## 5.2 Elements of Multibody Systems

A multibody system is composed of rigid bodies, of joints and of force elements. Another important element called the carrier body will be explained further below.

Joints and force elements have in common that they connect bodies and that they exert forces of equal magnitude and opposite direction on the two connected bodies. The difference between joints and force elements is the nature of these forces.

In a force element the force vector (direction and magnitude) is a *known* function of the positions and/or the velocities of the two bodies connected by the force element. The simplest force elements are springs and dampers. They are passive elements. Also active force elements (actuators) are admitted in which the force is determined by a *given* control law from observed position and velocity variables. The essential feature of force elements is that they do not create kinematical constraints.

In contrast, the force exerted on two bodies by a joint connecting these bodies is a pure kinematical constraint force. It is caused by frictionless rigid-body contacts. It cannot be expressed as function of position and velocity variables. Constraint forces do not enter the equations of motion because they have zero virtual work and zero virtual power. Note the following definition of joint. The joint connecting two bodies is the complete system of rigid-body contacts between these bodies. This definition has the consequence that two bodies cannot be connected by more than one joint. What this means is illustrated in Fig. 5.1. The two bodies are not connected by two spherical joints but by a single joint. This joint is a revolute joint with a single joint variable. Note also the following convention. A single joint cannot interconnect more than two bodies. What this means is illustrated in Figs. 5.2a,b. The three bodies in Fig. 5.2a are mounted on a single shaft. This shaft produces two revolute joints as is shown in Fig. 5.2b.

Depending on the nature of constraints the degree of freedom $f$ of a joint is any number $1 \leq f \leq 5$. Figures 5.3a–e show as examples five joints which have, in this order, the degrees of freedom $f = 1$, 2, 3, 4 and 5. In Fig. 5.3c two plane surfaces, one on each body, are in contact. In Fig. 5.3d one of the

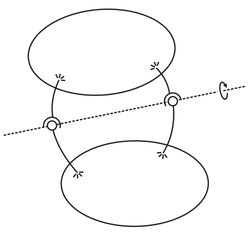


**Fig. 5.1.** Revolute joint connecting two bodies

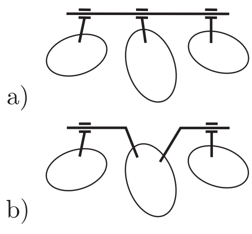



**Fig. 5.2.** The bodies in **(a)** are connected by two joints as is shown in **(b)**

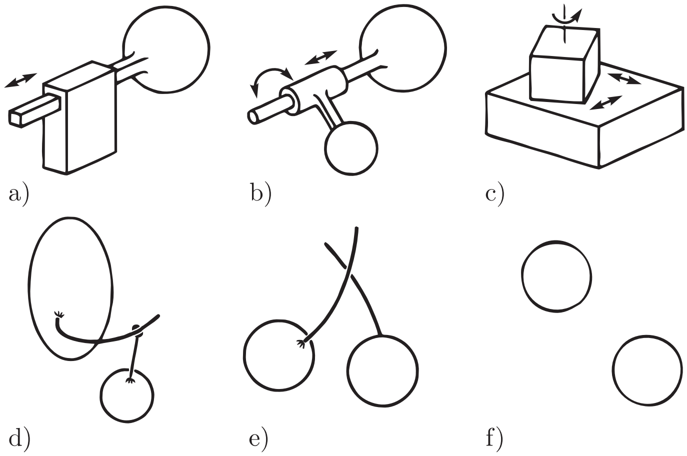



**Fig. 5.3a–f.** Joints with degrees of freedom 1, 2, 3, 4, 5 and 6

bodies is a pendulum whose suspension point is free to move along a guide which is fixed on the other body. In Fig. 5.3e each body has its own guide. The guides are constrained to touch each other but they are free to slip along each other.

In Fig. 5.3f two bodies without any material contact are shown. If it is decided to specify the position of one of the two bodies relative to the other by six variables then the two bodies are said to be connected by a *six-degree-of-freedom joint*. Such joints must be defined whenever without them the position of a single body or of a subsystem relative to the rest of the system would be unspecified.

*The carrier body:* Most multibody systems are connected by joints and/or by force elements to a frame which is fixed in inertial space. More general is the case when the system is connected to a moving carrier body the motion of which is *prescribed* as a function of time. Typical examples are the

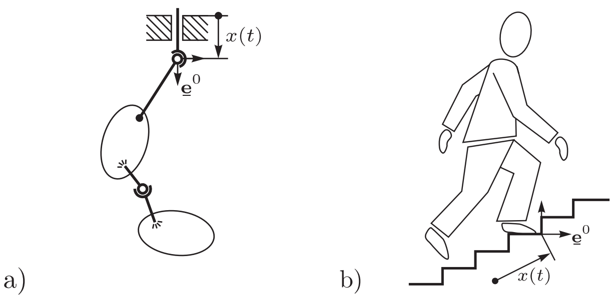
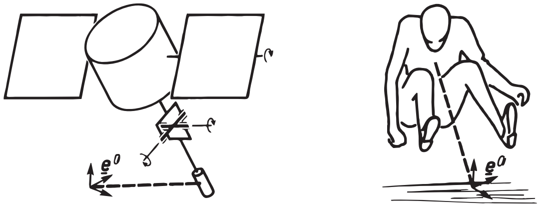



**Fig. 5.4.** Two systems with tree structure coupled to a carrier body, the motion of which is prescribed




a) b)

**Fig. 5.5.** Two systems with tree structure without kinematical constraints to inertial space

double-pendulum with a moving suspension point shown in Fig. 5.4a and the human figure on a moving escalator shown in Fig. 5.4b. It is obvious that the dimensions and inertia properties of the carrier body are irrelevant since its motion is prescribed. It is represented by a moving base $\underline{\mathbf{e}}^0$. The prescribed motion of $\underline{\mathbf{e}}^0$ as well as the properties of joints and of force elements between $\underline{\mathbf{e}}^0$ and the multibody system enter the equations of motion to be developed.

The multibody satellite in Fig. 5.5a and the human figure in Fig. 5.5b are examples of multibody systems without kinematical constraints to inertial space. For describing the position of such systems in inertial space it is necessary to define a six-degree-of-freedom joint connecting one arbitrarily chosen body of the system to a reference base $\underline{\mathbf{e}}^0$. In the two figures this joint is indicated by a broken line. Also in these cases the base $\underline{\mathbf{e}}^0$ is referred to as carrier body. When six-degree-of-freedom joints are taken into account then every multibody system is a *connected* system. By this is meant that between any two bodies of the system including body 0 there exists at least one path along a sequence of bodies and of joints such that no joint is passed more than once. A system is said to have *tree structure* if the path between *any two bodies* of the system is uniquely defined. The systems in Figs. 5.4a,b and 5.5a,b have tree structure. Tree-structured systems have the important property that the joint variables of all joints are kinematically unconstrained. This has the consequence that the total degree of freedom of the entire system equals the sum of the degrees of freedom of the individual joints.

In a system without tree structure the path between two bodies is not uniquely defined for all pairs of bodies. As an example, consider the system in Fig. 5.4b when both feet are in contact with the escalator. The legs, the trunk, the escalator and the connecting ankle, knee and hip joints form what is called a *closed kinematic chain*. The closure of this chain establishes constraints for the joint variables of the joints in the closed chain (and of these joints only). This has the consequence that the total degree of freedom of the entire system is smaller than the sum of the degrees of freedom of the individual joints. Constraint equations must be formulated for every closed kinematic chain individually. From these remarks it is seen that systems with tree structure are more easily analyzed than systems without tree structure.

Furthermore, it is seen that a system without tree structure can be analyzed by adding constraint equations to a system with tree structure. For this reason tree-structured systems are investigated first. They are the subject of Sect. 5.5. Section 5.6 is devoted to the formulation and to the incorporation of constraint equations for closed kinematic chains.

A final remark: Most multibody systems found in engineering have closed kinematic chains. However, systems with tree structure are not as exceptional as the Figs. 5.4a,b and 5.5a,b might suggest. In a road vehicle, for example, the engine is connected to the chassis not by joints but by bushings. A bushing is a force element with an internal force which is known as function of displacement and of velocity of the two connected bodies relative to one another. The same is true for the tires connecting the vehicle to the road. Neither bushings nor tires create closed kinematic chains.

## 5.3 Interconnection Structure of Multibody Systems

In this section mathematical tools are introduced for the description of interconnection structures of multibody systems (Wittenburg [98]). Because of their abstract nature these tools are applicable to interconnections by joints alone, to interconnections by force elements alone and also to interconnections by joints and by force elements in combination. In what follows joints and force elements in combination are considered. As illustrative example the system in Fig. 5.6 is used. The bodies are labeled $0, \ldots, n$ and the connections are labeled $1, \ldots, m$. In the example $n = 7$ and $m = 10$. Both labelings are arbitrary except that body 0 represents the carrier body. The connections labeled $1, \ldots, 7$ symbolize joints of unspecified nature whereas the connections 8, 9, 10 are drawn as force elements. For what follows this distinction is not important, however. The figure points to the fact that two bodies may be connected by more than a single force element.

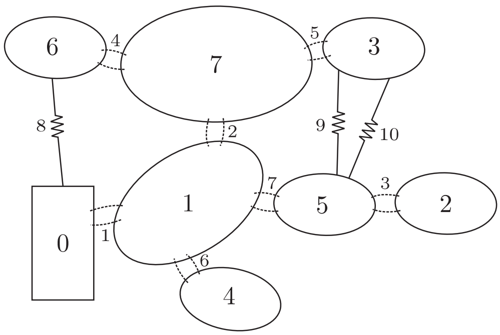



**Fig. 5.6.** Multibody system with joints and force elements. Carrier body 0

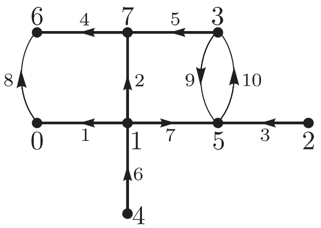
### 5.3.1 Directed System Graph. Associated Matrices

The basic idea is to display the interconnection structure of a multibody system by a graph. The graph consists of points called vertices and of lines connecting the vertices called arcs. The vertices $0, \ldots, n$ represent the bodies of the system, and the arcs $1, \ldots, m$ represent the connections. Since a graph displays neither locations nor physical properties of bodies or connections the vertices can be placed arbitrarily and the arcs can be drawn as straight or as curved lines. Figure 5.7a shows the graph for the system of Fig. 5.6. The graph is connected since the multibody system is connected by its joints (see the text following Figs. 5.5a,b).

To each arc of the graph an arbitrary sense of direction is assigned. It is indicated by the arrows in Fig. 5.7a. The resulting graph is called a directed graph. The sense of direction allows to distinguish the two vertices connected by an arc. This is necessary for two reasons. When formulating the kinematics of motion of two joint-connected bodies relative to one another it must be specified unambiguously which motion relative to which body is meant. Forces produced by a force element act with opposite signs on the two connected bodies. When formulating system dynamics it must be specified unambiguously on which body a force is acting with a positive sign and on which with a negative.

In the previous section it has been shown that the kinematics of multibody systems is simplest if the interconnection by joints is tree-structured. For this reason graphs with tree structure are given special attention. A graph is called tree-structured if between any two vertices there exists a unique minimal chain of arcs and vertices connecting the two vertices. This chain is called the path connecting the two vertices. In a tree-structured graph the identity $m = n$ holds. Proof: Starting with the single vertex 0 one must add one arc every time one vertex is added to the graph.

From a connected graph with $m > n$ arcs a graph with tree structure is produced by deleting $m - n$ suitably chosen arcs. In general, this can be done in more than one way. In Fig. 5.7a the arcs 8, 9 and 10 are deleted. These arcs are drawn with thin lines. The remaining arcs drawn with bold lines constitute what is called a *spanning tree* of the complete graph. In Fig. 5.7b this spanning tree is shown separately. The tree arcs are labeled $1, \ldots, n$ in an

a)


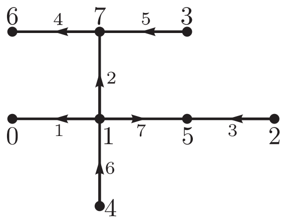

b)



**Fig. 5.7.** Directed graph **(a)** and a spanning tree **(b)** for the system of Fig. 5.6

**Table 5.1.** Integer functions of the directed graph in Fig. 5.7a

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $i^+(a)$ | 1 | 1 | 2 | 7 | 3 | 4 | 1 | 0 | 3 | 5 |
| $i^-(a)$ | 0 | 7 | 5 | 6 | 7 | 1 | 5 | 6 | 5 | 3 |

arbitrary order (in the present case this desired labeling was arranged from the outset). The deleted arcs are called chords. They are labeled $n+1, \ldots, m$ in an arbitrary order. In order to simplify reading indices $i$ and $j$ refer to vertices, indices $a$ and $b$ to arcs (both tree arcs and chords) and the index $c$ to chords alone.

In what follows the complete graph in Fig. 5.7a is considered, again. Each arc $a = 1, \ldots, m$ is incident with two vertices. Let $i^+(a)$ and $i^-(a)$ be the labels of the two vertices at the starting point and at the terminating point of arc $a$, respectively. Thus, $i^+(a)$ and $i^-(a)$ are the names of two integer functions with integer arguments. Both functions can be read from the directed graph. For the directed graph of Fig. 5.7a the two functions are given in Table 5.1. Columns 1 to 7 are associated with the spanning tree. The directed graph is easily reconstructed from its functions $i^+(a)$ and $i^-(a)$ ($a = 1, \ldots, m$). First, $n+1$ vertices labeled $0, \ldots, n$ are marked on a sheet of paper. Then, for every $a = 1, \ldots, m$ an arc is drawn pointing from vertex $i^+(a)$ to vertex $i^-(a)$. The result of this procedure is the original directed graph.

In what follows matrices are defined for directed graphs. The first matrix called *incidence matrix* is defined for the complete graph. It has rows $0, \ldots, n$ and columns $1, \ldots, m$. The rows correspond to vertices and the columns to arcs. The matrix elements are denoted $S_{ia}$ ($i = 0, \ldots, n;\ a = 1, \ldots, m$). They are defined as follows:

$$S_{ia} = \begin{cases} +1 & \text{(arc } a \text{ is incident with and pointing away from vertex } i) \\ -1 & \text{(arc } a \text{ is incident with and pointing toward vertex } i) \\ \phantom{+}0 & \text{(arc } a \text{ is not incident with vertex } i) \end{cases}$$
$$(i = 0, \ldots, n;\ a = 1, \ldots, m)\ . \tag{5.3}$$

This can be expressed in the form

$$S_{ia} = \begin{cases} +1 & (i = i^+(a)) \\ -1 & (i = i^-(a)) \\ \phantom{+}0 & \text{(else)} \end{cases} \qquad (i = 0, \ldots, n;\ a = 1, \ldots, m)\ . \tag{5.4}$$

Still simpler is the formula employing the Kronecker delta

$$S_{ia} = \delta_{i,i^+(a)} - \delta_{i,i^-(a)} \qquad (i = 0, \ldots, n;\ a = 1, \ldots, m)\ . \tag{5.5}$$

The incidence matrix is partitioned into the row matrix $\underline{S}_0$ which corresponds to vertex 0 and the $(n \times m)$-matrix $\underline{S}$ composed of the elements $S_{ia}$ ($i = 1, \ldots, n;\ a = 1, \ldots, m$). Both these matrices are further partitioned into submatrices $\underline{S}_{0\mathrm{t}}$ and $\underline{S}_{\mathrm{t}}$ associated with the spanning tree (columns $a = 1, \ldots, n$) and submatrices $\underline{S}_{0\mathrm{c}}$ and $\underline{S}_{\mathrm{c}}$ associated with chords (columns $a = n+1, \ldots, m$):

$$\begin{aligned} \underline{S}_0 &= [\ \underline{S}_{0\mathrm{t}} \quad \underline{S}_{\mathrm{c}}\ ]\,, \\ \underline{S} &= [\ \underline{S}_{\mathrm{t}} \quad \underline{S}_{\mathrm{c}}\ ]\,. \end{aligned} \tag{5.6}$$

Example: For the directed graph of Fig. 5.7a the matrices are

$$\underline{S}_0 = \left[\begin{array}{ccccccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \end{array}\right], \tag{5.7}$$

$$\underline{S} = \left[\begin{array}{ccccccc|ccc} +1 & +1 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & -1 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{5.8}$$

The submatrices to the left of the partitioning lines are associated with the spanning tree. From the definition (5.4) it follows that every column of $\underline{S}_0$ and $\underline{S}$ together contains exactly one element $+1$ and one element $-1$. Hence, the sum of rows $0, \ldots, n$ is a row of zeros. Expressed in matrix form this is the equation

$$\underline{S}_0 + \underline{1}^T \underline{S} = \underline{0} \tag{5.9}$$

with the row matrix $\underline{1}^T = [1\ 1\ \ldots\ 1]$.

In every row $j$ ($j = 0, \ldots, n$) the number of nonzero elements equals the number of arcs which are incident with vertex $j$. If a row $j$ of $\underline{S}$ has a single nonzero element $S_{jb}$ then the vertex $j$ is incident with arc $b$ alone. This means that the vertex $j$ is a terminal vertex of the graph. Example: The graph in Fig. 5.7a has the terminal vertices 2 and 4, and the spanning tree in Fig. 5.7b has the terminal vertices 2, 3, 4 and 6 (these are the rows of $\underline{S}_{\mathrm{t}}$ with a single nonzero element). The matrix $\underline{S}$ alone suffices to reconstruct the directed graph and, hence, the functions $i^+(a)$ and $i^-(a)$ ($a = 1, \ldots, m$). The matrix $\underline{S}_0$ is not required, because a single nonzero element in a column $b$ of $\underline{S}$ indicates that arc $b$ is incident with vertex 0.

The second matrix called *path matrix* $\underline{T}$ is defined for tree-structured directed graphs only. Also this matrix has elements $+1$, $-1$ and zero. Like $\underline{S}_{\mathrm{t}}$ it is an $(n \times n)$-matrix. The elements are denoted $T_{ai}$. The letters $a$ and $i$ indicate that in this matrix rows correspond to arcs and columns to vertices. The elements are defined as follows:

$$T_{ai} = \begin{cases} +1 & \text{(arc } a \text{ is on the path between vertices 0 and } i \\ & \quad \text{and is directed toward vertex 0)} \\ -1 & \text{(arc } a \text{ is on the path between vertices 0 and } i \\ & \quad \text{and is directed toward vertex } i) \\ 0 & \text{(arc } a \text{ is not on the path between vertices 0 and } i) \end{cases}$$

$$(i, a = 1, \ldots, n)\,. \tag{5.10}$$

There is no column corresponding to vertex 0. Example: The spanning tree in Fig. 5.7b has the path matrix

$$\underline{T} = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} . \tag{5.11}$$

From the definition (5.10) it follows that in every row of $\underline{T}$ all nonzero elements are identical. Every row has at least one nonzero element. If in row $b$ the element $T_{bj}$ is the only nonzero element then vertex $j$ is a terminal vertex incident with arc $b$.

Since the spanning tree is uniquely determined by the matrix $\underline{S}_{\mathrm{t}}$ also the path matrix $\underline{T}$ is uniquely determined by $\underline{S}_{\mathrm{t}}$. Proposition 1: One matrix is the inverse of the other:

$$\underline{T} = \underline{S}_{\mathrm{t}}^{-1} . \tag{5.12}$$

Proposition 2: The matrices $\underline{T}$ and $\underline{S}_{0\mathrm{t}}$ are related through the equation

$$\underline{S}_{0\mathrm{t}}\underline{T} = -\underline{1}^T \tag{5.13}$$

(the matrix $\underline{1}^T = [1\ 1\ \dots\ 1]$ is known from (5.9)).

For proving (5.12) it suffices to show that $\underline{T}\,\underline{S}_{\mathrm{t}}$ is the unit matrix. This product is an $(n \times n)$-matrix with elements $(\underline{T}\,\underline{S}_{\mathrm{t}})_{ab} = \sum_{i=1}^{n} T_{ai}S_{ib}$ $(a, b = 1, \dots, n)$. According to (5.4) $S_{ib}$ is equal to $+1$ for $i = i^+(b)$, equal to $-1$ for $i = i^-(b)$ and zero otherwise. Therefore, $(\underline{T}\,\underline{S}_{\mathrm{t}})_{ab} = T_{ai^+(b)} - T_{ai^-(b)}$. First, the case $b = a$ is considered. Arc $a$ is either directed toward vertex 0 or away from vertex 0. If the former is true then $T_{a,i^+(a)} = 1$, $T_{a,i^-(a)} = 0$. If the latter is true then $T_{a,i^+(a)} = 0$, $T_{a,i^-(a)} = -1$. Hence, in either case $(\underline{T}\,\underline{S}_{\mathrm{t}})_{aa} = 1$. Next, the case $b \neq a$ is investigated. Consider the path between the vertices 0 and $i^+(b)$ and the path between the vertices 0 and $i^-(b)$. Arc $a$ belongs either to both paths or to none of them. In either case, $T_{a,i^+(b)} = T_{a,i^-(b)}$ and, hence, $(\underline{T}\,\underline{S}_{\mathrm{t}})_{ab} = 0$. End of proof.

For proving (5.13) it must be shown that $\sum_{a=1}^{n} S_{0a}T_{ai} = -1$ for $i = 1, \dots, n$. This is, indeed the case, since for every vertex $i$ a single arc $b(i)$ satisfies the condition $S_{0b}T_{bi} \neq 0$ (arc $b$ is on the path between vertices 0 and $i$ and incident with vertex 0). Furthermore, $S_{0b}T_{bi} = -1$ independent of the sense of direction of this arc $b$. End of proof.

The existence of the path matrix $\underline{T}$ for tree-structured graphs and the two equations (5.12) and (5.13) are the mathematical reasons why multibody systems with tree structure are simpler than systems without tree structure.

In contrast to the matrix $\underline{S}_{\mathrm{t}}$ the matrix $\underline{T}$ is not easily determined directly from the two functions $i^+(a)$ and $i^-(a)$ of the spanning tree. It is equally difficult to reconstruct $i^+(a)$ and $i^-(a)$ from $\underline{T}$. An efficient method is described in Sect. 5.3.3 on regular labeling.

**Problem 5.1.** Give a direct proof for the statement $\underline{S}_{\rm t}\underline{T} = \underline{I}$ in (5.12).

**Problem 5.2.** Draw a tree-structured directed graph as follows. Arc $a$ is directed from vertex $a$ toward vertex 0 $(a = 1, \ldots, n)$. Determine the functions $i^+(a)$ and $i^-(a)$ and the matrices $\underline{S}_{0\rm t}$, $\underline{S}_{\rm t}$ and $\underline{T}$.

**Problem 5.3.** Delete in Fig. 5.7a arcs $1, \ldots, 7$ and give to arcs 8, 9, 10 the new labels 1, 2, 3, respectively. The *unconnected* directed graph thus defined has functions $i^+(a)$ and $i^-(a)$ $(a = 1, 2, 3)$ and an $(8 \times 3)$ incidence matrix with elements $S_{ia}$ $(i = 0, \ldots, 7\,,\ a = 1, 2, 3)$ defined by (5.3). Determine this incidence matrix.

For connected directed graphs without tree structure two more matrices are defined. As illustrative example the graph in Fig. 5.7a is used, again. As before it is referred to as complete graph in contrast to its spanning tree shown in bold lines and separately in Fig. 5.7b.

Each arc of the spanning tree defines a *cutset* of the complete graph. It consists of the tree arc itself and of the minimal set of chords which must be cut in order to split the complete graph in two subgraphs. The cutset associated with arc $a$ is also called cutset $a$. Example: The cutset 7 of the graph in Fig. 5.7a consists of the tree arc 7 and of the chords 9 and 10. A chord belonging to cutset $a$ is said to be positively directed (in the cutset) if it points toward the same subgraph as arc $a$ does. Otherwise it is negatively directed.

Each chord defines a *circuit* of the complete graph. It consists of the chord itself and of the minimal set of tree arcs creating a circuit. The circuit associated with chord $c$ is also called circuit $c$. Example: The circuit 8 of the graph in Fig. 5.7a consists of the chord 8 and of the tree arcs 1, 2 and 4. A tree arc belonging to circuit $c$ is said to be positively directed (in the circuit) if its sense of direction around the circuit is the same as that of chord $c$. Otherwise it is negatively directed.

After this introduction the $(n \times m)$ cutset matrix $\underline{P}$ and the $[(m-n) \times m]$ circuit matrix $\underline{U}$ are defined. Their elements are denoted $P_{ab}$ and $U_{ca}$, respectively. Each row of $\underline{P}$ corresponds to a cutset and each row of $\underline{U}$ corresponds to a circuit. The columns of both matrices correspond to the arcs $a = 1, \ldots, m$ of the complete graph. The matrix elements are defined as follows:

$$P_{ab} = \begin{cases} +1 & \text{(arc } b \text{ belongs to cutset } a \text{ and is positively directed)} \\ -1 & \text{(arc } b \text{ belongs to cutset } a \text{ and is negatively directed)} \\ \phantom{+}0 & \text{(arc } b \text{ does not belong to cutset } a\text{)} \end{cases}$$
$$(a = 1, \ldots, n;\ b = 1, \ldots, m)\ , \tag{5.14}$$

$$U_{ca} = \begin{cases} +1 & \text{(arc } a \text{ belongs to circuit } c \text{ and is positively directed)} \\ -1 & \text{(arc } a \text{ belongs to circuit } c \text{ and is negatively directed)} \\ \phantom{+}0 & \text{(arc } a \text{ does not belong to circuit } c\text{)} \end{cases}$$
$$(c = n+1, \ldots, m;\ a = 1, \ldots, m)\ . \tag{5.15}$$

Like the matrices $\underline{S}_0$ and $\underline{S}$ also $\underline{P}$ and $\underline{U}$ are partitioned into submatrices $\underline{P}_\mathrm{t}$, $\underline{P}_\mathrm{c}$ and $\underline{U}_\mathrm{t}$, $\underline{U}_\mathrm{c}$ associated with the spanning tree and with chords, respectively. From the definitions it follows that $\underline{P}_\mathrm{t}$ and $\underline{U}_\mathrm{c}$ are both unit matrices (of different dimensions). Thus

$$\underline{P} = [\,\underline{I} \quad \underline{P}_\mathrm{c}]\,, \qquad \underline{U} = [\underline{U}_\mathrm{t} \quad \underline{I}\,]\,. \tag{5.16}$$

For the directed graph and its spanning tree shown in Figs. 5.7a and b the matrices are

$$\underline{P} = \left[\begin{array}{ccccccc|ccc} +1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & +1 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & +1 & -1 \end{array}\right]\,, \tag{5.17}$$

$$\underline{U} = \left[\begin{array}{ccccccc|ccc} +1 & -1 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 & -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & +1 & 0 & +1 & 0 & 0 & +1 \end{array}\right]\,. \tag{5.18}$$

Between the matrices $\underline{P}$, $\underline{U}$, $\underline{S}$ and $\underline{T}$ there exist numerous relationships. First, the orthogonality relationship

$$\underline{S}\,\underline{U}^T = \underline{0}\,. \tag{5.19}$$

Proof: A single element of the product is

$$(\underline{S}\,\underline{U}^T)_{ic} = \sum_{a=1}^{m} S_{ia}U_{ca} = \sum_{a=1}^{n} S_{ia}U_{ca} + \sum_{a=n+1}^{m} S_{ia}\underbrace{U_{ca}}_{\delta_{ca}} = \sum_{a=1}^{n} S_{ia}U_{ca} + S_{ic} \tag{5.20}$$

($i = 1,\ldots,n;\ c = n+1,\ldots,m$). Two cases must be distinguished. Case 1: Vertex $i$ is incident with chord $c$ ($S_{ic} \neq 0$). Then, vertex $i$ is incident with exactly one tree arc, and the sum over $a$ is equal to $-S_{ic}$ independent of the senses of direction of chord $c$ and of this single tree arc. Case 2: Vertex $i$ is not incident with chord $c$ ($S_{ic} = 0$). Then, vertex $i$ is incident with two tree arcs, and the sum over $a$ is equal to zero independent of whether these two tree arcs belong to circuit $c$ or not. End of proof.

The next relationship is

$$\underline{T}\,\underline{S}_\mathrm{c} = \underline{P}_\mathrm{c}\,. \tag{5.21}$$

Proof: A single element of the product is

$$(\underline{T}\,\underline{S}_\mathrm{c})_{ac} = \sum_{i=1}^{n} T_{ai}S_{ic} = T_{a,i^+(c)} - T_{a,i^-(c)} \quad (a = 1,\ldots,n;\ c = n+1,\ldots,m)\,. \tag{5.22}$$

Two cases must be distinguished. Case 1: Chord $c$ belongs to cutset $a$. Then, independent of the senses of direction of arc $a$ and of chord $c$, $T_{a,i^+(c)} - T_{a,i^-(c)} = P_{ac} \neq 0$. Case 2: Chord $c$ does not belong to cutset $a$. Then, $T_{a,i^+(c)} = T_{a,i^-(c)}$ and, hence, $T_{a,i^+(c)} - T_{a,i^-(c)} = 0 = P_{ac}$. End of proof.

Equations (5.12) and (5.21) together establish the relationship

$$\underline{T}\,\underline{S} = \underline{P}\,. \tag{5.23}$$

When this is postmultiplied by $\underline{U}^T$ one gets, due to (5.19), the equation

$$\underline{P}\,\underline{U}^T = \underline{0}\,. \tag{5.24}$$

This represents another orthogonality relationship. Using the partitioning of (5.16), the equation takes the form $\underline{U}_\mathrm{t}^T + \underline{P}_\mathrm{c} = \underline{0}$ or

$$\underline{U}_\mathrm{t}^T = -\underline{P}_\mathrm{c}\,. \tag{5.25}$$

The matrices $\underline{S}$, $\underline{P}$ and $\underline{U}$ were known to mathematicians for a long time (see Busacker/Saaty [12]). The path matrix $\underline{T}$ was first defined by Branin [8] for electrical networks and independently by Roberson/Wittenburg [66] for multibody systems.

### 5.3.2 Directed Graphs with Tree Structure

For graphs with tree structure a few more definitions are introduced. For every arc $a = 1, \ldots, n$ a number $\sigma_a$ is defined:

$$\sigma_a = \begin{cases} +1 & \text{(arc } a \text{ is directed toward vertex 0)} \\ -1 & \text{(arc } a \text{ is directed away from vertex 0)} \end{cases} \qquad (a = 1, \ldots, n)\,. \tag{5.26}$$

Examples: The graph in Fig. 5.7b has $\sigma_3 = +1$ and $\sigma_4 = -1$.

For every pair of arcs $a, b$ a set $\kappa_{ab}$ of vertices is defined as follows. Cutting two arcs $a$ and $b$ $(a, b = 1, \ldots, n)$ results either in three subgraphs $(a \neq b)$ or in two subgraphs $(a = b)$. In the case $a \neq b$ $\kappa_{ab}$ is the set of vertices of that subgraph which contains no vertex which is incident with arc $b$. In the case $a = b$ $\kappa_{aa}$ is the set of vertices of that subgraph which does not contain vertex 0. Examples: In the graph of Fig. 5.7b $\kappa_{25}$ is the set of vertices 0, 1, 2, 4, 5 and $\kappa_{55}$ contains only vertex 3.

For arcs as well as for vertices weak ordering relationships are defined. For two arcs $a$ and $b \neq a$ the ordering relationship arc $a$ < arc $b$ means that arc $a$ is on the path from vertex 0 to vertex $i_{i^+(b)}$ (and also on the path from vertex 0 to vertex $i_{i^-(b)}$). Note that two arcs $a$ and $b$ located on different branches of the tree as seen from vertex 0 satisfy neither the relationship arc $a$ < arc $b$ nor the relationship arc $b$ < arc $a$.

Similarly, the relationship vertex $i$ < vertex $j$ means that vertex $i$ is on the path from vertex 0 to vertex $j$, but that it is not vertex $j$. In some places this is written in the short form $\mathrm{v}_i < \mathrm{v}_j$. Two vertices $i$ and $j$ located on different branches as seen from vertex 0 satisfy neither the relationship $\mathrm{v}_i < \mathrm{v}_j$ nor the relationship $\mathrm{v}_j < \mathrm{v}_i$.

Next, the inboard arc of a vertex and the inboard vertex of a vertex are defined. The inboard arc of a vertex $j \neq 0$ is the arc which is located on the path between the vertices 0 and $j$ and which, furthermore, is incident with vertex $j$. The inboard vertex of a vertex $j \neq 0$ is the vertex which is connected with vertex $j$ by the inboard arc of vertex $j$. Example: In the graph of Fig. 5.7b arc 7 and vertex 1 are the inboard arc and the inboard vertex, respectively, of vertex 5.

**Problem 5.4.** For the graph in Fig. 5.7b specify the sets $\kappa_{52}$ and $\kappa_{22}$.

**Problem 5.5.** For the graph in Fig. 5.7b specify the sets of all vertices $i$ which satisfy the following conditions (one at a time) for $k = 3$ and for $k = 5$
1. $\mathrm{v}_i < \mathrm{v}_k$ , 2. $\mathrm{v}_k < \mathrm{v}_i$.

### 5.3.3 Regular Tree Graphs

In Fig. 5.7b the labeling of vertices and arcs and the sense of directions of the arcs were intentionally unsystematic in order to show that (5.12) relating the matrix $\underline{S}_\mathrm{t}$ and the path matrix $\underline{T}$ is universally valid for directed tree-structured graphs. In what follows a regular labeling and regular arc directions are defined.

A tree-structured graph is regularly directed if all arcs $1, \ldots, n$ are pointing toward vertex 0. A labeling is called regular if the following two conditions are satisfied

- along every branch starting from vertex 0 the sequence of vertex numbers is monotonically increasing
- the inboard arc of vertex $j$ $(j = 1, \ldots, n)$ is arc $j$.

In general, there is more than one way in which numbers can be assigned satisfying these conditions. Any such labeling is called regular. A directed tree graph is called regular if it has regular labeling and regular arc directions. Then, the two functions $i^+(a)$ and $i^-(a)$ have the properties

$$i^+(a) = a \; , \qquad i^-(a) < a \qquad (a = 1, \ldots, n) \; . \tag{5.27}$$

Under these conditions the function $i^-(a)$ alone suffices for defining the interconnection structure. Furthermore, the matrices $\underline{S}_\mathrm{t}$ and $\underline{T}$ are both upper triangular matrices. Both matrices have elements $+1$ along the diagonal, and all nonzero elements of $\underline{T}$ are $+1$.

In what follows an algorithm is explained which starts out from given functions $i^+(a)$ and $i^-(a)$ of an unsystematically labeled and unsystematically directed graph. The vertices and arcs are regularly relabeled and the arcs are regularly redirected. Between the original and the new labeling a one-to-one relationship is established. The method is explained by taking as example

**Table 5.2.** Conversion from old labeling to new regular labeling

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 arc $a$; old labeling | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 vertex $i^+(a)$; old labeling | 1 | 1 | 2 | 7 | 3 | 4 | 1 |
| 3 vertex $i^-(a)$; old labeling | 0 | 7 | 5 | 6 | 7 | 1 | 5 |
| 4 vertex; old labeling | 1 | 7 | 2 | 6 | 3 | 4 | 5 |
| 5 vertex and inboard arc $j$; new labeling | 1 | 2 | 7 | 4 | 3 | 5 | 6 |
| 6 inboard vertex $i^-(j)$; new labeling | 0 | 1 | 6 | 2 | 2 | 1 | 1 |

the graph in Fig. 5.7b. Its functions $i^+(a)$ and $i^-(a)$ are copied from Table 5.1 into rows 1, 2 and 3 of Table 5.2. Imagine that rows 4, 5 and 6 of the table are still empty. Following Table 5.1 it has been explained that a terminal vertex together with its inboard arc and inboard vertex is identified by the fact that the vertex number occurs only once in rows 2 and 3. Furthermore, if this vertex number occurs in row 2 (in row 3) then the inboard arc $a$ is pointing toward vertex 0 (away from vertex 0). The numbers occurring only once are 2, 3, 4 and 6. Arbitrarily, the number 2 in column $a = 3$ is chosen. Arc 3 is pointing toward vertex 0. To the vertex 2 the new number 7 is given ($n = 7$ is the highest number available). In column 3 the numbers 2 and 7 are repeated in rows 4 and 5, respectively. In row 6 no entry is made at this point. Following this procedure column 3 in rows 1, 2 and 3 is deleted. This means that one terminal body and its inboard arc are removed from the graph. For the resulting smaller graph the same procedure is repeated. The vertex numbers occurring only once are 3, 4, 5 and 6. Arbitrarily, the number 5 in column $a = 7$ is chosen. To the vertex with the old number 5 the new number 6 is given (the highest number still available). The numbers 5 (old) and 6 (new) are filled into rows 4 and 5, respectively, of column 7. This procedure is repeated until only the vertex number 1 is left. By the same procedure it is given the new number 1. As final step row 6 is filled in. Consider column 3 again. The vertex labeled 2 (old) and 7 (new) is connected by its inboard arc $a = 3$ (old) to its inboard vertex 5 (old). According to rows 4 and 5 this inboard vertex has the new number 6. This number 6 is the entry in row 6. The same procedure is repeated in every column.

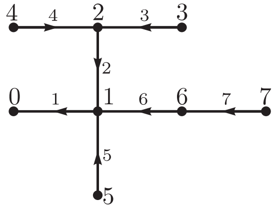



**Fig. 5.8.** Directed graph of Fig. 5.7b regularly relabeled and regularly redirected

Rows 4 and 5 of the table relate old to new vertex numbers and vice versa. Rows 1 and 5 relate old to new arc numbers and vice versa. Rows 5 and 6 together define the function $i^-(j)$ of the regularly labeled and regularly directed graph. This graph is shown in Fig. 5.8. The matrix $\underline{S}_{\mathrm{t}}$ and the path matrix $\underline{T}$ of this graph are

$$\underline{S}_{\mathrm{t}} = \begin{bmatrix} +1 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & +1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix}, \qquad \underline{T} = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ 0 & +1 & +1 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix}. \tag{5.28}$$

The matrix $\underline{T}$ is constructed from rows 5 and 6 of Table 5.2 as follows. For arbitrary $j$ ($j = 1, \ldots, n$) the arcs on the path from $\mathrm{s}_j$ to $\mathrm{s}_0$ have the numbers $j$, $i^-(j)$, $i^-(i^-(j))$, $i^-(i^-(i^-(j)))$, $\ldots, 1$. In column $j$ of $\underline{T}$ the elements in these rows are $+1$. All other elements are zero. Example: $j = 7$ yields the sequence 7, $i^-(7) = 6$ and $i^-(6) = 1$. Hence, $T_{77} = T_{67} = T_{17} = +1$. This is in accordance with (5.28).

**Problem 5.6.** For a system of bodies $i = 0, \ldots, n$ interconnected by joints $a = 1, \ldots, m$ the following quantities are defined

- absolute angular velocity $\boldsymbol{\omega}_i$ of body $i$ ($i = 1, \ldots, n$) ; $\boldsymbol{\omega}_0 = \mathbf{0}$
- angular velocity $\mathbf{\Omega}_a$ of body $i^-(a)$ relative to body $i^+(a)$ in joint $a$ ($a = 1, \ldots, m$)
- internal forces $+\mathbf{F}_a$ and $-\mathbf{F}_a$ produced by a spring in joint $a$ ($a = 1, \ldots, m$); $+\mathbf{F}_a$ is applied to body $i^+(a)$ and $-\mathbf{F}_a$ to body $i^-(a)$
- resultant force $\mathbf{F}_{i_{\mathrm{res}}}$ on body $i$ produced by the springs in all joints on body $i$ ($i = 1, \ldots, n$).

Use the incidence matrix for expressing $\mathbf{\Omega}_a$ ($a = 1, \ldots, m$) in terms of $\boldsymbol{\omega}_i$ ($i = 1, \ldots, n$) and $\mathbf{F}_{i_{\mathrm{res}}}$ ($i = 1, \ldots, n$) in terms of $\mathbf{F}_a$ ($a = 1, \ldots, m$).

Consider the special case of a system with tree structure with bodies $i = 0, \ldots, n$ and joints $a = 1, \ldots, n$. Define the column matrices $\underline{\boldsymbol{\omega}} = [\boldsymbol{\omega}_1 \ \ldots \ \boldsymbol{\omega}_n]^T$, $\underline{\mathbf{\Omega}} = [\mathbf{\Omega}_1 \ \ldots \ \mathbf{\Omega}_n]^T$, $\underline{\mathbf{F}} = [\mathbf{F}_1 \ \ldots \ \mathbf{F}_n]^T$ and $\underline{\mathbf{F}}_{\mathrm{res}} = [\mathbf{F}_{1_{\mathrm{res}}} \ \ldots \ \mathbf{F}_{n_{\mathrm{res}}}]^T$ and write the two sets of $n$ equations each in matrix form. Use the path matrix for resolving these equations for $\underline{\boldsymbol{\omega}}$ and for $\underline{\mathbf{F}}$.

Denote by $\mathbf{c}_{i^+(a),a}$ the vector from the center of mass of body $i^+(a)$ to the point of application of the spring force $+\mathbf{F}_a$ on this body and, likewise, by $\mathbf{c}_{i^-(a),a}$ the vector from the center of mass of body $i^-(a)$ to the point of application of the spring force $-\mathbf{F}_a$ on this body. Define, furthermore, the vectors $\mathbf{C}_{ia} = S_{ia}\mathbf{c}_{ia}$ ($i, a = 1, \ldots, n$) and the $(n \times n)$-matrix $\underline{\mathbf{C}}$ with these vectors as elements. It is a weighted incidence matrix. Express with this matrix the column matrix $\underline{\mathbf{M}}_{\mathrm{res}} = [\mathbf{M}_{1_{\mathrm{res}}} \ \ldots \ \mathbf{M}_{n_{\mathrm{res}}}]^T$ of resultant spring torques on the bodies $i = 1, \ldots, n$.

## 5.4 Principle of Virtual Power for Multibody Systems

Dynamics equations of motion can be obtained either by analytical methods or by the synthetical method starting from Newton's and Euler's equations for isolated bodies. In this section the analytical method based on the principle of virtual power is chosen. In Sect. 3.5 the principle of virtual power has been formulated for a single rigid body (see (3.44)). For an arbitrary system of $n$ rigid bodies the principle has the form

$$\sum_{i=1}^{n} [\delta\dot{\mathbf{r}}_i \cdot (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i) + \delta\boldsymbol{\omega}_i \cdot (\mathsf{J}_i \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathsf{J}_i \cdot \boldsymbol{\omega}_i - \mathbf{M}_i)] = 0 \,. \tag{5.29}$$

Each quantity carries the index $i$ of the individual body. The carrier body 0 is excluded from the sum because its virtual velocity change is zero. The index C referring to the body center of mass has been omitted. So, $\mathbf{r}_i$ is the position vector of the body $i$ center of mass, $\mathbf{F}_i$ is the resultant force acting on body $i$ and applied at the body $i$ center of mass, and $\mathbf{M}_i$ is the resultant torque about the body $i$ center of mass. Contributions to $\mathbf{F}_i$ and to $\mathbf{M}_i$ are made by gravity, by force elements, by sliding friction and by other forces which contribute to virtual power. Constraint forces caused by ideal kinematical constraints in joints do not contribute because for any pair of constraint forces, say $\mathbf{F}_1 = +\mathbf{F}$ and $\mathbf{F}_2 = -\mathbf{F}$ (actio = reactio) the term $\delta\dot{\mathbf{r}}_1 \cdot \mathbf{F}_1 + \delta\dot{\mathbf{r}}_2 \cdot \mathbf{F}_2$ is equal to zero.

In (5.29) the quantities $\delta\dot{\mathbf{r}}_i$, $\delta\boldsymbol{\omega}_i$, $\ddot{\mathbf{r}}_i$ and $\dot{\boldsymbol{\omega}}_i$ appear in linear form. For this reason the following matrix formulation of the equation is possible:

$$\delta\underline{\dot{\mathbf{r}}}^T \cdot (\underline{m}\,\underline{\ddot{\mathbf{r}}} - \underline{\mathbf{F}}) + \delta\underline{\boldsymbol{\omega}}^T \cdot (\underline{\mathsf{J}} \cdot \underline{\dot{\boldsymbol{\omega}}} - \underline{\mathbf{M}}^*) = 0 \tag{5.30}$$

(column matrices $\underline{\mathbf{r}} = [\mathbf{r}_1 \ldots \mathbf{r}_n]^T$, $\underline{\boldsymbol{\omega}} = [\boldsymbol{\omega}_1 \ldots \boldsymbol{\omega}_n]^T$, $\underline{\mathbf{F}} = [\mathbf{F}_1 \ldots \mathbf{F}_n]^T$, diagonal mass matrix $\underline{m}$, diagonal matrix $\underline{\mathsf{J}}$ of inertia tensors). The column matrix $\underline{\mathbf{M}}^*$ is introduced for abbreviation. It has the elements

$$\mathbf{M}_i^* = \mathbf{M}_i - \boldsymbol{\omega}_i \times \mathsf{J}_i \cdot \boldsymbol{\omega}_i \qquad (i = 1, \ldots, n) \,. \tag{5.31}$$

### 5.4.1 Systems Without Constraints to Inertial Space

If a system is free of constraints to inertial space such as an orbiting spacecraft or a flying or freely falling system then Newton's equation of motion for the composite system center of mass C can be decoupled from the remaining equations. This is done as follows. The radius vector of the composite system center of mass C is called $\mathbf{r}_\mathrm{C}$ and the vector from C to the body $i$ center of mass is called $\mathbf{R}_i$. Thus, by definition

$$\mathbf{r}_i = \mathbf{r}_\mathrm{C} + \mathbf{R}_i \quad (i = 1, \ldots, n) \,, \qquad \sum_{i=1}^{n} \mathbf{R}_i m_i = \mathbf{0} \,. \tag{5.32}$$

The expression for $\mathbf{r}_i$ is substituted into (5.29). Multiplying out and using the second Eq. (5.32) one obtains the equation

$$\delta\dot{\mathbf{r}}_\mathrm{C}\cdot\left(M\ddot{\mathbf{r}}_\mathrm{C}-\sum_{i=1}^{n}\mathbf{F}_i\right)+\sum_{i=1}^{n}\left[\delta\dot{\mathbf{R}}_i\cdot(m_i\ddot{\mathbf{R}}_i-\mathbf{F}_i)+\delta\boldsymbol{\omega}_i\cdot(\mathsf{J}_i\cdot\dot{\boldsymbol{\omega}}_i-\mathbf{M}_i^*)\right]=0 \tag{5.33}$$

with $M$ being the total system mass. The special property of a system without constraints to inertial space is the independence of $\delta\dot{\mathbf{r}}_\mathrm{C}$. From this follows the equation

$$M\ddot{\mathbf{r}}_\mathrm{C}=\sum_{i=1}^{n}\mathbf{F}_i\ . \tag{5.34}$$

This is Newton's law for the composite system center of mass. The rest of the equation is written in the matrix form

$$\delta\underline{\dot{\mathbf{R}}}^T\cdot(\underline{m}\,\underline{\ddot{\mathbf{R}}}-\underline{\mathbf{F}})+\delta\underline{\boldsymbol{\omega}}^T\cdot(\underline{\mathsf{J}}\cdot\underline{\dot{\boldsymbol{\omega}}}-\underline{\mathbf{M}}^*)=0\ . \tag{5.35}$$

The formal difference between this equation and (5.30) for arbitrary systems is that $\underline{\mathbf{R}}$ replaces $\underline{\mathbf{r}}$. Between $\underline{\mathbf{R}}$ and $\underline{\mathbf{r}}$ exists a simple relationship. By definition, the radius vector of the composite system center of mass is

$$\mathbf{r}_\mathrm{C}=\frac{1}{M}\sum_{j=1}^{n}m_j\mathbf{r}_j\ . \tag{5.36}$$

Substitution into the first Eq. (5.32) yields

$$\mathbf{R}_i=\sum_{j=1}^{n}\left(\delta_{ij}-\frac{m_j}{M}\right)\mathbf{r}_j\qquad(i=1,\ldots,n)\ . \tag{5.37}$$

Let $\underline{\mu}$ be the dimensionless constant matrix with elements

$$\mu_{ij}=\delta_{ij}-\frac{m_i}{M}\qquad(i,j=1,\ldots,n)\ . \tag{5.38}$$

Then, the matrix form of all $n$ Eqs. (5.37) is

$$\underline{\mathbf{R}}=\underline{\mu}^T\underline{\mathbf{r}}\ . \tag{5.39}$$

The matrix $\underline{\mu}$ has remarkable properties. It satisfies the three equations

$$\underline{\mu}^T\underline{1}=\underline{0}\ ,\qquad\underline{\mu}\,\underline{\mu}=\underline{\mu}\ ,\qquad\underline{\mu}\,\underline{m}=\underline{m}\,\underline{\mu}^T=\underline{\mu}\,\underline{m}\,\underline{\mu}^T\ . \tag{5.40}$$

The first equation states that the sum of all rows is a row of zeros which means that $\underline{\mu}$ is singular. Hence, (5.39) cannot be resolved for $\underline{\mathbf{r}}$. This is obvious for physical reasons. The positions $\underline{\mathbf{r}}$ of the body centers of mass in inertial space cannot be determined if only the positions relative to the composite system center of mass are known. For a proof of the other two equations[1] one must show that $(\underline{\mu}\,\underline{\mu})_{ij} = \mu_{ij}$ and that $(\underline{\mu}\,\underline{m}\,\underline{\mu}^T)_{ij} = (\underline{\mu}\,\underline{m})_{ij} = \mu_{ij}m_j$. For this purpose one calculates

$$(\underline{\mu}\,\underline{\mu})_{ij} = \sum_{k=1}^{n} \mu_{ik}\mu_{kj} = \sum_{k=1}^{n} \mu_{ik}\Big(\delta_{kj} - \frac{m_k}{M}\Big) = \mu_{ij} - \frac{1}{M}\sum_{k=1}^{n} \mu_{ik}m_k \ ,$$

$$\begin{aligned}(\underline{\mu}\,\underline{m}\,\underline{\mu}^T)_{ij} &= \sum_{k=1}^{n} \mu_{ik}m_k\mu_{jk} = \sum_{k=1}^{n} \mu_{ik}m_k\Big(\delta_{jk} - \frac{m_j}{M}\Big) \\ &= \mu_{ij}m_j - \frac{m_j}{M}\sum_{k=1}^{n} \mu_{ik}m_k \ .\end{aligned}$$

The sum $\sum_{k=1}^{n} \mu_{ik}m_k$ appearing in both equations equals zero. End of proof.

Note: Newton's law (5.34) for the composite system center of mass is valid not only for rigid-body systems but for arbitrary systems. So are the definition (5.36) of the composite system center of mass and the relationship (5.39).

### 5.4.2 Generalized Coordinates

Let $\underline{q} = [q_1, \ldots, q_N]^T$ be an arbitrary set of generalized coordinates which are suitable for specifying the location and the orientation of a multibody system. The coordinates may be either joint variables or variables of position relative to inertial space or a combination of the two. At this point it is also not necessary to know whether $\underline{q}$ represents a minimal set of variables equal in number to the degree of freedom of the entire system or whether $N$ exceeds the degree of freedom so that there exist constraint equations for the variables.

The radius vectors $\mathbf{r}_i$ of the body $i$ centers of mass ($i = 1, \ldots, n$) can be expressed as some more or less complicated nonlinear functions $\mathbf{r}_i(q_1, \ldots, q_N, t)$ of the chosen variables and of time $t$. Time $t$ appears explicitly because the position of the carrier body 0 is prescribed as function of time. Differentiation with respect to time produces equations of the general matrix forms

$$\dot{\underline{\mathbf{r}}} = \underline{\mathbf{a}}_1\dot{\underline{q}} + \underline{\mathbf{a}}_{10} \ , \qquad \delta\dot{\underline{\mathbf{r}}} = \underline{\mathbf{a}}_1\delta\dot{\underline{q}} \ , \qquad \ddot{\underline{\mathbf{r}}} = \underline{\mathbf{a}}_1\ddot{\underline{q}} + \underline{\mathbf{b}}_1 \ . \tag{5.41}$$

Here, $\underline{\mathbf{a}}_1$ is an $(n \times N)$-matrix of as yet unknown vectors which depend on $q_1, \ldots, q_N$. The elements of the column matrix $\underline{\mathbf{a}}_{10}$ are the partial derivatives $\partial\mathbf{r}_i(q_1, \ldots, q_N, t)/\partial t$. They depend on $q_1, \ldots, q_N$ and on $t$. The column matrix $\underline{\mathbf{b}}_1$ depends on $q_1, \ldots, q_N$, on $t$ and, in addition, on $\dot{q}_1, \ldots, \dot{q}_N$. If the carrier

[1] A matrix having the property $\underline{\mu}\,\underline{\mu} = \underline{\mu}$ is said to be idempotent (Gantmacher [19]). Every idempotent matrix can be expressed in the form $\underline{A}\,\underline{\Delta}\,\underline{A}^{-1}$ where $\underline{\Delta}$ is a diagonal matrix with elements 0 and 1 along the diagonal. From this it follows that the unit matrix is the only nonsingular idempotent matrix.

body is inertial space then time $t$ does not appear explicitly, whence follows, in particular, that $\underline{\mathbf{a}}_{10} = \underline{\mathbf{0}}$. The formula given for $\delta\dot{\underline{\mathbf{r}}}$ is explained by the fact, that time $t$ as well as $q_1, \ldots, q_N$ are held fixed.

For the vectors from the composite system center of mass to the body centers of mass the kinematical relationship (5.39), $\underline{\mathbf{R}} = \underline{\mu}^T \underline{\mathbf{r}}$, has been established. From this it follows that

$$\dot{\underline{\mathbf{R}}} = \underline{\mu}^T (\underline{\mathbf{a}}_1 \dot{\underline{q}} + \underline{\mathbf{a}}_{10}) \,, \qquad \delta\dot{\underline{\mathbf{R}}} = \underline{\mu}^T \underline{\mathbf{a}}_1 \delta\dot{\underline{q}} \,, \qquad \ddot{\underline{\mathbf{R}}} = \underline{\mu}^T (\underline{\mathbf{a}}_1 \ddot{\underline{q}} + \underline{\mathbf{b}}_1) \,. \tag{5.42}$$

In analogy to (5.41) there exist relationships of the forms

$$\underline{\boldsymbol{\omega}} = \underline{\mathbf{a}}_2 \dot{\underline{q}} + \underline{\mathbf{a}}_{20} \,, \qquad \delta\underline{\boldsymbol{\omega}} = \underline{\mathbf{a}}_2 \delta\dot{\underline{q}} \,, \qquad \dot{\underline{\boldsymbol{\omega}}} = \underline{\mathbf{a}}_2 \ddot{\underline{q}} + \underline{\mathbf{b}}_2 \tag{5.43}$$

with other matrices $\underline{\mathbf{a}}_2$, $\underline{\mathbf{a}}_{20}$ and $\underline{\mathbf{b}}_2$.

The expressions (5.43) and (5.41) are substituted into (5.30) of the principle of virtual power. The terms $\delta\dot{\underline{\mathbf{r}}}^T = \delta\dot{\underline{q}}^T \underline{\mathbf{a}}_1^T$ and $\delta\underline{\boldsymbol{\omega}}^T = \delta\dot{\underline{q}}^T \underline{\mathbf{a}}_2^T$ allow factoring out $\delta\dot{\underline{q}}^T$. Ordering of terms results in the equation

$$\delta\dot{\underline{q}}^T \Big\{ \big( \underline{\mathbf{a}}_1^T \cdot \underline{m}\, \underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2^T \cdot \underline{\mathbf{J}} \cdot \underline{\mathbf{a}}_2 \big) \ddot{\underline{q}} - \big[ \underline{\mathbf{a}}_1^T \cdot (\underline{\mathbf{F}} - \underline{m}\, \underline{\mathbf{b}}_1) + \underline{\mathbf{a}}_2^T \cdot (\underline{\mathbf{M}}^* - \underline{\mathbf{J}} \cdot \underline{\mathbf{b}}_2) \big] \Big\} = 0 \tag{5.44}$$

or abbreviated

$$\delta\dot{\underline{q}}^T (\underline{A} \ddot{\underline{q}} - \underline{B}) = 0 \tag{5.45}$$

with the matrices

$$\left. \begin{aligned} \underline{A} &= \underline{\mathbf{a}}_1^T \cdot \underline{m}\, \underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2^T \cdot \underline{\mathbf{J}} \cdot \underline{\mathbf{a}}_2 \,, \\ \underline{B} &= \underline{\mathbf{a}}_1^T \cdot (\underline{\mathbf{F}} - \underline{m}\, \underline{\mathbf{b}}_1) + \underline{\mathbf{a}}_2^T \cdot (\underline{\mathbf{M}}^* - \underline{\mathbf{J}} \cdot \underline{\mathbf{b}}_2) \,. \end{aligned} \right\} \tag{5.46}$$

For systems without constraints to inertial space such as orbiting spacecraft and freely falling systems the principle of virtual power has the form (5.35). Substitution of the expressions (5.42) and (5.43) into this equation results in the equation

$$\delta\dot{\underline{q}}^T (\hat{\underline{A}} \ddot{\underline{q}} - \hat{\underline{B}}) = 0 \tag{5.47}$$

with matrices (replace in (5.46) $\underline{\mathbf{a}}_1$ by $\underline{\mu}^T \underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_1$ by $\underline{\mu}^T \underline{\mathbf{b}}_1$)

$$\left. \begin{aligned} \hat{\underline{A}} &= \underline{\mathbf{a}}_1^T \underline{\mu} \cdot \underline{m}\, \underline{\mu}^T \underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2^T \cdot \underline{\mathbf{J}} \cdot \underline{\mathbf{a}}_2 \,, \\ \hat{\underline{B}} &= \underline{\mathbf{a}}_1^T \underline{\mu} \cdot (\underline{\mathbf{F}} - \underline{m}\, \underline{\mu}^T \underline{\mathbf{b}}_1) + \underline{\mathbf{a}}_2^T \cdot (\underline{\mathbf{M}}^* - \underline{\mathbf{J}} \cdot \underline{\mathbf{b}}_2) \,. \end{aligned} \right\} \tag{5.48}$$

In what follows it is assumed that the variables $q_1, \ldots, q_N$ represent a minimal set of independent variables equal in number to the degree of freedom of the system under consideration. Then, the variations $\delta\dot{\underline{q}}$ in (5.45) and (5.47) are independent. Hence, two minimal sets of differential equations are obtained in the forms

$$\underline{A}\, \ddot{\underline{q}} = \underline{B} \,, \tag{5.49}$$

$$\hat{\underline{A}} \ddot{\underline{q}} = \hat{\underline{B}} \,. \tag{5.50}$$

The second set of equations is valid only for systems without constraints to inertial space, while the first set is valid for arbitrary systems. The column matrices $\underline{B}$ and $\underline{\hat{B}}$ depend explicitly on $q_1, \ldots, q_N$, on $t$ and, in addition, on $\dot{q}_1, \ldots, \dot{q}_N$. The matrices $\underline{A}$ and $\underline{\hat{A}}$ depend explicitly on $q_1, \ldots, q_N$ only. They are symmetric. They are also positive definite. For $\underline{A}$ this is shown in the case when the carrier body is inertial space ($\underline{\mathbf{a}}_{10} = \underline{\mathbf{0}}$, $\underline{\mathbf{a}}_{20} = \underline{\mathbf{0}}$). The total kinetic energy $T$ of the system is

$$
\begin{aligned}
2T &= \sum_{i=1}^{n} \left( m_i \dot{\mathbf{r}}_i^2 + \boldsymbol{\omega}_i \cdot \mathsf{J}_i \cdot \boldsymbol{\omega}_i \right) = \underline{\dot{\mathbf{r}}}^T \cdot \underline{m}\, \underline{\dot{\mathbf{r}}} + \underline{\boldsymbol{\omega}}^T \cdot \underline{\mathsf{J}} \cdot \underline{\boldsymbol{\omega}} \\
&= \underline{\dot{q}}^T \left( \underline{\mathbf{a}}_1^T \cdot \underline{m}\, \underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2^T \cdot \underline{\mathsf{J}} \cdot \underline{\mathbf{a}}_2 \right) \underline{\dot{q}} = \underline{\dot{q}}^T \underline{A}\, \underline{\dot{q}} \,.
\end{aligned} \tag{5.51}
$$

Thus, the matrix $\underline{A}$ is the coefficient matrix of the total kinetic energy. Since the kinetic energy is positive definite also the matrix $\underline{A}$ is.

As a conclusion of this section it can be stated that a minimal set of equations of motion for a multibody system is explicitly available as soon as the kinematical matrices $\underline{\mathbf{a}}_1$, $\underline{\mathbf{a}}_{10}$, $\underline{\mathbf{b}}_1$ and $\underline{\mathbf{a}}_2$, $\underline{\mathbf{a}}_{20}$, $\underline{\mathbf{b}}_2$ in (5.41) and (5.43) are known in terms of a minimal set of independent variables. Compact expressions can be formulated most easily for multibody systems with tree structure. The next section is devoted to such systems. Based on formulations for tree-structured systems also systems without tree structure can be handled. This is shown in Sect. 5.6.

## 5.5 Systems with Tree Structure

Systems with tree structure are simple for two reasons which have already been explained. The first reason is the kinematical independence of joint variables. For this reason equations of motion are formulated for joint variables. The second reason is the existence of the path matrix as inverse of the incidence matrix. The kinematics of individual joints in terms of joint variables is the subject of Sect. 5.5.1. In Sect. 5.5.2 the kinematics of entire systems is formulated with the help of path matrix and incidence matrix. Sects. 5.5.3–5.5.6 are devoted to various aspects of the resulting equations of motion.

### 5.5.1 Kinematics of Individual Joints

This section focuses on a single joint of a multibody system. In the directed system graph arc $a$ is pointing from vertex $i^+(a)$ toward vertex $i^-(a)$. In Fig. 5.9 a single joint $a$ is shown schematically without indication of its nature. The joint connects the bodies $i^+(a)$ and $i^-(a)$. The arrow indicates the sense of direction of arc $a$ in the directed graph. Body-fixed reference bases $\underline{\mathbf{e}}^{i^+(a)}$ and $\underline{\mathbf{e}}^{i^-(a)}$ are attached to the two bodies at the body centers of mass.

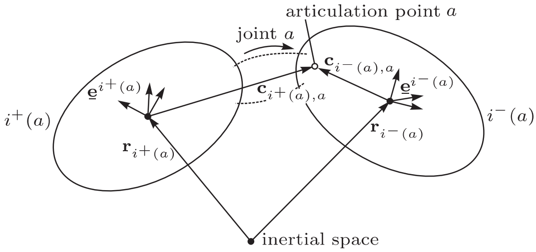



**Fig. 5.9.** Vectors on two bodies coupled by joint $a$ of unspecified nature

Exception: The center of mass of the carrier body 0 is of no interest. The base $\underline{\mathbf{e}}^0$ on this body is attached at some conveniently chosen point. Joint $a$ has a degree of freedom in the range $1 \leq f_a \leq 6$ (see Figs. 5.3a–f). An equal number of joint variables $q_{a\ell}$ $(\ell = 1, \ldots, f_a)$ is chosen. For the majority of joints the variables are rotation angles about certain axes or cartesian coordinates. Other types of variables are not ruled out, however. The following sign convention is adopted. The joint variables describe the position of body $i^-(a)$ relative to body $i^+(a)$.

Altogether six kinematical quantities are formulated for joint $a$ as functions of joint variables. Three of them are the angular orientation, the angular velocity and the angular acceleration of body $i^-(a)$ relative to body $i^+(a)$. First, the other three quantities are explained. These are the position, the velocity and the acceleration relative to body $i^+(a)$ of a single point fixed on body $i^-(a)$. How to choose this point will be explained later. The chosen point is referred to as *articulation point* $a$. Its constant position on body $i^-(a)$ is specified by the vector $\mathbf{c}_{i^-(a),a}$ in base $\underline{\mathbf{e}}^{i^-(a)}$ (see Fig. 5.9). The constant coordinates in this base are system parameters. The position vector of the articulation point in base $\underline{\mathbf{e}}^{i^-(a)}$ is denoted $\mathbf{c}_{i^+(a),a}$. It is a known function of some or of all joint variables of joint $a$. More precisely, the coordinates of the vector in base $\underline{\mathbf{e}}^{i^+(a)}$ are known functions. The articulation point is chosen such that these functions are as simple as possible. Examples: If joint $a$ is a spherical joint then the center of the sphere is chosen. If joint $a$ is a Hooke's joint then the intersection point of the two joint axes on the central cross is chosen. If joint $a$ is a revolute joint then an arbitrary point on the joint axis is chosen. In all three cases the articulation point is fixed on both bodies coupled by joint $a$, i.e. $\mathbf{c}_{i^+(a),a} = \text{const}$ on body $i^+(a)$.

The next two kinematical quantities are the velocity and the acceleration of the articulation point relative to body $i^+(a)$. They are denoted $\mathbf{v}_a$ and $\mathbf{a}_a$, respectively. They are the first and the second time derivatives of $\mathbf{c}_{i^+(a),a}$ in base $\underline{\mathbf{e}}^{i^+(a)}$. They have the forms

$$\mathbf{v}_a = \sum_{\ell=1}^{f_a} \mathbf{k}_{a\ell}\dot{q}_{a\ell}\ , \qquad \mathbf{a}_a = \sum_{\ell=1}^{f_a} \mathbf{k}_{a\ell}\ddot{q}_{a\ell} + \mathbf{s}_a\ . \tag{5.52}$$

A vector $\mathbf{s}_a \neq 0$ exists only if at least one of the vectors $\mathbf{k}_{a\ell}$ is not fixed on body $i^+(a)$.
Examples:

1. Spherical, Hooke's and revolute joints with articulation points chosen as described: $\mathbf{v}_a = \mathbf{0}$, $\mathbf{a}_a = \mathbf{0}$.
2. Cylindrical joint with articulation point on the joint axis, with unit vector $\mathbf{n}$ along the joint axis, with cartesian coordinate $q_{a1}$ along $\mathbf{n}$ and with rotation angle $q_{a2}$ about $\mathbf{n}$: $\mathbf{k}_{a1} = \mathbf{n}$, $\mathbf{k}_{a2} = \mathbf{0}$, $\mathbf{s}_a = \mathbf{0}$.

Next, the angular orientation, the angular velocity and the angular acceleration of body $i^-(a)$ relative to body $i^+(a)$ are expressed in terms of joint variables. The quantity determining the angular orientation is the direction cosine matrix $\underline{A}_a$ relating the bases $\underline{\mathbf{e}}^{i^+(a)}$ and $\underline{\mathbf{e}}^{i^-(a)}$. The definition is

$$\underline{\mathbf{e}}^{i^-(a)} = \underline{A}_a \underline{\mathbf{e}}^{i^+(a)} \; . \tag{5.53}$$

The matrix is a function of the angular variables among the joint variables of joint $a$. Example: In a cylindrical joint with variables $q_{a1}$ and $q_{a2}$ as before $\underline{A}_a$ is a function of $q_{a2}$ only.

The angular velocity and the angular acceleration of body $i^-(a)$ relative to body $i^+(a)$ are denoted $\boldsymbol{\Omega}_a$ and $\boldsymbol{\varepsilon}_a$, respectively. They have the forms

$$\boldsymbol{\Omega}_a = \sum_{\ell=1}^{f_a} \mathbf{p}_{a\ell}\dot{q}_{a\ell} \; , \qquad \boldsymbol{\varepsilon}_a = \sum_{\ell=1}^{f_a} \mathbf{p}_{a\ell}\ddot{q}_{a\ell} + \mathbf{w}_a \; . \tag{5.54}$$

Examples:

1. Prismatic joint: $\boldsymbol{\Omega}_a = \mathbf{0}$, $\boldsymbol{\varepsilon}_a = \mathbf{0}$.
2. Cylindrical joint with unit vector $\mathbf{n}$ and with variables $q_{a1}$ and $q_{a2}$ as before: $\mathbf{p}_{a1} = \mathbf{0}$, $\mathbf{p}_{a2} = \mathbf{n}$, $\mathbf{w}_a = \mathbf{0}$.
3. Hooke's joint with axial unit vectors $\mathbf{p}_{a1}$ fixed on body $i^+(a)$ and $\mathbf{p}_{a2}$ fixed on body $i^-(a)$: $\boldsymbol{\Omega}_a = \mathbf{p}_{a1}\dot{q}_{a1} + \mathbf{p}_{a2}\dot{q}_{a2}$, $\boldsymbol{\varepsilon}_a = \mathbf{p}_{a1}\ddot{q}_{a1} + \mathbf{p}_{a2}\ddot{q}_{a2} + \boldsymbol{\Omega}_a \times \mathbf{p}_{a2}\dot{q}_{a2}$, whence follows $\mathbf{w}_a = \mathbf{p}_{a1} \times \mathbf{p}_{a2}\dot{q}_{a1}\dot{q}_{a2}$.

Alternative formulation for spherical joints: It is known that three angular joint variables (Euler angles or Bryan angles) can be inconvenient. More convenient is the following choice of variables. The matrix $\underline{A}_a$ is expressed in the form (2.35) as function of Euler–Rodrigues parameters. Equations (5.54) are replaced by the equations

$$\boldsymbol{\Omega}_a = \sum_{\ell=1}^{3} \mathbf{p}_{a\ell}\Omega_{a\ell} \; , \qquad \boldsymbol{\varepsilon}_a = \sum_{\ell=1}^{3} \mathbf{p}_{a\ell}\dot{\Omega}_{a\ell} \; . \tag{5.55}$$

The scalars $\Omega_{a\ell}$ ($\ell = 1, 2, 3$) are the coordinates of $\boldsymbol{\Omega}_a$ in base $\underline{\mathbf{e}}^{i^-(a)}$, and the vectors $\mathbf{p}_{a\ell}$ ($\ell = 1, 2, 3$) are the base vectors themselves. The Euler–Rodrigues parameters and the coordinates $\Omega_{a\ell}$ are related through the kinematical differential equations (2.119).

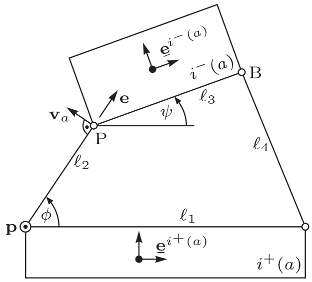



**Fig. 5.10.** Two massless rods creating a 1-d.o.f. joint

The chosen formulations for the six kinematical quantities of a joint are applicable not only to standard joints but to arbitrarily sophisticated joints. This is demonstrated by the joint shown in Fig. 5.10. The two bodies labeled $i^+(a)$ and $i^-(a)$ are coupled by two rods with revolute joints at both ends. The lengths $\ell_1, \ell_2, \ell_3, \ell_4$ form a planar fourbar. It is assumed that the rods are massless. This has the effect that the two rods together create a 1-d.o.f. joint connecting the bodies $i^+(a)$ and $i^-(a)$. The crank angle $\phi$ is chosen as joint variable $q_{a1}$ and the point P as articulation point. The figure explains the angle $\psi$, the unit vector $\mathbf{e}$ along the crank and the unit vector $\mathbf{p}$ normal to the plane. The angle $\psi$ is a function of $\phi$. It is left to the reader to show that it is determined by the equation[2] $A\cos\psi + B\sin\psi = C$ with coefficients $A = -2\ell_3(\ell_1 - \ell_2\cos\phi)$, $B = 2\ell_2\ell_3\sin\phi$, $C = 2\ell_1\ell_2\cos\phi - (\ell_1^2 + \ell_2^2 + \ell_3^2 - \ell_4^2)$. The six kinematical quantities are

$$\mathbf{c}_{i^+(a),a} = \ell_2\mathbf{e} + \text{const}\,, \qquad \mathbf{v}_a = \dot{\phi}\mathbf{p}\times\ell_2\mathbf{e}\,, \quad \mathbf{a}_a = \ddot{\phi}\mathbf{p}\times\ell_2\mathbf{e} - \dot{\phi}^2\ell_2\mathbf{e}\,,$$
$$\underline{A}_a = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Omega}_a = \mathbf{p}\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\,\dot{\phi}\,, \qquad \boldsymbol{\varepsilon}_a = \mathbf{p}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\,\ddot{\phi} + \frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2}\,\dot{\phi}^2\right).$$

The vectors $\mathbf{a}_a$ and $\boldsymbol{\varepsilon}_a$ have the forms (5.52) and (5.54), respectively, with vectors $\mathbf{k}_{a1} = \ell_2\mathbf{p}\times\mathbf{e}$, $\mathbf{s}_a = -\dot{\phi}^2\ell_2\mathbf{e}$, $\mathbf{p}_{a1} = \mathbf{p}\,\mathrm{d}\psi/\mathrm{d}\phi$ and $\mathbf{w}_a = \mathbf{p}\dot{\phi}^2\,\mathrm{d}^2\psi/\mathrm{d}\phi^2$. The coordinates of these vectors in base $\underline{\mathbf{e}}^{i^+(a)}$ are functions of $\phi$.

In preparation for the following section (5.52) and (5.54) are written in the matrix forms

$$\mathbf{v}_a = \underline{\mathbf{k}}_a^T\underline{\dot{q}}_a\,, \qquad \mathbf{a}_a = \underline{\mathbf{k}}_a^T\underline{\ddot{q}}_a + \mathbf{s}_a\,, \tag{5.56}$$
$$\boldsymbol{\Omega}_a = \underline{\mathbf{p}}_a^T\underline{\dot{q}}_a\,, \qquad \boldsymbol{\varepsilon}_a = \underline{\mathbf{p}}_a^T\underline{\ddot{q}}_a + \mathbf{w}_a \tag{5.57}$$

with row matrices $\underline{\mathbf{k}}_a^T = [\mathbf{k}_{a1} \;\ldots\; \mathbf{k}_{af_a}]$ and $\underline{\mathbf{p}}_a^T = [\mathbf{p}_{a1} \;\ldots\; \mathbf{p}_{af_a}]$. For a multibody system with joints $a = 1,\ldots,n$ column matrices $\underline{\mathbf{v}}$, $\underline{\mathbf{a}}$, $\underline{\mathbf{s}}$, $\underline{\boldsymbol{\Omega}}$,

[2] Cartesian coordinates $x_\mathrm{B}$ and $y_\mathrm{B}$ of the point B are functions of $\varphi$ and $\psi$. These expressions are substituted into the constraint equation $(\ell_1 - x_\mathrm{B})^2 + y_\mathrm{B}^2 = \ell_4^2$. Each angle $\phi$ is associated with two (not necessarily real) angles $\psi$.

$\underline{\boldsymbol{\varepsilon}}$ and $\underline{\mathbf{w}}$ of $n$ vectors each are defined, for example $\underline{\mathbf{v}} = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]^T$ and $\underline{\mathbf{w}} = [\mathbf{w}_1 \ \ldots \ \mathbf{w}_n]^T$. In terms of these matrices the four sets of $n$ equations each are combined in the matrix forms

$$\underline{\mathbf{v}} = \underline{\mathbf{k}}^T \underline{\dot{q}} \,, \qquad \underline{\mathbf{a}} = \underline{\mathbf{k}}^T \underline{\ddot{q}} + \underline{\mathbf{s}} \,, \tag{5.58}$$

$$\underline{\boldsymbol{\Omega}} = \underline{\mathbf{p}}^T \underline{\dot{q}} \,, \qquad \underline{\boldsymbol{\varepsilon}} = \underline{\mathbf{p}}^T \underline{\ddot{q}} + \underline{\mathbf{w}} \,. \tag{5.59}$$

The matrices $\underline{\mathbf{k}}^T$ and $\underline{\mathbf{p}}^T$ have block-diagonal form with the row matrices $\underline{\mathbf{k}}_a^T$ and $\underline{\mathbf{p}}_a^T$ along the diagonal, and $\underline{\ddot{q}}$ is the column matrix composed of the blocks $\underline{\ddot{q}}_a$ $(a = 1, \ldots, n)$.

### 5.5.2 Kinematics of Entire Systems

As illustrative example for the general formalism to be developed the tree-structured system shown in Fig. 5.11 is used. It is the system from Fig. 5.6 without force elements. Its directed graph is shown in Fig. 5.7b. For the associated incidence and path matrices see (5.7), (5.8) and (5.11). Each joint is of the general form shown in Fig. 5.9. Dots stand for the articulation points of joints. For joint $a = 2$ the vectors $\mathbf{c}_{72} = \mathbf{c}_{i^+(a),a}$ and $\mathbf{c}_{12} = \mathbf{c}_{i^-(a),a}$ are shown as examples. The carrier body 0 happens to be connected to a single body. The formalism to be developed is not restricted to this special case. Body 0 can be connected to several tree-structured subsystems. As in Fig. 5.7b the labeling of the bodies $1, \ldots, n$ and of the joints $1, \ldots, n$ as well as the directions of arcs in the graph are arbitrary. The following convention is adopted, however. All arcs incident with vertex 0 are directed toward vertex 0. This has the following consequences. The articulation points of all joints located on body 0 are fixed on body 0. The associated vectors $\mathbf{c}_{0a}$ are fixed in base $\underline{\mathbf{e}}^0$ the origin of which has, in inertial space, the prescribed position vector $\mathbf{r}_0(t)$. The position vector $\mathbf{r}_0(t) + \mathbf{c}_{0a}$ of the articulation point in inertial space is a prescribed function of time, too, and so are the absolute velocity

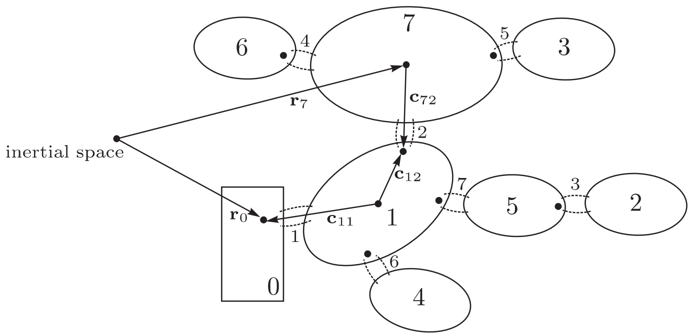



**Fig. 5.11.** Multibody system with tree structure and with joints of unspecified nature. Dots in joints and vectors are explained in Fig. 5.9

$\dot{\mathbf{r}}_0(t) + \boldsymbol{\omega}_0(t) \times \mathbf{c}_{0a}$ and the absolute acceleration

$$\ddot{\mathbf{r}}_0(t) + \dot{\boldsymbol{\omega}}_0(t) \times \mathbf{c}_{0a} + \boldsymbol{\omega}_0(t) \times [\boldsymbol{\omega}_0(t) \times \mathbf{c}_{0a}] \,. \tag{5.60}$$

Without loss of generality let it be assumed that the origin of the base $\underline{\mathbf{e}}^0$ with the prescribed position vector $\mathbf{r}_0(t)$ is one of the articulation points fixed on body 0. Then, the associated vector $\mathbf{c}_{0a}$ is zero. The advantage of this assumption is that systems with a single joint on body 0 do not have nonzero vectors $\mathbf{c}_{0a}$. This is the situation shown in Fig. 5.11 with the vector $\mathbf{c}_{01} = \mathbf{0}$ in joint 1.

The goal of this section is to express the matrices $\underline{\mathbf{a}}_1$, $\underline{\mathbf{a}}_{10}$, $\underline{\mathbf{b}}_1$, $\underline{\mathbf{a}}_2$, $\underline{\mathbf{a}}_{20}$ and $\underline{\mathbf{b}}_2$ in the relationships $\underline{\dot{\mathbf{r}}} = \underline{\mathbf{a}}_1\underline{\dot{q}} + \underline{\mathbf{a}}_{10}$, $\underline{\ddot{\mathbf{r}}} = \underline{\mathbf{a}}_1\underline{\ddot{q}} + \underline{\mathbf{b}}_1$ and $\underline{\boldsymbol{\omega}} = \underline{\mathbf{a}}_2\underline{\dot{q}} + \underline{\mathbf{a}}_{20}$, $\underline{\dot{\boldsymbol{\omega}}} = \underline{\mathbf{a}}_2\underline{\ddot{q}} + \underline{\mathbf{b}}_2$ (see (5.41) and (5.43)) in terms of joint variables and of time derivatives of joint variables. First, angular velocities are considered. Figure 5.11 shows that the absolute angular velocity $\boldsymbol{\omega}_i$ of an arbitrary body $i$ is the sum of $\boldsymbol{\omega}_0$ and of all vectors $\boldsymbol{\Omega}_a$ (some positive and some negative) along the path from body 0 to body $i$. The formula for $\boldsymbol{\omega}_i$ is

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_0 - \sum_{a=1}^{n} T_{ai}\boldsymbol{\Omega}_a \qquad (i = 1, \ldots, n) \,. \tag{5.61}$$

The elements $T_{ai}$ of the path matrix sort out the direct path from body 0 to body $i$ and they provide the correct signs as well. Differentiation with respect to time yields

$$\dot{\boldsymbol{\omega}}_i = \dot{\boldsymbol{\omega}}_0 - \sum_{a=1}^{n} T_{ai}(\boldsymbol{\varepsilon}_a + \underbrace{\boldsymbol{\omega}_{i^-(a)} \times \boldsymbol{\Omega}_a}_{\mathbf{f}_a}) \qquad (i = 1, \ldots, n) \,. \tag{5.62}$$

The matrix forms of these equations are[3]

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega}_0\underline{1} - \underline{T}^T\underline{\boldsymbol{\Omega}} \,, \qquad \underline{\dot{\boldsymbol{\omega}}} = \dot{\boldsymbol{\omega}}_0\underline{1} - \underline{T}^T\underline{\boldsymbol{\varepsilon}} - \underline{T}^T\underline{\mathbf{f}} \,. \tag{5.63}$$

Into these equations the expressions (5.59) are substituted. This results in the equations

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega}_0\underline{1} - \underline{T}^T\underline{\mathbf{p}}^T\underline{\dot{q}} \,, \qquad \underline{\dot{\boldsymbol{\omega}}} = \dot{\boldsymbol{\omega}}_0\underline{1} - \underline{T}^T\underline{\mathbf{p}}^T\underline{\ddot{q}} - \underline{T}^T(\underline{\mathbf{w}} + \underline{\mathbf{f}}) \tag{5.64}$$

or finally

$$\underline{\boldsymbol{\omega}} = -(\underline{\mathbf{p}}\,\underline{T})^T\underline{\dot{q}} + \boldsymbol{\omega}_0\underline{1} \,, \qquad \underline{\dot{\boldsymbol{\omega}}} = -(\underline{\mathbf{p}}\,\underline{T})^T\underline{\ddot{q}} + \dot{\boldsymbol{\omega}}_0\underline{1} - \underline{T}^T(\underline{\mathbf{w}} + \underline{\mathbf{f}}) \,. \tag{5.65}$$

[3] The definition $\mathbf{f}_a = \boldsymbol{\omega}_{i^-(a)} \times \boldsymbol{\Omega}_a$ is equivalent to $\mathbf{f}_a = \boldsymbol{\omega}_{i^+(a)} \times \boldsymbol{\Omega}_a$.
The expression for $\underline{\boldsymbol{\omega}}$ can be obtained without making use of Fig. 5.11. Figure 5.9 yields $\boldsymbol{\Omega}_a = \boldsymbol{\omega}_{i^-(a)} - \boldsymbol{\omega}_{i^+(a)}$ or, with the definition of the incidence matrix, $\boldsymbol{\Omega}_a = -\sum_{i=0}^{n} S_{ia}\boldsymbol{\omega}_i \quad (a = 1, \ldots, n)$. This is written in the matrix form $\underline{\boldsymbol{\Omega}} = -\boldsymbol{\omega}_0\underline{S}_0^T - \underline{S}^T\underline{\boldsymbol{\omega}}$. Multiplication from the left by $\underline{T}^T$ and application of (5.12) and (5.13) result in (5.63).

These equations have the desired forms $\underline{\boldsymbol{\omega}} = \underline{\mathbf{a}}_2\underline{\dot{q}} + \underline{\mathbf{a}}_{20}$ and $\underline{\dot{\boldsymbol{\omega}}} = \underline{\mathbf{a}}_2\underline{\ddot{q}} + \underline{\mathbf{b}}_2$. The matrices are

$$\underline{\mathbf{a}}_2 = -(\underline{\mathbf{p}}\,\underline{T})^T\,, \qquad \underline{\mathbf{a}}_{20} = \boldsymbol{\omega}_0\underline{1}\,, \qquad \underline{\mathbf{b}}_2 = \dot{\boldsymbol{\omega}}_0\underline{1} - \underline{T}^T(\underline{\mathbf{w}} + \underline{\mathbf{f}})\,. \tag{5.66}$$

Next, the matrices $\underline{\mathbf{a}}_1$, $\underline{\mathbf{a}}_{10}$ and $\underline{\mathbf{b}}_1$ are formulated. Figure 5.11 shows that the position vector $\mathbf{r}_i$ of body $i$ is

$$\mathbf{r}_i = \mathbf{r}_0 - \sum_{a=1}^{n} T_{ai}(\mathbf{c}_{i^+(a),a} - \mathbf{c}_{i^-(a),a}) \qquad (i = 1,\ldots,n)\,. \tag{5.67}$$

Of the two vectors $\mathbf{c}_{i^+(a),a}$ and $\mathbf{c}_{i^-(a),a}$ the latter one is fixed on body $i^-(a)$. This explains the expressions for the first and for the second time derivative of the difference vector:

$$\left.\begin{aligned} \dot{\mathbf{c}}_{i^+(a),a} - \dot{\mathbf{c}}_{i^-(a),a} &= -\mathbf{c}_{i^+(a),a} \times \boldsymbol{\omega}_{i^+(a)} + \mathbf{v}_a \\ &\quad + \mathbf{c}_{i^-(a),a} \times \boldsymbol{\omega}_{i^-(a)}\,, \\ \ddot{\mathbf{c}}_{i^+(a),a} - \ddot{\mathbf{c}}_{i^-(a),a} &= -\mathbf{c}_{i^+(a),a} \times \dot{\boldsymbol{\omega}}_{i^+(a)} + \mathbf{a}_a + \mathbf{h}_a \\ &\quad + \mathbf{c}_{i^-(a),a} \times \dot{\boldsymbol{\omega}}_{i^-(a)} \end{aligned}\right\}(a = 1,\ldots,n)\,. \tag{5.68}$$

The vectors $\mathbf{v}_a$ and $\mathbf{a}_a$ are known from (5.52), and $\mathbf{h}_a$ is

$$\begin{aligned} \mathbf{h}_a &= \boldsymbol{\omega}_{i^+(a)} \times (\boldsymbol{\omega}_{i^+(a)} \times \mathbf{c}_{i^+(a),a}) - \boldsymbol{\omega}_{i^-(a)} \times (\boldsymbol{\omega}_{i^-(a)} \times \mathbf{c}_{i^-(a),a}) \\ &\quad + 2\boldsymbol{\omega}_{i^+(a)} \times \mathbf{v}_a \qquad (a = 1,\ldots,n)\,. \end{aligned} \tag{5.69}$$

From the definition (5.4) of the incidence matrix it follows that the vector differences in (5.67) and (5.68) can be written in the forms

$$\mathbf{c}_{i^+(a),a} - \mathbf{c}_{i^-(a),a} = \sum_{i=0}^{n} S_{ia}\mathbf{c}_{ia}\,, \tag{5.70}$$

$$\dot{\mathbf{c}}_{i^+(a),a} - \dot{\mathbf{c}}_{i^-(a),a} = -\sum_{i=0}^{n} S_{ia}\mathbf{c}_{ia} \times \boldsymbol{\omega}_i + \mathbf{v}_a\,, \tag{5.71}$$

$$\ddot{\mathbf{c}}_{i^+(a),a} - \ddot{\mathbf{c}}_{i^-(a),a} = -\sum_{i=0}^{n} S_{ia}\mathbf{c}_{ia} \times \dot{\boldsymbol{\omega}}_i + \mathbf{a}_a + \mathbf{h}_a \tag{5.72}$$

$(a = 1,\ldots,n)$. These formulations suggest to define the vectors

$$\mathbf{C}_{ia} = S_{ia}\mathbf{c}_{ia} \qquad (i = 0,\ldots,n;\ a = 1,\ldots,n) \tag{5.73}$$

and to construct a matrix with these vectors as elements[4]. This matrix represents a weighted incidence matrix. Like the incidence matrix it is partitioned

[4] To be precise one must define that $\mathbf{c}_{ia} = \mathbf{0}$ if $S_{ia} = 0$ $(i = 0,\ldots,n;\ a = 1,\ldots,n)$.

into the row matrix $\underline{\mathbf{C}}_0$ with elements $\mathbf{C}_{0a}$ $(a = 1, \ldots, n)$ and the $(n \times n)$-matrix $\underline{\mathbf{C}}$ with elements $\mathbf{C}_{ia}$ $(i, a = 1, \ldots, n)$. With these matrices the $n$ Eqs. (5.67) are written in the matrix form $\underline{\mathbf{r}} = \mathbf{r}_0\underline{1} - \underline{T}^T\underline{\mathbf{C}}_0^T - \underline{T}^T\underline{\mathbf{C}}^T\underline{1}$ or[5]

$$\underline{\mathbf{r}} = \mathbf{r}_0\underline{1} - (\underline{\mathbf{C}}_0\underline{T})^T - (\underline{\mathbf{C}}\,\underline{T})^T\underline{1}\,. \tag{5.74}$$

For the first and for the second time derivative (5.71) and (5.72) yield the expressions

$$\dot{\underline{\mathbf{r}}} = \dot{\mathbf{r}}_0\underline{1} - \boldsymbol{\omega}_0 \times (\underline{\mathbf{C}}_0\underline{T})^T + (\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\boldsymbol{\omega}} - \underline{T}^T\underline{\mathbf{v}}\,, \tag{5.75}$$

$$\ddot{\underline{\mathbf{r}}} = \ddot{\mathbf{r}}_0\underline{1} - \dot{\boldsymbol{\omega}}_0 \times (\underline{\mathbf{C}}_0\underline{T})^T + (\underline{\mathbf{C}}\,\underline{T})^T \times \dot{\underline{\boldsymbol{\omega}}} - \underline{T}^T(\underline{\mathbf{a}} + \underline{\mathbf{h}}) \tag{5.76}$$

with column matrices $\underline{\mathbf{v}} = [\mathbf{v}_1 \ldots \mathbf{v}_n]^T$, $\underline{\mathbf{a}} = [\mathbf{a}_1 \ldots \mathbf{a}_n]^T$ and $\underline{\mathbf{h}} = [\mathbf{h}_1 \ldots \mathbf{h}_n]^T$. For $\underline{\mathbf{v}}$ and $\underline{\mathbf{a}}$ the expressions (5.58) are substituted and for $\underline{\boldsymbol{\omega}}$ and $\dot{\underline{\boldsymbol{\omega}}}$ the expressions $\underline{\boldsymbol{\omega}} = \underline{\mathbf{a}}_2\dot{\underline{q}} + \boldsymbol{\omega}_0\underline{1}$ and $\dot{\underline{\boldsymbol{\omega}}} = \underline{\mathbf{a}}_2\ddot{\underline{q}} + \underline{\mathbf{b}}_2$ (see (5.65)). This yields

$$\begin{aligned} \dot{\underline{\mathbf{r}}} &= [(\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\mathbf{a}}_2 - (\underline{\mathbf{k}}\,\underline{T})^T]\dot{\underline{q}} \\ &\quad + \dot{\mathbf{r}}_0\underline{1} - \boldsymbol{\omega}_0 \times [(\underline{\mathbf{C}}_0\underline{T})^T + (\underline{\mathbf{C}}\,\underline{T})^T\underline{1}]\,, \end{aligned} \tag{5.77}$$

$$\begin{aligned} \ddot{\underline{\mathbf{r}}} &= [(\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\mathbf{a}}_2 - (\underline{\mathbf{k}}\,\underline{T})^T]\ddot{\underline{q}} \\ &\quad + \ddot{\mathbf{r}}_0\underline{1} - \dot{\boldsymbol{\omega}}_0 \times (\underline{\mathbf{C}}_0\underline{T})^T + (\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\mathbf{b}}_2 - \underline{T}^T(\underline{\mathbf{s}} + \underline{\mathbf{h}})\,. \end{aligned} \tag{5.78}$$

These equations have the desired forms $\dot{\underline{\mathbf{r}}} = \underline{\mathbf{a}}_1\dot{\underline{q}} + \underline{\mathbf{a}}_{10}$ and $\ddot{\underline{\mathbf{r}}} = \underline{\mathbf{a}}_1\ddot{\underline{q}} + \underline{\mathbf{b}}_1$. The matrices are

$$\underline{\mathbf{a}}_1 = (\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\mathbf{a}}_2 - (\underline{\mathbf{k}}\,\underline{T})^T\,, \tag{5.79}$$

$$\underline{\mathbf{a}}_{10} = \dot{\mathbf{r}}_0\underline{1} - \boldsymbol{\omega}_0 \times [(\underline{\mathbf{C}}_0\underline{T})^T + (\underline{\mathbf{C}}\,\underline{T})^T\underline{1}]\,, \tag{5.80}$$

$$\underline{\mathbf{b}}_1 = \ddot{\mathbf{r}}_0\underline{1} - \dot{\boldsymbol{\omega}}_0 \times (\underline{\mathbf{C}}_0\underline{T})^T + (\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\mathbf{b}}_2 - \underline{T}^T(\underline{\mathbf{s}} + \underline{\mathbf{h}})\,. \tag{5.81}$$

With these matrices and with the matrices in (5.66),

$$\underline{\mathbf{a}}_2 = -(\underline{\mathbf{p}}\,\underline{T})^T\,, \qquad \underline{\mathbf{a}}_{20} = \boldsymbol{\omega}_0\underline{1}\,, \qquad \underline{\mathbf{b}}_2 = \dot{\boldsymbol{\omega}}_0\underline{1} - \underline{T}^T(\underline{\mathbf{w}} + \underline{\mathbf{f}})\,, \tag{5.82}$$

the final goal of the analysis has been achieved.

### 5.5.3 Equations of Motion

The matrices (5.79)–(5.82) determine the matrices (cf. (5.46))

$$\left.\begin{aligned} \underline{A} &= \underline{\mathbf{a}}_1^T \cdot \underline{m}\,\underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2^T \cdot \underline{\mathbf{J}} \cdot \underline{\mathbf{a}}_2\,, \\ \underline{B} &= \underline{\mathbf{a}}_1^T \cdot (\underline{\mathbf{F}} - \underline{m}\,\underline{\mathbf{b}}_1) + \underline{\mathbf{a}}_2^T \cdot (\underline{\mathbf{M}}^* - \underline{\mathbf{J}} \cdot \underline{\mathbf{b}}_2) \end{aligned}\right\} \tag{5.83}$$

[5] The expression for $\underline{\mathbf{r}}$ can be obtained without making use of Fig. 5.11. Figure 5.9 yields $\mathbf{r}_{i+(a)} - \mathbf{r}_{i-(a)} = \mathbf{c}_{i-(a),a} - \mathbf{c}_{i+(a),a}$ $(a = 1, \ldots, n)$ or $\sum_{i=0}^n S_{ia}\mathbf{r}_i = -\sum_{i=0}^n \mathbf{C}_{ia}$ $(a = 1, \ldots, n)$. This is written in the matrix form $\mathbf{r}_0\underline{S}_0^T + \underline{S}^T\underline{\mathbf{r}} = -\underline{\mathbf{C}}_0^T - \underline{\mathbf{C}}^T\underline{1}$. Multiplication from the left by $\underline{T}^T$ yields (5.74).

in the equations of motion (5.49) of tree-structured systems:

$$\underline{A}\,\underline{\ddot{q}} = \underline{B}\,. \tag{5.84}$$

The prescribed motion of the carrier body is represented by the vectors $\dot{\mathbf{r}}_0(t)$, $\ddot{\mathbf{r}}_0(t)$, $\boldsymbol{\omega}_0(t)$ and $\dot{\boldsymbol{\omega}}_0(t)$ which enter the right-hand side $\underline{B}$ of the equations. These terms are zero if the carrier body is inertial space. The term $\ddot{\mathbf{r}}_0\underline{1} - \dot{\boldsymbol{\omega}}_0 \times (\underline{\mathbf{C}}_0\underline{T})^T$ in (5.81) accounts for accelerations of articulation points fixed on the carrier body. Following (5.60) it has been said that $\underline{\mathbf{C}}_0$ equals zero if the carrier body 0 is connected to the system via a single joint and if $\mathbf{r}_0$ is defined as position vector of the articulation point chosen for this joint.

The equations of motion are particularly simple for systems which have revolute joints only and which are mounted on a stationary body 0. Many robots are systems of this kind. Each joint $a$ has a single axial unit vector $\mathbf{p}_a$ and a single rotation angle $q_a$ around this vector. The matrix $\underline{\mathbf{p}}$ is the diagonal matrix of the vectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$. As articulation points points on the joint axes are chosen. This has the consequence that not only the vectors $\mathbf{c}_{i^-(a),a}$ but also the vectors $\mathbf{c}_{i^+(a),a}$ and, hence, all vectors in the matrix $\underline{\mathbf{C}}$ are body-fixed vectors. Furthermore, $\underline{\mathbf{k}} = \underline{\mathbf{0}}$, $\underline{\mathbf{w}} = \underline{\mathbf{0}}$ and $\underline{\mathbf{s}} = \underline{\mathbf{0}}$. The essential kinematics equations have the special forms

$$\left.\begin{array}{ll} \underline{\dot{\mathbf{r}}} = \underline{\mathbf{a}}_1\underline{\dot{q}}\,, & \underline{\ddot{\mathbf{r}}} = \underline{\mathbf{a}}_1\underline{\ddot{q}} + \underline{\mathbf{b}}_1\,, \\ \underline{\boldsymbol{\omega}} = \underline{\mathbf{a}}_2\underline{\dot{q}}\,, & \underline{\dot{\boldsymbol{\omega}}} = \underline{\mathbf{a}}_2\underline{\ddot{q}} + \underline{\mathbf{b}}_2 \end{array}\right\} \tag{5.85}$$

with matrices

$$\left.\begin{array}{ll} \underline{\mathbf{a}}_1 = (\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\mathbf{a}}_2\,, & \underline{\mathbf{b}}_1 = (\underline{\mathbf{C}}\,\underline{T})^T \times \underline{\mathbf{b}}_2 - \underline{T}^T\underline{\mathbf{h}}\,, \\ \underline{\mathbf{a}}_2 = -(\underline{\mathbf{p}}\,\underline{T})^T\,, & \underline{\mathbf{b}}_2 = -\underline{T}^T\,\underline{\mathbf{f}}\,. \end{array}\right\} \tag{5.86}$$

In Sect. 5.7 systems are investigated in which all joints are spherical joints, and in Sect. 5.8 the special case of planar motions of systems with revolute joints is considered.

For systems without kinematical constraints to inertial space equations of motion have the special form (5.50):

$$\underline{\hat{A}}\underline{\ddot{q}} = \underline{\hat{B}}\,. \tag{5.87}$$

According to (5.48) the matrices $\underline{\hat{A}}$ and $\underline{\hat{B}}$ are obtained from (5.83) if $\underline{\mathbf{a}}_1$ is replaced by $\underline{\mu}^T\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_1$ by $\underline{\mu}^T\underline{\mathbf{b}}_1$. These matrices are

$$\underline{\mu}^T\underline{\mathbf{a}}_1 = (\underline{\mathbf{C}}\,\underline{T}\,\underline{\mu})^T \times \underline{\mathbf{a}}_2 - (\underline{\mathbf{k}}\,\underline{T}\,\underline{\mu})^T\,, \tag{5.88}$$

$$\underline{\mu}^T\underline{\mathbf{b}}_1 = (\underline{\mathbf{C}}\,\underline{T}\,\underline{\mu})^T \times \underline{\mathbf{b}}_2 - (\underline{T}\,\underline{\mu})^T(\underline{\mathbf{s}} + \underline{\mathbf{h}})\,, \quad \underline{\mathbf{b}}_2 = -\underline{T}^T(\underline{\mathbf{w}} + \underline{\mathbf{f}})\,. \tag{5.89}$$

The terms representing the motion of body 0 are eliminated because of the first Eq. (5.40).

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