
Contents

1	Introduction	1
1.1	A Brief History of Zeta Functions	1
1.1.1	Euler, Riemann	1
1.1.2	Dirichlet	3
1.1.3	Dedekind	4
1.1.4	Artin, Weil	5
1.1.5	Birch, Swinnerton-Dyer	6
1.2	Zeta Functions of Groups	6
1.2.1	Zeta Functions of Algebraic Groups	7
1.2.2	Zeta Functions of Rings	9
1.2.3	Local Functional Equations	10
1.2.4	Uniformity	11
1.2.5	Analytic Properties	12
1.3	p -Adic Integrals	14
1.4	Natural Boundaries of Euler Products	16
2	Nilpotent Groups: Explicit Examples	21
2.1	Calculating Zeta Functions of Groups	21
2.2	Calculating Zeta Functions of Lie Rings	23
2.2.1	Constructing the Cone Integral	23
2.2.2	Resolution	25
2.2.3	Evaluating Monomial Integrals	31
2.2.4	Summing the Rational Functions	32
2.3	Explicit Examples	32
2.4	Free Abelian Lie Rings	33
2.5	Heisenberg Lie Ring and Variants	34
2.6	Grenham's Lie Rings	38
2.7	Free Class-2 Nilpotent Lie Rings	40
2.7.1	Three Generators	40
2.7.2	n Generators	41
2.8	The 'Elliptic Curve Example'	42

2.9	Other Class Two Examples	43
2.10	The Maximal Class Lie Ring M_3 and Variants	45
2.11	Lie Rings with Large Abelian Ideals	48
2.12	$F_{3,2}$	51
2.13	The Maximal Class Lie Rings M_4 and Fil_4	52
2.14	Nilpotent Lie Algebras of Dimension ≤ 6	55
2.15	Nilpotent Lie Algebras of Dimension 7	62
3	Soluble Lie Rings	69
3.1	Introduction	69
3.2	Proof of Theorem 3.1	71
3.2.1	Choosing a Basis for $\mathfrak{t}_n(\mathbb{Z})$	71
3.2.2	Determining the Conditions	72
3.2.3	Constructing the Zeta Function	74
3.2.4	Transforming the Conditions	74
3.2.5	Deducing the Functional Equation	75
3.3	Explicit Examples	77
3.4	Variations	78
3.4.1	Quotients of $\mathfrak{t}_n(\mathbb{Z})$	78
3.4.2	Counting All Subrings	82
4	Local Functional Equations	83
4.1	Introduction	83
4.2	Algebraic Groups	83
4.3	Nilpotent Groups and Lie Rings	83
4.4	The Conjecture	84
4.5	Special Cases Known to Hold	86
4.6	A Special Case of the Conjecture	87
4.6.1	Projectivisation	88
4.6.2	Resolution	89
4.6.3	Manipulating the Cone Sums	91
4.6.4	Cones and Schemes	93
4.6.5	Quasi-Good Sets	95
4.6.6	Quasi-Good Sets: The Monomial Case	97
4.7	Applications of Conjecture 4.5	98
4.8	Counting Subrings and p -Subrings	102
4.9	Counting Ideals and p -Ideals	103
4.9.1	Heights, Cocentral Bases and the π -Map	104
4.9.2	Property (\dagger)	107
4.9.3	Lie Rings Without (\dagger)	119

5	Natural Boundaries I: Theory	121
5.1	A Natural Boundary for $\zeta_{\mathrm{GSp}_6}(s)$	121
5.2	Natural Boundaries for Euler Products	123
5.2.1	Practicalities	134
5.2.2	Distinguishing Types I, II and III	136
5.3	Avoiding the Riemann Hypothesis	139
5.4	All Local Zeros on or to the Left of $\Re(s) = \beta$	142
5.4.1	Using Riemann Zeros	143
5.4.2	Avoiding Rational Independence of Riemann Zeros	145
5.4.3	Continuation with Finitely Many Riemann Zeta Functions	149
5.4.4	Infinite Products of Riemann Zeta Functions	150
6	Natural Boundaries II: Algebraic Groups	155
6.1	Introduction	155
6.2	$G = \mathrm{GO}_{2l+1}$ of Type B_l	159
6.3	$G = \mathrm{GSp}_{2l}$ of Type C_l or $G = \mathrm{GO}_{2l}^+$ of Type D_l	161
6.3.1	$G = \mathrm{GSp}_{2l}$ of Type C_l	162
6.3.2	$G = \mathrm{GO}_{2l}^+$ of Type D_l	165
7	Natural Boundaries III: Nilpotent Groups	169
7.1	Introduction	169
7.2	Zeta Functions with Meromorphic Continuation	169
7.3	Zeta Functions with Natural Boundaries	170
7.3.1	Type I	171
7.3.2	Type II	171
7.3.3	Type III	173
7.4	Other Types	177
7.4.1	Types IIIa and IIIb	177
7.4.2	Types IV, V and VI	177
A	Large Polynomials	179
A.1	\mathcal{H}^4 , Counting Ideals	179
A.2	$\mathfrak{g}_{6,4}$, Counting All Subrings	180
A.3	T_4 , Counting All Subrings	180
A.4	$L_{(3,2,2)}$, Counting Ideals	181
A.5	$\mathcal{G}_3 \times \mathfrak{g}_{5,3}$, Counting Ideals	182
A.6	$\mathfrak{g}_{6,12}$, Counting All Subrings	183
A.7	$\mathfrak{g}_{1357\mathrm{G}}$, Counting Ideals	184
A.8	$\mathfrak{g}_{1457\mathrm{A}}$, Counting Ideals	186
A.9	$\mathfrak{g}_{1457\mathrm{B}}$, Counting Ideals	187
A.10	$\mathrm{tt}_6(\mathbb{Z})$, Counting Ideals	188
A.11	$\mathrm{tt}_7(\mathbb{Z})$, Counting Ideals	188

XII Contents

B Factorisation of Polynomials Associated to Classical Groups	191
References	201
Index	205
Index of Notation	207

Zeta Functions of Groups and Rings

du Sautoy, M.; Woodward, L.

2008, XII, 212 p., Softcover

ISBN: 978-3-540-74701-7