Preface

Random Schrödinger operators are models for the quantum mechanical description of disordered media. The main aim of the analysis of such models is the understanding of the (charge) transport properties of the material. It turns out that many of the properties of interest of random Schrödinger operators are related to a quantity called *integrated density of states*. It measures or 'counts' the number of electron energy levels of the Hamiltonian per unit volume. An alternative name for the integrated density of states is *spectral distribution function* since it is the distribution function of a spectral measure associated to the random family of operators. Many features of this quantity have an intuitive physical interpretation, others play a prominent role in proofs of key theorems. Moreover, the spectral distribution function is an object of study in other fields of mathematics, like differential geometry, group and von Neumann algebras, and homological algebra.

What are the properties of the integrated density of states which have been studied in the literature? It would be hard to give an exhaustive answer, but there are several classes of questions that have drawn the attention of many authors.

The first class is concerned with the definition and construction of the integrated density of states. Can it be expressed as a limit of a sequence of distribution functions associated to the spectra of 'simpler' operators? These operators are usually restrictions of the original Schrödinger operator to some finite volume set. There are various ways how to choose the approximation sequence of operators. Thus another question comes up naturally: Does this choice influence the final outcome, or does one obtain the same distribution function, independently of the approximation procedure? Furthermore, is there a closed formula for the integrated density of states? If there are various such formulas, are some of them better suited for certain applications than others?

Another circle of ideas concerns the continuity properties of the integrated density of states and its set of points of increase: Can one characterise the location and size of the discontinuities? What is the structure of the sets of constancy of the integrated density of states? Since it is a spectral measure distribution, these questions are intimately related to the spectrum of the random operator. Can one prove quantitative regularity properties of the integrated density of states: Is it log-Hölder, Hölder, or Lipschitz continuous? Does it exhibit even stronger regularity properties like differentiability or analyticity? Is it possible to give upper and lower bounds on the derivative?

Finally, one is interested in the behaviour of the integrated density of states as the energy variable approaches a spectral boundary. The most studied case is the infimum of the spectrum, although the behaviour at very high energies, or at internal spectral edges is of interest, too. Can one give a characteristic law of the asymptotics of the integrated density of states at the boundaries? Is it polynomial, is it exponential? Can one specify or estimate the characteristic exponents?

Of course, the answers to the above questions often depend on the type of random operator one is considering. Thus one my also ask: How do the parameters entering the model influence the abovementioned features? Is there a universal behaviour or some phase transition phenomenon?

We used above the term *random Schrödinger operator* although this terminology refers to just one instance among many types of equivariant Hamiltonians in various geometric settings for which the integrated density of states may be defined. A substantial body of papers is devoted to operators acting on combinatorial graphs, the simplest being the integer lattice $\mathbb{Z}^d$. Others consider operators acting on $L^2(\mathbb{R}^d)$, which includes Schrödinger operators. For models on $\mathbb{R}^d$, one may require a $\mathbb{R}^d$ or a $\mathbb{Z}^d$-equivariance condition. In the latter case, there is still a discrete structure present in the random operator, albeit it acts on continuous space. All settings mentioned so far concern the spaces $\mathbb{Z}^d$ or $\mathbb{R}^d$, and thus Euclidean geometry. Going beyond these, there are interesting related models on covering manifolds, finitely generated groups, as well as combinatorial and metric graphs with a quasi-transitive structure.

In the remainder of the preface we describe briefly the potential audience of the book, the recommended prerequisites, the approach taken to present the material, the selection of topics and the structure of the text.

The aim of the text is to give researchers interested in the subject of random Schrödinger operators an overview of known results and methods. Specialists may find it useful as a guide to further reading. The subject matter of the book draws on various mathematical disciplines. For that reason it was not possible to include all the background material, but the reader can find detailed descriptions of the relevant facts using the references to textbooks and monographs. Thus, the text should be accessible to graduate students who have a working knowledge of selfadjoint operators and quadratic forms, possibly from a course on linear operators in Hilbert space or an advanced functional analysis class. For students without this background any one of the following books is recommended as a reading companion: [18, 19] by Akhiezer and Glazman, [47] by Birman and Solomyak, [110] by Davies, [494] or [495] by Weidmann, or [497] by Werner, the last two references being in German. The reader will find the relevant material also in the treatises [140, 141, 142] by Dunford and Schwartz, [239] by Kato, or [407, 408, 409, 410] by Reed and Simon. For the reader who wants to know more about the physical background of the models discussed here we recommend the monographs [53, 145, 340, 312, 143] where properties of disordered systems are discussed from the point of view of theoretical physics.

As already mentioned, the integrated density of states can be defined in various geometric settings and for operators with various equivariance types. If one aims at discussing all such models in a text, one could treat them subsequently one by one, or order the text according to various results and properties and discuss each time all models. One could also first develop a general approach which covers all models, and prove theorems on an abstract level. Although all these seem viable options, here we choose a different and more modest way: We consider just one model in detail and refer in remarks to sources in the literature where the proof for other variants may be found. More precisely, we concentrate here on operators on a continuous configuration space with a discrete group structure. The most important example is the alloy-type model, a $\mathbb{Z}^d$-ergodic operator acting on $L^2(\mathbb{R}^d)$, but operators on Riemannian manifolds with non-abelian group actions are also considered.

While most of the relevant aspects of the spectral theory of random Schrödinger operators figure in the text, the presentation is centred around the integrated density of states. A broader view is taken in the monographs [81] by Carmona and Lacroix and [389] by Pastur and Figotin which describe the state of the art at the beginning of the 1990s. There are several other text of a survey nature on the subject from the second half of the 1980s, including the introductory article [247] by Kirsch, a section on random Jacobi matrices in [102] by Cycon, Froese, Kirsch and Simon, and the Lifshitz memorial issue [335]. In recent years there have been two more monographs treating related topics. The theory of Anderson localisation for random Schrödinger operators is exposed in detail in [458] by Stollmann. Many features of the spectral distribution function in the context of geometry, group theory and K-theory are discussed in Lück's book [346].

Let us mention a few recent overview articles which discuss certain aspects of the theory covered only marginally in the present book. A survey of localisation results for one-dimensional random models is provided in [459] by Stolz. A detailed account of $\mathbb{R}^d$-ergodic random Schrödinger operators, which model amorphous media, is given by Leschke, Müller and Warzel in [331]. The present text emphasises the construction of the integrated density of states and its continuity properties, while its asymptotic behaviour at the infimum of the spectrum is discussed only in remarks. An overview of the results devoted to the last mentioned topic can be found in the recent [259] by Kirsch and Metzger. There also spectral properties of random surface models are discussed. We mentioned above that it is possible to introduce the integrated density of states in a general, abstract framework applicable to various types of equivariant operators and geometric settings. Such an approach is taken for instance in [328], which covers e.g. Hamiltonians on Euclidean space and lattices, on covering manifolds, Delone sets, and on percolation graphs.

Of course, there are many other excellent sources in the literature not mentioned here in the preface. We hope to have them adequately quoted in the main text.

Here is a sketch of the contents and the structure of the book: The introduction explains how one arrives at random Schrödinger operators starting from the quantum mechanical theory of disordered solids. There we also fix frequently used notation, recall basic facts about selfadjoint operators, and define our models. Finally, the introduction explains the relation between spectral and transport properties of Schrödinger operators as well as the notion of spectral fluctuation boundaries.

The second chapter presents two alternative proofs of the approximation of the integrated density of states by its finite volume analoga. One of the approaches is general enough to be applicable to random Schrödinger and Laplace-Beltrami operators on manifolds.

The third chapter explains the relevance of Wegner estimates and regularity properties of the integrated density of states for other aspects of the theory of random Schrödinger operators. A prominent example would be the use of a Wegner bound in the multiscale proof of localisation.

The last two chapters present each a proof of Wegner's estimate for the alloy type or continuum Anderson model on $L^2(\mathbb{R}^d)$. The reason to present two different methods is that each of them has its own advantages when applied to models exhibiting various non-trivial features. We consider alloy type models with long-range or negative correlations, as well as singular and non-monotone dependence on the coupling constants. Several remarks are devoted to similar results for operators on graphs and manifolds mentioned above. Finally, an appendix is devoted to some facts from the theory of the spectral shift function which are used in the main text. More details can be found in the table of contents.

The present text is a revised version of the thesis [486] prepared for the habilitation at the Department of Mathematics of the Technische Universität Chemnitz, which in turn is based on [483]. W. König, P. Stollmann, and S. Teufel have kindly accepted the request of the Department to act as referees of the thesis and I would like to thank them at this occasion. The material presented here draws to a large extent on joint work with colleagues: I have greatly profited from discussions with T. Antunović, D. Borisov, M. Gruber, M. Helm, D. Hundertmark, R. Killip, W. Kirsch, S. Kondej, V. Kostrykin, D. Lenz, P. Müller, S. Nakamura, N. Peyerimhoff, O. Post and P. Stollmann and enjoyed working with them. This work has been made possible through the financial support of the Deutsche Forschungsgemeinschaft. I thank the staff of Springer in charge of the LNM series for their flexibility and efficiency in the course of the preparation of the manuscript.

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