Chapter 2
Vector Bundles

The notion of vector bundle is a basic extension to the geometric domain of the fundamental idea of a vector space. Given a space X, we take a real or complex finite dimensional vector space V and make V the fibre of a bundle over X, where each fibre is isomorphic to this vector space. The simplest way to do this is to form the product $X \times V$ and the projection $pr_X : X \times V \to X$ onto the first factor. This is the product vector bundle with base X and fibre V.

On first sight, the product bundle appears to have no special features, but it contains other vector bundles which often reflect the topology of X in a strong way. This happens, for example, for the tangent bundle and the normal bundle to the spheres which are discussed in 1(2.2) and 1(2.3). All bundles of vector spaces that we will consider will have the local triviality property, namely, they are locally isomorphic to a product bundle. The product bundle is also basic because most of the vector bundles we will be considering will be subvector bundles of a product vector bundle of higher dimension. In some cases, they will be so twisted that they can only live in an infinite dimensional product vector bundle.

We will begin by formulating the concept of bundles of vector spaces over X. These will not be necessarily locally trivial, but they form a well-defined concept and category. Then, a vector bundle is a locally trivial bundle of vector spaces. The point of this distinction is that being a vector bundle is a bundle of vector spaces with an additional axiom and not an additional structure. The local charts which result are *not* new elements of structure but only a property, but to state the property, we need the notion of bundle of vector spaces. After this is done, we will be dealing with just vector bundles.

Chapter 3 of *Fibre Bundles* (Husemöller 1994) is a reference for this chapter.

1 Bundles of Vector Spaces and Vector Bundles

All the vector spaces under consideration are defined either over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We need to use the scalars explicitly, and if either number system applies, we will use the symbol F to denote either the field of real or the field of complex numbers.

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1.1. Definition A bundle of vector spaces over B is a bundle $p : E \to B$ with two additional structures

$$E \times_B E \longrightarrow E \quad \text{and} \quad F \times E \longrightarrow E$$

defined over B called addition and scalar multiplication, respectively. As maps over B, they restrict to each fibre

$$E_b \times E_b = (E \times_B E)_b \longrightarrow E_b \quad \text{and} \quad F \times E_b = (F \times E)_b \longrightarrow E_b,$$

and the basic axiom is that they define a vector space structure over F on each fibre. We require further that the function which assigns to each $b \in B$ the unique zero $0_b \in E_b$ is a continuous section.

Although it is not so necessary, we will usually require that the fibres are finite dimensional, hence the subspace topology on the fibres will be the usual vector space topology. In the infinite dimensional case, the main difference is that one has to preassign a topology on the vector space compatible with addition and scalar multiplication. Bundles of infinite dimensional vector spaces are treated in more detail in Chap. 20.

1.2. Example Let V be a finite dimensional F-vector space with the usual topology, and let $p : B \times V \to B$ be the product bundle. Then, the usual vector space structure on V defines a bundle of vector space structure on the product bundle by

$$(b,v') + (b,v'') = (b, v' + v'') \quad \text{and} \quad k(b,v) = (b,kv), \quad \text{for} \quad k \in F.$$

If $p : E \to B$ is a bundle of vector spaces and if $A \subset B$ is a subspace, then the restriction $q : E|A \to A$ is a bundle of vector spaces with the restriction of the globally defined addition and scalar multiplication. Now, this can be generalized as in 1(2.1) to subbundles of vector spaces.

1.3. Definition Let $p : E' \to B$ and $p : E'' \to B$ be two bundles of vector spaces over B. A morphism $u : E' \to E''$ of bundles of vector spaces over B is a morphism of bundles such that the restriction to each fibre $u|E'_b : E'_b \to E''_b$ is a linear map.

1.4. Remark The composition of morphisms of bundles of vector spaces over B is again a morphism of bundles of vector spaces over B. Hence, it defines a category.

1.5. Example Let $p : E \to B$ be a bundle of vector spaces, and let A be a subspace of B. Then, the restriction to a subspace $p|(E|A) : E|A \to A$ is a bundle of vector spaces.

Now, we are in a position to make the main definition of this chapter.

1.6. Definition A vector bundle $p : E \to B$ is a bundle of vector spaces such that every point $b \in B$ has an open neighborhood U with the restriction $p|(E|U) : E|U \to U$ isomorphic to the product bundle of vectors spaces $\mathrm{pr}_1 : U \times V \to U$.

Now, we will always be working with vector bundles and observe that the concept of bundle of vector spaces was only introduced as a means of defining the local triviality property of vector bundles. Of course, this could have been done more directly, but in this way, we try to illustrate the difference between structure and axiom.

1.7. Definition A morphism $u : E' \to E''$ of vector bundles over B is a morphism of the bundles of vector spaces from E' to E''.

1.8. Remark The composition of morphisms of vector bundles over B is again a morphism of vector bundles over B. Hence, it defines a category of vector bundles over B. A trivial vector bundle is one which is isomorphic to a product vector bundle.

1.9. Example Let $E' \to B$ and $E'' \to B$ be two bundles of vector spaces over B. The fibre product $E' \times_B E''$, also denoted by $E' \oplus E''$, is a bundle of vector spaces. If E' and E'' are vector bundles, then $E' \oplus E''$ is also a vector bundle. For the product bundles $B \times V'$ and $B \times V''$, the fibre product or Whitney sum is given by $(B \times V') \oplus (B \times V'') = B \times (V' \oplus V'')$. The terminology of Whitney sum comes from the direct sum and the notation $E' \oplus E''$ from the direct sum of vector spaces.

1.10. Definition Let E', E'' and E be three vector bundles over X. A vector bundle morphism $\beta : E' \oplus E'' = E' \times_X E'' \to E$ is bilinear provided $\beta|(E' \oplus E'')_x : (E' \oplus E'')_x \to E_x$ is a bilinear map of vector spaces. The tensor product $E' \otimes E''$ of E' and E'' is a specific choice of a vector bundle with a bilinear morphism $\theta : E' \oplus E'' \to E' \otimes E''$ which has the universal property that every bilinear morphism $\beta : E' \oplus E'' \to E$ factors uniquely as $u\theta$, where $u : E' \otimes E'' \to E$ is a morphism of vector bundles. Note that this is the usual definition of the tensor product for X a point.

2 Isomorphisms of Vector Bundles and Induced Vector Bundles

In the next result, we use the vector bundle local triviality axiom.

2.1. Proposition *Let $u : E' \to E''$ be a morphism of vector bundles over a space B such that $u_b : E'_b \to E''_b$ is an isomorphism for each $b \in B$. Then, u is an isomorphism.*

Proof. The inverse $v : E'' \to E'$ of u exists fibrewise as a function. The only question is its continuity, and this we check on an open covering of the form $u : E'|U \to E''|U$, where E' and E'' are each trivial bundles. In this case, the restriction of u with inverse v has the form $u : U \times F^n \to U \times F^n$ with formula $u(b,y) = (b, T(b)y)$, where $T : U \to GL_n(F)$ is a continuous map. Then, $v(b,z) = (b, T(b)^{-1}(z))$ is also continuous. This proves the proposition.

2.2. Proposition *Let $p : E \to B$ be a bundle of vector spaces over B, and let $f : B' \to B$ be a continuous map. Then, the induced bundle $f^{-1}E \to B'$ has the structure of a bundle of vector spaces such that the natural f-morphism $w : f^{-1}E \to E$ over each $b' \in B'$ on the fibre*

$$w_{b'} : (f^{-1}E)_{b'} \longrightarrow E_{f(b')}$$

is a vector space isomorphism. If E is a vector bundle, then $f^{-1}E$ is a vector bundle.

Proof. Recall that the induced bundle $f^{-1}E$ is a subspace of $B' \times E$ consisting of all (b', x) with $f(b') = p(x)$. Then, we use $f^{-1}(E \times_B E) = f^{-1}(E) \times_{B'} f^{-1}(E)$, and the sum function on the bundle of vector spaces must be of the form $(b', x) + (b', y) = (b', x + y)$. Then, scalar multiplication must be of the form $a(b', y) = (b', ay)$. In both cases, these functions are continuous, and moreover, on the fibre $w_{b'} : (f^{-1}E)_{b'} \to E_{f(b')}$ is a vector space isomorphism.

Finally, if E is a product bundle, then $f^{-1}E$ is a product bundle, and if E is locally trivial, then $f^{-1}E$ is locally trivial, for it is trivial over open sets $f^{-1}(W)$, where E is trivial over W. This proves the proposition.

2.3. Proposition *Let $(u, f) : (p' : E' \to B') \to (p : E \to B)$ be a morphism of bundles, where p' and p are vector bundles such that over each $b' \in B'$ on the fibre $u_{b'} : E'_{b'} \to E_{f(b')}$ is a morphism of vector spaces. Then, u factors by a morphism $v : E' \to f^{-1}E$ of vector bundles over B' followed by the natural f-morphism $w : f^{-1}E \to E$. Moreover, if over each $b' \in B'$ on the fibre $u_{b'} : E'_{b'} \to E_{f(b')}$ is a isomorphism of vector spaces, then $v : E' \to f^{-1}E$ is an isomorphism of vector bundles over B'.*

Proof. The factorization of u is given by $v(x') = (p'(x'), u(x'))$, and it is a vector bundle morphism since u is linear on each fibre. Moreover, $wv(x') = u(x')$ shows that it is a factorization. If $u_{b'}$ is an isomorphism on the fibre of E' at b', then

$$v_{b'}(x') = (b', u_{b'}(x'))$$

is an isomorphism $E_{f(b')} \to f^{-1}(E)_{b'} = \{b'\} \times E_{f(b')}$. Thus, we can apply (2.1) to obtain the last statement. This proves the proposition.

3 Image and Kernel of Vector Bundle Morphisms

3.1. Remark The vector bundle morphism $w : [0,1] \times F \to [0,1] \times F$ defined by the formula $w(t, z) = (t, tz)$ has a fibrewise kernel and a fibrewise image, but neither is a vector subbundle because the local triviality condition is not satisfied at $0 \in [0,1]$, where the kernel jumps from zero and the image reduces to zero.

In order to study this phenomenon, we recall some elementary conditions on rank of matrices.

3.2. Notation Let $M_{q,n}(F)$ denote the F-vector space of $q \times n$ matrices over F, that is, q rows and n columns of scalars from F. For $q = n$, we abbreviate the notation $M_{n,n}(F) = M_n(F)$ and denote by $GL_n(F)$ the group of invertible $n \times n$ matrices under matrix multiplication. We give these spaces as usual the natural Euclidean topology. We have $M_{q,n}(\mathbb{C}) = M_{q,n}(\mathbb{R}) + M_{q,n}(\mathbb{R})i$, and using blocks of matrices, we have the natural inclusion $M_{q,n}(\mathbb{R}) + M_{q,n}(\mathbb{R})i \subset M_{2q,2n}(\mathbb{R})$, where $A + Bi$ is mapped to $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M_{2q,2n}(\mathbb{R})$.

3.3. Definition The rank filtration $R_k M_{q,n}(F)$ on $M_{q,n}(F)$ is the increasing filtration of all $A \in M_{q,n}(F)$ with rank $(A) \le k$. We denote by $R_{=k} M_{q,n}(F) = R_k M_{q,n}(F) - R_{k-1} M_{q,n}(F)$.

Clearly, we have for the rank filtration

$$\{0\} = R_0 M_{q,n}(F) \subset \ldots \subset R_k M_{q,n}(F) \subset \ldots \subset R_{\min\{q,n\}} M_{q,n}(F) = M_{q,n}(F).$$

For square matrices, we have $R_{=n} M_n(F) = GL_n(F)$, the group of invertible n by n matrices with coefficients in F.

3.4. Remark The sets $R_k M_{q,n}(F)$ in the rank filtration are closed sets which for $F = \mathbb{R}$ or $\mathbb{C}$ can be described as consisting of all matrices A for which all $(k+1) \times (k+1)$ subdeterminants are zero. The group $GL_n(F)$ is the group of all square matrices A with $\det(A) \ne 0$. Since the determinant is a polynomial function, these subsets are algebraic varieties. In particular, the terms of the filtration $R_k M_{q,n}(F)$ are closed subsets of $M_{q,n}(F)$, and $GL_n(F)$ is an open subset of $M_n(F)$. The same is true for $F = \mathbb{H}$, and we will see that it depends on only knowing that $GL_n(\mathbb{H})$ is an open subset of $M_n(\mathbb{H})$ by the subdeterminant characterization of rank.

3.5. Proposition *Let $u : E' \to E''$ be a morphism of vector bundles over a space B. If rank (u_b) is locally constant, then the bundles of vector spaces* $\ker(u)$ *and* $\mathrm{im}(u)$ *are vector bundles.*

Proof. We form the subspace $\bigcup_{b \in B} \ker(u_b) = \ker(u) \subset E'$ and the subspace $\bigcup_{b \in B} \mathrm{im}(u_b) = \mathrm{im}(u) \subset E''$. With the restriction of the projections $E' \to B$ and $E'' \to B$, the subbundles $\ker(u)$ and $\mathrm{im}(u)$ are bundles of vector spaces. In order to show that they are vector bundles under the rank hypothesis, we can restrict to an open neighborhood U of any point of B, where both E' and E'' are trivial and u_b has constant rank q.

Choose an isomorphism to the product bundle, and then u can be represented by a vector bundle morphism $w : U \times F^n \to U \times F^m$ which has the form $w(b,y) = (b, T(b)y)$. As a continuous function $T : U \to M_{m,n}$, the matrix $T(b)$ has rank q over U. For each $x \in U$, we can choose a change of coordinates so that $T(b) = \begin{pmatrix} A(b) & B(b) \\ C(b) & D(b) \end{pmatrix}$ such that

$$A(x) \in GL_q(F),\ B(x) = 0,\ C(x) = 0,\ D(x) = 0,$$

and choosing a subopen neighborhood of x in U, called again U, we can assume that $A(b)$ is invertible for all $b \in U$. Here, we use that $GL_q(F) \subset M_q(F)$ is open. Let $\pi : U \times F^n \to U \times F^{n-q}$ denote the projection on the second factor, and note that for $x \in U$, it is the projection onto the $\ker(T(x))$. Since $T(b)$ has constant rank q and

$A(b)$ is invertible, the restriction $\pi|\ker(w) = \pi|U \times \ker(T(b))$ is an isomorphism $U \times \ker(T(b)) \to U \times F^{n-q}$. Hence, we see that $\ker(w)$ is locally trivial and thus a vector bundle.

If $w^t : U \times F^m \to U \times F^n$ is the morphism given by the transpose matrix $T(b)^t$ and the formula $w^t(b,y) = (b, T(b)^t y)$, then $\mathrm{im}(w) = \ker(w^t)$ is also a vector bundle. This proves the proposition.

A very useful case where we know the rank is locally constant and used in the next chapter is in the following proposition.

3.6. Proposition *Let $e = e^2 : E \to E$ be an idempotent endomorphism of a vector bundle $p : E \to B$ over B. Then, the set of $b \in B$ with rank$(e_b) = q$ is open and closed in B. Hence,* $\ker(e)$ *and* $\mathrm{im}(e)$ *are vector bundles, and* $E = \ker(e) \times_B \mathrm{im}(e) = \ker(e) \oplus \mathrm{im}(e)$.

Proof. Since on each fibre the identity $1 = e_b + (1 - e_b)$ is the sum of two complementary projections, we have rank (e_b)+rank$(1 - e_b) = n$. The set of all $b \in B$ with rank$(e_b) = q$ is at the same time the set of all $b \in B$ with rank $(e_b) \le q$ and rank $(1 - e_b) \le n - q$, which is a closed set, and the set of all $b \in B$ with rank$(e_b) \ge q$ and rank$(1 - e_b) \ge n - q$, which is an open set. Now apply (3.5). This proves the proposition.

4 The Canonical Bundle Over the Grassmannian Varieties

4.1. Definition Let X be a union of an increasing family of subspaces

$$X_0 \subset \ldots \subset X_N \subset X_{N+1} \subset \ldots \subset \varinjlim_N X_N = X$$

with the weak topology, that is a subset $M \subset X$ is closed if and only if $X_m \cap M$ is closed in X_m for all $m \ge 0$.

Another name for the weak topology is the inductive limit topology.

The following inductive limits have the weak topology which start with the inclusions

$$F^N \subset F^{N+1} \subset \ldots \subset \varinjlim_N F^N = F^\infty.$$

4.2. Definition Let $P_n(F^N)$ denote the subspace of $(F^N)^n$ consisting of linearly independent n-tuples of vectors in F^N. Let $Gr_n(F^N)$ denote the quotient space of $P_n(F^N)$ which assigns to a linearly independent n-tuple the subspace F^N of dimension n of which the n-tuple is basis. The space $Gr_n(F^N)$ is called the Grassmann variety of n-dimensional subspaces of F^N, and $P_n(F^N)$ is called the Stiefel variety of linearly independent frames in N-dimensional space.

4.3. Remark The quotient morphism $q : P_n(F^N) \to Gr_n(F^N)$ has the structure of a principal $GL_n(F)$ bundle with right action $P_n(F^N) \times GL_n(F) \to P_n(F^N)$ given by right multiplication of an n-tuple of vectors and by $n \times n$ matrix of scalars

$$(x_1,\dots,x_n).(a_{i,j}) = (y_1,\dots,y_n),$$

where $\sum_{i=1}^{n} x_i a_{i,j} = y_j$ for $i,j = 1,\dots,n$. The subject of principal bundles is taken up in Chap. 5 and we will return to this example.

There are natural inclusions

$$P_n(F^N) \subset P_n(F^{N+1}) \ \text{ and } \ Gr_n(F^N) \subset Gr_n(F^{N+1})$$

induced by the inclusion of $F^N \subset F^{N+1} = F^N \oplus F$ as zero in the last coordinate.

4.4. Definition The product bundle $Gr_n(F^N) \times F^N$ has two subvector bundles: $E^n(N)$ consisting of all $(W,v) \in Gr_n(F^N) \times F^N$ with $v \in W$ and ${}^{\perp}E^{N-n}(N)$ consisting of all $(W,v) \in Gr_n(F^N) \times F^N$ with $v \perp W$, that is, v is orthogonal to all vectors in W. The vector bundle $E^n(N)$ is called the universal vector bundle over $Gr_n(F^N)$.

4.5. Remark If p_W denotes the orthogonal projection of F^N onto W, then we have an isomorphism $\theta : Gr_n(F^N) \times F^N \to E^n(N) \oplus {}^{\perp}E^{N-n}(N)$ onto the Whitney sum, where $\theta(W,v) = (W, p_W(v)) \oplus (W, v - p_W(v))$. Over the natural inclusion $Gr_n(F^N) \subset Gr_n(F^{N+1})$, there is the natural inclusion of product vector bundles $Gr_n(F^N) \times F^N \subset Gr_n(F^{N+1}) \times F^{N+1}$ which under θ induces natural inclusions of the Whitney sum factors $E^n(N) \subset E^n(N+1)$ and ${}^{\perp}E^{N-n}(N) \subset {}^{\perp}E^{N+1-n}(N+1)$.

Now, we consider the inductive limit spaces and vector bundles as N goes to infinity.

4.6. Remark The inductive limit construction of (4.1) yields the n-dimensional vector bundle

$$E^n = \varinjlim_{N \geq n} E^n(N) \longrightarrow \varinjlim_{N \geq n} Gr_n(F^N) = Gr_n(F^\infty).$$

This vector bundle has the universal property saying that every reasonable vector bundle is induced from it and under certain circumstances, from the subbundles on the finite Grassmann varieties as we see in the next section.

5 Finitely Generated Vector Bundles

5.1. Theorem *The following six properties of a vector bundle E over X of dimension n and given $N \geq n$ are equivalent:*

(1) There is a continuous $w : E \to F^N$ with $w|E_x : E_x \to F^N$ a linear monomorphism for each $x \in X$.
(2) There is a vector bundle morphism $(u,f) : E \to E^n(N)$ which is an isomorphism on each fibre of E.
(3) There is an isomorphism $E \to f^*(E^n(N))$ over X for some continuous map $f : X \to Gr_n(F^N)$.

(4) There is a vector bundle E' over X and an isomorphism $\phi : E \oplus E' \to X \times F^N$ to the trivial N-dimensional bundle over X.
(5) There exists a surjective vector bundle morphism $\psi : X \times F^N \to E$ which is surjective on each fibre.
(6) There exist continuous sections $s_1, \dots, s_N \in \Gamma(X,E)$ such that the vectors $s_1(x)$, $\dots, s_N(x)$ generate the vector space E_x for each $x \in X$.

Proof. We begin with a circle of implications.

(1) implies (2): Given $w : E \to F^N$ as in (1), we define the morphism of vector bundles $(u, f) : E \to E^n(N)$ by $f(x) = w(E_x)$ and $u(v) = (E_{\pi(v)}, w(v)) \in E^n(N)$ for $v \in E$ and $\pi : E \to X$ the projection.

(2) implies (3): Given (u, f) as in (2), we factor u by the induced bundle $E \xrightarrow{u'} f^*(E^n(N)) \xrightarrow{u''} E^n(N)$, where $u' : E \to f^*(E^n(N))$ is a fibrewise isomorphism, hence the desired isomorphism in (3).

(3) implies (4): Given the isomorphism $E \to f^*(E^n(N))$ as in (3), we define $E' = f^*({}^{\perp}E^{N-n}(N))$ and take the Whitney sum leading to an isomorphism $E \oplus E' \to f^*(E^n(N)) \oplus f^*({}^{\perp}E^{N-n}(N)) = f^*(E^n(N) \oplus^{\perp} E^{N-n}(N))$. Since $E^n(N) \oplus^{\perp} E^{N-n}(N) = Gr_n(F^N) \times F^N$, the product bundle, and a map induces the product bundle to a trivial bundle, we have an isomorphism $E \oplus E' \to X \times F^N$ as in (4).

(4) implies (1): Given an isomorphism $v : E \oplus E' \to X \times F^N$ over X. Using the injection $j : E \to E \oplus E'$ and the projection $pr_2 : X \times F^N \to F^N$, we form the composite $w = (pr_2)vj : E \to F^N$, and this is the desired map in (1), where the restriction $w|E_x : E_x \to F^N$ is a linear monomorphism for each $x \in X$.

(4) implies (5): We use the inverse $X \times F^N \to E \oplus E'$ to the isomorphism in (4) composed with the projection $E \oplus E' \to E$ to obtain a vector bundle morphism $v : X \times F^N \to E$ which is surjective on each fibre.
Conversely, (5) implies (4) by considering the kernel $E' = \ker(\psi)$ of ψ which is a subvector bundle of rank $N - n$ and its orthogonal complement E'' which is the subvector bundle of all (x, v) with $v \perp \ker(\psi)_x$. The restriction $\psi|E'' : E'' \to E$ is a fibrewise isomorphism and hence an isomorphism of vector bundles. The desired morphism is the sum of $\phi = (\psi|E'')^{-1} \oplus j : E \oplus \ker(\psi) = E \oplus E' \to X \times F^N$ in (4).

(5) and (6) are equivalent. For the natural basic sections $\sigma_i(x) = (x, e_i)$ of the trivial bundle $X \times F^N$ have images $s_i = \psi\sigma_i$ given by ψ with (5) for each $i = 1, \dots, N$ and conversely ψ is defined by a set of N sections s_i in (6) by the condition $\psi(x, a_1, \dots, a_N) = a_1 s_1(x) + \cdots + a_N s_N(x)$. Finally, the equivalence of (5) and (6) follows by observing that $\psi(x, \ )$ is surjective over $x \in X$ if and only if $s_1(x), \dots, s_N(x)$ generates E_x. This proves the theorem.

5.2. Definition A vector bundle satisfying any of the equivalent conditions of (5.1) is called finitely generated.

For $N = n$, we have the following corollary.

5.3. Corollary *The following six properties of a vector bundle E over X of dimension n are equivalent:*

(1) There is a continuous $w : E \to F^n$ *with* $w|E_x : E_x \to F^n$*, a linear isomorphism for each* $x \in X$.
(2) There is a vector bundle morphism $(u, f) : E \to E^n(n) = \{*\} \times F^n$.
(3) There is an isomorphism $E \to f^*(E^n(n))$ *over* X *for the continuous map* $f : X \to Gr_n(F^n) = \{*\}$*, a point.*
(4) or (5) There is an isomorphism $E \to X \times F^n$ *or* $X \times F^n \to E$ *between* E *and the product n-dimensional bundle over* X.
(6) There exist continuous sections $s_1, \ldots, s_n \in \Gamma(X, E)$ *such that the vectors* $s_1(x), \ldots, s_n(x)$ *form a basis of* E_x *for each* $x \in X$.

6 Vector Bundles on a Compact Space

6.1. Theorem *Every vector bundle over a compact space is finitely generated.*

Proof. Let $p : E \to X$ be a F-vector bundle over X. Let $V_1, \ldots, V_m$ be a finite open covering of X such that $E|V_i$ is trivial for all $i = 1, \ldots, m$. Choose open sets $U_1 \subset V_1, \ldots, U_m \subset V_m$ and continuous functions $\xi_i : X \to [0, 1] \subset F$ such that

(1) the $U_1, \ldots, U_m$ is an open covering of X and
(2) $\mathrm{supp}(\xi_i) \subset V_i$ and $\xi_i|U_i = 1$ for $i = 1, \ldots, m$. This is possible since X is normal. We define a Gauss map $w : E \to (F^n)^m$ by choosing trivializing Gauss maps $w_i : E|V_i \to F^n$ and forming the map $w(v) = (\xi_i(p(v))w_i(v))_{1 \le i \le m} \in (F^n)^m$, where this means $\xi_i(p(v))w_i(v) = 0$ if $v \in E - E|V_i$. The function w is continuous by the support condition (2) on the ξ_i, and the restriction $w|E_x$ is a linear monomorphism for each $x \in X$. Thus, there exists a Gauss map for E, and hence E is finitely generated.

6.2. Remark If $p : E \to X$ is a vector bundle over a space and if (U_i, V_i) is a sequence of normal pairs with continuous functions $\xi_i : X \to [0, 1] \subset F$ such that the $U_1, \ldots, U_m, \ldots$ is an open covering of X, then using the properties $\mathrm{supp}(\xi_i) \subset V_i$ and $\xi_i|U_i = 1$ for $i \ge 1$, we have a Gauss map $w : E \to (F^n)^\infty = F^\infty$ given by the same formula $w(v) = (\xi_i(p(v))w_i(v))_{1 \le i} \in (F^n)^\infty = F^\infty$. This Gauss map defines a vector bundle morphism $(u, f) : E \to E^n(\infty)$ which is an isomorphism on each fibre of E by $f(x) = w(E_x)$ and $u(v) = (E_{p(v)}, w(v))$ for $v \in E$. Then, there is an isomorphism $u' : E \to f^*(E^n(\infty))$, that is, such a vector bundle is induced from the universal bundle over $Gr_n(F^\infty)$.

7 Collapsing and Clutching Vector Bundles on Subspaces

There are two topological operations on vector bundles which play a basic role in the geometric considerations related to vector bundles.

7.1. Definition Let A be a closed subspace of X and form the quotient $q : X \to X/A$, where A is collapsed to a point. Let $p : E \to X$ be a vector bundle with a trivialization $t : E|A \to A \times F^n$ over the subspace A. The collapsed vector bundle $E/t \to X/A$ is the unique vector bundle defined in terms of q-morphism $u : E \to E/t$ of vector bundles such that $t = t_* u$ for an isomorphism of the fibre $t_* : (E/t)_* \to F^n$.

7.2. Definition Let $X = A' \cup A''$ be the union of two closed subspaces with $A = A' \cap A''$. Let $E' \to A'$ and $E'' \to A''$ be two vector bundles, and $\alpha : E'|A \to E''|A$ be an isomorphism of vector bundles over A. The clutched vector bundle $E' \cup_\alpha E''$ is the unique vector bundle $E \to X$ together with isomorphisms $u' : E' \to E|A'$ and $u'' : E'' \to E|A''$ such that on A we have $u' = u''\alpha$.

In the case of both of the previous definitions, a direct quotient process gives a bundle of vector spaces, and as for the question of local triviality, we have the following remarks.

7.3. Local Considerations in the Previous Two Definitions For the local triviality of the bundles E/t and $E' \cup_\alpha E''$, we have to be able to extend the trivializing t on A or the clutching isomorphism α on A to an open neighborhood of A. Here, we must assume that either A is a closed subspace in a compact X and then use the Tietze extension theorem or assume that A is a subcomplex of a CW-complex X. For this, see *Fibre bundles* (Husemöller 1994) p.123, 135, for more details.

7.4. Functoriality Both the collapsing and clutching are functorial for maps $w : E' \to E''$ such that $t''w = t'$ over $A \subset X$ inducing $E'/t' \to E''/t''$ and morphisms $f' : E' \to F'$ over A' and $f'' : E'' \to F''$ over A'' commuting with clutching data $f''\alpha = \beta f'$ inducing a morphism $f : E' \cup_\alpha E'' \to F' \cup_\beta F''$ of vector bundles.

7.5. Remark Let A be a closed subspace of X, and let E' be a vector bundle on X/A. The induced bundle $q^*(E') = E$ has a natural trivialization t over A, and there is a natural isomorphism $E/t \to E'$ from the collapsed vector bundle to the original vector bundle E'. Let $X = A' \cup A''$ and $A = A' \cap A''$, and let E be a vector bundle over X. For $E' = E|A'$ and $E'' = E|A''$, and α the identity on $E|A$. Then, there is a natural isomorphism

$$E' \cup_\alpha E'' \longrightarrow E$$

of the clutched vector bundle to the original vector bundle.

7.6. Commutation with Whitney Sum Over the quotient X/A, we have a natural isomorphism of the Whitney sum

$$(E'/t') \oplus (E''/t'') \longrightarrow (E' \oplus E'')/(t' \oplus t'').$$

Over $X = A' \cup A''$ and $A = A' \cap A''$, we have a natural isomorphism of the Whitney sum of clutched vector bundles

$$(E' \cup_\alpha E'') \oplus (F' \cup_\beta F'') \longrightarrow (E' \oplus F') \cup_{\alpha \oplus \beta} (E'' \oplus F'').$$

7.7. Commutation with Induced Bundles Let $f : (Y,B) \to (X,A)$ be a map of pairs, and let E be a vector bundle with trivialization $t : E|A \to A \times F^n$. For the natural morphism $w : E \to E/t$ over $q : X \to X/A$, we have a trivialization $tw : f^*(E)|B \to B \times F^n$ and a natural morphism of $f^*(E)/tw \to f^*(E/t)$ induced by w.

Let $f : (Y;B',B'',B) \longrightarrow (X;A',A'',A)$ be a map of coverings as in (7.2), let E' (resp. E'') be a vector bundle over A' (resp. A''), and let $\alpha : E'|A \to E''|A$ be an isomorphism. Then there is an natural isomorphism

$$(f|A')^*(E') \cup_\beta (f|A'')^*(E'') \to f^*(E' \cup_\alpha E''),$$

where $\beta = f^*(\alpha)$.

8 Metrics on Vector Bundles

Let $\bar{z}$ denote the conjugation which on $\mathbb{R}$ is the identity, on $\mathbb{C}$ is complex conjugation, and on $\mathbb{H}$ is the quaternionic conjugation.

8.1. Definition Let V be a left vector space over F. An inner product is a function $\beta : V \times V \to F$ such that

(1) For $a,b \in F$ and $x,x',y,y' \in V$, the sesquilinearity of β is

$$\beta(ax+bx',y) = a\beta(x,y) + b\beta(x',y)$$

$$\beta(x,ay+by') = \bar{a}\beta(x,y) + \bar{b}\beta(x,y').$$

(2) For $x,y \in V$, we have conjugate symmetry $\beta(x,y) = \overline{\beta(y,x)}$.
(3) For $x \in V$, we have $\beta(x,x) \geq 0$ in $\mathbb{R}$ and $\beta(x,x) = 0$ if and only if $x = 0$.

Two vectors $x,y \in V$ are perpendicular provided $\beta(x,y) = 0$. For a subspace W of V, the set $W^\perp$ of all $y \in V$ with y perpendicular to all $x \in W$ is a subspace of V and $V = W \oplus W^\perp$.

8.2. Definition An inner product on a vector bundle E over X is a map $\beta : E \oplus E \to F$ such that the restriction for $x \in X$ to each fibre $\beta_x : E_x \times E_x \to F$ is an inner product on E_x.

8.3. Example On the trivial bundles $X \times F^n$ or even $X \times F^\infty$, there exists a natural inner product with

$$\beta(x,u_1,\ldots,u_n,\ldots,x,v_1,\ldots,v_n,\ldots) = \sum\nolimits_{1\leq j} u_j \bar{v}_j$$

(always a finite sum). This formula holds both in the finite and in the infinite case where vectors have only finitely many nonzero components. By restriction, every inner product on a vector bundle gives an inner product on each subbundle. For a map $f: Y \to X$ and a vector bundle E with inner product β, we define a unique inner product on $f^*(E)$ such that the natural f-morphism $w: f^*(E) \to E$ is fibrewise an isomorphism of vector spaces with inner product $f^*(\beta)$ by the formula

$$f^*(\beta)(y,z';y,z'') = \beta(z',z'') \quad \text{for} \quad (y,z'),(y,z'') \in f^*(E).$$

In particular, every bundle which is induced from the universal bundle over $G_n(F^n)$ has a metric.

8.4. Remark If E' is a subbundle of a bundle E with a metric β, then the fibrewise union of $E_x'^{\perp}$ is a vector bundle, denoted by $E'^{\perp}$, and the natural $E' \oplus E'^{\perp} \to E$ is an isomorphism.

Reference

Husemöller, D.: *Fibre Bundles*, 3rd ed. Springer-Verlag, New York (1994)

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