1

Examples and Preliminaries

We begin with an example from everyday life, which will serve as a vehicle for an informal introduction to the main concepts of media theory. Several other examples follow, chosen for the sake of diversity, after which we briefly review some standard mathematical concepts and notation. The chapter ends with a short historical notice and the related bibliography. Our purpose here is to motivate the developments and to build up the reader's intuition, in preparation for the more technical material to follow.

1.1 A Jigsaw Puzzle

1.1.1 Gauss in Old Age. Figure 1.1(a) shows a familiar type of jigsaw puzzle, made from a portrait of Carl Friedrich Gauss in his old age. We call a *state* of this puzzle any partial solution, formed by a linked subset of the puzzle pieces in their correct positions. Four such states are displayed in Figure 1.1(a), (b), (c) and (d). Thus, the completed puzzle is a state. We also regard as states the initial situation (the empty board), and any single piece appropriately placed on the board. A careful count gives us 41 states (see Figure 1.1.2). To each of the six pieces of the puzzle correspond exactly two *transformations* which consist in placing or removing a piece. In the first case, a piece is placed either on an empty board, or so that it can be linked to some pieces already on the board. In the second case, the piece is already on the board and removing it either leaves the board empty or does not disconnect the remaining pieces. By convention, these two types of transformations apply artificially to all the states in the sense that placing a piece already on the board or removing a piece that is not on the board leaves the state unchanged.

This is our first example of a 'medium', a concept based on a pair $(\mathcal{S}, \mathcal{T})$ of sets: a set $\mathcal{S}$ states, and a collection $\mathcal{T}$ of transformations capable, in some cases, of converting a state into a different one. The formal definition of such a structure relies on two constraining axioms (see Definition 2.2.1).

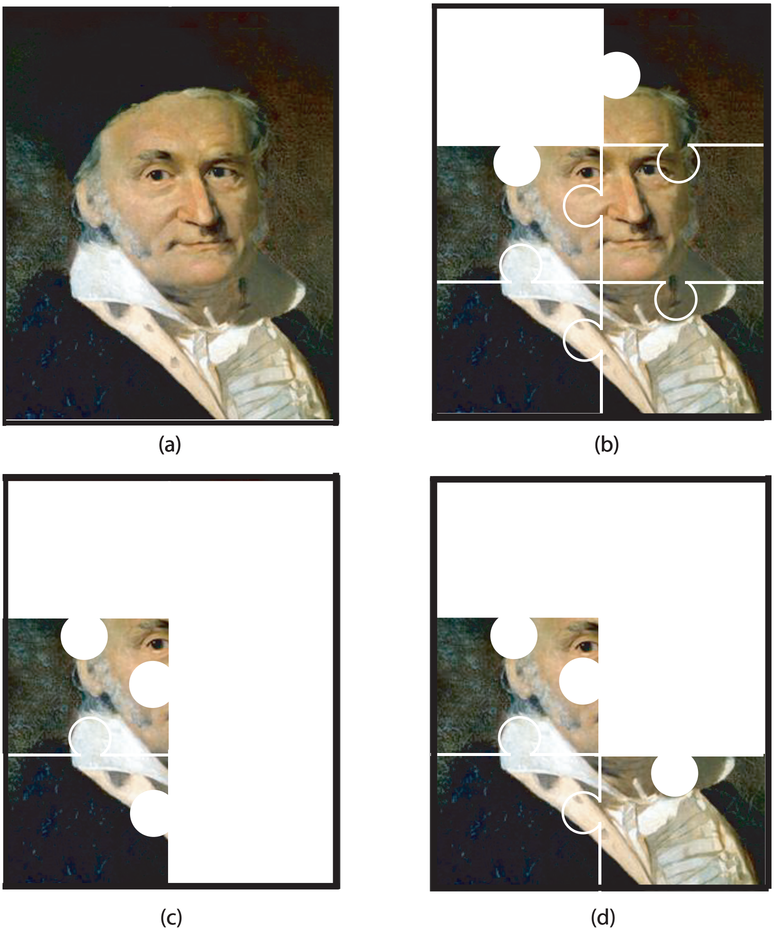

Figure 1.1. Four states of a medium represented by the jigsaw puzzle: Carl Friedrich Gauss in old age. The full medium contains 41 states (see Figure 1.2).

By design, none of these transformations is one-to-one. For instance, applying the transformation "adding the upper left piece of the puzzle"—the left part of Gauss's hat and forehead—to either of the states pictured in Figure 1.1(c) or (d) results in the same state, namely (a). In the first case, we have thus a loop. Accordingly, the two transformations associated with each piece are not mutual inverses. However, each of the transformations in a pair can undo the action of the other. We shall say that these transformations are 'reverses' of one another. For a formal definition of 'reverse' in the general case, see 2.1.1.

1.1.2 The Graph of Gauss's Puzzle. When the number of states is finite, it may be convenient to represent a medium by its graph and we shall often do so. The medium of Gauss's puzzle has its graph represented in Figure 1.2 below. As usual, we omit loops.

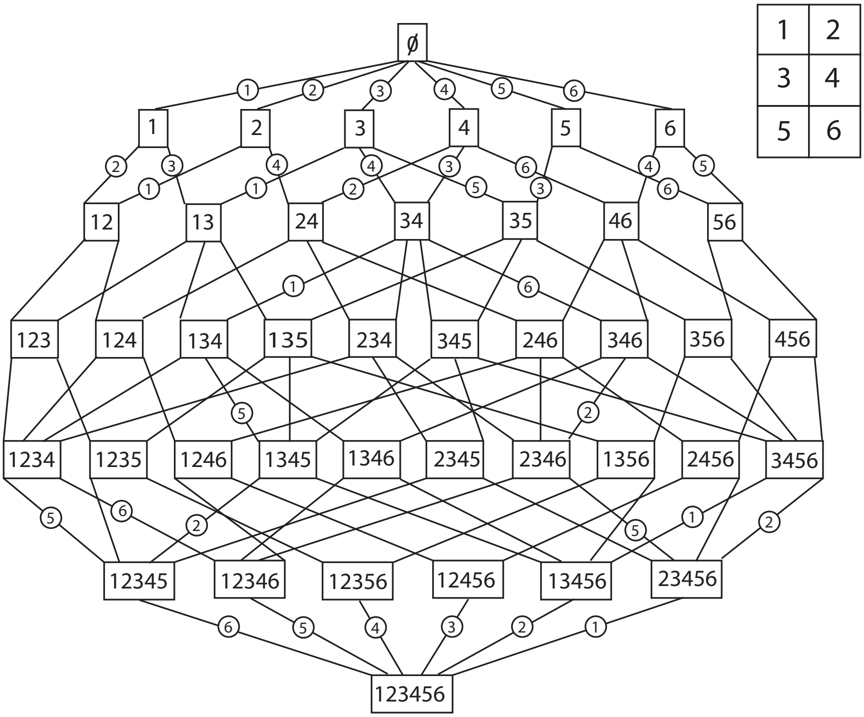

Figure 1.2. Graph of the Gauss puzzle medium. A schematic of the puzzle is a the upper right of the graph, with the six pieces numbered 1,. . . , 6. Each of the 41 vertices of the graph represent one state of the medium, that is, one partial solution of the puzzle symbolized by a rectangle containing the list of its pieces. Each edge represents a pair of mutually reverse transformations, one adding a piece, and the other removing it. To avoid cluttering the figure, only some of the edges are labeled (by a circle).

An examination of this graph leads to further insight. For any two states S and T, it is possible to find a sequence of transformations whose successive applications from S results in forming T. This 'path' from S to T never strays from the allowed set of states, and can be made minimally short, that is: its length is equal to the number of pieces which are not common to both states. Moreover, any two such paths from S to T will involve exactly the same transformations, but they may be applied in different orders[1]. As an illustration, we have marked in Figure 1.2 two such paths from state $\boxed{34}$ to the completed puzzle by coloring their edges in red and blue, respectively. These two paths are

$$\boxed{34} \overset{1}{\longmapsto} \boxed{134} \overset{5}{\longmapsto} \boxed{1345} \overset{2}{\longmapsto} \boxed{12345} \overset{6}{\longmapsto} \boxed{123456} \tag{1.1}$$

$$\boxed{34} \overset{6}{\longmapsto} \boxed{346} \overset{2}{\longmapsto} \boxed{2346} \overset{5}{\longmapsto} \boxed{23456} \overset{1}{\longmapsto} \boxed{123456}\,. \tag{1.2}$$

A medium could be defined from any (standard) jigsaw puzzle according to the rules laid out here[2]. Such media have the remarkable property that their set of transformations is naturally partioned into two classes of equal sizes, namely, one corresponding to the addition of the pieces to the puzzle, and the other one to their removal. In view of this asymmetry which also arises in other situations, we shall talk about 'orientation' to describe such a bipartition (see Definition 2.8.1). Thus, in medium terminology, a transformation is in a given class if and only if its reverse belongs to the other class. There are important cases, however, in which no such natural orientation exists. Accordingly, this concept is not an integral part of the definition of a medium (cf. 2.2.1).

In fact, the next two examples involve media in which no natural orientation of the set of transformations suggests itself.

1.2 A Geometrical Example

1.2.1 An Arrangement of Hyperplanes. Let $\mathcal{A}$ be some finite collection of hyperplanes in $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus (\cup \mathcal{A})$ is the union of the open, convex polyhedral regions bounded by the hyperplanes, some (or all) of which may be unbounded. We regard each polyhedral region as a state, and we denote by $\mathcal{P}$ the finite collection of all the states. From one state P in $\mathcal{P}$, it is always possible to move to another adjacent state by crossing some hyperplane including a facet of P. (We suppose that a single hyperplane is crossed at one time.) We formalize these crossings in terms of transformations of the states. To every hyperplane H in $\mathcal{A}$ corresponds the two ordered pairs (H^-, H^+) and (H^+, H^-) of the two open half spaces H^- and H^+ separated by the hyperplane. These ordered pairs generate two transformations τ_H^+ and τ_H^- of the states, where τ_H^+ transforms a state to an adjacent state in H^+, if possible, and leaves it unchanged otherwise, while τ_H^- transforms a state to an adjacent state in H^-, if possible, and leaves it unchanged otherwise. More formally, applying τ_H^+ to some state P results in some other state Q if $P \subseteq H^-$, $Q \subseteq H^+$,

[1] In this particular case, the two paths represented in (1.1) and (1.2) have their transformations in the exact opposite order, but not all pairs of different paths are reversed in this way (Problem 1.1).

[2] In the case of larger puzzles (for instance, 3×3), there might be states with holes: all the pieces are interconnected but there are pieces missing in the middle.

and the polyhedral regions P and Q share a facet which is included in the hyperplane separating H^- and H^+; otherwise, the application of τ_H^+ to P does not change P. The transformation τ_H^- is defined symmetrically. Clearly, the application of τ_H^+ cancels the action of τ_H^- whenever the latter was effective in modifying the state. However, as in the preceding example, τ_H^+ and τ_H^- are not mutual inverses. We say in such a case that τ_H^+ and τ_H^- are mutual reverses. Denoting by $\mathcal{T}$ the set of all such transformations, we obtain a pair $(\mathcal{P}, \mathcal{T})$ which is another example of a medium. A case of five straight lines in $\mathbb{R}^2$ defining fifteen states and ten pairs of mutually reverse transformations is pictured in Figure 1.3. The proof that, in the general case, an arbitrary locally finite hyperplane arrangement defines a medium is due to Ovchinnikov (2006) (see Theorem 9.1.8 in Chapter 9 here).

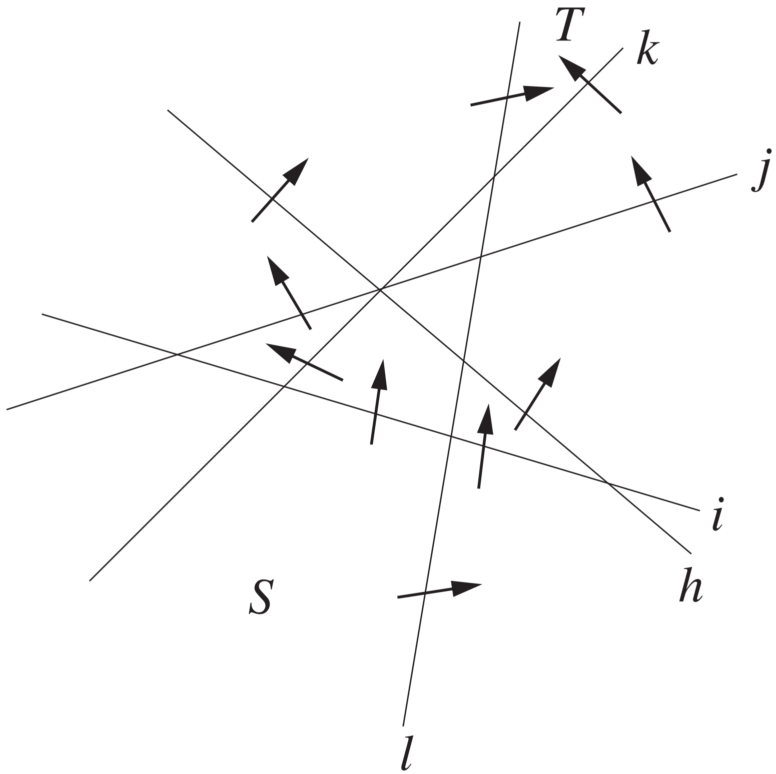

Figure 1.3. A line arrangement in the case of five straight lines in $\mathbb{R}^2$ delimiting fifteen states with ten pairs of transformations. Two direct paths from state S to state T cross the same lines in two different orders: *lihjk* and *ikjhl*.

1.3 The Set of Linear Orders

In this example, each of the $24 = 4!$ linear orders on the set $\{1, 2, 3, 4\}$ is regarded as a state. A transformation consists of transposing two adjacent numbers: the transformation τ_{ij} replaces an adjacent pair ji by the pair ij, or does nothing if ji does not form an adjacent pair in the initial state. There are thus $6 = \binom{4}{2}$ pairs of transformations τ_{ij}, τ_{ji}. Three of these transformations are 'effective' for the state 3142, namely:

$$3142 \overset{\tau_{13}}{\longmapsto} 1342$$
$$3142 \overset{\tau_{41}}{\longmapsto} 3412$$
$$3142 \overset{\tau_{24}}{\longmapsto} 3124\,.$$

As in the preceding example, no natural orientation arises here.

1.3.1 The Permutohedron. The graph of the medium of linear orders on $\{1, 2, 3, 4\}$ is displayed in Figure 1.4. Such a graph is sometimes referred to as a *permutohedron* (cf. Bowman, 1972; Gaiha and Gupta, 1977; Le Conte de Poly-Barbut, 1990). Again, we omit loops, as we shall always do in the sequel. The edges of this polyhedron can be gathered into six families of parallel edges; the edges in each family correspond to the same pair of transformations.

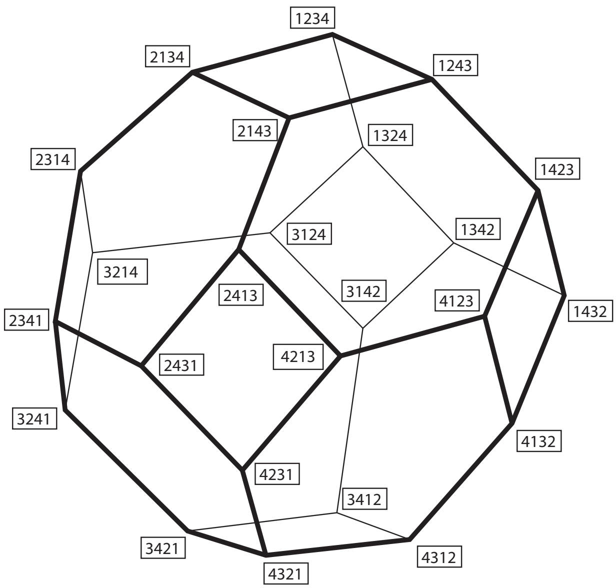

Figure 1.4. Permutohedron of $\{1, 2, 3, 4\}$. Graph of the medium of the set of linear orders on $\{1, 2, 3, 4\}$.

1.3.2 Remark. The family $\mathcal{L}$ of all linear orders on a particular finite set is characteristically associated with the group of permutations on that family. However, as illustrated by the graph of Figure 1.4, in which each set of parallel edges represents a particular the pair of mutually reverse transpositions of two adjacent objects, the concept of a medium is just as compelling as an algebraic structure canonically associated to $\mathcal{L}$.

1.4 The Set of Partial Orders

Consider an arbitrary finite set S. The family $\mathcal{P}$ of all strict partial orders (asymmetric, transitive, cf. 1.8.3, p. 14) on S enjoys a remarkable property: any partial order P can be linked to any other partial order P' by a sequence of steps each of which consists of changing the order either by adding one ordered pair of elements of S (imposing an ordering between two previously-incomparable elements) or by removing one ordered pair (causing two previously related elements to become incomparable), without ever leaving the family $\mathcal{P}$. Moreover, this can always be achieved in the minimal number of steps, which is equal to the 'symmetric difference' between P and P' (cf. Definition 1.8.1; see Bogart and Trotter, 1973; Doignon and Falmagne, 1997, and Chapter 5). To cast this example as a medium, we consider each partial order as a state, with the transformations consisting in the addition or removal of some pair. This medium is thus equipped with a natural orientation, as in the case of the jigsaw puzzle of 1.1.1.

The graph of such a medium is displayed in Figure 1.5 for the family of all partial orders on the set $\{a, b, c\}$. Only the edges corresponding to the transformation $P \mapsto P + \{ba\}$ are indicated. (Note that we sometimes use '+' to denote disjoint union; cf. 1.8.1.) Certain oriented media satisfy an important property: they are 'closed' with respect to their orientation. This property is conspicuous in the graph of Figure 1.5: if P, $P + \{xy\}$ and $P + \{zw\}$ are three partial orders on $\{a, b, c\}$, then $P + \{xy\} + \{zw\}$ is also such a partial order (however, see Problem 1.9).

This medium also satisfies the parallelism property observed in the permutohedron example: each set of parallel edges represents the same pair of mutually reverse transformations of adjacent objects. This property of certain media is explored in Definition 2.6.4 and Theorem 2.6.5.

Chapter 5 contains a discussion of this and related examples of families of relations, such as 'biorders' and 'semiorders' from the standpoint of media theory. (For the partial order example, see in particular Definition 1.8.3 and Theorem 5.3.5.).

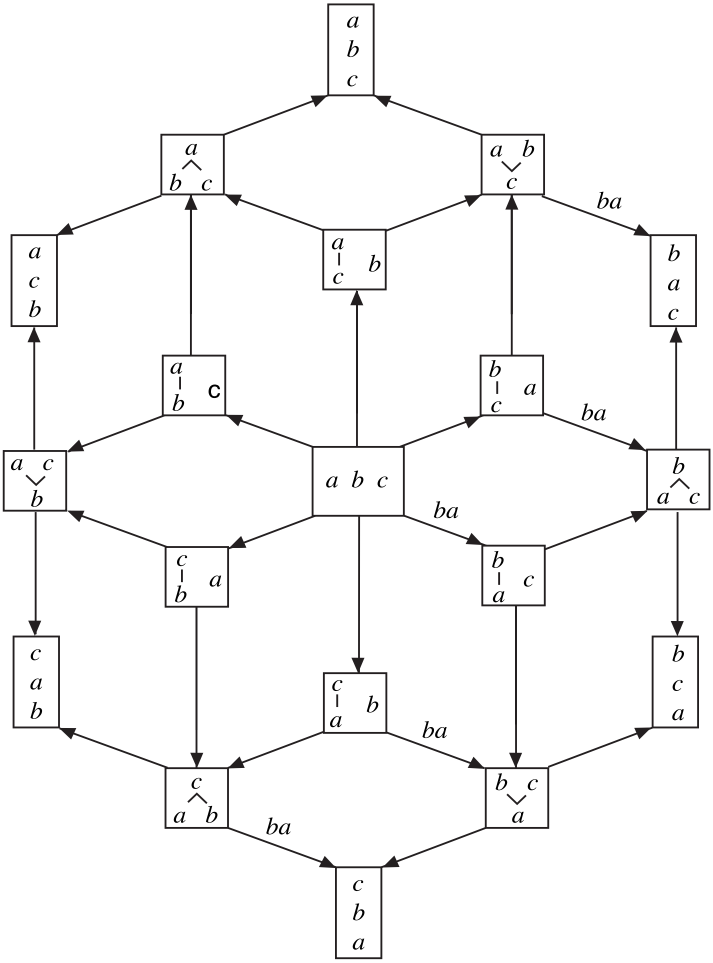

Figure 1.5. Graph of the medium of the set of all partial orders on $\{a, b, c\}$. The orientation of the edges represents the addition of a pair to a partial order. Only one class of edges is labelled, corresponding to the addition of the pair ba (see 1.8.2 for this notation).

1.5 An Isometric Subgraph of $\mathbb{Z}^n$

Perhaps the most revealing geometric representation of a finite oriented medium is as an isometric subgraph of the n-dimensional integer lattice $\mathbb{Z}^n$, for n minimal. By 'isometric' we mean that the distance between vertices in the subgraph is the same as that in $\mathbb{Z}^n$. Such a representation is always possible (cf. Theorems 3.3.4, 7.1.4, and 8.2.2), and algorithms are available for the construction (see Chapter 10).

Let $\mathcal{M}$ be a finite oriented medium and let $\mathcal{G} \subset \mathbb{Z}^n$ be its representing subgraph. Each state of the medium $\mathcal{M}$ is represented by a vertex of $\mathcal{G}$, and each pair $(\tau, \tilde{\tau})$ of mutually reverse transformations is associated with a hyperplane $\mathcal{H}$ orthogonal to one of the coordinate axes of $\mathbb{Z}^n$, say q_j. Suppose that $\mathcal{H}$ intersects q_j at the point $(i_1, \ldots, i_j, \ldots, i_n)$. Let us identify $\mathcal{M}$ with $\mathcal{G}$ (thus, we set $\mathcal{M} = \mathcal{G}$). The restriction of the transformation τ to $\mathcal{H} \cap \mathcal{G}$ is a 1-1 function from $\mathcal{H} \cap \mathcal{G}$ onto $\mathcal{H}' \cap \mathcal{G}$, where $\mathcal{H}'$ is a hyperplane parallel to $\mathcal{H}$ and intersecting q_j at the point $(i_1, \ldots, i_j + 1, \ldots, i_n)$. Thus, τ moves $\mathcal{H} \cap \mathcal{G}$ one unit upward. The restriction of τ to $\mathcal{G} \setminus \mathcal{H}$ is the identity function on that set. The reverse transformation $\tilde{\tau}$ moves $\mathcal{H}' \cap \mathcal{G}$ one unit downward, and is the identity on $\mathcal{G} \setminus \mathcal{H}'$.

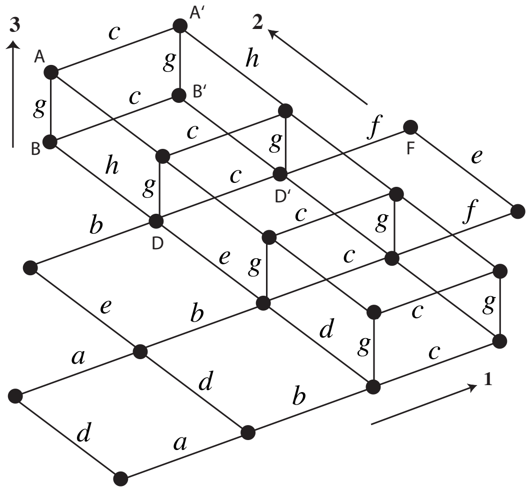

Figure 1.6. An isometric subgraph $\mathcal{D}$ of $\mathbb{Z}^3$. The orientation of the induced medium corresponds to the natural order of the integers and is indicated by the three arrows. To avoid cluttering the graph, the labeling of some edges is omitted.

The oriented graph $\mathcal{D}$ of Figure 1.6, representing a medium with 23 states and 8 pairs of mutually reverse transformations is a special case of this situation. The arrows labeled 1, 2 and 3 indicate the orientations of the axes q_1, q_2 and q_3 of $\mathbb{Z}^3$. The plane $<A, B, D>$ defined by the vertices A, B and D is orthogonal to q_1. The 8 edges marked c correspond to the pair of transformations $(\tau_c, \tilde{\tau}_c)$. The transformation τ_c moves $<A, B, D> \cap \mathcal{D}$ one unit to the right upward, that is, onto $<A', B', D'> \cap \mathcal{D}$, and is represented by loops elsewhere. The transformation $\tilde{\tau}_c$ is the reverse of τ_c.

The medium represented by $\mathcal{D}$ is not closed: the two transformations τ_f and τ_c transform D' into F and B', respectively. But applying τ_c to F gives a loop, and so does the application of τ_f to B'. Note that the subgraph $\mathcal{D}$ is not the unique representation of the medium $\mathcal{M}$ in $\mathbb{Z}^3$. A counter clockwise rotation could render the four edges marked a or b parallel to the third coordinate axis without altering the accuracy of the representation. However, because the graph is oriented we cannot apply a similar treatment to two edges marked f.

The last two examples of this chapter deal with empirical applications[3]. In both of these cases, the medium is equipped with a natural orientation.

1.6 Learning Spaces

1.6.1 Definition. Doignon and Falmagne (1999) formalize the concept of a *knowledge structure* (with respect to a topic) as a family $\mathcal{K}$ of subsets of a basic set Q of items[4] of knowledge. Each of the sets in $\mathcal{K}$ is a *(knowledge) state*, representing the competence of a particular individual in the population of reference. It is assumed that $\varnothing, Q \in \mathcal{K}$. Two compelling learning axioms are:

[K1] If $K \subset L$ are two states, with $|L \setminus K| = n$, then there is a chain of states

$$K_0 = K \subset K_1 \subset \cdots \subset K_n = L$$

such that $K_i = K_{i-1} + \{q_i\}$ with $q_i \in Q$ for $1 \leq i \leq n$. (We use '+' to denote disjoint union.) In words, intuitively: *If the state K of the learner is included in some other state L then the learner can reach state L by learning one item at a time.*

[K2] If $K \subset L$ are two states, with $K \cup \{q\} \in \mathcal{K}$ and $q \notin L$, then $L \cup \{q\} \in \mathcal{K}$. In words: *If item q is learnable from state K, then it is also learnable from any state L that can be reached from K by learning more items.*

A knowledge structure $\mathcal{K}$ satisfying Axioms [K1] and [K2] is called a *learning space* (cf. Cosyn and Uzun, 2005). To cast a learning space as a medium, we take any knowledge state to be a state of the medium. The transformations consist in adding (or removing) an item $q \in Q$ to (from) a state; thus, they take the form of the two functions: $\tau_q : \mathcal{K} \to \mathcal{K} : K \mapsto K + \{q\}$ and $\tilde{\tau}_q : \mathcal{K} \to \mathcal{K} : K \mapsto K \setminus \{q\}$. This results in a 'closed rooted medium' (see Definition 4.1.2, and Theorem 4.2.2). The study of media is thus instrumental in our understanding of learning spaces as defined by [K1] and [K2]. Note that a learning space is known in the combinatorics literature as an 'antimatroid', a structure introduced by Dilworth (1940) (cf. also Edelman and Jamison, 1985; Korte et al., 1991; Welsh, 1995; Björner et al., 1999). An empirical application of these concepts in the schools is reviewed in Section 13.1.

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[3] In particular, learning spaces provide the theoretical foundation for a widely used internet based system for the assessment of mathematical knowledge.

[4] In a scholarly context, an 'item' might be a type of problem to be solved, such as 'long division' in arithmetic.

1.7 A Genetic Mutations Scheme

The last example of this chapter is artificial. The states of the medium are linear arrangements of genes on a small portion of a chromosome[5]. We consider four pairs of transformations, corresponding to mutations producing chromosomal aberrations observed, for example in the *Drosophila melanogaster* (cf. Villee, 1967). We take the normal state to be the sequence A-B-C, where A, B and C are three genetic segments. The four mutations are listed below.

Table 1.1. Normal state and four types of mutations

Genetic arrangements	Names
A-B-C	normal state
A-B	deletion of segment C
A-B-C-C	duplication of segment C
A-B-C-X	translocation[a] of segment X
B-A-C	inversion of segment AB

[a] From another chromosome.

1.7.1 Mutation Rules. These mutations occur in succession, starting from the normal state A-B-C, according to the five following (fictitious) rules:

[IN] The segment A-B can be inverted whenever C is not duplicated.

[TR] The translocation of the segment X can only occur in the case of a two segment (abnormal) state.

[DE] A single segment C can always be deleted (from any state).

[DU] The segment C can be duplicated (only) in the normal state.

[RE] All the reverses of these four mutations exist, but no other mutations are permitted.

The resulting graph in Figure 1.7 is the graph of a medium if we admit the possibility of reverse mutations in all cases. If such reverse mutations are rare, one can assume, in the framework of the random walk process described in Chapter 12, that some or all the reverse mutations occur with a very low positive probability.

[5] This example is inspired by biogenetic theory but cannot be claimed to be fully faithful to it. Our goal here is only to suggest potential applications.

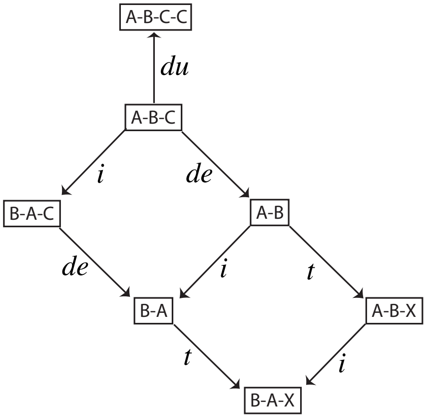

Figure 1.7. Oriented graph of the medium induced by the four mutations listed in Table 1.1, according to the five rules of 1.7.1. In the labelling of the edges, i, du, de and t stand for 'inversion of A-B', 'duplication of C', 'deletion of C' and 'translocation of X', respectively.

1.8 Notation and Conventions

We briefly review the primary mathematical notations and conventions employed throughout this book. A glossary of notation is given on page 309.

1.8.1 Set Theory. Standard logical and set theoretical notation is used throughout. We write $\Leftrightarrow$, as usual, for logical equivalence and $\Rightarrow$ for implication. The notation $\subseteq$ stands for the inclusion of sets, and $\subset$ for the proper (or strict) inclusion. We sometimes denote the union of disjoint sets by $+$ or by the summation sign $\sum$. The union of all the sets in a family $\mathcal{F}$ of subsets is symbolized by

$$\cup\mathcal{F} = \{x \,|\, x \in Y \text{ for some } Y \in \mathcal{F}\}, \tag{1.3}$$

and the intersection of all those sets by

$$\cap\mathcal{F} = \{x \,|\, x \in Y \text{ for all } Y \in \mathcal{F}\}. \tag{1.4}$$

Defined terms and statement of results are set in *slanted* font. The *complement* of a set Y with respect to some fixed ground set $\mathcal{X}$ including Y is the set $\overline{Y} = \mathcal{X} \setminus Y$.

The set of all the subsets, or *power set*, of a set $\mathcal{Z}$ is denoted by $\mathfrak{P}(\mathcal{Z})$. From (1.3) and (1.4), we get $\cup\mathfrak{P}(\mathcal{Z}) = \mathcal{Z}$ and $\cap\mathfrak{P}(\mathcal{Z}) = \varnothing$ (because $\varnothing \in \mathfrak{P}(\mathcal{Z})$). Note that we also write $\mathfrak{P}_{\mathrm{F}}(\mathcal{Z})$ for the set of all **finite** subsets of $\mathcal{Z}$.

The size (or cardinality, or cardinal number) of a set X is written as $|X|$. Two sets having the same cardinal numbers are said to be *equipollent*. The *symmetric difference* of two sets X and Y is the set

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X).$$

The *symmetric difference distance* of two sets X and Y is defined by

$$d(X,Y) = |X \triangle Y|. \tag{1.5}$$

If $\mathcal{Z}$ is finite and the function d is defined by (1.5) for all $X, Y \in \mathfrak{P}(\mathcal{Z})$, then $(\mathfrak{P}(\mathcal{Z}), d)$ is a metric space. We recall that a *metric space* is a pair $(\mathcal{X}, d)$ where $\mathcal{X}$ is a set, and d is a real valued function on $\mathcal{X} \times \mathcal{X}$ satisfies the three conditions: for all x, y and z in $\mathcal{X}$,

[D1] $d(x,y) \geq 0$, with $d(x,y) = 0$ if and only if $x = y$ (positive definiteness);

[D2] $d(x,y) = d(y,x)$ (symmetry);

[D3] $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

The *Cartesian product* of two sets X and Y is defined as

$$X \times Y = \{(x,y) \,|\, x \in X \,\&\, y \in Y\}$$

where (x,y) denotes an ordered pair and & means the logical connective 'and.' Writing $\Leftrightarrow$ for 'if and only if,' we thus have

$$(x,y) = (z,w) \quad \Longleftrightarrow \quad (x = z \,\&\, y = w).$$

More generally, $(x_1, \ldots, x_n)$ denotes the ordered n-tuple of the elements $x_1,\ldots, x_n$, and we have

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \,|\, x_1 \in X_1, \ldots, x_n \in X_n\}.$$

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ stand for the sets of natural numbers, integers, rational numbers, and real numbers, respectively; $\mathbb{N}_0$ and $\mathbb{R}_+$ denote the sets of nonnegative integers and nonnegative real numbers respectively.

1.8.2 Binary Relations, Relative Product. A set R is a *binary relation* if there are two (not necessarily distinct) sets X and Y such that $R \subseteq X \times Y$. Thus, a binary relation is a set of ordered pairs $xy \in X \times Y$, where xy is an abbreviation of (x,y). In such a case, we often write xRy to mean $xy \in R$. The qualifier 'binary' is often omitted. If $R \subseteq X \times X$, then R is said to be a binary relation *on* X. The (*relative*) *product* of two relations R and S is the relation

$$RS = \{xz \,|\, \exists y, xRySz\}$$

(in which $\exists$ denotes the existential quantifier). If $R = S$, we write $R^2 = RR$, and in general $R^{n+1} = R^n R$ for $n \in \mathbb{N}$. By convention, if R is a relation on X, then R^0 denotes the *identity relation* on X:

$$xR^0y \quad \Longleftrightarrow \quad x = y.$$

Note that when xRy and yRz, we sometimes write $xRyRz$ for short. Elementary properties of relative products are taken for granted. For example: if R, S, T and M are relation, then:

$$S \subseteq M \Longrightarrow RST \subseteq RMT \tag{1.6}$$

and

$$R(S \cup T) \subseteq RS \cup RT. \tag{1.7}$$

1.8.3 Order Relations. A relation is a *quasi order* on a set X if it is *reflexive* and *transitive* on X, that is, for all x, y, and z in X,

$$\begin{array}{lll} xRx & & \text{(reflexivity)} \\ xRy \;\&\; yRz \quad \Longrightarrow \quad xRz & & \text{(transitivity)} \end{array}$$

(where '$\Rightarrow$' means 'implies' or 'only if'). A quasi order R is a *partial order* on X if it is *antisymmetric* on X, that is, for all x and y in X

$$xRy \;\&\; yRx \quad \Longrightarrow \quad x = y.$$

If R is a partial order on a set X, then the pair (X, R) is referred to as a *partially ordered set.* A relation R is a *strict partial order* on X if it is transitive and *asymmetric* on X, that is, for all x, y in X,

$$xRy \quad \Longrightarrow \quad \neg(yRx)\,,$$

where $\neg$ stands for the logical 'not.' The *Hasse diagram* or *covering relation* of a partial order (X, R) is the relation $\breve{R} \subseteq R$ such that, for all x, y and z in X, $z\breve{R}x$ together with $zRyRx$ implies either $x = y$ or $y = z$. We say then that x *covers* z. If X is infinite, the Hasse diagram may be empty (see Problem 1.8). Otherwise, $\breve{R}$ provide a faithful and economical summary of R. Indeed, we have

$$R = \cup_{n=0}^{\infty} \breve{R}^n = \breve{R}^0 \cup \breve{R} \cup \cdots \cup \breve{R}^n \cup \cdots \tag{1.8}$$

The r.h.s. (right hand side) of (1.8) is called the *transitive closure* of $\breve{R}$. The Hasse diagram of a relation R is the 'smallest relation' (see Problem 1.4) the transitive closure of which gives back R. In general, we write Q^* for the transitive closure of a relation Q. Thus, we can rewrite the first equality in (1.8) in the compact form $R = \breve{R}^*$. The Hasse diagram of a strict partial order can also be defined (see Problem 1.8).

A partial order L on X is a *linear order* if it is is *strongly connected*, that is, for all x, y in X,

$$xLy \quad \text{or} \quad yLx\,. \tag{1.9}$$

A relation L on X is a *strict linear order* if it is a strict partial order which is *connected*, that is, (1.9) holds for all distinct x, y in X.

Suppose that L is a strict linear order on X. A *L-minimal* element of $Y \subseteq X$ is a point $x \in Y$ such that $\neg(yLx)$ for any $y \in Y$. A strict linear order L on X is a *well-ordering* of X if every nonempty $Y \subseteq X$ has a L-minimal element. In such case, we may say that L *well-orders* X.

We follow Roberts (1979) and call a *strict weak order* on a set X a relation $\prec$ on X satisfying the condition: for all $x, y, z \in X$,

$$x \prec y \quad \Rightarrow \quad \begin{cases} \neg(y \prec x) \text{ and} \\ \text{either } x \prec z \text{ or } z \prec y \text{ (or both).} \end{cases} \tag{1.10}$$

For the definition of a weak order, see Problem 1.10.

1.8.4 Equivalence Relations, Partitions. A binary relation R is an *equivalence relation* on a set X if it is reflexive, transitive, and *symmetric* on X, that is, for all x, y in X, we have

$$xRy \quad \Longleftrightarrow \quad yRx.$$

The following construction is standard. Let R be a quasi order on a set X. Define the relation $\sim$ on X by the equivalence

$$x \sim y \quad \Longleftrightarrow \quad (xRy \;\&\; yRx). \tag{1.11}$$

It is easily seen that the relation $\sim$ is reflexive, transitive, and symmetric on X, that is, $\sim$ is an equivalence relation on X. For any x in X, define the set $\langle x \rangle = \{y \in X \,|\, x \sim y\}$. The family $\mathcal{Z} = X/\!\sim\, = \{\langle x \rangle \,|\, x \in X\}$ of subsets of X is called the *partition* of X *induced* by $\sim$. Any partition $\mathcal{Z}$ of X satisfies the following three properties

[P1] $Y \in \mathcal{Z}$ implies $Y \neq \varnothing$;
[P2] $Y, Z \in \mathcal{Z}$ and $Y \neq Z$ imply $Y \cap Z = \varnothing$;
[P3] $\cup\mathcal{Z} = X$.

Conversely, any family $\mathcal{Z}$ of subsets of a set X satisfying [P1], [P2], and [P3] is a partition of X. The elements of $\mathcal{Z}$ are called the *classes* of the partition. A partition containing just two classes is sometimes referred to as a *bipartition*.

An example of a bipartition was provided by the family $\{\mathcal{T}^+, \mathcal{T}^-\}$ of our puzzle Example 1.1.1, where $\mathcal{T}^+$ contains all the transformations consisting in adding pieces to the puzzle, and $\mathcal{T}^-$ those containing their reverses, that is, removing those pieces.

1.8.5 Graphs. The language of graph theory is coextensive with that of relations, with the former applying naturally when geometrical representations are used. We will use either of them as appropriate to the situation.

A *directed graph* or *digraph* is a pair (V, A) where V is a set and $A \subseteq V \times V$. The elements of V are referred to as *vertices* and the ordered pairs in A as *arcs* or *directed edges*. The pair (V, A) may be referred to as a *digraph on* V. Several pictorial representations of graphs have been given already. In Figure 1.6, for example, the vertices are all the partial orders on the set $\{a, b, c\}$, and each of the arcs corresponds to the addition of an ordered pair to a partial order, forming a new partial order. This illustrates the usual convention of representing arcs by arrows and vertices by points or small circles. When the arc vw exists, we say that 'there is an arc from v to w.' Two distinct vertices v,w are called *adjacent* when one or both of the two arcs vw or wv exists. In pictorial representations of graphs, loops—circular arrows joining a vertex to itself—are routinely omitted when they provide redundant information.

As will be illustrated by many examples in this monograph, there are digraphs satisfying the condition: there is an arc from v to w whenever there is an arc from w to v and vice versa. (In the terminology of relations introduced in 1.8.4, we say then that the set of arcs regarded as a relation is symmetric.) In such cases, the digraph is called a *graph* and each pair of arcs (vw, wv) is referred as an *edge* of the graph which we denote by $\{v, w\}$. The permutohedron of Figure 1.4 is a geometric representation of a graph. Note that, as is customary or at least frequent, a single line between two points representing adjacent vertices replaces the pair of arrows picturing the two arcs.

Let $\boldsymbol{s}_n$ be a sequence $v_0, v_1, \ldots, v_n$ of vertices in a digraph such that $v_i v_{i+1}$ is an arc, for $0 \leq i \leq n-1$. Such a sequence is called a *walk* from v_0 to v_n. A *segment* of the walk $\boldsymbol{s}_n$ is a subsequence $s_j, s_{j+1}, \ldots, s_{j+k}$, with $0 \leq j \leq j+k \leq n$. The walk $\boldsymbol{s}_n$ is *closed* if $v_0 = v_n$, and *open* otherwise. A walk whose vertices are all distinct, is a *path*. A closed path is a *circuit* or a *cycle*. The *length* of a walk $\boldsymbol{s}_n$ (whether open or closed), is assigned to be n. Notice that the concepts of walk, path and circuit apply to a graph, with a sequence of edges $\{v_i, v_{i+1}\} = (v_i v_{i+1}, v_{i+1} v_i)$, $0 \leq i \leq n-1$. When the digraph is a graph, there is a path of length n from v to w if and only if there is a path of length n from w to v. In such a case, we define the *(graph theoretical) distance* $\delta(v, w)$ between two distinct vertices v and w to be the length of the shortest path from v to w, if there is such a path, and infinity otherwise. We also define $\delta(v, v) = 0$, for all vertices v. It is easily shown (cf. Problem 1.16) that the pair (V, δ) is a metric space, that is, the function δ satisfies Conditions [D1], [D2] and [D3] in 1.8.1 (with $\delta = d$). To avoid ambiguity when more than one graph is under discussion, we may index the distance function by the graph and write $\delta_G(v, w)$ for the distance between the vertices v and w in the graph $G = (V, A)$. We sometimes need to focus on part of a graph. A graph $H = (W, B)$ is a *subgraph* of a graph $G = (V, A)$ if $W \subseteq V$ and $B \subseteq A$; it is an *induced subgraph* of G if $W \subseteq V$ and $B = (W \times W) \cap A$. A graph $H = (W, B)$ is an *isometric* subgraph of a graph $G = (V, A)$ if it is a subgraph of G and moreover

$$\delta_H(v, w) = \delta_G(v, w) \qquad (v, w \in W).$$

Two graphs (V, A) and (W, B) are *isomorphic* if there is a bijection $\varphi : V \to W$ such that

$$\{\varphi(v), \varphi(w)\} \in B \iff \{v, w\} \in A,$$

for all $v, w \in V$. If (W, B) is a subgraph of a graph (U, C), then we say that φ is an *embedding* of (V, A) into (U, C). This embedding is an *isometric embedding* if (W, B) is an isometric subgraph of (U, C).

In the permutohedron graph of Figure 1.4, the subgraph defined by the subset of the six vertices

$$W = \{2134, 2143, 2413, 2431, 2341, 2314\}$$

(forming an hexagonal face of the permutohedron) is isometric. But removing a single point of W would define a subgraph that would not be isometric (see Problem 1.15 in this connection).

A graph $G = (V, A)$ is *connected* if any two distinct vertices of G are endpoints of a path of G. It is *bipartite* if its vertex set can be partitioned into $V = V_1 + V_2$, such that every edge of G connects a vertex in V_1 with a vertex in V_2. A classical result from König (1916) is that a graph is bipartite if and only if it contains no odd circuit, that is, a circuit with an odd number of edges. A *tree* is a connected graph without cycles. A vertex in a tree is a *leaf* if it is adjacent to a single vertex.

For graph terminology and results, see Busacker and Saaty (1965), Bondy and Murphy (1976), Roberts (1984), or Bondy (1995).

1.9 Historical Note and References

The concept of a medium was introduced by Falmagne (1997), as a generalization of conditions satisfied by certain families of relations, such as the family $\mathcal{P}$ of all (strict) partial orders on a finite set S. A key property of such a family, wellgradedness, already encountered in this chapter is that, for any two partial orders $\prec$ and $\prec'$, there necessarily exists a sequence

$$\prec_1 = \prec, \prec_1, \ldots, \prec_n = \prec' \tag{1.12}$$

of partial orders in $\mathcal{P}$ such that any two consecutive partial orders in (1.12) differ by exactly one pair,

$$|\prec_i \,\triangle \prec_{i+1}| = 1 \qquad (i = 1, \ldots, n-1), \tag{1.13}$$

and, moreover, such a path between $\prec$ and $\prec'$ in $\mathcal{P}$ is minimal in the sense that:

$$|\prec \,\triangle \prec'| = n. \tag{1.14}$$

This property, which is a focus of Chapter 5, is also satisfied by other families of relations, such as the semiorders and the biorders (cf. Definition 5.1.1, Formulas (5.13) and (5.9)). This was shown by Doignon and Falmagne

(1997), who referred to it as the 'wellgradedness' of such families. On hindsight, it should perhaps have evoked the condition of transitivity of certain semigroups. Curiously, however, the connection between wellgradedness and the axioms specifying a medium was made via the development of random walk models based on such families in view of some applications in the social sciences. In computing the asymptotic probabilities of the states of these random walks (which are formed by the relations in each of these families; see e.g. Falmagne, 1996; Falmagne and Doignon, 1997) it was realized that essentially—but not exactly—the same limit theorem had to be proven. The axiomatization of the concept of a medium was the natural step taken in Falmagne (1997), which also contains a number of basic results. This work was further extended by Falmagne and Ovchinnikov (2002) and Ovchinnikov (2006) (see also Ovchinnikov and Dukhovny, 2000).

Media were later investigated from an algorithmic standpoint by Eppstein and Falmagne (2002). One of their results concerns the existence of a tight bound for the shortest 'reset sequence' (cf. Ginsburg, 1958; Moore, 1956) for a medium. They also describe a near-linear time algorithm for testing whether an orientation is closed, and a polynomial time one for finding a closed orientation if one exists. Those results are contained in Chapter 10.

Concepts essentially equivalents to media were investigated much earlier by graph theorists under the name of 'partial cubes', that is, isometric subgraphs of hypercubes, beginning with Graham and Pollak (1971). Partial cubes were characterized by Djoković (1973) and Winkler (1984) (see Imrich and Klavžar, 2000, which also contains references to other characterizations of partial cubes). The ties between partial cubes and media are described in Chapter 7. Some critical differences between partial cubes and media lie in the language and notation which, in the case of media theory, are akin to automata theory. Obviously, the language and notation one uses are strongly suggestive and may lead to different types of results, as exemplified in this volume.

The random walk models mentioned above were used for the analysis of opinion polls, in particular those polls concerning the presidential election in the US. Such polls are typically performed several times on the same large sample of respondents. The results are referred to as 'panel data.' Among other queries, the respondent are always asked to provide a type of ranking of the candidates. Assuming that the panel data consist in k polls taken at times $t_1, \ldots, t_k$, this means that each respondent has provided a finite sequence $\prec_{t_1}, \ldots, \prec_{t_k}$ of some kind of order relations, such as weak orders. The form of such data suggests an interpretation in terms of visits, by each respondent, of the states of some random walk on the family of all such order relations. In such a framework, these visits take place in real time and $\prec_{t_1}, \ldots, \prec_{t_k}$ is regarded as a sequence of 'snapshots' revealing an aspect of the opinion of a respondent at times $t_1, \ldots, t_k$. Various random walk models have been used to analyze the results of the 1992 US presidential election opposing Clinton, Bush and Perot (Falmagne et al., 1997; Regenwetter et al., 1999; Hsu and

Regenwetter, in press). The general theory of such random walks on media is developed in Falmagne (1997) and Falmagne et al. (2007). Our last two chapters include an exposition of this work.

A major application of media theory is to learning spaces[6], which were defined on p. 10. As is established in Theorem 4.2.2, a learning space is essentially a 'rooted, closed medium.' Learning spaces provide a model for the structural organization of the feasible knowledge states in a topic, such as algebra or chemistry. The particular learning space associated with a topic is at the core of an algorithm for the assessment of knowledge in that topic. It also guide and monitors the students' learning. Such a system is currently used by many schools and colleges in the US and abroad. Further discussion of this topic will be found in Chapter 13.

In view of the wide diversity of the examples covered in this chapter and later in this book, it seems likely that media have other useful applications.

Problems

Many of the questions asked below will be dealt with in depth in the chapters of this book. Some of the problems cannot be solved in formal sense without using the axioms specifying a medium, which can be found in Definition 2.2.1. In such cases the reader should rely on the intuitive conception of a medium developed in this chapter to analyze the problem and attempt a formalization. We propose such exercises as a useful preparation for the rest of this volume.

1.1 In the medium of the Gauss puzzle 1.1.1, identify two states S and T, and two distinct minimally short paths from S to T that do not have their transformations in exact opposite orders. What are conditions (necessary and sufficient) that garantee that the transformations will be in exact opposite orders?

1.2 A facet of the permutohedron on four elements is either a square or a regular hexagon. Describe the facets of a permutohedron on five elements.

1.3 Certain oriented media, such as that of our first example of the puzzle, are closed for their orientation: suppose that S is any state of the medium, and that $\tau : S \mapsto S\tau$ and $\mu : S \mapsto S\mu$ are any two 'positive' transformations, then whenever $S\tau$ and $S\mu$ are defined as states distinct from S, then $(S\tau)\mu = (S\mu)\tau = S\tau\mu = S\mu\tau$ is also a state of the medium. Such a medium is said to be 'closed' (cf. Chapter 4). Under which condition is the medium induced by a finite set of straight line in the plane (in the sense of Section 1.2.1 and Figure 1.3) a closed medium? Can you prove your response?

[6] The exact connection between learning spaces and media was recognized only recently.

1.4 Let R be a strict partial order. Suppose that $\mathcal{H}$ is a family of relations such that, for each $Q \in \mathcal{H}$, the transitive closure Q^* of Q is equal to R. Verify that we have then $\cap\mathcal{H} = \breve{R}$. (So, the phrase 'the smallest relation the transitive closure of which gives back R' makes sense in 1.8.3.)

1.5 Verify the implication (1.6).

1.6 Prove the inclusion (1.7). Why don't we have the equality?

1.7 The medium represented by its graph in Figure 1.7 is equipped with an orientation. This medium is not closed under that orientation. Also, it does not contain the state $\boxed{\text{A-B-C-X}}$. Modify the rules so as to obtain a closed medium containing the eight states of Figure 1.7, the state $\boxed{\text{A-B-C-X}}$, plus at most two other states.

1.8 Define the Hasse diagram of a strict partial order. Exactly when is the Hasse diagram of a partial order or strict partial order empty?

1.9 We have seen that the medium of all partial orders on the set $\{a, b, c\}$ was closed with respect to the natural orientation of that medium, namely: if P, $P + \{xy\}$ and $P + \{zw\}$ are partial orders in the same medium, then so is $P + \{xy\} + \{zw\}$. Prove or disprove (by a counterexample) that that closedness property holds for all families of partial orders on a finite set.

1.10 In the spirit of the distinction between a partial order and a strict partial order (Definition 1.8.3), define the concept of a weak order $\precsim$ on a set X.

1.11 Let $\mathcal{H}$ be the collection of all the Hasse diagrams of the collection of all the partial order on a given finite set. Is $\mathcal{H}$ well-graded? Hint: Examine Equations (1.12), (1.13) and (1.14) in this connection.

1.12 Let $\mathcal{F}$ and $\mathcal{G}$ be the families of all the partial orders on the sets $\{a, b, c\}$ and $\{a, b, x\}$. Describe $\mathcal{F} \setminus \mathcal{G}$. Does it form a medium?

1.13 Let $\mathcal{F}$ and $\mathcal{G}$ be the families of all the partial orders on two distinct overlapping sets. Describe $\mathcal{F} \cap \mathcal{G}$. Does it form a medium?

1.14 Consider a planar political map of the world (indicating the countries' boundaries). In the style of the hyperplane arrangement of 1.2.1, define the states to be the countries, and let the transformations be the crossings of a single border between two countries. Does this construction define a medium? Why or why not?

1.15 Define a isometric subgraph (W, B) of the permutohedron (L, A) of Figure 1.4, with $W \subset L$, having a maximal number of point. (Thus, adding a single point to W would destroy the isometricity.)

1.16 Let (A, V) be a graph, and let δ be the distance on that graph as defined in 1.8.5. Prove that the pair (A, δ) is a metric space, that is, verify that the three conditions [D1], [D2] and [D3] are satisfied.

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