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Correlations and Anomalous Transport Models

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2.1 Introduction

At present, the major obstacle on the way to the realization of controlled thermonuclear fusion in closed magnetic configuration devices is commonly attributed to the existence of anomalous energy losses due to particle and energy transport across a confining magnetic field. The anomalous transport of particles is usually related to the turbulent character of plasma behavior [1–5]. In spite of considerable effort, this problem still remains to be solved [6–12].

The equations describing diffusion phenomena [13–16] are among the key tools for investigating transport processes in plasmas. The ever-increasing complexity of the problems requires the development of more and more elaborate and diverse diffusion models [17–22]. The relation between heat conduction and random walk processes was established as early as the beginning of the 20th century [13]. At the first stage of research in this field, the main problem was that of calculating the diffusion coefficient (thermal conductivity). The investigation of turbulent diffusion in the atmosphere had led to the use of scalings, correlation functions, and new equations that differ substantially in structure from the conventional diffusion equation [18–20]. The research on atmospheric turbulence alone allows one to understand the hierarchic nature of turbulent scales and the importance of accounting for anisotropy. The intensive investigation of processes in strongly magnetized plasma, which was started in the middle of 20th century, essentially expands the notion about both transport processes and the nature of turbulence. Thus, transport models in stochastic magnetic fields and two-dimensional turbulent models were developed. It was revealed that transport processes in turbulent plasmas are often nondiffusive in nature. New forms of equations describing transport processes have constantly been searched for since the first studies on quasi-linear theory [23, 24]. The description of diffusion under strongly nonequilibrium conditions in highly turbulent plasma required the use of equations that take into account memory effects and the nonlocal nature of transport processes [18–20].

The objective of this paper is to consider various methods for constructing such equations, ranging from those in the quasi-linear approximation [23, 24] to those with fractional derivatives [19–22]. The topics to be discussed include the telegraph equation, the Levy–Khintchine distribution, and the Kohlrausch slow relaxation law and continuous time random walks. Use will be made of some important notions belonging to theoretical probabilistic analysis: the return probability, the self-intersection probability, and the probability of staying in a trap.

Another important aspect of the problem of describing turbulent transport is using the correlation methods of analysis [25–30]. Even from the common point of view, one can see that the correlation function is a more relevant tool to investigate a constant diffusion coefficient. Long-range correlations are responsible for anomalous transport. The methods of direct calculations and the diffusive approximation of the correlation effects are represented. One can see that the analysis of correlation effects and the interrelation between the diffusion coefficient and the autocorrelation function have been of major importance. It would therefore be instructive to trace the relation between Taylor's paper [31] that introduced the autocorrelation function and the papers on percolation diffusion [17], which are new trends of the turbulent transport theory.

It is well known that scaling representation, which was initially developed by Richardson and Kolmogorov, plays an important role in turbulence. A large number of researchers have used the ideas of scaling laws and fractality [32–48] to describe properties of turbulent transport. This is not surprising, because turbulent diffusion models differ significantly from one-dimensional transport models. Thus, the presence of vortex structures in turbulent flows and plasma requires the consideration of hierarchies of spatial and correlation scales. The system of convective cells is one of the typical examples of quasi-regular vortex structures. On the one hand, the space between convective rolls is responsible for convective transport. On the other hand, in vortex structures trapping leads to subdiffusive transport. Therefore, obtaining the expression for the effective diffusion coefficient is a nontrivial problem. Often, several different types of transport are present simultaneously in turbulent diffusion [38, 39, 48], making it important to take into account initial diffusivity and anisotropy. The anisotropy of the medium is thought to be due to the presence of a strong magnetic field or shear convective flows.

The sealing approach to describe long-range correlation was essentially developed in papers on the theory of phase transitions and critical phenomena [49–57]. Thus, the power form dependence of correlation scale on the small parameter, which describes the closeness of a system to the percolation threshold, is a fairly universal model to describe anomalous transport. In such an approach, the existence of very long (percolation) streamlines in two-dimensional random flows allows one to use the well-developed mathematical methods of transport analysis in percolation systems [17]. A detailed analysis of the more important results obtained in this field is presented in this chapter.

The review is structured as follows. It contains essentially seven parts. The first part covers Sects. 2.1 and 2.2. Here, the diffusion equations for the description of nonlocal effects are considered.

Section 2.1 is devoted to the consideration of models using the conventional representation of the diffusion equations and the definition of the correlation representation of the turbulent diffusion coefficient. Thus, the models by Richardson [58] and Batchelor [59] are based on diffusion coefficient approximations that are in agreement with the scaling for relative diffusion [58]. The Davydov idea [60] to describe the turbulent transport of a passive scalar by a telegraph equation is treated in this section.

In Sect. 2.2, the importance of using integral representation for the description of nonlocal effects is pointed out. The Levy–Khintchine approach based on the power representation of the Fourier-transformation of the nonlocal functional kernel is considered [61]. The Monin idea [62] of agreement between the transport equations in the Levy–Khintchine form and the Kolmogorov law [63] for isotropic turbulence is discussed in Sect. 2.2.

The second part consists of Sects. 2.3–2.8. Here, we deal with the initial diffusivity effects and the quasi-linear approximation of the nonlinear equations.

The Corrsin conjecture to describe the relationship between Lagrangian and Eulerian correlation functions is introduced in Sect. 2.3 [28]. It is an important aspect of the problem that the Corrsin conjecture relates correlation effects to the seed diffusion nature of transport. Later, this idea was developed in Dupree's and Kadomtsev–Pogutse's papers [64–67]. Then in Sect. 2.4 we consider the effects of molecular diffusion, which lead to the power approximation of the correlation function [68]. Another important aspect is anisotropy effects. In this connection, the double diffusion regime in a stochastic magnetic field is considered [69]. The Howell representation [73] of the effective coefficient of diffusion is observed.

In Sect. 2.5, the heuristic Taylor method to obtain such an equation is considered. The approximation suggested in [117] has become an important step in the development of description methods of anomalous transport and complex correlation effects.

In Sect. 2.6, we consider anomalous transport in the system of random shear flows [72] where the nontrivial character of diffusivity in an anisotropic system is manifested. The quasi-linear equations are derived in Sect. 2.7 [23, 24]. We then discuss the short-range and long-range effects in terms of the quasi-linear approach [20]. The possibility of using the quasi-linear approximation for the description of stochastic magnetic field diffusion is discussed [67, 70].

Section 2.8 treats the diffusive renormalization of correlation effects. We will consider the direct calculations of the correlation function [71], the Corrsin conjecture [72], and the renormalized quasi-linear equations [67]. The focus of Sect. 2.9 is the derivation of the expression for the effective diffusion coefficient, which is based on the balance of convective and diffusive fluxes, in the convective cell system.

The third part covers Sect. 2.10, where the problems of relations between stochastic instability [74] and transport effects in the stochastic magnetic field are analyzed on the basis of Rechester and Rosenbluth's models [75]. The relations between the Rechester–Rosenbluth model and the Kadomtsev–Pogutse approach are treated here.

The fourth part consists of Sects. 2.11 and 2.12. This part deals with the fractal and percolation approaches to describe the transport effects.

Several important definitions from fractal theory [56, 57] are introduced in Sect. 2.11. We then consider the fractal interpretation of Richardson's [58] and Kolmogorov's [63] laws by using the notion of fractal dimensionality [76–78].

Section 2.12 is devoted to the consideration of percolation methods for describing transport effects. Here we discuss the fractal representations of important formulas of transport theory, the percolation renormalization technique [79], and the convective cells problem [33, 80, 81] as the simplest examples to describe transport effects in the presence of the structure.

The fifth part covers Sects. 2.13–2.17. Here, percolation methods for describing the anomalous transport in random two-dimensional flows are observed on the grounds of both the monoscale and multiscale approaches [17]. We point out the importance and universality of renormalization methods to describe turbulent transport in terms of percolation theory. The model of steady flow percolation [82], time-dependent percolation [83], and the influence of drift effects [84] are considered.

Section 2.17 of this review deals with the multiscale approach [85, 86] that is applied to the description of percolation effects. The relationships between the exponents (the hull exponent, the correlation exponent, the Hurst exponent) are considered [17].

The sixth part consists of Sects. 2.18–2.20. Here, the memory and trap effects are represented on the basis of both fractal and continuous time random walk approaches.

Section 2.18 treats the problem of subdiffusive regimes and the trap approximation to describe anomalous transport effects [18–20]. The Balagurov–Vaks trap model is considered [87]. The simplest fractal representation of subdiffusive behavior is explored, as are comb structures. Section 2.19 describes the continuous time random walk approach for the description of nonlocal and memory effects [18–20]. Using the relaxation function in the power form leads to the consideration of transport fractional differential equations. Fractional differential equations with the correlation exponent as a parameter are derived and the relationships between the correlation exponent and the Hurst exponent are obtained in Sect. 2.20.

The last part consists of Sect. 2.21. Here, we discuss the relationship between the conventional space approach to transport and the phase-space approach. The Hamiltonian approach gives the advantage of using degrees of freedom to treat nonlocality and memory effects in the framework of phase-space. The kinetic model provides the possibility of describing ballistic modes and establishing the relationship between different exponents and distributions [96]. We consider the phase-space modification of the Corrsin conjecture, sticky island exponent, and nonlocal velocity distribution function.

2.2 Turbulent Diffusion and Transport

In spite of considerable progress in the understanding of anomalous transport, many aspects of the first papers in this region remain present-day. Thus, at the first stages of research on turbulent diffusion processes it was proposed using correlation functions, modifying the conventional diffusion equation, and searching scaling laws that describe nondiffusive transport. In this section we will discuss the aforementioned ideas using the classical papers by Taylor [31], Richardson [58], Davydov [60] and Batchelor [59].

2.2.1 The Correlation Function and the Taylor Diffusivity

In this section we briefly consider defining the turbulent diffusion coefficient. Taylor published a paper [31] (1921) in which he suggested a formula showing a direct relationship between the diffusion coefficient and the autocorrelation function of velocity. Actually, a new "tool" was suggested for the analysis of diffusion processes.

Following ideas in Langevin's and Einstein's papers [88, 89], Taylor wrote a stochastic equation of motion of a test Lagrangian particle in a random field,

$$x(t) = \int_0^t V(x_0, \tau)\, d\tau, \tag{2.1}$$

where $x(t)$ is the coordinate of the particle at time t, $V(x_0, t)$ is the random function of Lagrangian velocity, and x_0 is the initial coordinate of the Lagrangian particle. The purpose of his calculation was the mean square of a random displacement of the particle

$$\langle x^2 \rangle = \langle x(t)x(t) \rangle = \left\langle \int_0^t V(t_1)\, dt_1 \int_0^t V(t_2)\, dt_2 \right\rangle. \tag{2.2}$$

The brackets $\langle\rangle$ indicate an average over the ensemble of Lagrangian trajectory. Here, we omit the calculations described in detail in [27–29]. The final result of the calculations can be written in the form

$$\langle x^2 \rangle = 2 \int_0^t dt_1 \int_0^{t_1} C(\tau)\, d\tau, \tag{2.3}$$

where $C(\tau)$ is the Lagrange correlation function,

$$C(\tau) = \langle V(x_0, t) V(x_0, t+\tau) \rangle. \tag{2.4}$$

A somewhat different form of this formula was suggested by Kampe de Feriet in [90]:

$$\langle x^2 \rangle = 2 \int_0^t (t-\tau) C(\tau)\, d\tau. \tag{2.5}$$

Here, the symmetry of integral expression (2.3) is used to simplify the representation of the formula. Estimates of the turbulent diffusion coefficient in the Taylor approach lead to the expression

$$D_T = \frac{1}{2}\frac{d}{dt}\langle x^2 \rangle = \frac{d}{dt} \int_o^t (t-t')C(t')\, dt' = \int_0^t C(\tau)\, d\tau \approx V_0^2 \tau. \tag{2.6}$$

Here, V_0 is the characteristic velocity and τ is the correlation time. The specific form of the expression for the turbulent diffusion coefficient $D_T(t)$ depends on the behavior of the correlation function $C(t)$. This differs significantly from the "graded" representation of the familiar Fick's Law [7–9] with $D_0 \propto \Delta_{\mathrm{COR}}^2/\tau$. Here, Δ_{COR} is the spatial correlation scale. Usually, the exponential form is used:

$$C(t) = V_0^2 \exp\left(-\frac{|t|}{\tau}\right), \tag{2.7}$$

where V_0 is the characteristic velocity and τ is the characteristic correlation time. Such a representation of the correlation function for $t \gg \tau$ is in agreement with the rigorous results from the stochastic equation theory [13, 14, 16].

There are two asymptotic cases of significance. In the first, when $t \gg \tau$, upon substitution of (2.7) into (2.5) and simple transformations one can easily obtain

$$\left\langle x^2 \right\rangle = 2V_0^2 t\tau - 2\int_0^\infty \tau C(\tau)\, d\tau \approx 2V_0^2 \tau t. \tag{2.8}$$

This expression coincides with the well-known Einstein law for the root-mean-square displacement, $R^2 \propto t$.

In the second case, when $t \to 0$, we can use the simplest approximation of the correlation function in the form

$$C(t) \approx V_0^2 \left(1 - \frac{t^2}{\tau^2}\right). \tag{2.9}$$

This is an important correction of representation (2.7), from a formal point of view, because we have to take into account the rigorous condition of applicability of correlation approximations [27],

$$\left.\frac{d}{dt} C(t)\right|_{t \to 0} \to 0. \tag{2.10}$$

Upon substitution of (2.9) into formula (2.5), one obtains the ballistic motion law in the form

$$\left\langle x^2 \right\rangle = 2V_0^2 t^2. \tag{2.11}$$

Another important relationship, which will be used in the subsequent discussions, is the expression

$$\left.\frac{d^2}{dt^2}\left\langle x^2 \right\rangle\right|_{t=\tau} = 2C(\tau). \tag{2.12}$$

This formula can be obtained by differentiation of expression (2.6).

Even from general considerations, it is clear that the correlation function is a more relevant tool of investigation than the constant diffusion coefficient. In the next sections, we will show that the development of correlation ideas had an essential influence on the form of diffusion equations.

2.2.2 The Richardson Law

The problem formulated by Taylor [31] appears to be especially actual in relation to the Richardson investigations of turbulent diffusion that were carried out in 1926 [58]. The author of [58] discovered that the laws of atmospheric diffusion essentially differ from conventional expression (2.8). An analysis of experimental results has led to the expression for the mean square separation of a pair of marked particles,

$$\frac{1}{2}\frac{d}{dt}\left\langle Y^2(t)\right\rangle \approx \mathrm{const}\cdot\left\langle Y^2(t)\right\rangle^{2/3}. \tag{2.13}$$

Here, $Y(t)$ is the separation between two particles that are situated at the points $x_1(t)$ and $x_2(t)$,

$$Y(t) = x_2(t) - x_1(t). \tag{2.14}$$

Dependence (2.13) shows the accelerating character of particle relative motion. The expression can be represented in the scaling form:

$$\left\langle Y^2(t)\right\rangle \propto t^3. \tag{2.15}$$

Result (2.15) is not trivial because it differs significantly even from the ballistic scaling (2.11). Indeed, from the formal standpoint we can expect that

$$\left\langle Y^2(t)\right\rangle = \left\langle x_1^2(t)\right\rangle - 2\left\langle x_1(t)x_2(t)\right\rangle + \left\langle x_2^2(t)\right\rangle. \tag{2.16}$$

Destroying correlations in time leads to the result that is in accord with the following estimates:

$$\left\langle Y^2(t)\right\rangle \approx 2(2D_T)t. \tag{2.17}$$

However, from the point of view of the scaling law (2.13) we deal with the dependence

$$D_R \approx \left\langle Y^2(t)\right\rangle^{2/3}. \tag{2.18}$$

This expression, in fact, mirrors the nonlocal character of transport effects in the conditions of atmospheric turbulence, since the separation between the diffusing particles significantly changes only under the influence of eddies comparable in size to interparticle separation.

Richardson suggested using the diffusion equation for the description of the probability density evolution F to find two initially close particles at a distance l from one another at the moment t:

$$\frac{\partial F(l,t)}{\partial t} = \frac{\partial}{\partial l} D_R \frac{\partial F(l,t)}{\partial l}. \tag{2.19}$$

In the framework of the offered scaling law (2.13), the expression for $D_R(l)$ takes the form

$$D_R(l) \approx l^{4/3}. \tag{2.20}$$

This result was later confirmed in the framework of the theory of uniform isotropic turbulence [63]. Kolmogorov and Obuchov showed in their articles [63, 91] that the rate of energy dissipation ε_K is the only dimensional characteristic in a wide interval of scales l. Then it is possible to compose the scaling laws based on the dimensional character of the value $\varepsilon_K = [L^2/T^3]$ and the variable k that characterizes the spatial scale $k \approx 1/l(k) = [1/L]$. Simple calculations then yield the dimensional estimate for the Richardson coefficient:

$$D_R(l) = \left[\frac{L^2}{T}\right] \approx \varepsilon_K^{1/3} \frac{k^{2/3}}{k^2} \approx \varepsilon_K^{1/3} \frac{1}{k^{4/3}} \approx \varepsilon_K^{1/3} l^{4/3}. \tag{2.21}$$

Thus, the idea of describing turbulence by the hierarchy of eddies of different scales [63] has obtained its first experimental confirmation.

It is important to note that expression (2.13) suggested by Richardson is in accord with the experimental data in a wide spectrum of parameters. This quite justifies the usage of the expression "the Richardson law". The papers by Taylor and Richardson undoubtedly opened up a fundamentally new avenue for research and had a profound effect on the subsequent development of the theory of transport processes.

2.2.3 The Davydov Model of Turbulent Diffusion

The nonlocal character of transport that has been investigated by Richardson is manifested not only for the relative diffusion of particles. The problem of the diffusion description of a single test particle in the field of turbulence also leads to the necessity to take into account the interaction between different scales $l(k)$. Such an approach naturally requires considerable modification of Fick's diffusion equation:

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2}. \tag{2.22}$$

Here, n is the particle density and D is the conventional coefficient of diffusion. One of the first models to describe turbulent diffusion is the Davydov model [60], which is based on the telegraph equation

$$\frac{1}{\tau} \frac{\partial n}{\partial t} + \frac{\partial^2 n}{\partial t^2} = V^2 \frac{\partial^2 n}{\partial x^2}, \tag{2.23}$$

where V is the velocity scale and τ is the correlation time. From the dimensional standpoint, the use of this expression permits one to obtain scaling laws for the mean square displacement of a particle in the ballistic form

$$\langle x^2 \rangle \propto t^2. \tag{2.24}$$

Note that Maxwell [18] was the first to suggest the hyperbolic model of heat-conductivity for the description of the finite velocity of perturbation spreading. This corresponds fairly well to his investigations of electromagnetic theory.

Davydov used the phenomenological set of equations for the particle density $n(x,t)$:

$$\frac{\partial n(x,t)}{\partial t}+\frac{\partial q(x,t)}{\partial x}=0; \tag{2.25}$$

$$\frac{\partial q}{\partial t}=\frac{q_0-q}{\tau}. \tag{2.26}$$

Here, q is the particle flux. It is natural to use the classical expression for the particle initial flux:

$$q_0(x,t)=-D\frac{\partial n(x,t)}{\partial x}. \tag{2.27}$$

Formal manipulations with this set of equations yield telegraph equation (2.23). The author of [60] suggested using (2.23) to take into account the finite particle velocity V during the molecular diffusion. The classical parabolic diffusion equation follows from telegraph equation (2.23) in the limit

$$\tau\to 0;\qquad D\approx V^2\tau\to \text{const}. \tag{2.28}$$

The physical meaning of the representation proposed by Davydov for the particle flux q can be easily clarified by writing the formal solution

$$q(x,t)=\int_0^t q_0(x,t')\exp\bigl(-(t-t')\bigr)\frac{dt'}{\tau}=-\int_0^t D\frac{\partial n}{\partial x}\exp\bigl(-(t-t')\bigr)\frac{dt'}{\tau}. \tag{2.29}$$

Obviously, such an expression for the particle flux contains memory effects. After Davydov, this formula was generalized in many studies in such a way as to replace the exponential function by an arbitrary memory function $M(t-t')$,

$$q(x,t)=\int_0^t q_0(x,t')M(t-t')\frac{dt'}{\tau}. \tag{2.30}$$

In this general case one obtains the diffusion equation in the form

$$\frac{\partial n(x,t)}{\partial t}=\int_0^t D\frac{\partial^2 n(x,t')}{\partial x^2}M(t-t')\frac{dt'}{\tau}. \tag{2.31}$$

Later, the telegraph equation in the form (2.23) was often applied to describe turbulent diffusion [27, 92–95].

From the modern point of view such an approach looks fairly naive. However, in essence, the idea of using the additional derivative in the equations describing the anomalous character of turbulent diffusion was clearly formulated by Davydov as early as 1934 [60]. At present, not only are conventional partial derivatives used, but fractional derivatives are also used, better mirroring the essence of the nonlocality and memory effects because they have the integral character of the operator [18–22]

$$\frac{\partial^\xi n}{\partial x^\xi},\qquad \frac{\partial^\varsigma n}{\partial t^\varsigma},\qquad \ldots. \tag{2.32}$$

Here, ξ and ζ are the fractional parameters of the problem. Moreover, this approximation method is also applied to the description of strong nonequilibrium processes in the framework of kinetic equation [96, 97] for the distribution function $f(V, x, t)$. Here, V is the velocity. Thus, it is suggested replacing the Fokker–Planck diffusive operator $\partial^2 f/\partial V^2$ by the fractional derivatives $\partial^\xi f/\partial V^\xi$ that reflect the nonlocal character of relaxation in the phase space. In the next sections of the paper we will discuss these problems in detail.

2.2.4 The Batchelor Approximation for the Diffusion Coefficient

The approximation suggested by Richardson (2.20) corresponds to his ideas about the hierarchical and nonlocal character of turbulent transport. Thus, he related nonlocality to the increasing scale of eddies taking part in transport processes. Therefore, in his approach the diffusion coefficient D_R is the function of the interparticle distance l. However, there exist alternative possibilities. Batchelor [59] considered the problem from a different point of view. In his model the diffusion coefficient D_R is the result of statistical averaging over the ensemble of different scales. Hence, he proposed using the temporal dependence for the definition of D_R. In the framework of this approach the dimensional consideration yields the expression

$$D_R(t) \approx \frac{\langle Y^2(t)\rangle}{t} \approx \left\langle l^2(t)\right\rangle^{2/3} \propto t^2. \tag{2.33}$$

The equation for the probability density then takes the following form, which is similar to the conventional one (2.22) but with the time-dependent coefficient of diffusion:

$$\frac{\partial F(l,t)}{\partial t} = D_R(t)\frac{\partial}{\partial l}\frac{\partial F}{\partial l}. \tag{2.34}$$

After the simplest analysis it becomes clear that the Richardson model and the Batchelor model lead to different results in spite of the underlying law (2.13). Thus, in the conventional diffusion equation (2.22) the law of temporal relaxation of the function F in the Fourier form corresponds to

$$\tilde{F}_k(t) \propto \exp(-t), \tag{2.35}$$

whereas in the Batchelor model we deal with stronger damping:

$$\tilde{F}_k(t) \propto \exp\left(-t^3\right). \tag{2.36}$$

Here, $\tilde{F}_k(t)$ is the Fourier transformation of the function $F(x,t)$ over the variable x. It is obvious that the characters of the solutions suggested by Richardson and Batchelor describing the probability density evolution are also different. Considering the model with a point-source of particles, one can obtain for the Richardson model [58]

$$F(l,t) = \frac{8}{315\pi^{8/2}}\left(\frac{9}{4t}\right)^{9/2} \exp\left(-\frac{9l^{2/3}}{4t}\right). \tag{2.37}$$

Under analogous conditions (the model with a point-source) for the Batchelor model one obtains [59]

$$F(l,t) = \left(\frac{1}{2\pi \langle l^2(t)\rangle}\right)^{3/2} \exp\left(-\frac{l^2}{2\langle l^2(t)\rangle}\right). \tag{2.38}$$

Note that the arguments in favor of one type or another of the diffusion coefficient have a qualitative character in both these cases. Moreover, the "combination" of both these approaches is possible, if one supposes that D_R can depend on both the inter-particle distance l and time t:

$$D_R(t,l) \approx t^{\phi} l^{\varphi}. \tag{2.39}$$

To save the Richardson law we need to take into account the relationship between exponents ϕ and φ:

$$2\phi + 3\varphi = 4. \tag{2.40}$$

Then, the case $\phi = 0$, $\varphi = 4/3$ corresponds to the Richardson law and the case $\phi = 2$, $\varphi = 0$ corresponds to the Batchelor supposition. Thus, Okubo [98] suggested a mixed algebraic representation for the diffusion coefficient:

$$D_R(t,l) \approx t l^{2/3}. \tag{2.41}$$

The three-dimensional solution for the point-source at $t = 0$ is given by [98]

$$F(l,t) = \mathrm{const} \cdot t^{-\frac{3(1+\phi)}{2-\varphi}} \exp\left(-\mathrm{const} \cdot \frac{l^{2-\varphi}}{t^{1+\phi}}\right). \tag{2.42}$$

Unfortunately, it is impossible to decide what is a correct equation, if one looks at this problem from the conventional diffusion point of view, because the physical arguments from Kolmogorov and Obukhov lead to an explanation in terms of the hierarchy of scales, whereas Richardson and Batchelor deal with the local diffusive equation with partial differentials. However, these classical papers [58–60] formulated problems that allow us to develop theoretical methods of anomalous transport description that are based on the analysis of correlation effects and scaling laws.

2.3 Nonlocal Effects and Diffusion Equations

The nonlocal nature of relative diffusion has stimulated the search for equations that differ significantly from conventional diffusion equations. An elegant integral equation corresponding to this problem was suggested by Einstein [89]. The use of this equation in combination with the scaling ideas has led to the necessity to consider a distribution function that differs essentially from the Gauss function. A new type of distribution, called the Levy–Khintchine distribution, is now one of the basic tools for researching anomalous transport.

2.3.1 The Functional Equation for Random Walks

For the Richardson law Kolmogorov and Obuchov [63, 91] obtained dimensional estimates, which give qualitative explanations of the nonlocality transport effects of turbulent diffusion in terms the interaction of different scales. However, the nonlocal effects can also be described by means of the random walk model. Thus, besides the different phenomenological methods of improvement of the diffusion equation, there exists a possibility to use the integral equation to describe the random walk processes.

As early as 1905, Albert Einstein obtained a functional equation for the particle density solely on the basis of the general ideas about the process of random walk [89]:

$$n(x, t+\tau) = \int_{-\infty}^{+\infty} W(y) n(x-y, t)\, dy, \tag{2.43}$$

where $W(y)$ is the probability density of undergoing a jump y. This fundamentally nonlocal equation can be made local by reducing it to a diffusion equation. Assuming that the time scale τ is short and the jump y is small, Einstein arrived at the classical diffusion equation. In this way, he used the expansions

$$n(x, t+\tau) = n(x, t) + \frac{\partial n}{\partial t}\tau + \cdots, \tag{2.44}$$

$$n(x+y, t) = n(x, t) + \frac{\partial n}{\partial x} y + \frac{y^2}{2}\frac{\partial^2 n}{\partial x^2} + \cdots. \tag{2.45}$$

Simple calculations yield

$$n + \frac{\partial n}{\partial t}\tau = n \int_{-\infty}^{\infty} W(y)\, dy + \frac{\partial n}{\partial x}\int_{-\infty}^{\infty} W(y) y\, dy + \frac{\partial^2 n}{\partial x^2}\int_{-\infty}^{\infty} W(y)\frac{y^2}{2} + \cdots. \tag{2.46}$$

Assuming that the function W is symmetric, $W(y) = W(-y)$, and specifying the normalization condition

$$\int_{-\infty}^{+\infty} W(y)\, dy = 1, \tag{2.47}$$

one obtains the conventional diffusion equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}, \quad \text{where } D = \frac{1}{\tau}\int_{-\infty}^{+\infty} W(y)\frac{y^2}{2}\, dy. \tag{2.48}$$

Note that the number of terms in expansions (2.44), (2.45) was chosen in a physically meaningful way. Based on the relationship characterizing the average behavior of Brownian particles, $\langle x^2 \rangle \approx R^2 \propto t$. We can estimate the orders of the terms for $t \to \infty$ in the expansions as follows: $n \propto 1/R$. This corresponds to the one-dimensional case. Then one can obtain

$$\begin{aligned} &n \propto t^{-1/2}, \qquad \frac{\partial n}{\partial t} \propto \frac{n}{t} \propto t^{-3/2}, \qquad \frac{\partial^2 n}{\partial t^2} \propto \frac{n}{t^2} \propto t^{-5/2},\\ &\frac{\partial n}{\partial x} \propto \frac{n}{R} \propto t^{-1}, \qquad \frac{\partial^2 n}{\partial x^2} \propto t^{-3/2}. \end{aligned} \tag{2.49}$$

Retaining only two terms in expansions (2.44) and (2.45) each results in a telegraph equation. However, this does not indicate that the telegraph equation is invalid. The reason is that, in this case, the effects of the finite propagation velocity of the perturbations come into play, which are absent in the classical diffusion model.

The integral approach was further developed in the papers by Smoluchowski, Chapman, and Kolmogorov [99–101]. A key element in their approach is Markov's postulate [13, 14] that the length of the jump y is independent of the prehistory of motion. To describe the nonlocal effects, just the integral form of the equations is important.

Using expansion (2.44) of functional (2.43), we can readily obtain the Smoluchowski equation [99]

$$\frac{\partial n(x,t)}{\partial t} = \int_{-\infty}^{+\infty} \left[K(x',x)n(x',t) - K(x,x')n(x,t) \right] dx'. \tag{2.50}$$

Here, $K(x,x')\,dx\,dx'$ is the probability for a particle at position x at time t to pass over to the interval $x' + dx'$ during the time interval dt. We introduce the functional

$$G(x',x) = K(x',x) - \delta(x-x') \int_{-\infty}^{+\infty} K(x,x')\,dx'. \tag{2.51}$$

For a uniform isotropic medium, we have $G(x'-x) = G(|x-x'|)$. In the simplest case under consideration, this functional has the form

$$\frac{\partial n}{\partial t} = \int_{-\infty}^{+\infty} G(x-x')n(t,x')\,dx'. \tag{2.52}$$

This representation reflects the nonlocal character of transport and at the same time it has a close relation to the conventional diffusion equation (2.22). It is possible to consider several analytical functions $G(x)$ to find some solution of this equation [15]. As an example, one can form such an approximation on the basis of Poisson's probabilistic law [13, 14]. But there is another good way that leads to new and very fruitful research trends, which are especially relevant for turbulent diffusion problems.

2.3.2 Nonlocality and the Levy Distribution

Functional (2.52) is linear and it is more convenient to switch to the Fourier representation for $n(x,t)$ with respect to the variable x. Formal manipulations yield

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = \tilde{G}_k \tilde{n}_k(t), \tag{2.53}$$

which indicates the absence of memory effects for the Fourier harmonics. Here, $\tilde{G}_k$ and $\tilde{n}_k(t)$ are the Fourier transformations of the functions $G(x)$ and $n(x,t)$ with respect to the variable x. The expression

$$\tilde{G}_k \tilde{n}_k = -Dk^2 \tilde{n}_k \tag{2.54}$$

corresponds to the classical diffusion equation, where D is the conventional diffusion coefficient. In the case of telegraph equation (2.23), the memory effects were taken into account (see (2.31)):

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = -k^2 \int_0^t \tilde{n}_k(t) M(t-t') \frac{dt'}{\tau} = -k^2 M(t) * \tilde{n}_k(t), \tag{2.55}$$

where the asterisk indicates the convolution operation.

Applying the Laplace transformation in time, we obtain the following expression for the telegraph equation with memory:

$$\tilde{\tilde{G}}(k,s) = -\frac{Dk^2}{1 - is\tau}. \tag{2.56}$$

Hereafter, $\tilde{\tilde{G}}(k,s)$ will denote both the Fourier and Laplace transformations of the function $G(x,t)$ with respect to the variables x and t. It is an easy matter to combine the memory and nonlocality effects into a common expression containing a convolution:

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = -k^2 \int_0^t \tilde{n}_k(t) \tilde{D}_k(k, t-t') \frac{dt'}{\tau} = -k^2 \tilde{D}_k(k,t) * \tilde{n}_k(t). \tag{2.57}$$

Performing the Laplace transformation in time gives the transition from the conventional result to the new one:

$$-Dk^2 \to -k^2 \tilde{\tilde{D}}_{k,s}(k,s). \tag{2.58}$$

In the theoretical probabilistic approach, however, this heuristic method is unsatisfactory. Below, we will consider this point in more detail.

The approach based on (2.53) was developed by Levy and Khintchine [61], who used the approximate equation of the form

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = -k^{\alpha_L} \tilde{n}_k(t), \quad 0 < \alpha_L \le 2. \tag{2.59}$$

It is easy to see that, for $\alpha_L = 2$, we are dealing with a Gaussian distribution (corresponding to a conventional diffusion equation). Some other analytic distributions are also known.

For the case $\alpha_L = 1$, we obtain the Cauchy distribution [102].

For the case $\alpha_L = 3/2$, one arrives at the familiar Holtsmark distribution [13].

For the case $\alpha_L = 1/2$, we have the Levy–Smirnov distribution [18].

For the case $\alpha_L = 2/3$, we obtain the Smirnov distribution [18]. In this context, it is important to note that all the probability densities with $\alpha_L < 2$ have power-law tails. Another important feature is that the second and higher order of moments of the distributions with $1 \le \alpha_L < 2$ and all moments of the distributions with $0 < \alpha_L < 1$ diverge.

Representation (2.59) is sufficient to consider the important models of anomalous diffusion, which are often described by the scaling law,

$$\left\langle x^2 \right\rangle^{1/2} \approx R \propto t^H. \tag{2.60}$$

Here, H is the Hurst exponent [18–22]. In the case of classical diffusive behavior we find $H = 1/2$. The cases $0 < H < 1/2$ correspond to the subdiffusive behavior. The cases $1/2 < H < 1$ correspond to the superdiffusive behavior. There is a relationship between the Hurst exponent H and the Levy–Khintchine exponent a_L that is the parameter of the power approximation (2.59):

$$H = \frac{1}{\alpha_L}. \tag{2.61}$$

There is also a very interesting result which follows from the Fourier representation of density $n(x,t)$:

$$\left\langle x^2 \right\rangle^{1/2} = -\frac{\partial}{\partial k}\left(\frac{\partial}{\partial k}\tilde{n}_k(t)\bigg|_{t=0}\right). \tag{2.62}$$

This expression is very useful for relating scaling laws to probabilistic approximations in the Levy–Khintchine term.

2.3.3 The Monin Fractional Differential Equation

Monin [62] used the Einstein–Smoluchowski functional given in (2.52) and (2.59) to describe turbulent diffusion in the atmosphere. That paper anticipated the development of modern ideas of using additional fractional partial derivatives in diffusion equations.

Monin was guided by Kolmogorov's ideas about the universal properties of well-developed isotropic turbulence [63]. In the corresponding formulation of the problem, all statistical parameters are determined exclusively by the scale length $l_k \approx 1/k \approx [L]$ and the mean energy dissipation rate $\varepsilon_K = [L^2/T^3]$. Here, L and T characterize the physical dimensionality of space and time. In the framework of Fourier's representation (2.59) there is a single "uncertain parameter" a_L, which defines the power form of the kernel of the nonlocal functional. Based on dimensional considerations, it is possible to compose the approximation for function $\tilde{G}(k)$, which has $[1/T]$ order that is in agreement with relaxation law (2.35). Monin obtained the following expression for the kernel of the nonlocal functional describing turbulent diffusion:

$$\tilde{G}(k) \propto \tilde{G}(\varepsilon_K, k) = \varepsilon_K^{1/3} k^{2/3}. \tag{2.63}$$

The expression used by Monin has a dimensionality that is inversely proportional to time. This fact reflects the essential difference of such a model from the Batchelor approach [59]. This representation is consistent with the results derived in 1926 by Richardson [58] under the assumption

$$\tilde{G}(k) = -D(k)k^2, \tag{2.64}$$

where $D(k) \approx \frac{l^2}{t} \propto l^{4/3} \propto k^{-4/3}$. Also, in modern terminology [20–22], the equation

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = -\varepsilon_K^{\frac{1}{3}} k^{\frac{2}{3}} \tilde{n}_k(t) \tag{2.65}$$

is the one with the fractional derivative with respect to x,

$$\frac{\partial^{\alpha_L} n}{\partial x^{\alpha_L}} \propto \frac{n}{(\Delta x)^{\alpha_L}} \propto k^{\alpha_L} n, \tag{2.66}$$

where $\alpha_L = 2/3$ [see formula (2.59)]. Monin was the first to obtain this equation for the probability density on the basis of physical considerations. He solved this equation and wrote the solution in terms of the Whittaker functions. The solution behaves asymptotically as $n(x \to \infty) \propto x^{-11/13}$. The problem of the relaxation in a self-similar regime was discussed in detail in [27, 62].

However, Monin was unsatisfied with the above form of the equation. In fact, he derived the following equation with fractional derivatives:

$$\frac{\partial n}{\partial t} = \varepsilon_K^{1/3} \frac{\partial^{2/3} n}{\partial x^{2/3}}. \tag{2.67}$$

It is only recently that the idea of using fractional derivatives has come to be recognized [18–22]. In an effort to derive an equation that would be as clear as the telegraph equation, Monin differentiated his equation twice with respect to time and obtained the expression

$$\frac{\partial^3 n}{\partial t^3} = \varepsilon_K \frac{\partial^2 n}{\partial x^2}. \tag{2.68}$$

Note that Monin suggested his equation to describe the diffusing particle probability density evolution n. However his idea can be used to describe the probability density evolution F, which describes relative diffusion. Such a version was considered in [103, 104]. But in those papers [103, 104] use was made of the modern terminology and the fractional differential is represented as nonlocal integral operator

$$\frac{\partial F(\vec{l}, t)}{\partial t} = \Gamma(2/3) \frac{\sqrt{3}}{4\pi^2} \Delta_L \int \frac{F(\vec{l}, t)}{|\vec{l} - \vec{l}'|^{5/3}} d^3\vec{l}'. \tag{2.69}$$

Here, Γ is the Gamma function and Δ_L is the Laplace operator.

It is natural that the use of nonlocal operator (2.69) leads to the distribution function, which differs significantly from the Richardson and Batchelor models. Nevertheless, convincing arguments in favor of the choice of the specific type of equation describing the behavior of the distribution function are absent and the search for adequate models and experimental proofs has been continued. Note that in spite of the assumptions of isotropy and the relative simplicity of experiments, these problems remain acute. From this standpoint, the absence of a detailed pattern of anomalous transport in high-temperature plasma does not look so catastrophic.

2.4 The Corrsin Conjecture

The classical correlation definition of Taylor's coefficient of turbulent diffusion does not contain any information on molecular diffusion. It is obvious that a serious problem arises when we analyze the passive tracer transport. In this section we will consider several important models in which the effects of molecular (seed) diffusion and correlation effects play a significant role. The scaling arguments are presented.

2.4.1 The Corrsin Independence Hypothesis

The definition of the correlation function suggested by Taylor [31] is based on using Lagrangian velocities $V(x_0, y)$, but their experimental determination is a serious problem. That is why use is made of the Eulerian representation for the correlation function, which takes into account the velocity correlation at points separated by a distance λ:

$$C_E(\lambda, t) = \langle u(x_0, T)u(x_0 + \lambda, T + t)\rangle. \tag{2.70}$$

This form of the correlation function is more convenient for experimenters. We can also express the Lagrangian correlation function through the Eulerian velocity,

$$C(t) = \big\langle u(x_0; T)u\big(x(x_0, T + t); T + t\big)\big\rangle. \tag{2.71}$$

Here, $U(x_0, T)$ is the Eulerian velocity at point x_0 and time T. However, there is no simple relation between the Lagrangian correlation function and the Eulerian one. Actually, there is no Lagrangian relation between the points x_0 and $x_0 + \lambda$ in expression (2.70). Here, λ is merely some arbitrary displacement.

Corrsin [28] suggested an approximation formula in terms of the randomization of the Lagrange correlation function with the probability density $\rho(x, t)$,

$$C(t) = \int_{-\infty}^{\infty} \rho(\lambda, t)C_E(\lambda, t)\, d\lambda, \tag{2.72}$$

in which he expressed the Lagrangian correlation function through the Eulerian one [25–30]. However, a more important point is the idea of the diffusion nature of the displacement λ, because for $\rho(\lambda, t)$ Corrsin used the classical solution of the diffusion equation in a three-dimensional space,

$$\rho(\lambda, t) = \frac{1}{(4\pi D_0 t)^{3/2}} \exp\left(-\frac{\lambda^2}{4D_0 t}\right). \tag{2.73}$$

This formula also includes the molecular diffusion coefficient D_0. Hence, one can consider both the turbulent transport and the molecular diffusion. Finally, Corrsin obtained the integral expression

$$C(t) = \int_{-\infty}^{\infty} \frac{C_E(\lambda, t)}{(4\pi D_0 t)^{3/2}} \exp\left(-\frac{\lambda^2}{4D_0 t}\right) d\lambda. \tag{2.74}$$

From this point of view, one can note that λ is the distance and the diffusive displacement at the same time. In fact, instead of formal averaging in the form

$$\big\langle V\big(x(0)\big)V\big(x(t)\big)\big\rangle = \int_{-\infty}^{\infty}\big\langle V(0)V(y)\delta\big(y-x(t)\big)\big\rangle\,dy, \tag{2.75}$$

the factorization approach was used (the "independence hypothesis"):

$$\big\langle V(0)V(y)\delta\big(y-x(t)\big)\big\rangle = \big\langle V(0)V(x)\big\rangle\big\langle\delta\big(y-x(t)\big)\big\rangle. \tag{2.76}$$

Moreover, Corrsin used Gaussian distribution (2.73) to describe trajectory correlations:

$$\big\langle\delta\big(y-x(t)\big)\big\rangle \approx \rho(y,t). \tag{2.77}$$

Using rigorous analysis, Weinstock [105] and Kraichnan [106] showed that the Corrsin conjecture is equivalent to a first-order truncation of the renormalization expansion, which can be considered systematically.

The Corrsin conjecture has been tested against kinetic simulations of two- and three-dimensional flows with an energy spectrum sharply peaked about one well-determined length scale [105, 107, 108] with the conclusion that it is valid for all times (not only for large times) provided that there is no helicity and that the flow is not frozen in time.

2.4.2 The Simplified Corrsin Conjecture

The interesting paper by Hay and Pasquill [109] was written almost simultaneously with the Corrsin paper [28]. They taken into account that Eulerian and Lagrangian correlation functions have similar shapes; but at the same time, their characteristic scales are different. Thus, the characteristic temporal scale, which corresponds to the Lagrangian correlation function, is defined by the expression

$$\tau_L = \frac{1}{\langle V^2\rangle}\int_0^{\infty} C_L(t)\,dt. \tag{2.78}$$

The characteristic Eulerian temporal and spatial scales are defined analogously

$$\tau_E = \frac{1}{\langle U^2\rangle}\int_0^{\infty} C_E(\Delta,t)\,dt, \tag{2.79}$$

$$l_E = \frac{1}{\langle U^2\rangle}\int_0^{\infty} C_E(\Delta,t)\,d\Delta. \tag{2.80}$$

The authors of [109] supposed that there exists a certain universal constant β_C that allows us to relate Lagrangian and Eulerian scales,

$$\tau_L = \beta_C\tau_E, \tag{2.81}$$

$$l_L(\beta_C t) = \beta_C l_E(t). \tag{2.82}$$

The variety of turbulence types leads to the fact that the values of β_C defined experimentally lie in a wide interval:

$$1 \leq \beta_C < 8.5. \tag{2.83}$$

In spite of the obvious simplicity of the approach suggested by Hay and Pasquill, in recent papers it was shown that in the framework of the consideration of one-particle vertical diffusion in strongly stratified turbulence, the Eulerian and Lagrangian velocity correlation functions are almost the same:

$$\langle V_i(0)V_j(t)\rangle = \langle U_i(x,0)U_j(x,t)\rangle. \tag{2.84}$$

Thus, Kaneda and Ishida [110] considered the Fourier transformation of the Corrsin conjecture (2.76) in the form

$$\langle V_i(t)V_j(t')\rangle = \int d^3\vec{k}\,\tilde{R}_{ij}(\vec{k},t,t')\langle e^{-i\vec{k}(x(t')-x(t))}\rangle, \tag{2.85}$$

where

$$\tilde{R}_{ij}(\vec{k},t,t') = \frac{1}{(2\pi)^3}\int \langle U_i(\vec{x}+\vec{r},t)U_j(\vec{x},t')e^{-i\vec{k}\vec{r}}\,d^3\vec{r}\rangle. \tag{2.86}$$

From the physical point of view the Eulerian velocity correlations must be dominated by large eddies (see [107]). This corresponds to small values of $\vec{k}$. Therefore, for $\vec{k} \approx 0$ one can expect that

$$\langle e^{-i\vec{k}(x(t')-x(t))}\rangle \approx 1. \tag{2.87}$$

This simple estimate gives the simplified Corrsin conjecture

$$\langle V_i(t)V_j(t')\rangle = \langle U_i(\vec{x},t)U_j(\vec{x},t')\rangle. \tag{2.88}$$

This new representation

$$C_L(\tau) \approx C_E(\lambda,\tau)|_{\lambda=0} \tag{2.89}$$

was first discussed by the author of [107]. In the framework of the Boussinesq approximation of strongly stratified flows, the validity of the simplified Corrsin conjecture (2.88) was checked by direct numerical simulations [110, 112]. It was shown that the simulation results agree well with the hypothesis (2.88).

2.4.3 The Correlation Function and Scalings

The Corrsin conjecture looks fairly formal; however, it allows us to visualize correlation effects and to take into account the effects of molecular diffusion. Note that the definition of Taylor's coefficient of turbulent diffusion does not contain any information on molecular diffusion. It is obvious that a serious problem arises when analyzing the passive tracer transport [27, 28]. Moreover, the Corrsin representation offers an additional possibility of developing the scaling approximation of transport by the power approximations of the Eulerian correlation function and different kinds of probability density [18–22].

The effective use of scaling laws will be considered here. Thus, the authors of [113] made a rigorous analysis of equations for a random noncompressible flow where the mean velocity is zero and the spatial correlation function decays as

$$C_E(\lambda) \propto \frac{1}{\lambda^{\alpha_C}}. \tag{2.90}$$

Koch and Brady used a continuum nonlocal advection–diffusion theory (the direct-interaction approximation) and obtained an expression which connects the Hurst exponent H that describes the transport character with the correlation exponent α_C:

$$H = \frac{2}{2+\alpha_C}. \tag{2.91}$$

Here, α_C describes the power behavior of the spatial correlation function of velocity

$$C(\lambda) = \langle V(x)V(x+\lambda)\rangle \propto V_0^2\left(\frac{\lambda_0}{\lambda}\right)^{\alpha_C}. \tag{2.92}$$

Here, V_0 and λ_0 are the dimensional parameters of the model. Note that this relationship can be obtained by simple calculations based on both the dimensional consideration of the correlation function

$$C \approx V^2 \approx \frac{\lambda^2}{t^2} \tag{2.93}$$

and the power dependence (2.92). Then, the comparison of (2.93) with (2.92) yields

$$\frac{\lambda^2}{t^2} \approx V_0^2\left(\frac{\lambda_0}{\lambda}\right)^{\alpha}. \tag{2.94}$$

If we suppose that the spatial scale λ is the correlation scale and the diffusive displacement at the same time (as in the Corrsin conjecture), then it is possible to treat (2.94) as the transport scaling. Now, one can obtain the diffusive estimate:

$$\lambda \propto \left(V_0^2\lambda_0^{\alpha_C}\right)^{\frac{1}{2+\alpha_C}} t^{\frac{2}{2+\alpha_C}} \tag{2.95}$$

and

$$H = \frac{2}{2+\alpha_C}, \tag{2.96}$$

where $0 < \alpha_C < 2$ since the result (2.95) was obtained for incompressible flow, where the subdiffusive transport is absent [17, 114]. Later, relationship (2.96) was repeatedly discussed in connection with the analysis of more complex models of turbulent transport [17, 85, 86].

2.5 Effects of Seed Diffusivity

Corrsin was one of the first to understand the importance of accounting for seed diffusivity effects to describe correlations. Further investigation of turbulent transport led the appearance of numerous estimates and scalings based on diffusive estimates. Thus, in the framework of diffusive approximations it is possible to consider not only transport of a passive tracer but also anomalous diffusion of particles in a braided magnetic field.

2.5.1 Seed Diffusivity and Correlations

It is well known that interactions both create and destroy correlations. There is a useful estimate which illustrates this in terms of the correlation function. It was assumed [16, 68] that the number of interactions N_I is proportional to the number of particles that are located in the correlation region W_D:

$$N_I \approx n W_D \approx n R_D^d. \tag{2.97}$$

Here, n is the density of particles in this region, R_D is the spatial scale of this region, and d is the space dimensionality. Then, from the dimensional point of view, the correlation effects can be expressed in the form

$$C(t) = \left\langle V(0)V(t)\right\rangle \approx V_0 \frac{V_0}{N_I} \approx \frac{V_0^2}{n W_D}, \tag{2.98}$$

where $V(t)$ is the velocity at the moment t and V_0 is the characteristic scale of the velocity. The estimate of W_D can be obtained from the conventional Gaussian distribution

$$\rho(x,t) = \frac{1}{(4\pi D_0 t)^{d/2}} \exp\left(-\frac{x^2}{4D_0 t}\right). \tag{2.99}$$

Here, D_0 is the molecular coefficient of diffusion. To derive the estimate it was supposed that correlation scale R_D has the diffusion nature

$$R_D \propto (D_0 t)^{1/2} \quad \text{for } t \to \infty. \tag{2.100}$$

This corresponds to the Corrsin assumptions. Simple calculations then yield

$$C(t) = \left\langle V(0)V(t)\right\rangle \approx \frac{V_0^2}{n(D_0 t)^{d/2}} \propto \frac{1}{t^{d/2}}. \tag{2.101}$$

In spite of the difference between this result and the exponential form (2.7), the obtained power approximation of the correlation function is not senseless. The "long tails" of correlation functions $C(t) \propto t^{-3/2}$ are being investigated in molecular dynamics and are related to "the collective" (hydrodynamic) nature of the evolution of a system [16, 68]. The correlation function is related to diffusion coefficient (2.6), which in our case leads to the estimate

$$C(t) \propto \frac{d}{dt} D_T \propto \frac{R^2}{t^2} \propto \frac{1}{t^{d/2}}. \tag{2.102}$$

This yields the transport scaling, which differs significantly from the classical diffusive one.

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