

# Contents

**Physical Background to the K-Theory Classification of D-Branes:  
Introduction and References . . . . . 1**

**Part I Bundles over a Space and Modules over an Algebra**

**1 Generalities on Bundles and Categories . . . . . 9**

1 Bundles Over a Space . . . . . 9

2 Examples of Bundles . . . . . 11

3 Two Operations on Bundles . . . . . 13

4 Category Constructions Related to Bundles . . . . . 14

5 Functors Between Categories . . . . . 16

6 Morphisms of Functors or Natural Transformations . . . . . 18

7 Étale Maps and Coverings . . . . . 20

References . . . . . 22

**2 Vector Bundles . . . . . 23**

1 Bundles of Vector Spaces and Vector Bundles . . . . . 23

2 Isomorphisms of Vector Bundles and Induced Vector Bundles . . . . . 25

3 Image and Kernel of Vector Bundle Morphisms . . . . . 26

4 The Canonical Bundle Over the Grassmannian Varieties . . . . . 28

5 Finitely Generated Vector Bundles . . . . . 29

6 Vector Bundles on a Compact Space . . . . . 31

7 Collapsing and Clutching Vector Bundles on Subspaces . . . . . 31

8 Metrics on Vector Bundles . . . . . 33

Reference . . . . . 34

**3 Relation Between Vector Bundles, Projective Modules, and  
Idempotents . . . . . 35**

1 Local Coordinates of a Vector Bundle Given by Global Functions  
over a Normal Space . . . . . 36

2 The Full Embedding Property of the Cross Section Functor . . . . . 37

3 Finitely Generated Projective Modules . . . . . 38

4 The Serre–Swan Theorem . . . . . 40

5	Idempotent Classes Associated to Finitely Generated Projective Modules .....	42
<b>4</b>	<b>K-Theory of Vector Bundles, of Modules, and of Idempotents .....</b>	<b>45</b>
1	Generalities on Adding Negatives .....	45
2	$K$ -Groups of Vector Bundles .....	47
3	$K$ -Groups of Finitely Generated Projective Modules .....	48
4	$K$ -Groups of Idempotents .....	50
5	$K$ -Theory of Topological Algebras .....	51
	References .....	54
<b>5</b>	<b>Principal Bundles and Sections of Fibre Bundles: Reduction of the Structure and the Gauge Group I .....</b>	<b>55</b>
1	Bundles Defined by Transformation Groups .....	55
2	Definition and Examples of Principal Bundles .....	57
3	Fibre Bundles .....	58
4	Local Coordinates for Fibre Bundles .....	58
5	Extension and Restriction of Structure Group .....	60
6	Automorphisms of Principal Bundles and Gauge Groups .....	62
	Reference .....	62
 <b>Part II Homotopy Classification of Bundles and Cohomology: Classifying Spaces</b>		
<b>6</b>	<b>Homotopy Classes of Maps and the Homotopy Groups .....</b>	<b>65</b>
1	The Space $\text{Map}(X, Y)$ .....	65
2	Continuity of Substitution and $\text{Map}(X \times T, Y)$ .....	66
3	Free and Based Homotopy Classes of Maps .....	67
4	Homotopy Categories .....	68
5	Homotopy Groups of a Pointed Space .....	69
6	Bundles on a Cylinder $B \times [0, 1]$ .....	72
<b>7</b>	<b>The Milnor Construction: Homotopy Classification of Principal Bundles .....</b>	<b>75</b>
1	Basic Data from a Numerable Principal Bundle .....	75
2	Total Space of the Milnor Construction .....	76
3	Uniqueness up to Homotopy of the Classifying Map .....	78
4	The Infinite Sphere as the Total Space of the Milnor Construction ..	80
	References .....	81
<b>8</b>	<b>Fibrations and Bundles: Gauge Group II .....</b>	<b>83</b>
1	Factorization, Lifting, and Extension in Square Diagrams .....	84
2	Fibrations and Cofibrations .....	85
3	Fibres and Cofibres: Loop Space and Suspension .....	88
4	Relation Between Loop Space and Suspension Group Structures on Homotopy Classes of Maps $[X, Y]_*$ .....	90

5	Outline of the Fibre Mapping Sequence and Cofibre Mapping Sequence .....	91
6	From Base to Fibre and From Fibre to Base .....	93
7	Homotopy Characterization of the Universal Bundle .....	95
8	Application to the Classifying Space of the Gauge Group .....	95
9	The Infinite Sphere as the Total Space of a Universal Bundle .....	96
	Reference .....	96
<b>9</b>	<b>Cohomology Classes as Homotopy Classes: CW-Complexes .....</b>	<b>97</b>
1	Filtered Spaces and Cell Complexes .....	98
2	Whitehead's Characterization of Homotopy Equivalences .....	99
3	Axiomatic Properties of Cohomology and Homology .....	100
4	Construction and Calculation of Homology and Cohomology .....	103
5	Hurewicz Theorem .....	105
6	Representability of Cohomology by Homotopy Classes .....	105
7	Products of Cohomology and Homology .....	106
8	Introduction to Morse Theory .....	107
	References .....	109
<b>10</b>	<b>Basic Characteristic Classes .....</b>	<b>111</b>
1	Characteristic Classes of Line Bundles .....	111
2	Projective Bundle Theorem and Splitting Principle .....	113
3	Chern Classes and Stiefel–Whitney Classes of Vector Bundles .....	114
4	Elementary Properties of Characteristic Classes .....	117
5	Chern Character and Related Multiplicative Characteristic Classes .....	118
6	Euler Class .....	121
7	Thom Space, Thom Class, and Thom Isomorphism .....	122
8	Stiefel–Whitney Classes in Terms of Steenrod Operations .....	122
9	Pontrjagin classes .....	125
	References .....	125
<b>11</b>	<b>Characteristic Classes of Manifolds .....</b>	<b>127</b>
1	Orientation in Euclidean Space and on Manifolds .....	127
2	Poincaré Duality on Manifolds .....	129
3	Thom Class of the Tangent Bundle and Duality .....	130
4	Euler Class and Euler Characteristic of a Manifold .....	131
5	Wu's Formula for the Stiefel–Whitney Classes of a Manifold .....	132
6	Cobordism and Stiefel–Whitney Numbers .....	133
7	Introduction to Characteristic Classes and Riemann–Roch .....	134
	Reference .....	135

<b>12 Spin Structures</b> .....	137
1 The Groups $Spin(n)$ and $Spin^c(n)$ .....	137
2 Orientation and the First Stiefel–Whitney Class .....	139
3 Spin Structures and the Second Stiefel–Whitney Class .....	140
4 $Spin^c$ Structures and the Third Integral Stiefel–Whitney Class .....	141
5 Relation Between Characteristic Classes of Real and Complex Vector Bundles .....	142
6 Killing Homotopy Groups in a Fibration .....	142

### Part III Versions of $K$ -Theory and Bott Periodicity

<b>13 G-Spaces, G-Bundles, and G-Vector Bundles</b> .....	149
1 Relations Between Spaces and G-Spaces: G-Homotopy .....	149
2 Generalities on G-Bundles .....	152
3 Generalities on G-Vector Bundles .....	153
4 Special Examples of G-Vector Bundles .....	155
5 Extension and Homotopy Problems for G-Vector Bundles for G a Compact Group .....	157
6 Relations Between Complex and Real G-Vector Bundles .....	158
7 $KR_G$ -Theory .....	159
References .....	161
<b>14 Equivariant <math>K</math>-Theory Functor <math>K_G</math> : Periodicity, Thom Isomorphism, Localization, and Completion</b> .....	163
1 Associated Projective Space Bundle to a $G$ -Equivariant Bundle .....	163
2 Assertion of the Periodicity Theorem for a Line Bundle .....	164
3 Thom Isomorphism .....	167
4 Localization Theorem of Atiyah and Segal .....	170
5 Equivariant $K$ -Theory Completion Theorem of Atiyah and Segal .....	172
References .....	173
<b>15 Bott Periodicity Maps and Clifford Algebras</b> .....	175
1 Vector Bundles and Their Principal Bundles and Metrics .....	175
2 Homotopy Representation of $K$ -Theory .....	176
3 The Bott Maps in the Periodicity Series .....	179
4 $KR_G^*(X)$ and the Representation Ring $RR(G)$ .....	180
5 Generalities on Clifford Algebras and Their Modules .....	181
6 $KR_G^{-q}(\ast)$ and Modules Over Clifford Algebras .....	184
7 Bott Periodicity and Morse Theory .....	185
8 The Graded Rings $KU^*(\ast)$ and $KO^*(\ast)$ .....	187
References .....	188

<b>16 Gram–Schmidt Process, Iwasawa Decomposition, and Reduction of Structure</b>	189
1 Classical Gram–Schmidt Process	189
2 Definition of Basic Linear Groups	190
3 Iwasawa Decomposition for $GL$ and $SL$	191
4 Applications to Structure Group Reduction for Principal Bundles Related to Vector Bundles	192
5 The Special Case of $SL_2(\mathbb{R})$ and the Upper Half Plane	193
6 Relation Between $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ with the Lorentz Groups	194
A Appendix: A Novel Characterization of the Iwasawa Decomposition of a Simple Lie Group (by <i>B. Krötz</i> )	195
References	201
<b>17 Topological Algebras: <math>G</math>-Equivariance and <math>KK</math>-Theory</b>	203
1 The Module of Cross Sections for a $G$ -Equivariant Vector Bundle	204
2 $G$ -Equivariant $K$ -Theory and the $K$ -Theory of Cross Products	205
3 Generalities on Topological Algebras: Stabilization	207
4 $\text{Ell}(X)$ and $\text{Ext}(X)$ Pairing with $K$ -Theory to $\mathbb{Z}$	209
5 Extensions: Universal Examples	212
6 Basic Examples of Extensions for $K$ -Theory	215
7 Homotopy Invariant, Half Exact, and Stable Functors	219
8 The Bivariant Functor $kk_*(A, B)$	220
9 Bott Map and Bott Periodicity	221
A Appendix: The Green–Julg Theorem (by <i>S. Echterhoff</i> )	223
References	226
<b>Part IV Algebra Bundles: Twisted <math>K</math>-Theory</b>	
<b>18 Isomorphism Classification of Operator Algebra Bundles</b>	229
1 Vector Bundles and Algebra Bundles	230
2 Principal Bundle Description and Classifying Spaces	231
3 Homotopy Classification of Principal Bundles	233
4 Classification of Operator Algebra Bundles	235
References	239
<b>19 Brauer Group of Matrix Algebra Bundles and <math>K</math>-Groups</b>	241
1 Properties of the Morphism $\alpha_n$	241
2 From Brauer Groups to Grothendieck Groups	243
3 Stability I: Vector Bundles	244
4 Stability II: Characteristic Classes of Algebra Bundles and Projective $K$ -Group	245
5 Rational Class Groups	246
6 Sheaf Theory Interpretation	247
Reference	249

<b>20</b>	<b>Analytic Definition of Twisted <math>K</math>-Theory</b>	251
1	Cross Sections and Fibre Homotopy Classes of Cross Sections	251
2	Two Basic Analytic Results in Bundle Theory and $K$ -Theory	252
3	Twisted $K$ -Theory in Terms of Fredholm Operators	253
<b>21</b>	<b>The Atiyah–Hirzebruch Spectral Sequence in <math>K</math>-Theory</b>	255
1	Exact Couples: Their Derivation and Spectral Sequences	255
2	Homological Spectral Sequence for a Filtered Object	256
3	$K$ -Theory Exact Couples for a Filtered Space	258
4	Atiyah–Hirzebruch Spectral Sequence for $K$ -Theory	260
5	Formulas for Differentials	262
6	Calculations for Products of Real Projective Spaces	263
7	Twisted $K$ -Theory Spectral Sequence	264
	Reference	264
<b>22</b>	<b>Twisted Equivariant <math>K</math>-Theory and the Verlinde Algebra</b>	265
1	The Verlinde Algebra as the Quotient of the Representation Ring	266
2	The Verlinde Algebra for $SU(2)$ and $\mathfrak{sl}(2)$	268
3	The $G$ -Bundles on $G$ with the Adjoint $G$ -Action	271
4	A Version of the Freed–Hopkins–Teleman Theorem	273
	References	274

## Part V Gerbes and the Three Dimensional Integral Cohomology Classes

<b>23</b>	<b>Bundle Gerbes</b>	277
1	Notation for Gluing of Bundles	277
2	Definition of Bundle Gerbes	280
3	The Gerbe Characteristic Class	281
4	Stability Properties of Bundle Gerbes	283
5	Extensions of Principal Bundles Over a Central Extension	284
6	Modules Over Bundle Gerbes and Twisted $K$ -Theory	284
	Reference	286
<b>24</b>	<b>Category Objects and Groupoid Gerbes</b>	287
1	Simplicial Objects in a Category	287
2	Categories in a Category	290
3	The Nerve of the Classifying Space Functor and Definition of Algebraic $K$ -Theory	293
4	Groupoids in a Category	295
5	The Groupoid Associated to a Covering	297
6	Gerbes on Groupoids	298
7	The Groupoid Gerbe Characteristic Class	300

<b>25</b>	<b>Stacks and Gerbes</b>	303
1	Presheaves and Sheaves with Values in Category	304
2	Generalities on Adjoint Functors	306
3	Categories Over Spaces (Fibred Categories)	309
4	Prestacks Over a Space	311
5	Descent Data	315
6	The Stack Associated to a Prestack	316
7	Gerbes as Stacks of Groupoids	318
8	Cohomological Classification of Principal $G$ -Sheaves	318
9	Cohomological Classification of Bands Associated with a Gerbe	320
	<b>Bibliography</b>	323
	<b>Index of Notations</b>	327
	<b>Notation for Examples of Categories</b>	333
	<b>Index</b>	335



Basic Bundle Theory and K-Cohomology Invariants

Husemöller, D.; JOACHIM, M.; Jurco, B.; Schottenloher, M.

2008, XV, 340 p., Hardcover

ISBN: 978-3-540-74955-4