are such that $u=\pm 1$, $v=\mp\rho$ at any instant.

Control $u(x,\dot{x})$ required in Problem 2.3 and the corresponding time τ can be found by obtaining the time-optimal control for (2.4.12) with boundary conditions (2.4.10) and (2.4.11). The corresponding time-optimal control problem is written in the form

$$\dot{x}_1=x_2, \quad \dot{x}_2=-f(x_2)+(1-\rho)u; \quad |u|\le 1, \quad 0\le\rho<1,$$

$$x_1(0)=\xi, \quad x_2(0)=\eta, \quad x_1(\tau)=x_2(\tau)=0, \quad \tau\to\min, \tag{2.4.13}$$

$$(x_1=x, \quad x_2=\dot{x}).$$

Time-optimal control

We will solve problem (2.4.13) by the maximum principle. The Hamiltonian for problem (2.4.13) has the form

$$H=p_1x_2+p_2[(1-\rho)u-f(x_2)], \quad |u|\le 1, \tag{2.4.14}$$

where p_1 and p_2 are conjugate variables.

The conjugate system has the form

$$\dot{p}_1=0, \quad \dot{p}_2=-p_1+f'(x_2)p_2. \tag{2.4.15}$$

Since (2.4.13) is an autonomous system, we have the first integral for our time-optimal control problem

$$H=p_1x_2+p_2[(1-\rho)u-f(x_2)]=h\ge 0, \tag{2.4.16}$$

where h is a constant.

By the maximum principle, we obtain from (2.4.14)

$$u = \operatorname{sign} p_2. \tag{2.4.17}$$

Let us consider the possibility of the existence of singular sections of the optimal trajectory on which $p_2 = 0$. On such a singular section, by the second equation in (2.4.15), we have also $p_1 = 0$. Therefore, if a singular section exists, we have $p_1 \equiv \text{const} = 0$ on the entire trajectory. But then the second equation in (2.4.15) is homogeneous with respect to p_2 on the entire trajectory, and since $p_2 = 0$ on the singular section, we have $p_2 \equiv 0$ on the entire trajectory. However, by the maximum principle, the conjugate vector does not vanish identically on the optimal trajectory. The contradiction proves that the optimal trajectory is free from singular sections. Thus, the equality $p_2 = 0$ may be observed only at isolated instants of time (switching points) and, by (2.4.17), we have $u = \pm 1$ almost everywhere.

Let us first consider the sections of the optimal trajectory where $p_2 > 0$ and $u = 1$. From (2.4.13), we obtain for these sections

$$\frac{dx_1}{dx_2} = x_2[(1-\rho)u - f(x_2)]^{-1}. \tag{2.4.18}$$

It follows from (2.4.18) that, in the (x_1, x_2)-plane, the sections of the optimal trajectory with $p_2 > 0$ are arcs of the curves

$$x_1 = \phi_\rho^+(x_2) + c^+, \tag{2.4.19}$$

where c^+ is an arbitrary constant and the function $\phi_\rho^+(x_2)$ is defined by the equality

$$\phi_\rho^+(y) = \int_0^y \frac{z dz}{1-\rho - f(z)}, \quad 0 \le \rho < 1. \tag{2.4.20}$$

Let us note some properties of the function $\phi_\rho^+(y)$ that follow from (2.4.20) and (2.4.8) and are needed in the sequel. As y varies from $-\infty$ to 0, the function ϕ_ρ^+ is positive and strictly decreasing, vanishing for $y = 0$. The point $y = 0$ is the unique extremum of the function $\phi_\rho^+(y)$ (its minimum). If the transcendental equation for z^+

$$f(z^+) = 1 - \rho \tag{2.4.21}$$

is unsolvable, i.e., if $f(z) < 1 - \rho$ for all z, then the function $\phi_\rho^+(y)$ is strictly increasing for all $y \ge 0$. In this case $\phi_\rho^+(y) > 0$ for all $y \ne 0$.

If, however, z^+ is a root of (2.4.21), then this root is positive and unique by conditions (2.4.8). In this case, the function $\phi_\rho^+(y)$ is strictly increasing from 0 to ∞ in the interval $y \in (0, z^+)$ and strictly decreasing for $y > z^+$. A typical curve of the dependence (2.4.19) in the (x_1, x_2)-plane for $c^+ = 0$ is shown in Fig. 2.12 for the case where (2.4.21) has a root $z^+ > 0$. The direction in which time t increases along the trajectory according to the first equation in (2.4.13) is shown by arrows in Fig. 2.12.

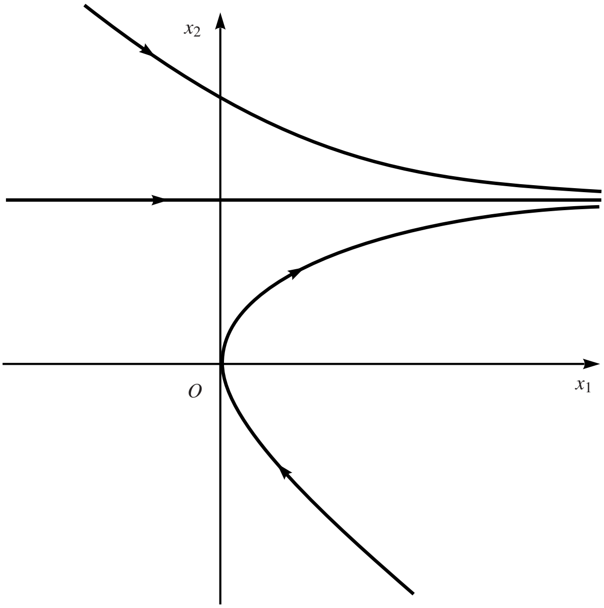

Fig. 2.12 Phase trajectory for $c^+ = 0$ and $z^+ > 0$

We similarly consider the sections of the trajectories with $p_2 < 0$. These sections are arcs of the curves

$$x_1 = \phi_\rho^-(x_2) + c^-. \tag{2.4.22}$$

Here, as in (2.4.19), c^- is an arbitrary constant and the function ϕ_ρ^- is defined by an equality similar to (2.4.20):

$$\phi_\rho^-(y) = \int_0^y \frac{zdz}{-(1-\rho) - f(z)}, \quad 0 \le \rho < 1. \tag{2.4.23}$$

We introduce a transcendental equation for z^- similar to (2.4.21):

$$f(z^-) = -(1-\rho). \tag{2.4.24}$$

If (2.4.24) does not have a solution z^-, i.e., if $f(z) > \rho - 1$ for all z, then the function $\phi_\rho^-(y)$ from (2.4.23) is strictly increasing for $y < 0$ and strictly decreasing for $y > 0$. Here, $\phi_\rho^-(y) < 0$ for all $y \neq 0$.

If z^- is a root of (2.4.24), then it is unique and negative ($z^- < 0$) by conditions (2.4.8). In this case, the function $\phi_\rho^-(y)$ is strictly decreasing for $y \in (-\infty, z^-)$, strictly increasing for $y \in (z^-, 0)$, and again strictly decreasing for $y \in (0, \infty)$. As $y \to z^-$, this function tends to $-\infty$, and for $y = 0$ it has a local zero maximum. A typical graph of the function $\phi_\rho^-(y)$ can be obtained from the graph of the function $\phi_\rho^+(y)$ in Fig. 2.12 by a central symmetry transformation (or, equivalently, by simultaneously reversing the directions of both axes x_1 and x_2).

The curves described above are the trajectories corresponding to $p_2 > 0$ and $p_2 < 0$ that pass through the origin in the $(x_1,\, x_2)$-plane. Other curves whose arcs may be sections of the optimal trajectories are obtained from these curves by parallel translation by c^+ and c^- along the x_1 axis [see (2.4.19) and (2.4.22)].

Note that if the transcendental equations (2.4.21) and (2.4.24) have solutions, then system (2.4.13) has the corresponding solutions

$$x_2 = z^+ \ (p_2 > 0), \quad x_2 = z^- \ (p_2 < 0). \tag{2.4.25}$$

In the (x_1, x_2)-plane, solutions (2.4.25) correspond to the phase trajectories in the form of straight lines parallel to the x_1-axis. These lines are the asymptotes of curves (2.4.19) and (2.4.22), respectively (see Fig. 2.12).

Thus, the required optimal trajectories consist of sections of curves (2.4.19) and (2.4.22) with various c^+ and c^- and also, possibly, segments of the straight lines (2.4.25), if the corresponding equations (2.4.21) and (2.4.24) are solvable.

We will now show that each optimal trajectory has at most one control switching point, i.e., the function $p_2(t)$ vanishes at most once.

Suppose that this is not so, and the function $p_2(t)$ vanishes at two instants t' and t'', being positive between them. Then

$$p_2(t) > 0, \quad t \in (t', t''); \quad p_2(t') = p_2(t'') = 0. \tag{2.4.26}$$

From the first integral (2.4.16) for t' and t'', we obtain by (2.4.26)

$$p_1 x_2(t') = p_1 x_2(t'') = h \geq 0. \tag{2.4.27}$$

If $p_1 = \text{const} = 0$, then, from (2.4.15), we obtain for $p_2(t)$ a linear homogeneous equation, which with zero conditions (2.4.26) at t' and t'' has an identically zero solution $p_2(t) \equiv 0$. But this contradicts to the maximum principle that asserts the existence of a nonzero conjugate vector. Therefore, $p_1 = \text{const} \neq 0$, and, from (2.4.27), we obtain $x_2(t') = x_2(t'')$. However, on all phase trajectories except the straight lines (2.4.25) the variable x_2 is either strictly increasing or strictly decreasing as time t increases. This follows from the previous analysis of the phase trajectories and is clear from Fig. 2.12. The equality $x_2(t') = x_2(t'')$ is therefore possible only if the relevant section of the trajectory is a segment of the straight line (2.4.25), i.e.,

$$x_2(t) \equiv z^+, \quad t \in (t', t''). \tag{2.4.28}$$

Substituting (2.4.28) into the second conjugate equation (2.4.15), we obtain a linear equation with constant coefficients

$$\dot{p}_2(t) = -p_1 + kp_2, \quad k = f'(z^+) > 0,$$

where $k > 0$ by (2.4.8). The general solution of this equation has the form

$$p_2(t) = \frac{p_1}{k} + Ce^{kt}, \tag{2.4.29}$$

where C is an arbitrary constant. But solution (2.4.29) is monotone in t and cannot satisfy conditions (2.4.26) for any $p_1 \neq 0$ and C. Thus, the section of the optimal trajectory, where conditions (2.4.26) hold, cannot be a straight segment of the line (2.4.28). We have thus shown that an optimal trajectory may not include sections of the form (2.4.26).

We can similarly prove that an optimal trajectory may not include sections such that the function $p_2(t)$ is negative inside the section and vanishes at its endpoints.

Therefore, on each optimal trajectory the function $p_2(t)$ vanishes at most once, i.e., the control may have at most one switching point.

The only phase trajectories that reach the origin as the time increases are the branch of curve (2.4.19) with $c^+ = 0$ which lies in the quadrant $x_1 \geq 0$, $x_2 \leq 0$ (Fig. 2.12) and the branch of curve (2.4.22) with $c^- = 0$ which lies in the quadrant $x_1 \leq 0$, $x_2 \geq 0$. These curve branches correspond to the controls $u = 1$ and $u = -1$, respectively. The collection of these branches form the switching curve, whose equation is written as

$$x_1 = \psi_\rho(x_2). \tag{2.4.30}$$

Here, we have introduced the notation

$$\psi_\rho(y) = \phi_\rho^+(y),\ y \leq 0; \quad \psi_\rho(y) = \phi_\rho^-(y),\ y \geq 0. \tag{2.4.31}$$

By the properties of functions (2.4.20) and (2.4.23), the function $\psi_\rho(y)$ defined by (2.4.31) is strictly decreasing for all y and vanishes for $y = 0$, where it has a point of inflection.

We can now easily describe the entire field of optimal trajectories. An optimal trajectory originating from any point of the phase plane (x_1, x_2) consists of a segment of one of the families (2.4.19) or (2.4.22) and a section of switching curve (2.4.30).

The field of optimal trajectories is qualitatively shown in Fig. 2.13 for the case where (2.4.21) and (2.4.24) have roots. The thick curve in Fig. 2.13 is the switching curve (2.4.30), and the arrows indicate the direction of increase of the time t. Note that this picture of the field of optimal trajectories is similar to the picture observed for the linear resistance, see Sect. 2.2.1, Fig. 2.2.

The optimal control corresponding to this field of phase trajectories may be represented in the form

$$\begin{gathered} u_\rho(x_1, x_2) = \operatorname{sign}[\psi_\rho(x_2) - x_1] \quad \text{for} \quad x_1 \neq \psi_\rho(x_2), \\ u_\rho(x_1, x_2) = \operatorname{sign} x_1 = -\operatorname{sign} x_2 \quad \text{for} \quad x_1 = \psi_\rho(x_2), \\ (x_1 = x, \quad x_2 = \dot{x}), \end{gathered} \tag{2.4.32}$$

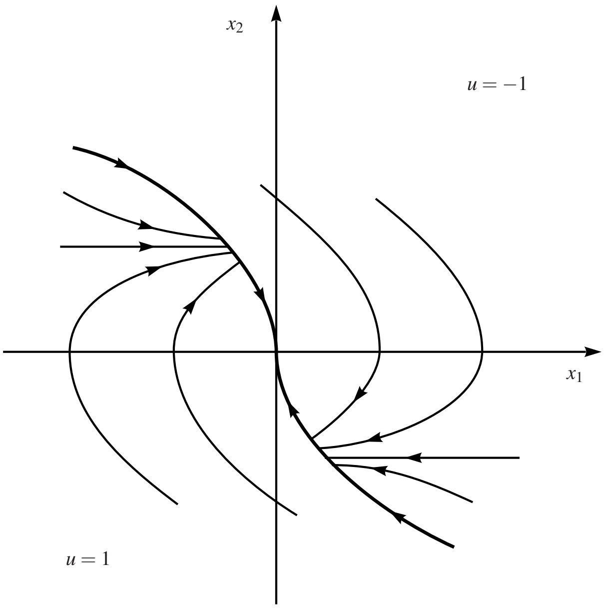

Fig. 2.13 Optimal phase trajectories

where the function ψ_ρ is defined by relationships (2.4.31), (2.4.20), and (2.4.23).

The control law (2.4.32) solves Problem 2.3. This solution may be called suboptimal, because it is time-optimal (unimprovable) when v is the "worst-case" disturbance, as assumed in the game-theoretical approach. With the worst-case disturbance $v = -\rho u$, the system moves along optimal trajectories, see Fig. 2.13. If the disturbance deviates from the worst case ($v \neq -\rho u$), which is usually so, the trajectories deviate from the optimal trajectories. The motion along the switching curve occurs in the sliding mode, and the time taken to reach the origin only diminishes.

2.4.3 Simplified control for the subsystem and comparative analysis

So far, we have assumed that the disturbance is unknown, but its maximum attainable value is known and essentially affects the feedback control. In dimensionless variables, the disturbance bound has the form $|v| \leq \rho$, see (2.4.9), and the feedback control (2.4.32) depends on the parameter ρ.

We can use a different approach to the control synthesis in the presence of disturbances, which simply ignores the disturbances (see Sect. 2.2.2).

In our case, this simplified approach means that the parameter ρ is set equal to zero during the control synthesis, and the disturbances are ignored. The control $u_0(x_1, x_2)$ obtained in this way is defined by relationships (2.4.32), (2.4.31), (2.4.20), and (2.4.23) with $\rho = 0$. The switching curve for the simplified control is given by (2.4.30) with $\rho = 0$. It is represented in Fig. 2.14 by the thick curve.

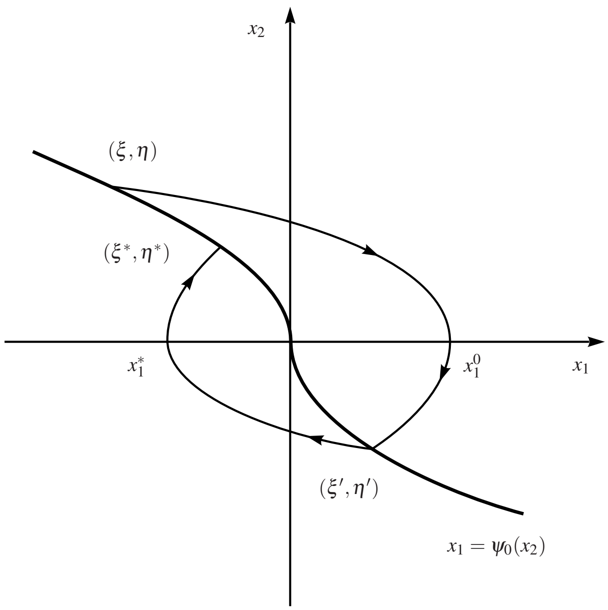

Fig. 2.14 Switching curve with $\rho = 0$ and a trajectory for simplified control

Let us compare the two control synthesis techniques — the game-theoretical and the simplified method. To this end, we will examine the dynamics of system (2.4.1) for some $\rho \in (0,1)$ under the action of the simplified control $u_0(x_1, x_2)$. We will represent this system in the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -f(x_2) + u_0(x_1, x_2) + v, \tag{2.4.33}$$

$$|v| \leq \rho < 1, \quad (x_1 = x,\ x_2 = \dot{x}).$$

For system (2.4.33), we consider the following auxiliary problem of finding the worst-case disturbance (see analogous Problem 2.2 in Sect. 2.2.2).

Problem 2.4. Find the optimal control $v(x_1, x_2)$ for system (2.4.33) that satisfies the constraint $|v| \leq \rho$ and such that the first intersection of any phase trajectory of this system with the switching curve $x_1 = \psi_0(x_2)$ lies as far as possible from the origin, i.e., at the maximum possible $|x_1|$ or, equivalently, the maximum possible $|x_2|$.

First assume that the starting point is in the region $x_1 \geq \psi_0(x_2)$. Then, by (2.4.32), we have $u_0 = -1$ for the given trajectory. The phase trajectory of system (2.4.33) first crosses that branch of the curve $x_1 = \psi_0(x_2)$, where $x_1 > 0, x_2 < 0$ [see Fig. 2.14, where it is assumed that the initial point (ξ, η) lies on the curve $x_1 = \psi_0(x_2)$]. Problem 2.4 is described by the relationships

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -f(x_2) - 1 + v, \quad |v| \le \rho < 1,$$
$$x_1(0) = \xi, \quad x_2(0) = \eta, \quad \xi \ge \psi_0(\eta), \tag{2.4.34}$$
$$x_1(\tau) = \phi_0(x_2(\tau)), \quad x_1(\tau) > 0, \quad x_2(\tau) < 0, \quad x_1(\tau) \to \max.$$

The instant τ when the process terminates is not fixed. Maximization of $x_1(\tau)$ is equivalent by (2.4.34) to the minimization of the integral functional

$$\int_0^\tau (-x_2)dt \to \min. \tag{2.4.35}$$

Applying the maximum principle to our problem defined by (2.4.34) and (2.4.35), we form the Hamiltonian

$$H = p_1 x_2 + p_2[v - 1 - f(x_2)] + x_2, \tag{2.4.36}$$

where p_1 and p_2 are the conjugate variables. They satisfy the conjugate system

$$\dot{p}_1 = 0, \quad \dot{p}_2 = f'(x_2)p_2 - p_1 - 1 \tag{2.4.37}$$

and the transversality conditions corresponding to the boundary conditions (2.4.34):

$$p_1\phi_0'(x_2) + p_2 = 0, \quad H = 0, \quad t = \tau. \tag{2.4.38}$$

From the first condition in (2.4.38), applying relationships (2.4.31) and (2.4.20) for $\rho = 0$ and noting that $x_2(\tau) < 0$ by (2.4.34), we obtain

$$p_1 = -p_2 \frac{1 - f(x_2)}{x_2}, \quad t = \tau. \tag{2.4.39}$$

Substituting (2.4.39) into (2.4.36) and using the second transversality condition in (2.4.38), we obtain after simplifications

$$H = p_2(v - 2) + x_2 = 0, \quad t = \tau.$$

Since $x_2(\tau) < 0$ and $|v| \le \rho < 1$, we obtain from this equality

$$p_2(\tau) < 0. \tag{2.4.40}$$

We find the optimal control by maximizing H from (2.4.36) over $|v| \le \rho$:

$$v = \rho \operatorname{sign} p_2. \tag{2.4.41}$$

Singular sections of the trajectory are ruled out. Indeed, if $p_2 \equiv 0$ in some time interval, then in this interval $p_1 = -1$ by the second equation (2.4.37). But $p_1 \equiv$ const, and therefore $p_1 = -1$ on the entire trajectory. Then the second equation in (2.4.37) becomes linear and homogeneous for p_2, and its solution with initial condition (2.4.40) does not vanish.

Thus, there are no singular sections, and (2.4.41) implies that control $v(t)$ has switching points when $p_2(t) = 0$.

Let us find the switching curve in the (x_1, x_2)-plane. Since system (2.4.34) is autonomous, its Hamiltonian (2.4.36) preserves a constant value along the optimal trajectory, and by (2.4.38) this constant value is zero:

$$H = (p_1 + 1)x_2 + p_2[v - 1 - f(x_2)] \equiv 0.$$

Hence, it follows that at the switching point, i.e., for $p_2 = 0$, we have either $p_1 = -1$ or $x_2 = 0$. But the equality $p_1 = -1$, as we have shown, implies that p_2 never vanishes. We thus have $x_2 = 0$ at the switching point, and the switching curve in this case is the ray $x_2 = 0$, $x_1 > 0$.

In order to determine the sign of the control for $x_2 < 0$ and $x_2 > 0$, it suffices to determine its sign at a single point. At the terminal time τ we have $x_2(\tau) < 0$ by (2.4.34) and $p_2(\tau) < 0$ by (2.4.40). Thus, $v = -\rho$ for $x_2 < 0$.

As a result,

$$v(x_1, x_2) = \rho \operatorname{sign} x_2. \tag{2.4.42}$$

We have obtained the optimal feedback control in the region $x_1 > \psi_0(x_2)$. To obtain the control in the region $x_1 < \psi_0(x_2)$, we note some symmetry properties. When $f(z)$ is replaced by $g(z) = -f(-z)$, we have by (2.4.20) and (2.4.23)

$$\phi_\rho^+(y) \to -\phi_\rho^-(-y), \quad \phi_\rho^-(y) \to -\phi_\rho^+(-y). \tag{2.4.43}$$

From (2.4.31) and (2.4.43), it follows that after this change

$$\psi_\rho(y) \to -\psi_\rho(-y). \tag{2.4.44}$$

Let us now make in (2.4.33) the change of variables

$$x_1 \to -x_1, \quad x_2 \to -x_2, \quad v \to -v, \quad f(z) \to -f(-z). \tag{2.4.45}$$

By (2.4.44) and (2.4.32), u_0 is changed to $-u_0$, and system (2.4.33) remains invariant. Hence, it follows that in the region $x_1 < \psi_0(x_2)$ the field of optimal trajectories and the optimal control are the same as in the region $x_1 > \psi_0(x_2)$, but with $f(z)$ replaced by $g(z) = -f(-z)$. Since synthesis (2.4.42) is independent of the specific form of function $f(z)$, it also applies in the region $x_1 < \psi_0(x_2)$. Thus, (2.4.42) defines the solution of Problem 2.2 in the entire (x_1, x_2)-plane.

Analysis of the phase trajectories

Consider the motion of system (2.4.33) under the action of the simplified control $u_0(x_1, x_2)$ defined by relationships (2.4.32), (2.4.31), (2.4.20), and (2.4.23) for $\rho = 0$ and the worst-case disturbance v from (2.4.42). Assume that the initial point ξ, η

lies on the branch of the switching curve $x_1 = \psi_0(x_2)$, where $x_1 < 0$ and $x_2 > 0$ (see Fig. 2.14). Let us investigate the phase trajectory until its next intersection with the same branch of the switching curve. This piece of the trajectory consists of four sections, each with constant u_0 and v. These sections have the following endpoints and controls (see Fig. 2.14):

$$\begin{array}{ll} 1)\ \ (\xi,\eta) \to (x_1^0,0), & u_0 = -1,\ v = \rho; \\ 2)\ (x_1^0,0) \to (\xi',\eta'), & u_0 = -1,\ v = -\rho; \\ 3)\ (\xi',\eta') \to (x_1^*,0), & u_0 = 1, \quad v = -\rho; \\ 4)\ (x_1^*,0) \to (\xi^*,\eta^*), & u_0 = 1, \quad v = \rho. \end{array} \tag{2.4.46}$$

The parameters of endpoints (2.4.46) satisfy relationships that reflect their position on the switching curve and on the coordinate axes (see Fig. 2.14):

$$\begin{gathered} \xi = \psi_0(\eta), \quad \eta > 0, \quad \xi < 0; \quad x_1^0 > 0; \\ \xi' = \psi_0(\eta'), \quad \eta' < 0, \quad \xi' > 0; \quad x_1^* < 0; \\ \xi^* = \psi_0(\eta^*), \quad \eta^* > 0, \quad \xi^* < 0. \end{gathered} \tag{2.4.47}$$

Substituting u_0 and v from (2.4.46) into (2.4.33) and integrating along the corresponding sections of the trajectory, we have

$$\xi' - \xi = \int_\eta^0 \frac{zdz}{-1+\rho-f(z)} + \int_0^{\eta'} \frac{zdz}{-1-\rho-f(z)},$$

$$\xi^* - \xi' = \int_{\eta'}^0 \frac{zdz}{1-\rho-f(z)} + \int_0^{\eta^*} \frac{zdz}{1+\rho-f(z)}.$$

Replacing ξ, ξ', and ξ^* by their expressions from (2.4.47) and using formulas (2.4.31), (2.4.20), and (2.4.23) for $\rho = 0$, we obtain

$$\begin{aligned} \int_0^{\eta'} \frac{zdz}{1-f(z)} - \int_0^{\eta} \frac{zdz}{-1-f(z)} &= \int_0^{\eta} \frac{zdz}{1-\rho+f(z)} - \int_0^{\eta'} \frac{zdz}{1+\rho+f(z)}, \\ \int_0^{\eta^*} \frac{zdz}{-1-f(z)} - \int_0^{\eta'} \frac{zdz}{1-f(z)} &= \int_0^{\eta'} \frac{zdz}{-1+\rho+f(z)} + \int_0^{\eta^*} \frac{zdz}{1+\rho-f(z)}. \end{aligned} \tag{2.4.48}$$

Recall that $\eta' < 0$, $\eta > 0$, and $\eta^* > 0$ by (2.4.47). Set $\eta' = -\eta^0$, $\eta^0 > 0$ and transform relationships (2.4.48) so that they contain integrals only over intervals lying on the positive half-axis. Simplifying, we obtain

$$\Phi_4(\eta^0) = \kappa^2(\rho)\Phi_1(\eta), \quad \Phi_2(\eta^*) = \kappa^2(\rho)\Phi_3(\eta^0). \tag{2.4.49}$$

Here,

$$\Phi_1(y) = \Phi^+(y; f), \quad \Phi_2(y) = \Phi^-(y; f),$$
$$\Phi_3(y) = \Phi^+(y; g), \quad \Phi_4(y) = \Phi^-(y; g),$$
$$\Phi^\pm(y; h) = \int_0^y \frac{z\,dz}{(1+h)[1 \pm (1 \mp \rho)^{-1} h]}, \tag{2.4.50}$$
$$f = f(z) \geq 0, \quad g = -f(-z) \geq 0,$$
$$\kappa(\rho) = \left[\frac{\rho(1+\rho)}{(1-\rho)(2+\rho)}\right]^{1/2}.$$

Consider the transcendental equations (2.4.49) that determine η^0 and η^* for given $\eta > 0$ and $\rho \in (0,1)$. To this end, we will note some properties of functions Φ_i, $i = 1,2,3,4$, from (2.4.50). Recall that by (2.4.8) $f(z) > 0$ for $z > 0$ and $f(z) \to 0$ as $z \to 0$.

The denominators in the integrands for functions Φ_1 and Φ_3 in (2.4.50) are positive for all $z \geq 0$. Therefore, functions Φ_1 and Φ_3 are defined and bounded for all $y \geq 0$.

If the equations

$$f(z_2) = 1+\rho, \quad g(z_4) = -f(-z_4) = 1+\rho \tag{2.4.51}$$

have solutions for z_2 and z_4, then the denominators of the integrands of the corresponding functions Φ_2 and Φ_4 in (2.4.50) vanish for finite z_2 and z_4 equal to the roots of equations (2.4.51). In this case, Φ_2 and Φ_4 are monotone increasing and go to infinity at $y = z_2$ and $y = z_4$, respectively. If (2.4.51) have no solutions, then the functions Φ_2 and Φ_4 are defined for all $y > 0$. In both cases, the denominators of the integrands for the functions Φ_2 and Φ_4 have maxima over $f \geq 0$ and $g \geq 0$, which are both equal to $(2+\rho)^2(1+\rho)^{-1}/4$. We thus have the inequalities

$$\Phi_2(y) \geq \frac{1}{2}\vartheta y^2, \quad \Phi_4 \geq \frac{1}{2}\vartheta y^2, \quad \vartheta = 4(1+\rho)(2+\rho)^{-2}.$$

The functions Φ_2 and Φ_4 are thus always positive and strictly increasing, taking all values from 0 to ∞ for $y \geq 0$.

Hence, it follows that the transcendental equations (2.4.49) for any $\eta > 0$ and $\rho \in (0,1)$ have unique positive solutions $\eta^0 > 0$ and $\eta^* > 0$. These solutions are continuous and monotone functions of η.

Let us differentiate equalities (2.4.49) with respect to η. After simple reductions we obtain

$$\frac{d\eta^*}{d\eta} = \frac{\kappa^2(\rho)\Phi_3'(\eta^0)}{\Phi_2'(\eta^*)}\frac{d\eta^0}{d\eta} = \frac{\kappa^4(\rho)\Phi_3'(\eta^0)\Phi_1'(\eta)}{\Phi_2'(\eta^*)\Phi_4'(\eta^0)}. \tag{2.4.52}$$

From relationships (2.4.50) and properties (2.4.8) of the function $f(z)$, we obtain the inequalities

$$\frac{\Phi_1'(y)}{\Phi_2'(y)} < 1, \quad \frac{\Phi_3'(y)}{\Phi_4'(y)} < 1, \quad y > 0.$$

Using the second inequality, we obtain from (2.4.52)

$$\frac{d\eta^*}{d\eta} < \kappa^4(\rho)\frac{\Phi_1'(\eta)}{\Phi_2'(\eta^*)}, \quad \eta > 0. \tag{2.4.53}$$

We can verify that the function $\kappa^2(\rho)$ from (2.4.50) is strictly increasing from 0 to ∞ on $\rho \in [0,1]$, and $\kappa = 1$ for ρ equal to (see Sect. 2.2.3)

$$\rho^* = \frac{1}{2}(\sqrt{5}-1) \approx 0.618. \tag{2.4.54}$$

First assume that $\rho < \rho^*$ and therefore $\kappa^2(\rho) < \alpha$, where $\alpha < 1$ is a positive number. Then, from (2.4.53), we have

$$\frac{d\eta^*}{d\eta} < \alpha^2\frac{\Phi_1'(\eta)}{\Phi_2'(\eta^*)}, \quad \eta > 0, \tag{2.4.55}$$

and hence

$$\Phi_2(\eta^*) < \alpha^2\Phi_1(\eta), \quad \eta > 0. \tag{2.4.56}$$

We will show that $\eta^* < \eta$. Assume that this is not so, specifically $\eta^* \geq \eta$. From (2.4.50), we obtain $\Phi_2(y) > \Phi_1(y)$ for all $y > 0$. Then, by the monotonicity of the function $\Phi_2(y)$, we obtain the chain of inequalities

$$\Phi_2(\eta^*) \geq \Phi_2(\eta) > \Phi_1(\eta),$$

which leads to the contradiction with inequality (2.4.56). Thus, $\eta^* < \eta$.

Let us transform inequality (2.4.55), substituting the expressions for the derivatives Φ_1' and Φ_2' from (2.4.50) and using the positivity of the function $f(z)$:

$$\frac{d\eta^*}{d\eta} < \frac{\alpha^2\eta[1+f(\eta^*)][1-(1+\rho)^{-1}f(\eta^*)]}{\eta^*[1+f(\eta)][1+(1-\rho)^{-1}f(\eta)]} < \frac{\alpha^2\eta[1+f(\eta^*)]}{\eta^*[1+f(\eta)]}, \quad \eta > 0.$$

We can simplify the last inequality, noting that $f(\eta^*) < f(\eta)$ by the monotonicity of $f(z)$ and by the inequality $\eta^* < \eta$. We obtain

$$\frac{d\eta^*}{d\eta} < \frac{\alpha^2\eta}{\eta^*}, \quad \eta > 0.$$

Integrating this inequality with $\eta^* = 0$ when $\eta = 0$, we obtain $(\eta^*)^2 < \alpha^2\eta^2$ or $\eta^*/\eta < \alpha$.

Thus, if $\rho < \rho^*$, where ρ^* is defined in (2.4.54), then $\eta^*/\eta < \alpha$, i.e., the phase trajectory approaches the origin. The distance from the origin diminishes at a rate not slower than geometric progression. The system, therefore, reaches the prescribed state in finite time, although after infinitely many control switchings.

Suppose that the system has reached a small neighbourhood of the origin, so that η is sufficiently small. Here, η^0 and η^* are also small in view of their continuous dependence on η. Since $f(z) \to 0$ as $z \to 0$ by (2.4.8), terms $f(z)$ and $g(z)$ can be omitted in integrals (2.4.50) for small y, and we have

$$\Phi_i(y) \sim \frac{1}{2}y^2, \quad y \to 0, \quad i = 1,2,3,4.$$

The transcendental equation (2.4.49) for small η thus takes the form

$$(\eta^0)^2 = \kappa^2(\rho)\eta^2, \quad (\eta^*)^2 = \kappa^2(\rho)(\eta^0)^2.$$

Hence, we obtain

$$\frac{\eta^*}{\eta} = \kappa^2(\rho). \tag{2.4.57}$$

Let $\rho > \rho^*$ and, therefore, $\kappa^2(\rho) > 1$. Then, by (2.4.57), we obtain $\eta^* > \eta$, and the phase trajectory, even if it has reached a small neighbourhood of the origin, eventually moves away from the origin. The system does not come to the prescribed state.

Thus, with an arbitrary function $f(z)$ that satisfies condition (2.4.8), the simplified approach produces control $u_0(x_1,x_2)$ that is defined by relationships (2.4.32) for $\rho = 0$ and has the following properties:

- If $\rho < \rho^* \approx 0.618$, then, for any admissible disturbance $|v| \le \rho$, the system reaches the origin. The time to reach the origin is finite, although the number of switchings in general is infinite.
- If $\rho > \rho^*$, there exists an admissible disturbance v defined by (2.4.42) for which the system never reaches the origin.

Therefore, simplified control guarantees a solution of Problem 2.3 only for $\rho < \rho^*$, i.e., when the ratio of the maximum allowed disturbance to the maximum allowed control does not exceed the golden section.

Specifying the form of the function $f(z)$, we can construct a more detailed picture of the phase motion. Note that the results presented here and obtained first in [28] generalize the results of [27] and [29], where the cases of zero and linear resistance, respectively, have been previously considered in detail, see Sects. 2.2.1–2.2.3.

Conclusions

The proposed control law (2.4.32) based on the game-theoretical approach takes the given system (2.4.7) to the origin in finite time for any non-linearity $f(z)$ and any uncertain disturbance, if $\rho < 1$. This control law does not require a knowledge of the disturbance; we only need to know the maximum allowed disturbance, which must not exceed the maximum control.

Let us stress the difference in the requirements imposed on functions $f(z)$ and $v(x,\dot{x},t)$. Both these functions may be arbitrary in the framework of the corresponding conditions: (2.4.8) for $f(z)$ and (2.4.9) for v. However, the non-linear resistance function $f(z)$ should be known in order to construct the control, while the disturbance $v(x,\dot{x},t)$ is not needed.

The simplified approach to the control synthesis, which totally ignores the disturbances, is less effective. It a priori takes the system to the origin only for $\rho < \rho^* \approx 0.618$. If $\rho > \rho^*$, then there exists a disturbance for which the system never reaches the origin.

Yet both approaches have a similar structure and differ only by their switching curves.

The proposed control technique is robust with respect to various disturbances and parameter variations. These factors can be easily incorporated in the analysis, if we increase the assumed level of allowed disturbances, i.e., parameter ρ, creating a certain safety margin by this parameter.

Note that the obtained feedback control is suboptimal in the sense that it is time-optimal for the worst-case disturbance.

Our results can be applied to various dynamic systems, e.g., to control the electric motors of robotic systems. This opens up the possibility of taking into account various resistance laws that are often encountered in practice.

2.5 Applications and numerical examples

2.5.1 Application to robotics

Let us consider applications of the results obtained to problems of robot dynamics. For this, we will see that the formulation of Problem 2.1 as well as conditions (2.1.12) and (2.1.13) are typical and are often satisfied for robots.

Let us consider a manipulation robot that has n degrees of freedom and consists of n links connected by cylindrical or prismatic joints. Each link of the robot is an absolutely rigid body. The position of the ith link relative to that of the $(i-1)$st one is characterized by the relative angle of rotation (in the case of a cylindrical joint) or by a relative displacement (in the case of a prismatic joint). We take these angles and displacements as generalized coordinates $(q_1,\ldots,q_n)$ determining the position of the robot. The equations of motion of the robot can be represented in the form of Lagrange's equations (2.1.1), where the kinetic energy has the form (2.1.2). The role of the generalized forces is played by the torques about the axes of cylindrical joints and by the forces in the directions of displacements in the case of prismatic joints. Here, the forces U_i in (2.1.1) are the control torques or forces caused by the motors (drives), and Q_i include all the other external and internal forces and torques: gravity, resistance, friction, various perturbations, etc.

Let us now look at the dynamics of the robot together with its drives. We suppose that each control torque or force U_i is produced by a separate direct-current electric motor, $i = 1,\ldots,n$, and forces Q_i can be represented in the form (2.3.5)–

(2.3.8). The kinetic energy of the robot T is made up of the kinetic energy of its links $T^1(q,\dot{q})$ and the kinetic energy of the rotors of the electric motors $T^2(q,\dot{q},N)$. Here, $N=(N_1,\ldots,N_n)$ are the gear ratios of the reduction gears, which are treated as parameters. We shall assume that $N_i \geq 1$ and neglect the inertia of the moving parts of the reduction gears. According to König's theorem, the kinetic energy of the ith rotor is equal to the sum of two terms: the kinetic energy of a point mass equal to the mass of the rotor and located at its center, and the kinetic energy of rotation of the rotor, that is

$$T_i^2(q,\dot{q},N_i) = T_i^v(q,\dot{q}) + T_i^\omega(q,\dot{q},N_i).$$

Suppose that J_i and J_i' are the moments of inertia of the ith rotor about its axis of rotation and an axis passing through the centre of inertia perpendicular to the axis of rotation. Then, if the angular velocity vector of the stator of the ith electric motor has a projection on the axis of rotation of the rotor equal to ω_i and a perpendicular component equal to ω_i', we have

$$T_i^\omega(q,\dot{q},N_i) = \frac{1}{2}\left[J_i(N_i\dot{q}_i+\omega_i)^2 + J_i'\omega_i'^2\right].$$

The angular velocities ω_i and ω_i' are linear combinations of the generalized velocities $\dot{q}_1,\ldots,\dot{q}_n$ with coefficients depending on q. The kinetic energy of the robot can therefore be represented in the form

$$T = \frac{1}{2}\sum_{j=1}^{n} J_j(N_j\dot{q}_j)^2 + \frac{1}{2}N_{\max}\langle B\dot{q},\dot{q}\rangle, \tag{2.5.1}$$

where $B(q,N)$ is a bounded matrix such that the inequality

$$|B(q,N)z| \leq \lambda|z|, \qquad \lambda = \text{const}, \tag{2.5.2}$$

is satisfied in the case of an arbitrary vector z.

The largest and smallest of the gear ratios $N_1,\ldots,N_n$ are henceforth denoted by $N_{\max}$ and $N_{\min}$, and λ is independent of N_i.

We substitute (2.5.1) into Lagrange's equations in the form of (2.1.1) and obtain

$$N_i^2 J_i\ddot{q}_i + N_{\max}[B(q,N)\ddot{q}]_i = U_i + S_i(q,\dot{q},t,N). \tag{2.5.3}$$

We divide the ith equation of (2.5.3) by N_i and make the change of variables

$$p_i = N_i q_i. \tag{2.5.4}$$

As a result, we obtain

$$J_i\ddot{p}_i + N_{\max}N_i^{-1}\sum_{j=1}^{n} B_{ij}N_j^{-1}\ddot{p}_j = N_i^{-1}(U_i+S_i). \tag{2.5.5}$$

Allowing for the fact that $N_i^{-1}U_i = M_i$, where M_i is the electromagnetic torque produced by the electric motor, we reduce system (2.5.5) to the form

$$(J+\tilde{B})\ddot{p} = M + S^*. \tag{2.5.6}$$

Here,

$$J = \mathrm{diag}(J_1,\dots,J_n), \quad \tilde{B} = N_{\max}H^{-1}BH^{-1}, \quad M = (M_1,\dots,M_n),$$
$$S^* = H^{-1}S, \qquad H = \mathrm{diag}(N_1,\dots,N_n). \tag{2.5.7}$$

Consequently, when account is taken of the change of variables (2.5.4) and notation (2.5.7), the equations of motion can be represented in the form of (2.1.3) and (2.3.2), and, by (2.5.2) and (2.5.7), we have the inequality

$$|\tilde{B}z| \le \mu|z|, \quad \mu = N_{\max}N_{\min}^{-2}\lambda, \tag{2.5.8}$$

that is analogous to constraint (2.3.3). The initial and terminal conditions can be represented in the form (2.1.7) and (2.1.8).

We will now consider different ways of formulating control problems.

1°. Suppose that the constraints

$$|M_i| \le M_i^0 \tag{2.5.9}$$

are imposed on the control torques M_i produced by the electric motors. In this case, the results obtained in the preceding sections and summarized in Theorem 2.3 can be used to construct the control. Inequality (2.3.34), rewritten in the notation of system (2.5.6), defines the permissible values of parameter μ. On substituting its value from (2.5.8) into this inequality instead of μ, we obtain a constraint on the possible values of the gear ratios of the reduction gears

$$\frac{N_{\min}^2}{N_{\max}} > \frac{\lambda}{J_{\min}}\left(1 + \frac{|M^0 + H^{-1}G^0|}{\min_i(M_i^0 - N^{-1}G_i^0)}\right),$$
$$M^0 = (M_1^0,\dots,M_n^0), \qquad G^0 = (G_1^0,\dots,G_n^0). \tag{2.5.10}$$

Here, $J_{\min}$ is the least of the moments of inertia of the rotors $J_1,\dots,J_n$, constants G_i^0 are introduced in (2.3.5)–(2.3.7); besides, we assume that

$$G_i^0 < N_i M_i^0$$

for all $i = 1,\dots,n$.

2°. Suppose that the voltages applied to the windings of the electric motors play the role of controls. We augment the equations of motion (2.5.6) with the balance equations for the voltages in the rotor circuits and relations associating torques M_i with the currents [50]:

$$L_i \frac{dj_i}{dt} + R_i j_i + k_i^E \dot{p}_i = u_i, \qquad M_i = k_i^M j_i - b_i \dot{p}_i. \tag{2.5.11}$$

Here, L_i is the coefficient of inductance, R_i is the electrical resistance, k_i^E and k_i^M are constant coefficients, u_i is the voltage in the rotor circuit of the ith motor, term $b_i \dot{p}_i$ is the moment due to mechanical resistance, and b_i is a positive constant coefficient. The first term in the first equation (2.5.11) is usually small compared with the remaining terms. Therefore, the expression

$$M_i = k_i^M R_i^{-1}(u_i - k_i^E \dot{p}_i) - b_i \dot{p}_i$$

is obtained from (2.5.11); when this is substituted into (2.5.6), we obtain

$$(J + \tilde{B})\ddot{p} = U^* + S^{**},$$

$$S^{**} = S^* - \Lambda \dot{p}, \quad \Lambda = \mathrm{diag}(k_1^M k_1^E R_1^{-1} + b_1, \ldots, k_n^M k_n^E R_n^{-1} + b_n), \tag{2.5.12}$$

$$U^* = (k_1^M R_1^{-1} u_1, \ldots, k_n^M R_n^{-1} u_n).$$

Suppose that the constraints

$$|u_i| \leq u_i^0 \tag{2.5.13}$$

are imposed on the control voltages. Constraints (2.5.13) are transformed into constraints on the components of vector U^* from (2.5.12):

$$|U_i^*| \leq U_i^{*0} = k_i^M R_i^{-1} u_i^0. \tag{2.5.14}$$

The equations of motion (2.5.12) are again reduced to the form (2.1.3) and (2.3.2). Inequalities (2.5.14) are of the same form as relations (2.1.6). It is obvious that in this case we can use the method of control considered above in Sect. 2.3. By Theorem 2.3, we obtain a constraint which is analogous to (2.5.10):

$$\frac{N_{\min}^2}{N_{\max}} > \frac{\lambda}{J_{\min}} \left(1 + \frac{|U^{*0} + H^{-1} G^0|}{\min_i (U_i^{*0} - N_i^{-1} G_i^0)} \right), \quad U^{*0} = (U_1^{*0}, \ldots, U_n^{*0}). \tag{2.5.15}$$

Thus, if the gear ratios of the drives and the parameters of the robot are such that inequalities (2.5.10) and (2.5.15) are satisfied, then it is possible to construct a control which transfers the system under consideration from an initial state to a specified state in finite time. The control takes account of the existence of perturbations and structural constraints.

Remark 2.2. Considering system (2.5.3) and rewriting in its terms condition (2.3.34), one can obtain constraints imposed on parameters of the system in another form. We have

$$\frac{\min_i (N_i^2 J_i)}{N_{\max} \lambda} > 1 + \frac{|HM^0 + G^0|}{\min_i (N_i M_i^0 - G_i^0)}$$

for the case 1° of the bounded electromagnetic torques and

$$\frac{\min_i(N_i^2 J_i)}{N_{\max}\lambda} > 1 + \frac{|HU^{*0} + G^0|}{\min_i(N_i U_i^{*0} - G_i^0)}$$

for the case 2° of the bounded electric voltages. It seems that these sufficient conditions for the method of control decomposition are more efficient in the case, where moments of inertia of the rotors J_i, $i = 1,\dots,n$ are essentially different from each other, but the difference between the effective moments of inertia $N_i^2 J_i$ is not very large.

Remark 2.3. If the elements of the matrix Λ are sufficiently large, then, in order to shorten the motion time, it is advisable to reduce system (2.5.12) to the form of (2.1.28). In this case, one should set matrix A_* equal to matrix J, and coefficients λ_i should be made equal to the corresponding elements of matrix Λ. After that, the approach described in Sects. 2.1.3–2.2.3 can be applied to the obtained subsystems with linear resistance.

3°. Recently, direct drives without gears are often used in robots. For such drives, we have $N_i = 1$ and $J_i = 0$, so that we set $J = 0$ and $H = E$ in (2.5.6) and (2.5.7). The equations of motion and the constraints are again reduced to the forms (2.1.3) and (2.1.6). However, it is no longer possible to choose matrix A_* in the form (2.3.2) and (2.3.3), since $J = 0$. This matrix must be chosen differently, for example, in the form $A_* = A(q^*)$ [see Remark 2.1 at the end of Sect. 2.1.2]. To apply the results obtained, it is necessary to verify conditions (2.1.12) or (2.1.13), and this has to be done in each specific case. In order to demonstrate this proposition, in the next subsection, the problem of the feedback control design for the two-link manipulator with direct drives is considered.

Thus, the results obtained can, under certain conditions, be used for constructing control for manipulation robots.

2.5.2 Feedback control design and modelling of motions for two-link manipulator with direct drives

The system considered in this subsection is a simplified model of a mechanical manipulation robot with two absolutely rigid links. The system can perform motions in a horizontal plane and is controlled by two torques produced by drives installed at its joints. Geometric constraints are imposed on the control torques. Using the decomposition method described in Sects. 2.3 and 2.5, we will construct the feedback control that brings the system to the prescribed terminal position.

Problem statement

Consider a mechanical two-link system (see Fig. 2.15) that consists of a stationary base G_0 and two absolutely rigid links G_1 and G_2. The elements of the system are connected by two revolute joints O_1 and O_2 such that both links can move only in horizontal plane.

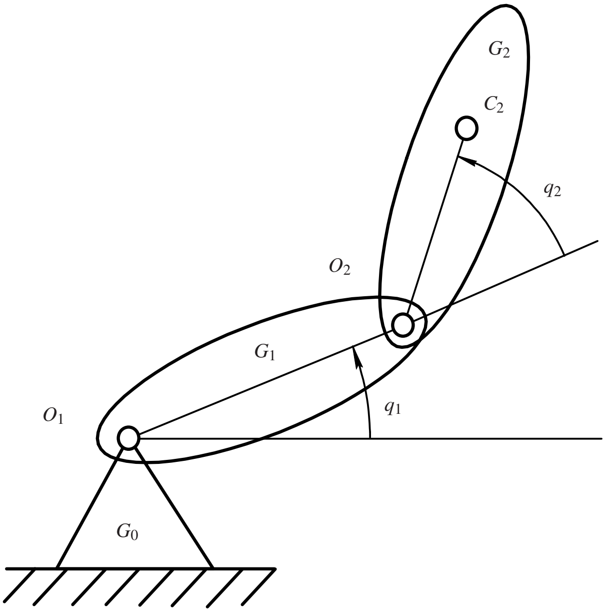

Fig. 2.15 Two-link manipulator

Lagrange's equations that describe the motion of this system are obtained are [41]:

$$
\begin{aligned}
&(m_2 l_1^2 + I_1 + I_2 + 2m_2 l_1 l_{g2} \cos q_2)\, \ddot{q}_1 + (I_2 + m_2 l_1 l_{g2} \cos q_2)\, \ddot{q}_2 \\
&-2m_2 l_1 l_{g2} \sin q_2\, \dot{q}_1 \dot{q}_2 - m_2 l_1 l_{g2} \sin q_2\, \dot{q}_2^2 = M_1 + Q_1, \qquad (2.5.16)\\
&(I_2 + m_2 l_1 l_{g2} \cos q_2)\, \ddot{q}_1 + I_2\, \ddot{q}_2 + m_2 l_1 l_{g2} \sin q_2\, \dot{q}_1^2 = M_2 + Q_2.
\end{aligned}
$$

Here, q_1 is the angle of rotation of link G_1 relative to base G_0, and q_2 is the angle between the straight lines O_1O_2 and O_2C_2; C_2 is a center of mass of link G_2. Angle q_2 defines the position of link G_2 relative to link G_1; l_1 is the length of segment O_1O_2; l_{g2} is the length of segment O_2C_2; m_2 is the mass of link G_2; I_i is the moment of inertia of the ith link relative to the axis of joint O_i; and M_i and Q_i are the control torque and moment of other forces applied at the joint O_i, respectively; here and below $i = 1, 2$.

The constraints

$$|M_i| \le M_i^0 \qquad (2.5.17)$$

are imposed on the control torques. Here, M_i^0 are given constants.

Let us turn to dimensionless variables

$$t' = \left[\frac{M_2^0}{m_2 l_1 l_{g2}}\right]^{1/2} t, \quad U_i = \frac{M_i}{M_2^0}, \quad U_i^0 = \frac{M_i^0}{M_2^0},$$
$$Q_i' = \frac{Q_i}{M_2^0}, \quad \alpha = \frac{I_1 + m_2 l_1^2}{m_2 l_1 l_{g2}}, \quad \beta = \frac{I_2}{m_2 l_1 l_{g2}}. \tag{2.5.18}$$

If we now drop the primes at t' and Q_i', then relations (2.5.16) take the form

$$(\alpha + \beta + 2\cos q_2)\,\ddot{q}_1 + (\beta + \cos q_2)\,\ddot{q}_2 - (2\dot{q}_1\dot{q}_2 + \dot{q}_2^2)\sin q_2 = U_1 + Q_1,$$
$$(\beta + \cos q_2)\,\ddot{q}_1 + \beta\,\ddot{q}_2 + \dot{q}_1^2 \sin q_2 = U_2 + Q_2, \tag{2.5.19}$$

and inequalities (2.5.17) coincide with (2.1.6). Note that the conditions $\alpha\beta > 1$ and $U_2^0 = 1$ are fulfilled due to notation (2.5.18).

Next, we consider Problem 2.1 (see Sect. 2.1.2) for system (2.5.19) with constraints (2.1.6) imposed on the new controls U_i. We suppose that the domain of possible motions is given by (2.3.1). In this subsection, we assume that all external forces and disturbances are missing, i.e.,

$$Q_1 = Q_2 = 0.$$

Simplifying assumptions and decomposition of the system

Let us resolve system (2.5.19) with respect to the second derivatives $\ddot{q}_1$ and $\ddot{q}_2$ and multiply the left-hand sides of the obtained relations by some positive coefficients J_1 and J_2. Then the system takes the form (2.3.9), where the functions V_i are equal to

$$V_1 = U_1\left(J_1 \frac{\beta}{\alpha\beta - \cos^2 q_2} - 1\right) - J_1 U_2 \frac{\beta + \cos q_2}{\alpha\beta - \cos^2 q_2}$$
$$+ J_1 \frac{\beta(\dot{q}_1 + \dot{q}_2)^2 \sin q_2 + \dot{q}_2^2 \sin q_2 \cos q_2}{\alpha\beta - \cos^2 q_2},$$
$$V_2 = U_2\left(J_2 \frac{\alpha + \beta + 2\cos q_2}{\alpha\beta - \cos^2 q_2} - 1\right) - J_2 U_1 \frac{\beta + \cos q_2}{\alpha\beta - \cos^2 q_2} \tag{2.5.20}$$
$$-J_2 \frac{(\beta + \cos q_2)(\dot{q}_1 + \dot{q}_2)^2 \sin q_2 + (\alpha + \cos q_2)\dot{q}_1^2 \sin q_2}{\alpha\beta - \cos^2 q_2}.$$

Suppose that the inequalities (2.3.12) hold. If we treat V_i as independent restricted disturbances, then the original nonlinear system is divided into two linear subsystems with one degree of freedom each.

Control for each of these subsystems can be given by relations (2.3.13) and (2.3.14). In what follows, we show that conditions (2.3.12) are really fulfilled under some restrictions on the parameters of the system and constants J_i.

Determination of the control parameters X_1 and X_2

Let us impose certain restrictions on the parameters of system (2.5.19) and show that there exist X_i entering control law (2.3.13) such that relations (2.3.12) are really fulfilled.

(a) Suppose the inequality

$$\beta < 1 \tag{2.5.21}$$

is true. For example, if link G_2 is a thin rod of length $l_2 < l_1$ with an arbitrary distribution of density $\rho(x)$, then

$$\beta = \frac{1}{m_2 l_1 l_{g2}} \int_0^{l_2} \rho(x) x^2 dx = \frac{1}{m_2 l_1 l_{g2}} \int_0^{l_2} x d\left(\int_0^x \rho(y) y dy\right)$$

$$= \frac{1}{m_2 l_1 l_{g2}} \left(m_2 l_2 l_{g2} - \int_0^{l_2} \int_0^x \rho(y) y dy dx \right) < 1.$$

(b) Let us require that the magnitudes of the angles q_2^- and q_2^+ in (2.3.1) are restricted:

$$-\arccos(-\beta) < q_2^-, \qquad q_2^+ < \arccos(-\beta). \tag{2.5.22}$$

From (2.5.22) and inequalities $q_2^- < q_2 < q_2^+$ that are satisfied under control (2.3.13) (see Sect. 2.3), it follows that

$$\cos q_2 > -\beta \tag{2.5.23}$$

during the whole control process.

(c) Suppose that the value U_1^0 that restricts control U_1 satisfies the inequalities

$$\frac{\beta+1}{\beta} < U_1^0 < \frac{\alpha+\beta+2}{\beta+1}. \tag{2.5.24}$$

Since

$$\frac{\alpha+\beta+2}{\beta+1} - \frac{\beta+1}{\beta} = \frac{\alpha\beta-1}{\beta(\beta+1)} > 0,$$

one can always ensure the fulfillment of relations (2.5.24) by imposing more rigid restrictions on one of torques M_i in (2.5.17). We point out that in view of (2.5.21) and (2.5.24), $U_1^0 > 2$.

(d) We choose constants J_i in system (2.3.9) so that the following inequalities hold:

$$J_1 \frac{\beta}{\alpha\beta-1} < 1, \qquad J_2 \frac{\alpha+\beta+2}{\alpha\beta-1} < 1. \tag{2.5.25}$$

Let us estimate magnitude of V_1 from (2.5.20) using assumptions (a)–(d). On the strength of inequalities (2.1.6), (2.5.23), and (2.5.25), we obtain

$$|V_1| \le U_1^0 \left| J_1 \frac{\beta}{\alpha\beta - \cos^2 q_2} - 1 \right| + J_1 U_2^0 \frac{|\beta + \cos q_2|}{\alpha\beta - \cos^2 q_2}$$

$$+ J_1 \frac{\beta(\dot q_1 + \dot q_2)^2 + \dot q_2^2}{\alpha\beta - 1} = U_1^0 + J_1 \frac{\beta + \cos q_2 - U_1^0 \beta}{\alpha\beta - \cos^2 q_2}$$

$$+ J_1 \frac{\beta(\dot q_1 + \dot q_2)^2 + \dot q_2^2}{\alpha\beta - 1}.$$

Now, using inequalities (2.3.24) and (2.5.24), we get

$$|V_1| \le U_1^0 + J_1 \frac{\beta + 1 - U_1^0 \beta}{\alpha\beta} + J_1 \frac{Y_2^2 + \beta\,(Y_1 + Y_2)^2}{\alpha\beta - 1}. \tag{2.5.26}$$

In a similar manner, we can obtain the following estimate on V_2

$$|V_2| \le 1 + J_2 \frac{\beta U_1^0 + (U_1^0 - 2)\cos q_2 - \alpha - \beta}{\alpha\beta - \cos^2 q_2}$$

$$+ J_2 \frac{(\beta + 1)(\dot q_1 + \dot q_2)^2 + (\alpha + 1)\dot q_1^2}{\alpha\beta - 1}.$$

By virtue of relations (2.3.24), (2.5.24), and $U_1^0 > 2$, we have

$$\begin{aligned} |V_2| &\le 1 + J_2 \frac{U_1^0(\beta + 1) - \alpha - \beta - 2}{\alpha\beta} \\ &+ J_2 \frac{(\alpha + 1)Y_1^2 + (\beta + 1)\,(Y_1 + Y_2)^2}{\alpha\beta - 1}. \end{aligned} \tag{2.5.27}$$

Let us replace values $|V_i|$ by their estimates (2.5.26) and (2.5.27) in inequalities (2.3.15). We obtain

$$\begin{aligned} X_1 + \frac{Y_2^2 + \beta\,(Y_1 + Y_2)^2}{\alpha\beta - 1} &\le \frac{U_1^0 \beta - \beta - 1}{\alpha\beta}, \\ X_2 + \frac{(\alpha + 1)Y_1^2 + (\beta + 1)\,(Y_1 + Y_2)^2}{\alpha\beta - 1} &\le \frac{\alpha + \beta + 2 - U_1^0(\beta + 1)}{\alpha\beta}. \end{aligned} \tag{2.5.28}$$

By virtue of (2.5.24), the expressions in the right-hand sides of inequalities (2.5.28) are positive. Let us choose Y_i according to (2.3.25). Then $Y_i \to 0$ as $X_i \to 0$. Hence, there always exist positive X_1 and X_2 that satisfy inequalities (2.5.28) and, thus, inequalities (2.3.12). Note that constants J_i do not appear in restrictions (2.5.28) directly; therefore, their specific values are not essential.

To sum up, we may state the following.

Let conditions (2.5.21), (2.5.22), and (2.5.24) be fulfilled. Then, the feedback control $U_i(q_i, \dot q_i)$ that solves Problem 2.1 for system (2.5.16) is given by relations

(2.3.13). In these relations, parameters X_i are chosen so that inequalities (2.5.28) should be fulfilled. This control carries system (2.5.16) from the initial position (2.1.7) to the terminal position (2.1.8), if, at the initial instant, restrictions (2.3.17) are satisfied. The trajectory of the system lies in the domain D defined by (2.3.1); time τ of the control process is not greater than value τ^* that is defined by expressions (2.3.26) and (2.3.27).

Let us show the way to choose admissible values X_i. We search for them in the form (2.3.35). In this case, inequalities (2.5.28) become

$$Z^2 \leq \frac{U_1^0\beta - \beta - 1}{\alpha\beta} \times \left[d_1 + 2\frac{d_2^2 + \beta\,(d_1+d_2)^2}{\alpha\beta - 1}\right]^{-1},$$

$$Z^2 \leq \frac{\alpha + \beta + 2 - U_1^0(\beta+1)}{\alpha\beta} \times \left[d_2 + 2\frac{(\alpha+1)d_1^2 + (\beta+1)\,(d_1+d_2)^2}{\alpha\beta - 1}\right]^{-1}.$$

Now, we find the maximum value Z that satisfies the both inequalities obtained and then determine the parameters X_i by formulas (2.3.35).

Note that the set of possible values X_i may be significantly extended. For this purpose, it is necessary to obtain more precise estimates of $|V_i|$ in (2.5.26) and (2.5.27).

Computer simulation

Calculations were carried out for the following dimensional characteristics of system (2.5.16):

$$\begin{gathered} l_1 = 1\,\mathrm{m}, \quad l_{g2} = 0.5\,\mathrm{m}, \quad I_1 = I_2 = 3.33\,\mathrm{kg}\cdot\mathrm{m}^2, \\ m_2 = 10\,\mathrm{kg}, \quad M_1^0 = 2.9\,\mathrm{N}\cdot\mathrm{m}, \quad M_2^0 = 1\,\mathrm{N}\cdot\mathrm{m}. \end{gathered} \tag{2.5.29}$$

For this example, we suppose that $Q_1 = Q_2 = 0$ (see Sect. 2.5.2). The initial and terminal conditions, and also values $q_i^{\pm}$ defining the permissible domain of motion, are given by:

$$\begin{gathered} q_1^- = q_1^0 = -0.1\,\mathrm{rad}, \quad q_2^- = q_2^0 = -0.05\,\mathrm{rad}, \\ \dot{q}_1^0 = \dot{q}_2^0 = q_1^* = q_2^* = q_1^+ = q_2^+ = 0. \end{gathered} \tag{2.5.30}$$

In this case, $\alpha = 2.66$ and $\beta = 0.66 < 1$; inequalities (2.5.22) and (2.5.24) become $-2.3 < q_2^-, q_2^+ < 2.3$ and $2.5 < U_1^0 < 3.2$. Obviously, the parameters of the system (2.5.29) and (2.5.30) satisfy these restrictions. Let us choose dimensionless values for X_i that satisfy inequalities (2.5.28). For $X_1 = 1.82 \times 10^{-2}$ and $X_2 = 9.13 \times 10^{-3}$, the dimensional estimate $\tau^* = 4.68$ s of the time of control and the real time of the process $\tau = 3.64$ s are obtained. Figure 2.16 demonstrates the time histories of angular velocities $\dot{q}_1$ and $\dot{q}_2$. At the final stage, $\dot{q}_1$ and $\dot{q}_2$ vary linearly that agrees with the motion of the phase points of subsystems (2.3.9) along the parabolic sections of the switching curves. The phase trajectories of the subsystems are depicted in Figs. 2.17 and 2.18. The termination of the motion occurs at different instants for two degrees of freedom.

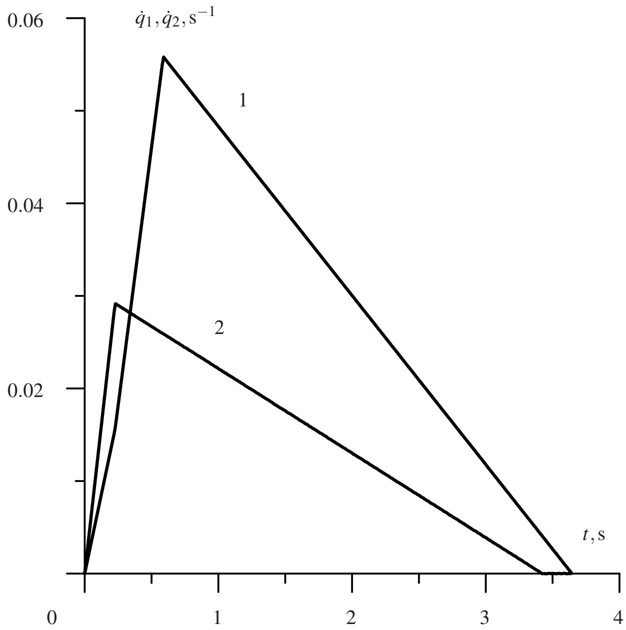

Fig. 2.16 Time history of the angular velocities

2.5.3 Modelling of motions of three-link manipulator

The three-link mechanism shown in Fig. 2.19 is chosen as an example of the control design by the method described in this chapter. This mechanism models the arm of the manipulation robot consisting of the upper and lower arms. The arm lies in the vertical plane and is connected to the vertical column supported by a fixed base.

The moment of inertia of the vertical column with respect to its axis of rotation is equal to I_1^Z. The links of the arm are the rods of the masses m_2, m_3 and lengths l_2, l_3, respectively. The centers of mass of the upper and lower arms are located exactly in the middle of the corresponding links. The principal central moments of inertia of the links with respect to the axes that are perpendicular to the rods and with respect to the longitudinal axes are equal to I_i^S and I_i^N, $i = 2, 3$, respectively.

The vertical column as well as the upper and lower arms are supplied by the actuators that include DC motors and reduction gears. For the sake of simplicity, we suppose that the axis and direction of rotation of the rotor in each electric drive coincide with the axis and direction of rotation of the corresponding joint. The masses

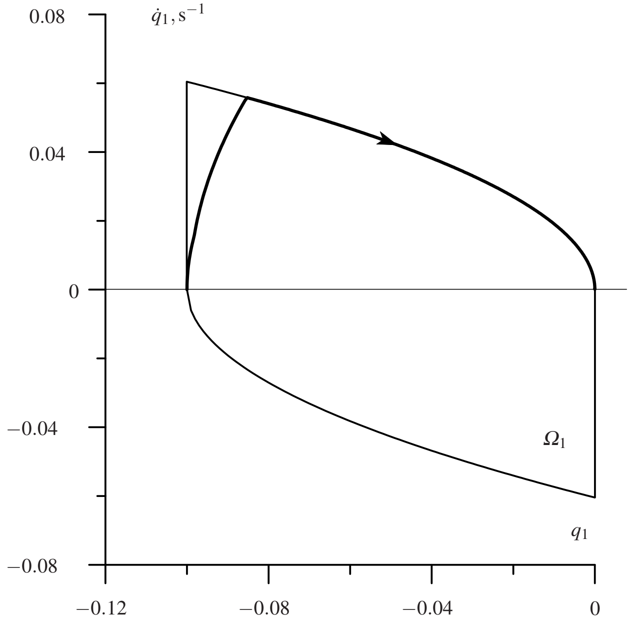

Fig. 2.17 Phase trajectory of the subsystem 1

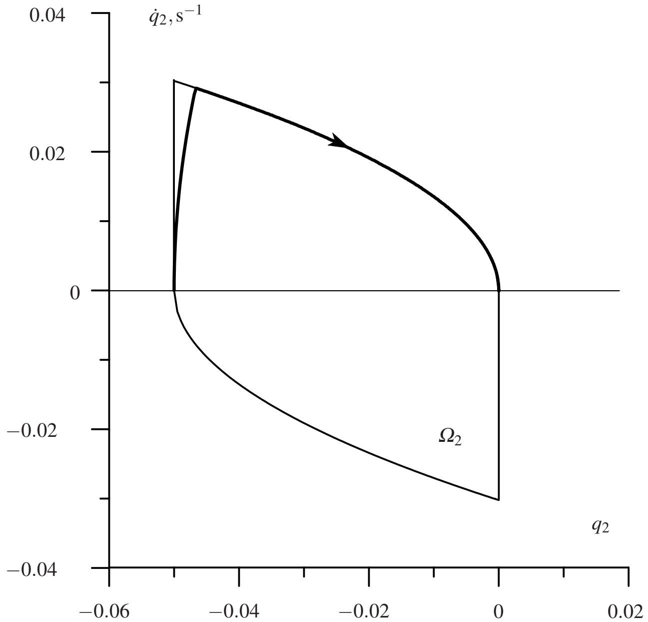

Fig. 2.18 Phase trajectory of the subsystem 2

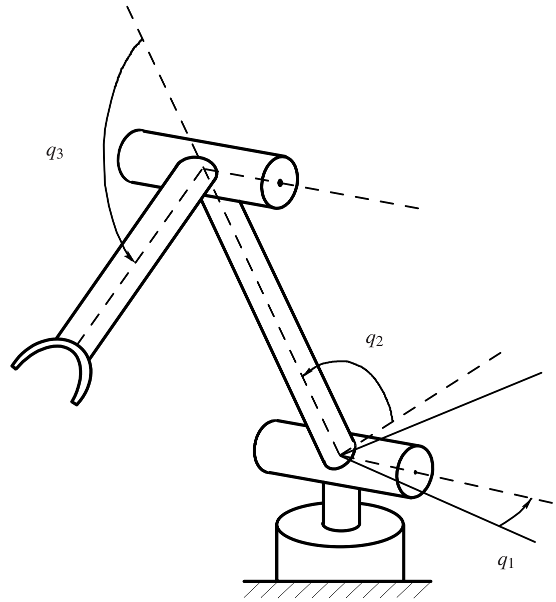

Fig. 2.19 Three-link manipulator

of the rotors of the motors are equal to m_i^R, $i = 1,2,3$. We neglect the inertia of the rotational parts of the reduction gears. The generalized coordinates q_1, q_2, and q_3 are the angles of rotation in the three cylindrical joints of the manipulator: angle q_1 of rotation of the vertical column about the vertical axis, angles q_2 and q_3 of rotation of the upper and lower arms about the corresponding horizontal axes (Fig. 2.19).

Under the assumptions made, let us obtain the elements of the matrix of the kinetic energy $A(q)$ from (2.1.2)

$$A(q) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

We have

$$a_{11} = J_1 N_1^2 + J_2' + J_3' + I_1^Z$$
$$+\frac{1}{2}\Big\{ (m_3 l_2^2 + I_2^S - I_2^N)\cos 2q_2 + (I_3^S - I_3^N)\cos 2(q_2+q_3)$$
$$+m_3 l_2 l_3 [\cos q_3 + \cos(q_3 + 2q_2)] + I_2^S + I_3^S + I_2^N + I_3^N + m_3 l_2^2 \Big\}$$
$$+\frac{1}{8}\Big\{ m_2 l_2^2 (1+\cos 2q_2) + m_3 l_3^2 [\cos 2(q_2+q_3) + 1] \Big\},$$

$$a_{22} = J_2N_2^2 + J_3 + I_2^S + I_3^S$$
$$+l_2^2\left(m_3^R + m_3 + \frac{1}{4}m_2\right) + m_3l_3\left(l_2\cos q_3 + \frac{1}{4}l_3\right),$$

$$a_{23} = a_{32} = J_3N_3 + I_3^S + \frac{1}{2}m_3l_3\left(l_2\cos q_3 + \frac{1}{2}l_3\right),$$
$$a_{33} = J_3N_3^2 + I_3^S + \frac{1}{4}m_3l_3^2.$$

Here, the notation for the moments of inertia J_i and J_i' of the rotors and gear ratios N_i of the reduction gears introduced in Sect. 2.5.1 is used.

As the generalized forces Q_i in (2.1.1), we will consider only torques due to the gravity in the joints (the forces of viscous and dry friction are not taken into account):

$$Q_1 = 0,$$
$$Q_2 = -9.81l_2\left(\frac{1}{2}m_2 + m_3^R + m_3\right)\cos q_2 - 9.81\cdot\frac{1}{2}m_3l_3\cos(q_2+q_3),$$
$$Q_3 = -9.81\cdot\frac{1}{2}m_3l_3\cos(q_2+q_3).$$

We will consider the case, where the constraints are imposed on the magnitude of the control electric voltages (see Sect. 2.5.1, case 2°).

Below, four variants of the simulation results (1–4) for the control of the considered system are presented. Input data for each case are shown in Tables 2.1–2.7, including the parameters of the links and electric drives, initial and terminal conditions, and domains of the possible motions. Output data are presented in Tables 2.3–2.7 and Figs. 2.20–2.24, including the control parameters, estimates of the motion time for each of three subsystems, real values of the motion time, time histories of the generalized velocities $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$, and phase trajectories of the subsystems. In addition, for the first set of the manipulation robot parameters, we give three variants of the simulation results (1a, 1b, and 1c) obtained by using the control method described in Sect. 2.3.4. While implementing the numerical simulations 1a, 1b, and 1c, the current states of the subsystems are determined at the discrete time instants (with finite time step). As a result, the motion along the switching curve takes place with a finite frequency of the sign change of the control; this motion approximates the sliding regime along the switching curve.

Table 2.1 Parameters of the links (variants 1–4)

i	m_i, kg	l_i, m	l_{gi}, m	I_i^S, kg×m^2	I_i^N, kg×m^2	I_i^Z, kg×m^2
1	–	–	–	–	–	0.2
2	5	0.8	0.4	0.25	0.01	–
3	5	0.8	0.4	0.25	0.01	–
1	–	–	–	–	–	0.2
2	5	0.8	0.4	0.25	0.01	–
3	4	0.64	0.32	0.20	0.01	–
1	–	–	–	–	–	0.2
2	5	0.8	0.4	0.25	0.01	–
3	4	0.64	0.32	0.17	0.086	–
1	–	–	–	–	–	0.2
2	5	0.8	0.4	0.25	0.01	–
3	4	0.74	0.37	0.18	0.009	–

Table 2.2 Parameters of the actuators (variants 1–4)

i	k_i^E, J/A	k_i^M, J/A	R_i, Ω	u_i, V	m_i^R, kg	J_i, kg×m^2	J_i', kg×m^2	N_i
1	0.04	0.04	1	27	0.5	0.00079	0.00041	160
2	0.04	0.04	1	27	0.5	0.00079	0.00041	250
3	0.04	0.04	1	27	0.5	0.00079	0.00041	150
1	0.04	0.04	0.7	27	0.4	0.00069	0.00036	120
2	0.04	0.04	0.6	27	0.25	0.00039	0.00022	180
3	0.04	0.04	0.6	27	0.25	0.00039	0.00022	150
1	0.113	0.109	0.7	42	0.4	0.00069	0.00036	150
2	0.1	0.09	0.6	36	0.25	0.00039	0.00022	250
3	0.1	0.09	0.6	36	0.25	0.00039	0.00022	200
1	0.08	0.07	0.7	27	0.4	0.00039	0.00022	120
2	0.06	0.06	0.6	27	0.25	0.00039	0.00022	180
3	0.06	0.05	0.6	27	0.25	0.00039	0.00022	150

Table 2.3 Variant 1: initial (q_i^0, $\dot{q}_i^0$) and terminal (q_i^*) conditions, domain of possible motions ($[q_i^-, q_i^+]$), control parameter (X_i), estimated (τ_i^*) and real(τ_i) motion times for the ith subsystem

i	q_i^0	$\dot{q}_i^0$, s^{-1}	q_i^*	q_i^-	q_i^+	X_i, s^{-2}	τ_i^*, s	τ_i, s
1	-1	1	0	-1	0	1.060	1.413	1.382
2	-0.3	0	0	-0.35	0.05	0.424	1.682	1.263
3	-1	0	0	-1	0	1.060	1.942	1.467

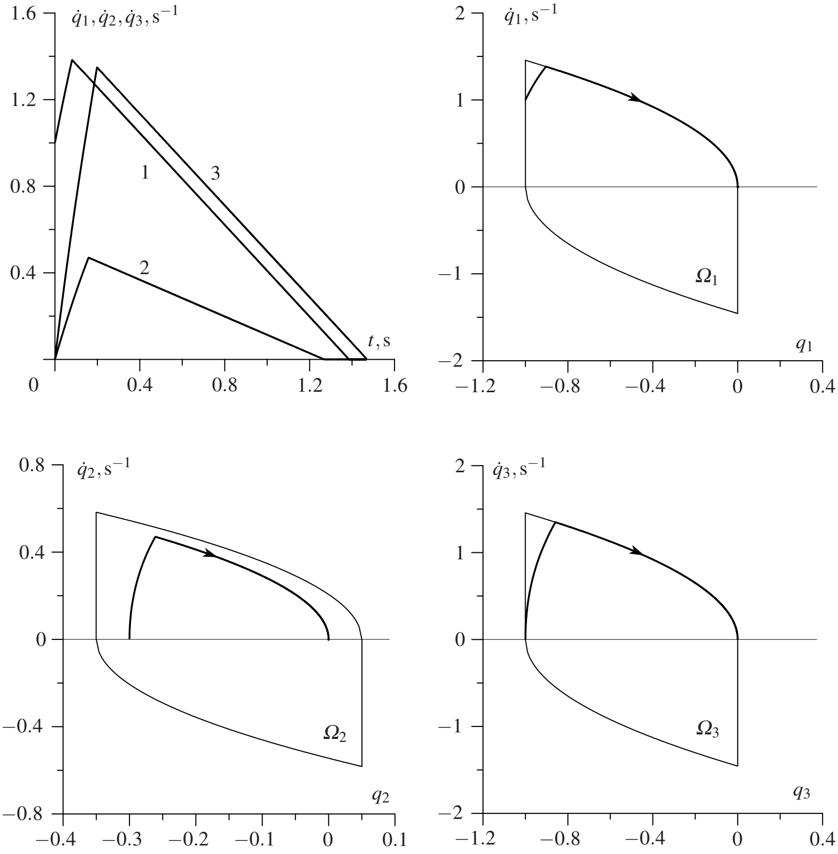

Fig. 2.20 Time histories of the generalized velocities and phase trajectories of the subsystems (variant 1)

Table 2.4 Variants 1a–1c: domain of possible motions ($[q_i^-, q_i^+]$), initial (q_i^0, $\dot{q}_i^0$) and terminal (q_i^*) conditions, and real motion time (τ_i) for the ith subsystem

i	q_i^-	q_i^+	q_i^0	$\dot{q}_i^0$, s^{-1}	q_i^*	τ_i, s
1	-1	0	-1	0	0	1.22
2	-0.3	0	-0.3	0	0	1.35
3	-0.9	0	-0.9	0	0	1.27
1	-1	0	-1	0.5	0	1.27
2	-0.35	0.05	-0.3	0	0	0.97
3	-1	0	-1	0	0	1.07
1	-1	0	-1	0.5	0	1.18
2	-0.35	0.05	-0.3	0	0	1.01
3	-1	0	1	-0.3	-0.1	1.20

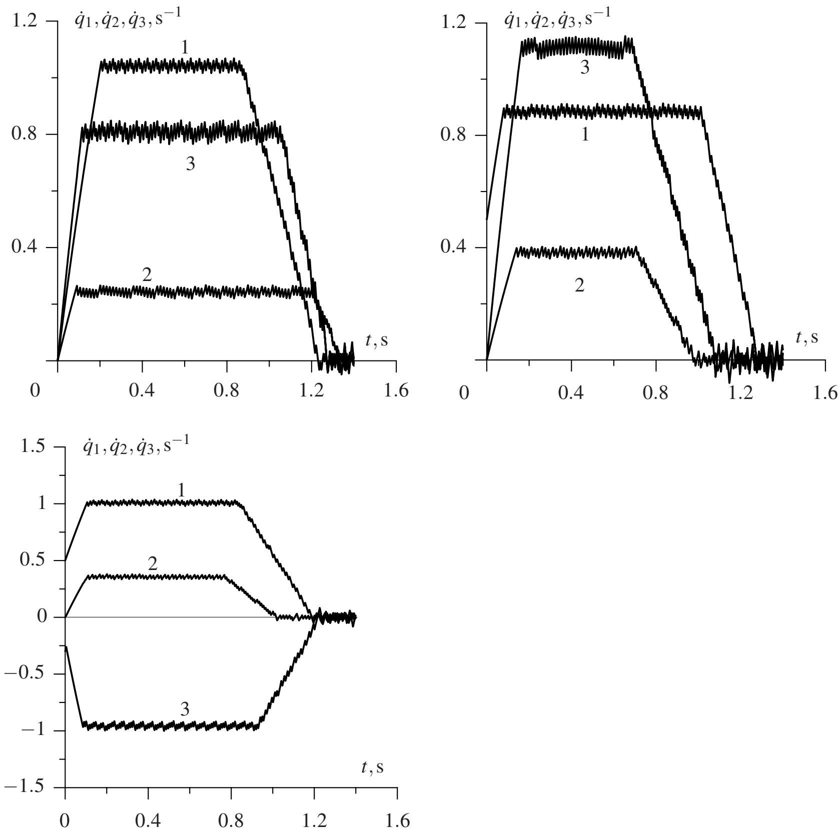

Fig. 2.21 Time histories of the generalized velocities (variants 1a–1c)

Table 2.5 Variant 2: initial (q_i^0, $\dot{q}_i^0$) and terminal (q_i^*) conditions, domain of possible motions ($[q_i^-, q_i^+]$), control parameter (X_i), estimated (τ_i^*) and real(τ_i) motion times for the ith subsystem

i	q_i^0	$\dot{q}_i^0$, s^{-1}	q_i^*	q_i^-	q_i^+	X_i, s^{-2}	τ_i^*, s	τ_i, s
1	-0.8	1	0	-0.85	0.05	1.392	1.145	1.109
2	-0.3	-0.2	0	-0.35	0.05	0.619	1.788	1.023
3	-1	0	0	-1	0	1.547	1.607	1.172

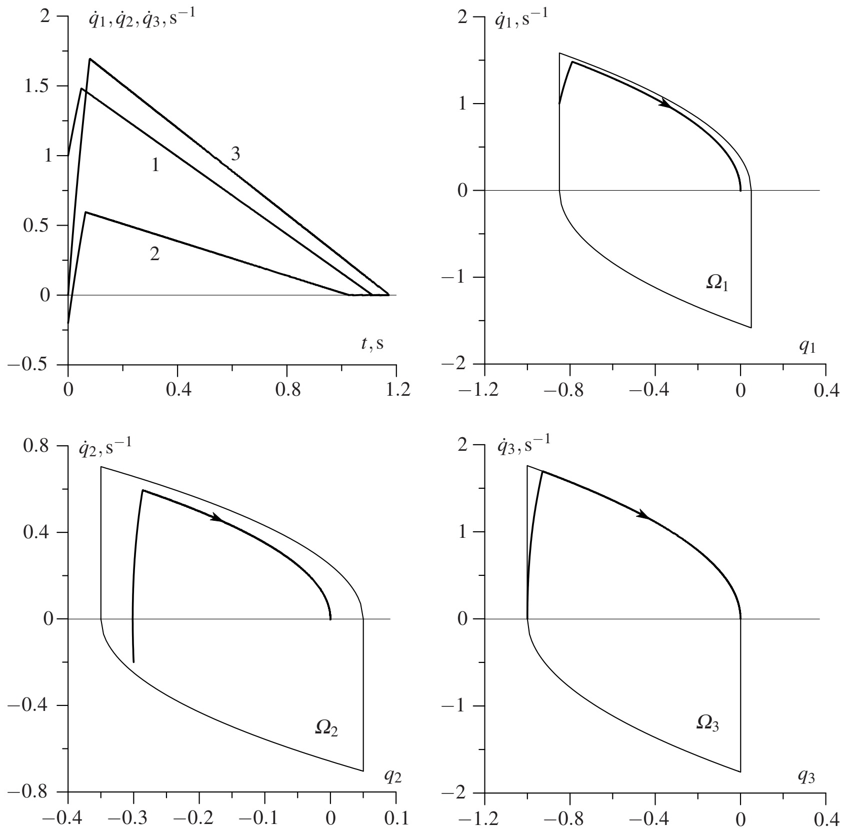

Fig. 2.22 Time histories of the generalized velocities and phase trajectories of the subsystems (variant 2)

Table 2.6 Variant 3: initial (q_i^0, $\dot{q}_i^0$) and terminal (q_i^*) conditions, domain of possible motions ($[q_i^-, q_i^+]$), control parameter (X_i), estimated (τ_i^*) and real(τ_i) motion times for the ith subsystem

i	q_i^0	$\dot{q}_i^0$, s^{-1}	q_i^*	q_i^-	q_i^+	X_i, s^{-2}	τ_i^*, s	τ_i, s
1	1	0	-0.2	-0.4	1.1	0.413	3.407	2.413
2	3	0	2	2	3	0.275	3.809	2.697
3	-0.25	0	0.3	-0.5	0.5	0.413	2.825	1.994

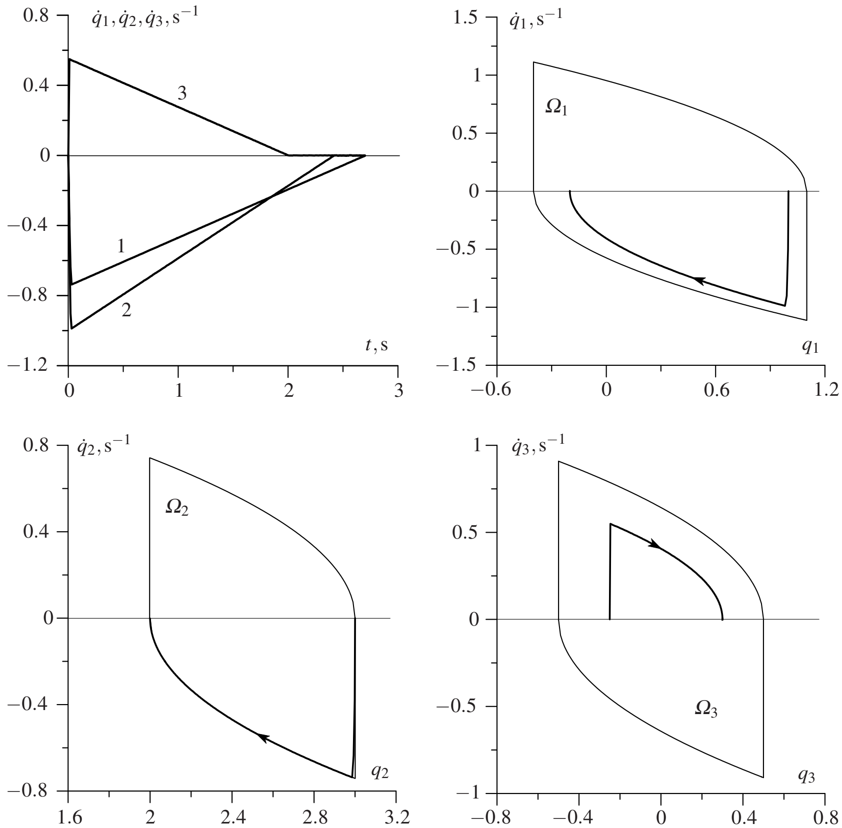

Fig. 2.23 Time histories of the generalized velocities and phase trajectories of the subsystems (variant 3)

Table 2.7 Variant 4: initial (q_i^0, $\dot{q}_i^0$) and terminal (q_i^*) conditions, domain of possible motions ($[q_i^-, q_i^+]$), control parameter (X_i), estimated (τ_i^*) and real(τ_i) motion times for the ith subsystem

i	q_i^0	$\dot{q}_i^0$, s^{-1}	q_i^*	q_i^-	q_i^+	X_i, s^{-2}	τ_i^*, s	τ_i, s
1	-1.5	0.7	-1	-1.6	-0.7	0.317	4.052	1.766
2	-0.3	-0.2	-0.2	-0.5	0	0.176	3.332	1.071
3	-0.5	0	-1	-1.3	-0.3	0.352	2.380	1.683

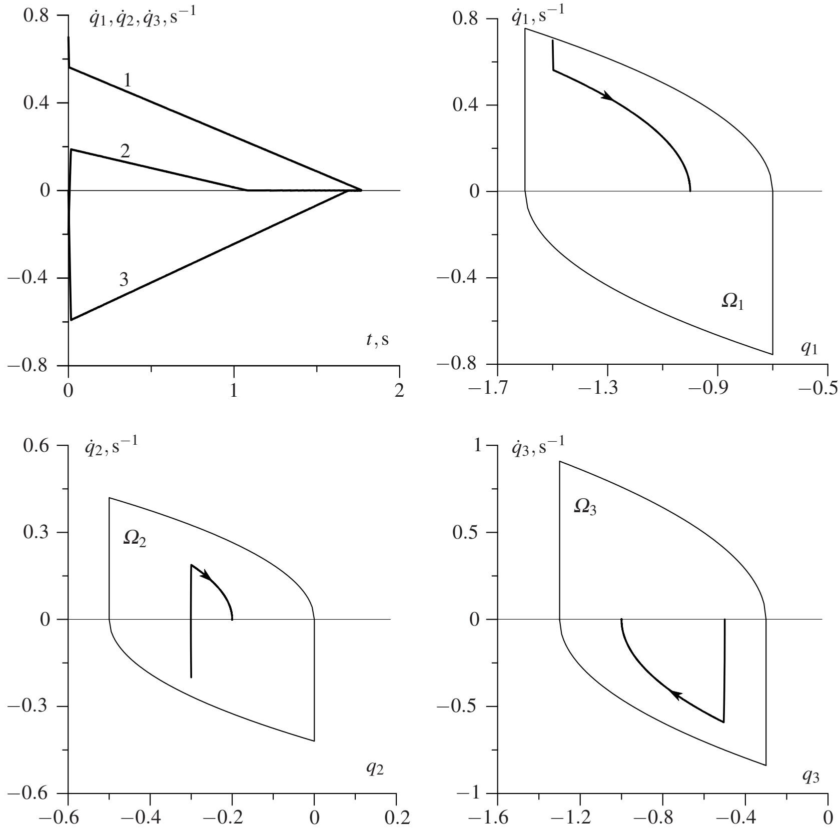

Fig. 2.24 Time histories of the generalized velocities and phase trajectories of the subsystems (variant 4)

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