# 2

# Stochastic Integration with Respect to fBm and Related Topics

## 2.1 Pathwise Stochastic Integration

### 2.1.1 Pathwise Stochastic Integration in the Fractional Sobolev-type Spaces

In this subsection we consider pathwise integrals $\int_0^T f(t)dB_t^H$ for processes $f$ from the fractional Sobolev type spaces $I_{a+}^{\alpha}(L^p)$ for some $p > 1$. This approach was developed by Zähle (Zah98), (Zah99), (Zah01).

Consider two nonrandom functions $f$ and $g$ defined on some interval $[a, b] \subset \mathbb{R}$ and suppose that the limits $f(u+) := \lim_{\delta \downarrow 0} f(u + \delta)$ and $g(u-) := \lim_{\delta \downarrow 0} g(u - \delta)$, $a \le u \le b$, exist. Put $f_{a+}(x) := (f(x) - f(a+))\mathbf{1}_{(a,b)}(x)$, $g_{b-}(x) := (g(b-) - g(x))\mathbf{1}_{(a,b)}(x)$. Suppose also that $f_{a+} \in I_{a+}^{\alpha}(L_p[a, b])$, $g_{b-} \in I_{b-}^{1-\alpha}(L_p[a, b])$ for some $p \ge 1, q \ge 1, 1/p + 1/q \le 1$, $0 \le \alpha \le 1$. Then, evidently, $D_{a+}^{\alpha} f_{a+} \in L_p[a, b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_q[a, b]$.

**Definition 2.1.1.** The *generalized* (*fractional*) *Lebesgue–Stieltjes integral* $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) := \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx + f(a+)(g(b-) - g(a+)).$$

**Lemma 2.1.2.** *Definition* 2.1.1 *does not depend on the possible choice of* $\alpha$.

*Proof.* Let $f_{a+} \in (I_{a+}^{\alpha} \cap I_{a+}^{\alpha+\beta})(L_p[a, b])$, $g_{b-} \in (I_{b-}^{1-\alpha} \cap I_{b-}^{1-\alpha-\beta})(L_q[a, b])$ for some $\alpha, \beta$ such that $0 \le \alpha \le 1$, $0 \le \alpha + \beta \le 1$, $1/p + 1/q \le 1$. Then, according to (1.1.5) (composition formula for fractional derivatives) and (1.1.6) (integration-by-parts formula),

$$\int_a^b (D_{a+}^{\alpha+\beta} f_{a+})(x)(D_{b-}^{1-\alpha-\beta} g_{b-})(x)dx$$

$$= \int_a^b (D_{a+}^{\beta} D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha-\beta} g_{b-})(x)dx$$

$$= \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{\beta} D_{b-}^{1-\alpha-\beta} g_{b-})(x)dx$$

$$= \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx.$$

□

Let $\alpha p < 1$. Then $f_{a+} \in I_{a+}^{\alpha}(L_p[a,b])$ if and only if $f \in I_{a+}^{\alpha}(L_p[a,b])$ and in this case we can simplify the formula for the generalized integral:

$$\begin{aligned} &\int_a^b f(x)dg(x) = \int_a^b \left((D_{a+}^{\alpha} f)(x) - \tfrac{1}{\Gamma(1-\alpha)} \cdot \tfrac{f(a+)}{(x-a)^{\alpha}}\right)(D_{b-}^{1-\alpha} g_{b-})(x)dx \\ &+ f(a+)(g(b-) - g(a+)) = \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx \\ &- f(a+) I_{b-}^{1-\alpha}(D_{b-}^{1-\alpha} g)(a) + f(a+)(g(b-) - g(a+)) \\ &= \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx. \end{aligned} \tag{2.1.1}$$

**Lemma 2.1.3.** *Let $g_{b-} \in I_{b-}^{1-\alpha}(L_q[a,b]) \cap C[a,b]$ for some $q > \frac{1}{1-\alpha}$ and $0 < \alpha < 1$. Then for any $a < c < d < b$*

$$\int_a^b (D_{a+}^{\alpha} \mathbf{1}_{[c,d)})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx = g(d) - g(c). \tag{2.1.2}$$

*Proof.* We have that

$$(D_{a+}^{\alpha} \mathbf{1}_{[c,d)})(x) = \begin{cases} 0, & x \le c, \\ \frac{(x-c)^{-\alpha}}{\Gamma(1-\alpha)}, & c < x \le d, \\ \frac{(x-c)^{-\alpha} - (x-d)^{-\alpha}}{\Gamma(1-\alpha)}, & d \le x \le b. \end{cases}$$

Therefore, by using (2.1.1), we obtain for $\alpha p < 1$, or $q > \frac{1}{1-\alpha}$, that

$$\begin{aligned} &\int_a^b (D_{a+}^{\alpha} \mathbf{1}_{[c,d)})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx = \tfrac{1}{\Gamma(1-\alpha)} \int_c^b (x-c)^{-\alpha}(D_{b-}^{1-\alpha} g_{b-})(x)dx \\ &- \tfrac{1}{\Gamma(1-\alpha)} \int_d^b (x-d)^{-\alpha}(D_{b-}^{1-\alpha} g_{b-})(x)dx = I_{b-}^{1-\alpha}(D_{b-}^{1-\alpha} g_{b-})(c) \\ &- I_{b-}^{1-\alpha}(D_{b-}^{1-\alpha} g_{b-})(d) = g(d) - g(c). \end{aligned}$$

□

**Corollary 2.1.4.** *Let the function $g \in C^{\lambda}[a,b]$ for some $\lambda \le 1$, then $g_{b-} \in I_{b-}^{1-\alpha}(L_p[a,b])$ for any $p \ge 1$ and $1 - \alpha < \lambda$. So, we can put $p > 2/\lambda$, $\alpha = 1 - \lambda/2$ and obtain for $g$ (2.1.2).*

**Corollary 2.1.5.** *For any step function $f_{\pi}(x) = \sum\limits_{k=0}^{n-1} c_k \mathbf{1}_{[x_k, x_{k+1})}(x)$ with $a = x_0 < \cdots < x_n = b$ and $g$ satisfying the conditions of Lemma 2.1.3, we have that $\int_a^b f(x)dg(x) = \sum\limits_{k=0}^{n-1} c_k(g(x_{k+1}) - g(x_k))$.*

Further we suppose that $g(b-) = g(b)$ and $g(a+) = g(a)$.

Denote by $BV[a,b]$ the class of functions of bounded variation on $[a,b]$.

**Lemma 2.1.6.** *Let the functions* $f_{a+} \in I^{\alpha}_{a+}(L_p[a,b])$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q[a,b]) \cap BV[a,b]$ *with* $p \geq 1, q \geq 1, 1/p + 1/q \leq 1$ *and*

$$\int_a^b I^{\alpha}_{a+}(|(D^{\alpha}_{a+}f)|)(x)|g|(dx) < \infty. \tag{2.1.3}$$

*Then*

$$\int_a^b f(x)dg(x) = \text{(L-S)} \int_a^b f(x)dg(x).$$

*Proof.* We have that

$$\begin{aligned}&\text{(L-S)} \int_a^b f(x)dg(x) = \text{(L-S)} \int_a^b I^{\alpha}_{a+}(D^{\alpha}_{a+}f)(x)dg(x)\\ &= \tfrac{1}{\Gamma(1-\alpha)}\text{(L-S)} \int_a^b (\int_a^x (x-y)^{\alpha-1}(D^{\alpha}_{a+}f)(y)dy)dg(x).\end{aligned} \tag{2.1.4}$$

Condition (2.1.3) together with Fubini theorem permits us to change the order of integration:

$$\begin{aligned}&\text{(L-S)} \int_a^b (\int_a^x (x-y)^{\alpha-1}(D^{\alpha}_{a+}f)(y)dy)dg(x)\\ &= \int_a^b (D^{\alpha}_{a+}f)(y)(\int_y^b (x-y)^{\alpha-1}dg(x))dy\\ &= (\alpha-1)\int_a^b (D^{\alpha}_{a+}f)(y)(\int_y^b (\int_x^{\infty}(z-y)^{\alpha-2}dz)dg(x))dy.\end{aligned} \tag{2.1.5}$$

Further, if $y \in (a,b)$ is the point of continuity of function $g$, then

$$\begin{aligned}&\int_y^b (\int_x^{\infty}(z-y)^{\alpha-2}dz)dg(x) = \int_y^b (\int_y^z dg(x))(z-y)^{\alpha-2}dz\\ &+ \int_b^{\infty}(\int_y^b dg(x))(z-y)^{\alpha-2}dz = \int_y^b \tfrac{g(z)-g(y)}{(z-y)^{2-\alpha}}dz\\ &+ \tfrac{g(b)-g(y)}{(\alpha-1)(b-y)^{\alpha-1}} = \tfrac{\Gamma(\alpha)}{\alpha-1}(D^{1-\alpha}_{b-}g_{b-})(y).\end{aligned} \tag{2.1.6}$$

Since set of discontinuity points of $g$ is at most countable , and taking (2.1.4)–(2.1.6) together, we obtain the proof. □

Now we consider the case of Hölder functions $f$ and $g$. The existence of (R-S) $\int_a^b fdg$ for $f \in C^{\lambda}[a,b]$, $g \in C^{\mu}[a,b]$ with $\lambda + \mu > 1$ was established by Kondurar (Kon37). Moreover, this integral coincides with $\int_a^b fdg$ , as the next theorem states.

Let $f \in C^{\lambda}[a,b]$ for some $0 < \lambda \leq 1$ and $|f(x) - f(y)| \leq c(\lambda)|x-y|^{\lambda}$, $x, y \in [a,b]$. Consider the following step function:

$$f_{\pi}(x) = \sum_{k=0}^{n-1} f(x_k)\mathbf{1}_{[x_k,x_{k+1})}(x),$$

where the partition $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

Evidently, $\lim_{|\pi|\to 0}\sup_{\pi}\|f_{\pi} - f\|_{L_{\infty}[a,b]} = 0$.

**Theorem 2.1.7.** *1) For any* $0 < \alpha < \lambda$

$$\lim_{|\pi|\to 0} \sup_{\pi} \left\| (D^{\alpha}_{a+} f_{\pi}) - (D^{\alpha}_{a+} f) \right\|_{L_1[a,b]} = 0.$$

*2) Let* $f \in C^{\lambda}([a,b])$, $g \in C^{\mu}[a,b]$ *with* $\lambda + \mu > 1$, *then* (R-S) $\int_a^b f dg$ *exists and*

$$\int_a^b f dg = \text{(R-S)} \int_a^b f dg.$$

*Proof.* 1) It is sufficient to prove that $\int_a^b \frac{|f_\pi(x) - f(x)|}{(x-a)^\alpha} dx \to 0$ and $\int_a^b \int_a^x (x-y)^{-\alpha-1} |f_\pi(x) - f(x) - f_\pi(y) + f(y)| dy\, dx \to 0$ as $|\pi| \to 0$. But $|f_\pi(x) - f(x)| \le |f(x_k) - f(x)| \le c(\lambda)|\pi|^\lambda$ for $x \in [\, x_k, x_{k+1}\, )$, therefore $\int_a^b \frac{|f_\pi(x) - f(x)|}{(x-a)^\alpha} dx \le c(\lambda)|\pi|^\lambda \frac{(b-a)^{1-\alpha}}{1-\alpha} \to 0$ as $|\pi| \to 0$. Also, for $x \in [\, x_k, x_{k+1}\, )$

$$\begin{aligned}
&A(x) := \textstyle\int_a^x (x-y)^{-\alpha-1} |f_\pi(x) - f(x) - f_\pi(y) + f(y)| dy \\
&= \sum_{i=0}^{k-1} \textstyle\int_{x_i}^{x_{i+1}} (x-y)^{-\alpha-1} |f(x_k) - f(x) - f(x_i) + f(y)| dy \\
&+ \textstyle\int_{x_k}^x (x-y)^{-\alpha-1} |f(y) - f(x)| dy \le 2c(\lambda) \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (x-y)^{-\alpha-1} dy \cdot |\pi|^\lambda \\
&+ c(\lambda) \textstyle\int_{x_k}^x (x-y)^{\lambda-\alpha-1} dy \le 2c(\lambda)|\pi|^\lambda \frac{(x-x_k)^{-\alpha}}{1-\alpha} + c(\lambda) \frac{(x-x_k)^{\lambda-\alpha}}{\lambda-\alpha} \\
&\le 3c(\lambda) \frac{|\pi|^{\lambda-\alpha}}{\lambda-\alpha},
\end{aligned}$$

which means that $\int_a^b A(x) dx \to 0$ as $|\pi| \to 0$.

2) We take $1 - \mu < \alpha < \lambda$, then the fractional derivatives $D^\alpha_{a+} f(x)$ and $(D^{1-\alpha}_{b-} g)_{b-}(x)$ exist, and, moreover,

$$\begin{aligned}
&|(D^{1-\alpha}_{b-} g)_{b-}(x)| \le \tfrac{1}{\Gamma(1-\alpha)} \left( \tfrac{|g(b) - g(x)|}{(b-x)^{1-\alpha}} + (1-\alpha) \textstyle\int_x^b \frac{|g(y) - g(x)|}{(y-x)^{2-\alpha}} dy \right) \\
&\le \tfrac{1}{\Gamma(1-\alpha)} \cdot c(\lambda)(b-x)^{\mu+\alpha-1} \left( 1 + \tfrac{1-\alpha}{\mu+\alpha-1} \right) \le C
\end{aligned}$$

for some constant $C$. Therefore, according to part 1) of the proof,

$$\begin{aligned}
&|\textstyle\int_a^b f_\pi dg - \int_a^b f dg| \le \int_a^b |(D^\alpha_{a+} f_\pi)(x) - (D^\alpha_{a+} f)(x)||(D^{1-\alpha}_{b-} g)_{b-}(x)| dx \\
&\le C \textstyle\int_a^b |(D^\alpha_{a+} f_\pi)(x) - (D^\alpha_{a+} f)(x)| dx \to 0,
\end{aligned} \tag{2.1.7}$$

as $|\pi| \to 0$.

Furthermore, according to Corollary 2.1.5,

$$\int_a^b f_\pi dg = \sum_{k=0}^{n-1} f(x_k)(g(x_{k+1}) - g(x_k)) \to \text{(R-S)} \int_a^b f dg, \tag{2.1.8}$$

and from (2.1.7)–(2.1.8) we obtain the desired equality. □

Now we establish the properties of generalized integral $\int_s^t f dg$ as the function of upper and lower boundaries.

**Lemma 2.1.8** ((Zah98)). *1) Let $a \le s < t \le b$ and the functions $f$ and $g$ satisfy the assumptions*
*(i)* $(f \cdot \mathbf{1}_{(s,t)}) \in I_+^{\alpha}(L_p[a,b])$, $g_{b-} \in I_-^{1-\alpha}(L_q[a,b])$ *for some* $0 < \alpha < 1$, $p \ge 1, q \ge 1, 1/p + 1/q \le 1$,
*(ii)* $f_{s+} \in I_+^{\alpha'}(L_{p'}[s,t])$, $g_{t-} \in I_-^{1-\alpha'}(L_{q'}[s,t])$ *for some* $0 < \alpha' < 1$, $p' \ge 1, q' \ge 1, 1/p' + 1/q' \le 1$. *Then*

$$\int_a^b \mathbf{1}_{(s,t)} f dg = \int_s^t f dg.$$

*2) The equality*

$$\int_s^t f dg + \int_t^u f dg = \int_s^u f dg$$

*holds for* $a \le s < t < u \le b$, *if all the integrals exist as generalized Lebesgue–Stieltjes integrals.*

*Proof.* 1) Let $\{\varphi_n(x), x \in \mathbb{R}\}$ be a sequence of smooth kernels, i.e. $\varphi_n \in C^\infty(\mathbb{R})$, $\varphi_n \ge 0$, $\varphi_n = 0$ outside $[-1/n, 0]$ and $\int_{-1/n}^0 \varphi_n(x)dx = 1$. More exactly, let $\varphi_n(x) = n\varphi(nx)$ for $\varphi \in C^\infty(\mathbb{R})$, $\varphi = 0$ outside of $[-1,0]$. Then we can approximate the function $g_{b-}$ by smooth functions $g_n := g_{b-} * \varphi_n$, and the following properties hold:

$$\begin{aligned}
&g_n(b-) = n \textstyle\int_{[x-b,x-a]\cap[-1/n,0]} (g(b-) - g(x-t))\varphi(nt)dt \,|_{x=b-} = 0;\\
&(D_{b-}^{1-\alpha} g_n)(x) = D_{b-}^{1-\alpha}(\textstyle\int_{\mathbb{R}} g_{b-}(x-t)\varphi_n(t)dt)\\
&= \mathbf{1}_{(a,b)}(x)(\Gamma(1-\alpha))^{-1}\Big(\textstyle\int_{\mathbb{R}} g_{b-}(x-t)\varphi_n(t)dt(b-x)^{\alpha-1}\\
&\quad + \alpha \textstyle\int_x^b (y-x)^{2-\alpha}(\int_{\mathbb{R}}(g_{b-}(x-t) - g_{b-}(y-t))\varphi_n(t)dt)dy\Big)\\
&= \tfrac{\mathbf{1}_{(a,b)}(x)}{\Gamma(1-\alpha)} \textstyle\int_{\mathbb{R}} \varphi_n(t)\left(\frac{g_{b-}(x-t)}{(b-x)^{1-\alpha}} + \alpha \int_x^b \frac{g_{b-}(x-t) - g_{b-}(y-t)}{(y-x)^{2-\alpha}} dy\right) dt\\
&= \mathbf{1}_{(a,b)}(x)((D_{b-}^{1-\alpha} g_{b-}) * \varphi_n)(x);
\end{aligned} \tag{2.1.9}$$

$$\begin{aligned}
&\left\|(D_{b-}^{1-\alpha} g_n) - (D_{b-}^{1-\alpha} g_{b-})\right\|_{L_q[a,b]}^q\\
&\left\|(D_{b-}^{1-\alpha} g_{b-}) * \varphi_n - (D_{b-}^{1-\alpha} g_{b-})\right\|_{L_q[a,b]}^q\\
&= \textstyle\int_a^b |\int_{-1}^0 ((D_{b-}^{1-\alpha} g_{b-})(x - \frac{t}{n}) - (D_{b-}^{1-\alpha} g_{b-})(x))\varphi(t)dt|^q dx\\
&\le C \textstyle\int_a^b \int_{-1}^0 |(D_{b-}^{1-\alpha} g_{b-})(\cdot - \frac{t}{n}) - (D_{b-}^{1-\alpha} g_{b-})(\cdot)|^q dt\, dx \to 0, \quad n \to \infty.
\end{aligned} \tag{2.1.10}$$

Therefore, from this $L_q$-convergence, from Lemma 2.1.2 and the properties of convolutions,

$$\begin{aligned}
&\textstyle\int_a^b \mathbf{1}_{(s,t)} f dg = \int_a^b (D_{a+}^{\alpha} \mathbf{1}_{(s,t)} f)(u)(D_{b-}^{1-\alpha} g_{b-})(u)du\\
&= \lim_{n\to\infty} \textstyle\int_a^b (D_{a+}^{\alpha} \mathbf{1}_{(s,t)} f)(u)(D_{b-}^{1-\alpha} g_n)(u)du\\
&= \lim_{n\to\infty} \textstyle\int_a^b (\mathbf{1}_{(s,t)} f)(u) g_n'(u)du = \lim_{n\to\infty} \int_s^t f(u)(g_{b-} * \varphi_n')(u)du.
\end{aligned}$$

Further, for any $c > 0$ $(c * \varphi_n')(u) = 0$, therefore

$$\begin{aligned}\int_s^t f(u)(g_{b-} * \varphi_n')(u)du &= \int_s^t f(u)(g * \varphi_n')(u)du \\ = \int_s^t f(u)(g_{t-} * \varphi_n')(u)du,\end{aligned} \tag{2.1.11}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \int_s^t f(u)(g_{b-} * \varphi_n')(u)du = \lim_{n\to\infty} \int_s^t f(u)(g_{t-} * \varphi_n')(u)du \\ &= \lim_{n\to\infty} \int_s^t f(u)(g_{t-} * \varphi_n)'(u)du.\end{aligned} \tag{2.1.12}$$

Thanks to Lemma 2.1.2, assumption (ii), (2.1.9) and (2.1.10), applied to $t$ instead of $b$,

$$\begin{aligned}&\lim_{n\to\infty} \int_s^t f(u)(g_{t-} * \varphi_n)'(u)du \\ &= \lim_{n\to\infty} \int_s^t (D_{s+}^{\alpha'} f_{s+})(u)(D_{t-}^{1-\alpha'}(g_{t-} * \varphi_n))(u)du \\ &= \lim_{n\to\infty} \int_s^t (D_{s+}^{\alpha'} f_{s+})(u)((D_{t-}^{1-\alpha'} g_{t-}) * \varphi_n)(u)du \\ &= \int_s^t (D_{s+}^{\alpha'} f_{s+})(u)(D_{t-}^{1-\alpha'} g_{t-})(u)du = \int_s^t fdg,\end{aligned} \tag{2.1.13}$$

and we obtain the first statement. The second one we obtain by using some of the equalities from (2.1.11):

$$\begin{aligned}&\int_s^t fdg + \int_t^u fdg = \lim_{n\to\infty} \int_s^t f(r)(g * \varphi_n')(r)dr \\ &+ \lim_{n\to\infty} \int_t^u f(r)(g * \varphi_n')(r)dr = \lim_{n\to\infty} \int_s^u f(r)(g * \varphi_n')(r)dr \\ &= \int_s^u fdg.\end{aligned}$$

□

### 2.1.2 Pathwise Stochastic Integration in Fractional Besov-type Spaces

In this subsection we consider the approach to pathwise stochastic integration in fractional Besov-type spaces, introduced by Nualart and Răşcanu (NR00) (see also (CKR93) and (NO03a)).

Consider the following functional spaces. Let for $0 < \beta < 1$ $\varphi_f^\beta(t) := |f(t)| + \int_0^t |f(t) - f(s)|(t-s)^{-\beta-1}ds$, and $W_0^\beta = W_0^\beta[0T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{0,\beta} := \sup_{t\in[0,T]} \varphi_f^\beta(t) < \infty.$$

Furthermore, let $W_1^\beta = W_1^\beta[0, T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{1,\beta} := \sup_{0\le s<t\le T} \left( \frac{|f(t) - f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} du \right) < \infty$$

and $W_2^\beta = W_2^\beta[0,T]$ be the space of real-valued measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{2,\beta} := \int_0^T \frac{|f(s)|}{s^\beta} ds + \int_0^T \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{\beta+1}} du < \infty.$$

Note that the spaces $W_i^\beta, i = 0, 2$ are Banach spaces with respect to corresponding norms and $\|f\|_{1,\beta}$ is not the norm in a usual sense.

Moreover, for any $0 < \varepsilon < \beta \wedge (1-\beta)$

$$C^{\beta+\varepsilon}[0,T] \subset W_i^\beta[0,T] \subset C^{\beta-\varepsilon}[0,T],\ i = 0, 1,\ C^{\beta+\varepsilon}[0,T] \subset W_2^\beta[0,T].$$

Therefore, the trajectories of fBm $B^H$ for a.a. $\omega \in \Omega$, any $T > 0$ and any $0 < \beta < H$ belong to $W_1^\beta[0,T]$.

Let $f \in W_1^\beta[0,T]$. Then its restriction to $[0,t] \subset [0,T]$ belongs to $I_-^\beta(L_\infty[0,t])$ and

$$\Lambda_\beta(f) := \sup_{0 \le s < t \le T} |(D_{t-}^\beta f_{t-})(s)| \le \frac{1}{\Gamma(1-\beta)} \|f\|_{1,\beta} < \infty.$$

The restriction of $f \in W_2^\beta[0,T]$ to $[0,t] \subset [0,T]$ belongs to $I_+^\beta(L_1[0,t])$.

Now, let $f \in W_2^\beta[0,T]$, $g \in W_1^{1-\beta}[0,T]$. Then for any $0 < t \le T$ there exists the Lebesgue integral $\int_0^t (D_{0+}^\beta f)(x)(D_{t-}^{1-\beta} g_{t-})(x)dx$, so we can define $\int_0^t f dg$ according to Definition 2.1.1 and formula (2.1.2). Moreover, for any $0 < t \le T$ $\int_0^t f dg = \int_0^T 1_{(0,t)} f dg$, and the integral $\int_0^t f dg$ admits an estimate

$$\begin{aligned} &|\textstyle\int_0^t f dg| \le \int_0^t |(D_{0+}^\beta f)(x)||(D_{t-}^{1-\beta} g_{t-})(x)|dx \\ &\le \Lambda_{1-\beta}(g)\|f\|_{2,\beta} \le (\Gamma(\beta))^{-1} \|g\|_{1,1-\beta} \|f\|_{2,\beta}. \end{aligned}$$

Further we fix some $0 < \beta < 1/2$.

**Lemma 2.1.9** ((NR00)). *1. Let $f \in W_0^\beta[0,T]$, $g \in W_1^{1-\beta}[0,T]$, $G_t(f) := \int_0^t f dg$, $t \in [0,T]$. Then*

$$\varphi_{G.(f)}^\beta(t) \le C_{\beta,T}^1 \Lambda_{1-\beta}(g) \int_0^t ((t-s)^{-2\beta} + s^{-\beta}) \varphi_f^\beta(s) ds.$$

*2. Let $f \in W_0^\beta[0,T]$, $g \in W_1^{1-\beta}[0,T]$. Then $G.(f) \in C^{1-\beta}[0,T]$ and*

$$\|G(f)\|_{1,1-\beta} \le C_{\beta,T}^2 \Lambda_{1-\beta}(g) \|f\|_{0,\beta}.$$

*Here $C_{\beta,T}^i, i = 1, 2$ depend only on $T$ and $\beta$.*

*Proof.* 1. It is not hard to check that for $f \in W_0^\beta[0,T]$ and $g \in W_1^{1-\beta}[0,T]$ condition 1) of Lemma 2.1.8 holds. Therefore, evidently,

$$\begin{aligned}&|G_t(f)-G_s(f)| = |\int_s^t f dg| \le \int_s^t |(D_{s+}^{\beta} f)(u)||(D_{t-}^{1-\beta} g_{t-})(u)|du \\ &\le \Lambda_{1-\beta}(g)\int_s^t \left(\frac{|f(u)|}{(u-s)^{\beta}} + \beta\int_s^u \frac{|f(u)-f(v)|}{(u-v)^{\beta+1}}dv\right)du.\end{aligned} \tag{2.1.14}$$

From (2.1.14) it follows that

$$\begin{aligned}&\int_0^t \frac{|G_t(f)-G_u(f)|}{(t-u)^{\beta+1}}du \le \Lambda_{1-\beta}(g)\Big(\int_0^t |f(u)|(\int_0^u (t-s)^{-\beta-1}(u-s)^{-\beta}ds)du \\ &+\int_0^t\int_0^u \frac{|f(u)-f(v)|}{(u-v)^{\beta+1}}(t-v)^{-\beta}dv\, du\Big).\end{aligned} \tag{2.1.15}$$

The first integral on the right-hand side of (2.1.15) can be estimated as $C\int_0^t |f(u)|(t-u)^{-2\beta}du$ with $C=\int_0^\infty (1+u)^{-\beta-1}u^{-\beta}du$, and the second one can be estimated as $\int_0^t (t-u)^{-\beta}\int_0^u \frac{|f(u)-f(v)|}{(u-v)^{\beta+1}}dv\, du$.

Since $(t-u)^{-2\beta} \ge (t-u)^{-\beta}T^{-\beta}$, we obtain from (2.1.15) that

$$\int_0^t \frac{|G_t(f)-G_u(f)|}{(t-u)^{\beta+1}}du \le \Lambda_{1-\beta}(g)(C+T^{\beta})\int_0^t (t-u)^{-2\beta}\varphi_f^{\beta}(u)du. \tag{2.1.16}$$

Further, from (2.1.14) it follows that

$$\begin{aligned}&|G_t(f)| \le \Lambda_{1-\beta}(g)\int_0^t \left(\frac{|f(u)|}{u^{\beta}} + \beta\int_0^u \frac{|f(u)-f(v)|}{(u-v)^{\beta+1}}dv\right)du \\ &\le \Lambda_{1-\beta}(g)(1+\beta T^{\beta})\int_0^t u^{-\beta}\varphi_f^{\beta}(u)du,\end{aligned} \tag{2.1.17}$$

and the proof follows from (2.1.16)–(2.1.17).

2. It follows from (2.1.14) that

$$|G_t(f)-G_s(f)| \le \Lambda_{1-\beta}(g)\frac{1+\beta T^{\beta}}{1-\beta}\|f\|_{0,\beta}(t-s)^{1-\beta},$$

and from (2.1.17) we obtain that

$$|G_t(f)| \le \Lambda_{1-\beta}(g)\frac{1+\beta T^{\beta}}{1-\beta}T^{1-\beta}\|f\|_{0,\beta},$$

whence the proof follows with $C^2_{\beta,T} = (1\vee T^{1-\beta})\frac{1+\beta T^{\beta}}{1-\beta}$. □

Similar but more simple estimates hold for the Lebesgue integral $F_t(f) = \int_0^t f(s)ds$, so we omit the proof of the following lemma.

**Lemma 2.1.10** ((NR00)). *1. Let $0<\beta<1$ and $f:[0,T]\to\mathbb{R}$ be a measurable function with $\sup_{t\in[0,T]}\int_0^t |f(s)|(t-s)^{-\beta}ds < \infty$.*

*Then*

$$\varphi^{\beta}_{F.(f)}(t) \le C^3_{\beta,T}\int_0^t |f(s)|(t-s)^{-\beta}ds,$$

*with $C^3_{\beta,T} = T^{\beta}+1/\beta$.*

*2. Let $f$ be bounded on $[0,T]$. Then $F(f)\in C^1[0,T]$ and $\|F(f)\|_{0,\beta} \le C^4_{\beta,T}f^*_T$, where $f^*_T := \sup_{t\in[0,T]}|f(t)|, C^4_{\beta,T}$ depends on $\beta$ and $T$.*

## 2.2 Pathwise Stochastic Integration w.r.t. Multi-parameter fBm

### 2.2.1 Some Additional Properties of Two-parameter Fractional Integrals and Derivatives

Throughout this section we consider two-parameter functions and fields. The first result can be proved similarly to the one-parameter case. Let the rectangle $\mathcal{P} = [a, b]$ be fixed.

**Lemma 2.2.1.** *1. Let $f \in I_{\pm}^{\beta_1\beta_2}(L_p(\mathcal{P}))$ for some $p > 1$. Then $\lim_{\beta_1\to 0,\beta_2\to 0} D_{a+(b-)}^{\beta_1\beta_2} f(x) = f(x)$, where the limit is in $L_p(\mathcal{P})$. 2. Let, in addition, the function $f$ be twice continuously differentiable in the neighborhood of the point $x$. Then $\lim_{\beta_1\to 1,\beta_2\to 1} D_{a+(b-)}^{\beta_1\beta_2} f(x) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x)$. So, we can put $D_{a+(b-)}^{00} f := f$, $D_{a+(b-)}^{11} f := f$.*

**Theorem 2.2.2.** *Let $0 < \beta_i < 1$ and $1 < p < \beta_1^{-1} \vee \beta_2^{-1}$. Then the operator $I_{a+}^{\beta_1\beta_2}$ is bounded from $L_p(\mathcal{P})$ into $L_q(\mathcal{P})$, where $1 < q < p((1-\beta_1 p)^{-1} \wedge (1-\beta_2 p)^{-1})$.*

*Proof.* Denote $r := p((1-\beta_1 p)^{-1} \vee (1-\beta_2 p)^{-1})$. Since $r > p$, it is sufficient to consider $q \in (p, r)$. Then for $\frac{1}{p'} + \frac{1}{p} = 1$, $\frac{1}{p_i'} + \frac{1}{r} = 1 - \beta_i$, from the generalized Hölder inequality, it holds that

$$
\begin{aligned}
|(I_{a+}^{\beta_1\beta_2} f)(x)| &\le C\Big(\int_{[a,x]} |f(u)|^p \prod_{i=1,2} (x_i - u_i)^{(\beta_i - 1)\gamma q} du\Big)^{\frac{1}{q}} \\
\times\Big(\int_{[a,x]} |f(u)|^p du\Big)^{\frac{1}{p}-\frac{1}{q}} &\Big(\int_{[a,x]} \prod_{i=1,2} (x_i - u_i)^{(\beta_i-1)(1-\gamma)p'} du_i\Big)^{\frac{1}{p'}} \\
&\le C \|f\|_{L_p(\mathcal{P})}^{1-\frac{p}{q}} \Big(\int_{[a,x]} |f(u)|^p \prod_{i=1,2} (x_i - u_i)^{(\beta_i - 1)\gamma q} du\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Here we choose $\gamma$ satisfying the inequalities $(1-\beta_i)\gamma q < 1$ and $(1-\beta_i)(1-\gamma)p' < 1$, which is equivalent to $1 - (p'(1-\beta_i))^{-1} < \gamma < (q(1-\beta_i))^{-1}$. Such a choice is possible, since the inequality $1 - (p'(1-\beta_i))^{-1} < (q(1-\beta_i))^{-1}$ is equivalent to $q < p(1-\beta_i p)^{-1}$, and this is evident under our suppositions. By integration over $\mathcal{P}$ we obtain that

$$
\begin{aligned}
&\left\| I_{a+}^{\beta_1\beta_2} f \right\|_{L_q(\mathcal{P})} \le C \|f\|_{L_p(\mathcal{P})}^{1-\frac{p}{q}} \Big(\int_{\mathcal{P}} |f(u)|^p du \cdot \int_{\mathcal{P}} \prod_{i=1,2} (x_i - u_i)^{(\beta_i-1)\gamma q} dx\Big)^{\frac{1}{q}} \\
&\le C \|f\|_{L_p(\mathcal{P})}.
\end{aligned}
$$

□

**Corollary 2.2.3.** *Let* $f \in L_p(\mathcal{P}), g \in L_q(\mathcal{P})$, $I_{a+}^{\beta_1\beta_2} g \in L_r(\mathcal{P})$ *for* $1/p+1/r = 1$ *and* $r < q((1-\beta_1 q)^{-1} \wedge (1-\beta_2 q)^{-1})$, *i.e.* $1/p+1/q < 1+\beta_1 \wedge \beta_2$. *Then*

$$\int_{\mathcal{P}} f(u) I_{a+}^{\beta_1\beta_2} g(u) du = \int_{\mathcal{P}} g(u) I_{b-}^{\beta_1\beta_2} f(u) du.$$

Evidently,

$$I_{\pm}^{\rho_1\rho_2} I_{\pm}^{\beta_1\beta_2} = I_{\pm}^{\rho_1+\beta_1 \rho_2+\beta_2} \quad \text{on} \quad L_1(\mathcal{P});$$

for $f \in I_{\pm}^{\rho_1+\beta_1\rho_2+\beta_2}(L_1(\mathcal{P}))$, $\rho_i, \beta_i \geq 0, \rho_i+\beta_i \leq 1$

$$D_{a+(b-)}^{\rho_1\rho_2} D_{a+(b-)}^{\beta_1\beta_2} f = D_{a+(b-)}^{\rho_1+\beta_1\rho_2+\beta_2} f;$$

for $f \in I_{a+(b-)}^{\rho_1\rho_2}(L_p(\mathcal{P}))$, $g \in I_{b-}^{\rho_1\rho_2}(L_q(\mathcal{P}))$, $p, q > 1$, $1/p+1/q < 1+\rho_1 \wedge \rho_2$

$$\int_{\mathcal{P}} D_{a+}^{\rho_1\rho_2} f(u) g(u) du = \int_{\mathcal{P}} f(u) D_{b-}^{\rho_1\rho_2} g(u) du.$$

### 2.2.2 Generalized Two-parameter Lebesgue–Stieltjes Integrals

We suppose that all the functions, considered on some rectangle $\mathcal{P} = [a, b]$, belong to the space $D(\mathcal{P})$, i.e. they have the limits in all the quadrants,

$$Q^{++}(x) = \{s \in \mathcal{P} | s \geq x\}, \qquad Q^{+-}(x) = \{s \in \mathcal{P} | s_1 \geq x_1, s_2 < x_2\},$$
$$Q^{-+}(x) = \{s \in \mathcal{P} | s_1 < x_1, s_2 \geq x_2\}, \; Q^{--}(x) = \{s \in \mathcal{P} | s < x\},$$

$f(x) = \lim_{s \to x, s \geq x} f(s)$, and on the sides of rectangle the limits that can be defined are supposed to exist and denoted as $f(x_1, b_2-)$, $f(b_1-, x_2)$, $f(b-)$. Denote $f_{a+}(x) = \Delta_a f(x)$, $x \in \mathcal{P}$, and $f_{b-}(x) := f(x) - f(x_1, b_2-) - f(b_1-, x_2) + f(b-)$.

**Definition 2.2.4.** Let $f, g : \mathcal{P} \to \mathbb{R}$. The generalized two-parameter Lebesgue–Stieltjes integral of $f$ w.r.t. $g$ is defined by

$$\begin{aligned}\int_{\mathcal{P}} f dg := &\int_{\mathcal{P}} (D_{a+}^{\beta_1\beta_2} f_{a+})(u)(D_{b-}^{1-\beta_1 1-\beta_2} g_{b-})(u) du \\ &+ \sum_{i=1,2} \int_{a_i}^{b_i} (D_{a_i+}^{\beta_i} f_{a_i+})(u, a_i)(D_{b_i-}^{1-\beta_i})(g_{b_i-}(u, b_i-) - g_{b_i-}(u, b_i-)) du \\ &+ f(a)\Delta_a g(b), \quad (2.2.1)\end{aligned}$$

under the assumption that all the integrals on the right-hand side exist.

A more convenient formula for $\int_{\mathcal{P}} f dg$ has a form

$$\int_{\mathcal{P}} f dg = \int_{\mathcal{P}} (D_{a+}^{\beta_1\beta_2} f)(u)(D_{b-}^{1-\beta_1 1-\beta_2} g_{b-})(u) du.$$

(We do not specify here the conditions ensuring the latter equality but it is very easy to do it, similarly to the one-parameter case.) The next results also can be proved similarly to the one-parameter case ((SKM93) and (Zah98)).

**Theorem 2.2.5.** *Definition* 2.2.4 *is correct, i.e. the right-hand side of* (2.2.1) *does not depend on the choice of* $\beta_i, i = 1, 2$.

**Theorem 2.2.6.** *Let* $f : \mathcal{P} \to \mathbb{R}, f \in C^{\lambda_1\lambda_2}(\mathcal{P})$ *and* $\lambda_i + \beta_i < 1, i = 1, 2$, $0 < \beta_i < 1$. *Then* $I^{\beta_1\beta_2}_{a+(b-)}(f_{a+(b-)}) \in C^{\lambda_1+\beta_1\lambda_2+\beta_2}(\mathcal{P})$.

**Theorem 2.2.7.** *Let the function* $f \in C^{\lambda_1\lambda_2}(\mathcal{P})$. *Then for any* $p \geq 1$ *and* $0 < \varepsilon_i < \lambda_i, i = 1, 2$

$$f_{a+(b-)} \in I^{\varepsilon_1\varepsilon_2}_{\pm}(L_p(\mathcal{P}))$$

*and*

$$D^{\varepsilon_1\varepsilon_2}_{a+(b-)} f_{a+(b-)} \in C^{\lambda_1-\varepsilon_1\lambda_2-\varepsilon_2}(\mathcal{P}).$$

**Theorem 2.2.8.** *Let* $f \in C(\mathcal{P})$, $g \in BV(\mathcal{P})$, $f \in I^{\beta_1\beta_2}_{+}(L_p(\mathcal{P}))$, $g_{b-} \in I^{1-\beta_1 1-\beta_2}_{-}(L_q(\mathcal{P}))$, $i = 1, 2$, $j = 3 - i$, $\frac{1}{p} + \frac{1}{q} \leq 1$, $0 \leq \beta_i \leq 1, i = 1, 2$. *Then the generalized two-parameter Lebesgue–Stieltjes integral* $\int_{\mathcal{P}} f dg$ *equals the Riemann–Stieltjes integral* $\int_{\mathcal{P}} f(x) dg(x)$.

**Theorem 2.2.9.** *1. Let* $g \in C^{\lambda_1\lambda_2}(\mathcal{P})$ *for some* $0 < \lambda_i \leq 1, i = 1, 2$. *Then for any* $\mathcal{P}_1 = [c, d) \subset \mathcal{P}$

$$\int_{\mathcal{P}} \mathbf{1}_{\mathcal{P}_1} dg = \Delta_c g(d).$$

*2. Let* $g \in C^{\lambda_1\lambda_2}(\mathcal{P})$ *and let the partition* $\pi = \pi^1 \times \pi^2$, *where* $\pi^i = \{a_i = x^i_0 < \cdots < x^i_{n_i} = b_i\}$ *be the partition of* $[a_i, b_i]$.
*Also, let* $f_\pi(x) = \sum\limits_{i=1,2} \sum\limits_{j_i=0}^{n_i-1} f_{j_1j_2} \mathbf{1}_{\mathcal{P}_{j_1j_2}}(x)$, *where* $\mathcal{P}_{j_1j_2} = \prod_{i=1,2}[x^i_{j_i}, x^i_{j_i+1})$.
*Then* $\int_{\mathcal{P}} f_\pi dg = \sum\limits_{i=1,2} \sum\limits_{j_i=0}^{n_i-1} f_{j_1j_2} \Delta_{x_j} g(x_{j+1})$, *where* $x_j = (x^1_{j_1}, x^2_{j_2})$.

Now, let $\pi_n$ be the sequence of partitions of rectangle $\mathcal{P}$, $\pi_n \subset \pi_{n+1}$ and $|\pi_n| = \max_{i=1,2} \max_{0 \leq j_i \leq n_{i,n}-1}(x^{i,n}_{j_i+1} - x^{i,n}_{j_i})$. Let $f : \mathcal{P} \to \mathbb{R}$, $f_{j_1j_2} = f(x^n_{j_i+1})$. We say that the partitions $\pi_n$ are uniform, if $n^{(n)}_1 = n^{(n)}_2$ and $x^{i,n}_{j_i+1} - x^{i,n}_{j_i} = \frac{b_i - a_i}{n^{(n)}_1}, i = 1, 2$.

**Theorem 2.2.10.** *1. Let* $f \in C^{\lambda_1\lambda_2}(\mathcal{P})$ *for some* $0 < \lambda_i \leq 1, i = 1, 2$. *Then*

$$\lim_{n\to\infty} \sup_{\pi_n} \|f_{\pi_n} - f\|_{L_\infty(\mathcal{P})} = 0,$$

*where* $\sup_{\pi_n}$ *is taken over all the sequences of partitions mentioned above.*

*2.* $\lim_{n\to\infty} \sup_{\pi'_n} \left\| D^{\beta_1\beta_2}_{a+}(f_{\pi'_n})_{a+} - D^{\beta_1\beta_2}_{a+} f_{a+} \right\|_{L_1(\mathcal{P})} = 0,$
*for any* $\beta_1 \vee \beta_2 < \lambda_1 \wedge \lambda_2$ *and all the sequences of uniform partitions of* $\mathcal{P}$.

*Proof.* The first statement is a direct consequence of uniform continuity $f$ on $\mathcal{P}$. Further, let $g_n(x) = f_{\pi'_n}(x) - f(x)$. For the second statement it is sufficient to prove that any of the following functions

$$
\begin{aligned}
G_1^n(x) &:= g_n(x)(x-a_1)^{-\beta_1}(y-a_2)^{-\beta_2},\\
G_2^n(x) &:= (x_2-a_2)^{-\beta_2}\int_{a_1}^{x_1}\big(g_n(x)-g_n(s_1,x_2)\big)(x_1-s_1)^{-1-\beta_1}ds_1,\\
G_3^n(x) &:= (x_1-a_1)^{-\beta_1}\int_{a_2}^{x_2}\big(g_n(x)-g_n(x_1,s_2)\big)(x_2-s_2)^{-1-\beta_2}ds_2,\\
G_4^n(x) &:= \int_{[a,x]}\Delta_s g_n(x)\prod_{i=1,2}(x_i-s_i)^{-1-\beta_i}ds
\end{aligned}
$$

tends to zero in $L_1(\mathcal{P})$. First, note that $|g_n(x)| \le C(|\pi_n|^{\lambda_1} + |\pi_n|^{\lambda_2})$, whence $\|G_1^n\|_{L_1(\mathcal{P})} \le C(|\pi_n|^{\lambda_1} + |\pi_n|^{\lambda_2})\prod\limits_{i=1,2}(b_i-a_i)^{1-\beta_i} \to 0, \quad n\to\infty$. Further, let the point $x \in \mathcal{P}_j^n := \prod_{i=1,2}[x_{j_i}^{i,n}, x_{j_i+1}^{i,n}) =: [x_j^n, x_{j+1}^n)$. Then it holds that

$$
\begin{aligned}
G_2^n(x) = (x_2-a_2)^{-\beta_2}\Bigg(\sum_{k=0}^{j_1-1}\int_{x_k^{1,n}}^{x_{k+1}^{1,n}}(x_1-s_1)^{-1-\beta_1}ds_1 \\
+\int_{x_{j_1}}^{x_1} g_n(x,x_j^n,s_1)(x_1-s_1)^{-1-\beta_1}ds_1\Bigg),
\end{aligned}
$$

where $g_n(x,x_j^n,s_1) = f(x_j^n) - f(x) - f(x_k^{1,n}, x_{j_2}^{2,n}) + f(s_1,x_2)$. Therefore,

$$
\begin{aligned}
&|G_2^n(x)|I\{x\in\mathcal{P}_j^n\}\\
&\le C(x_2-a_2)^{-\beta_2}\Bigg[\Big(\sum_{i=1,2}|x_{j_i}^{i,n}-x_i|^{\lambda_i}\Big)\int_{a_1}^{x_{j_1}^{1,n}}(x_1-s_1)^{-1-\beta_1}ds_1\\
&\quad+\sum_{k=0}^{j_1-1}\big((x_{k+1}^{1,n}-x_k^{1,n})^{\lambda_1}+(x_{j_2+1}^{2,n}-x_{j_2}^{2,n})^{\lambda_2}\big)\int_{x_k^{1,n}}^{x_{k+1}^{1,n}}(x_1-s_1)^{-1-\beta_1}ds_1\\
&\quad+\int_{x_{j_1}}^{x}(x_1-s_1)^{\lambda_1-1-\beta_1}ds_1\Bigg]\\
&\le C(x_2-a_2)^{-\beta_2}\Bigg[\sum_{i=1,2}(x_1-x_{j_1}^{1,n})^{\lambda_i}(x_1-x_{j_1}^{1,n})^{-\beta_1}\\
&\quad+\sum_{k=0}^{j_1-1}((x_{k+1}^{1,n}-x_k^{1,n})^{\lambda_1}+(x_{j_2+1}^{2,n}-x_{j_2}^{2,n})^{\lambda_2})\int_{x_k^{1,n}}^{x_{k+1}^{1,n}}(x_1-s_1)^{-1-\beta_1}ds_1\\
&\quad+(x_1-x_{j_1}^{1,n})^{\lambda_1-\beta_1}\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
&\|G_1^n\|_{L_1(\mathcal{P})} \le \sum_{j_1,j_2} \|G_1^n\|_{L_1(\mathcal{P}_j^n)} \le C \sum_{j_1,j_2} \Bigg( \int_{\mathcal{P}_j^n} \Big( (x_2-a_2)^{-\beta_2} (x_1 - x_{j_1}^{1,n})^{\lambda_1-\beta_1} \\
&+ (x_2-a_2)^{-\beta_2} (x_2 - x_{j_2}^{2,n})^{\lambda_2} (x_1 - x_{j_1}^{1,n})^{-\beta_1} \\
&+ \sum_{k=0}^{j_1-1} (x_{k+1}^{1,n} - x_k^{1,n})^{\lambda_1} (x_2-a_2)^{-\beta_2} \int_{x_k^{1,n}}^{x_{k+1}^{1,n}} (x_1-s_1)^{-1-\beta_1} ds_1 \\
&+ (x_2-a_2)^{-\beta_2} (x_{j_2+1}^{2,n} - x_{j_2}^{2,n})^{\lambda_1} \sum_{k=0}^{j_1-1} \int_{x_k^{1,n}}^{x_{k+1}^{1,n}} (x_1-s_1)^{-1-\beta_1} ds_1 \\
&+ (x_2-a_2)^{-\beta_2} (x_1 - x_{j_1}^{1,n})^{\lambda_1-\beta_1} \Big) dx \Bigg) \\
&\le C(b_2-a_2)^{1-\beta_2} \Bigg( |\pi_n|^{\lambda_1-\beta_1} + |\pi_n|^{\lambda_2} \sum_{j_1=1}^{n_1^{(n)}} (x_{j_1+1}^{1,n} - x_{j_1}^{1,n})^{1-\beta_1} \\
&+ \sum_{j_1=0}^{n_1^{(n)}-1} (x_{k+1}^{1,n} - x_k^{1,n})^{\lambda_1} \int_{x_k^{1,n}}^{x_{k+1}^{1,n}} \Big( \int_{x_{k+1}^{1,n}}^{b_1} (x_1-s_1)^{-1-\beta_1} dx_1 \Big) ds_1 \\
&+ |\pi_n|^{\lambda_2} \sum_{j_1=0}^{n_1^{(n)}-1} \int_{x_k^{1,n}}^{x_{k+1}^{1,n}} \int_{a_1}^{x_k^{1,n}} (x_1-s_1)^{-1-\beta_1} ds_1 dx_1 + |\pi_n|^{\lambda_1-\beta_1} \Bigg).
\end{aligned}
\tag{2.2.2}
$$

The first, third and fifth terms on the right-hand side of (2.2.2) are bounded from above by $C|\pi_n|^{\lambda_1-\beta_1} \to 0, \quad n \to \infty$, and it is true for any $\pi_n$. The second and fourth terms can be effectively estimated when $\pi_n = \pi_n'$ is uniform. In this case

$$
|\pi_n'|^{\lambda_2} \sum_{j_1=1}^{n_1^{(n)}} (x_{j_1+1}^{1,n} - x_{j_1}^{1,n})^{1-\beta_1} \le \frac{C}{(n_1^{(n)})^{\lambda_2-\beta_1}} \to 0, \quad n \to \infty,
$$

and

$$
\begin{aligned}
&|\pi_n'|^{\lambda_2} \sum_{j_1=0}^{n_1^{(n)}-1} \int_{x_k^{1,n}}^{x_{k+1}^{1,n}} \int_{a_1}^{x_k^{1,n}} (x_1-s_1)^{-1-\beta_1} ds_1 dx_1 \\
&\le |\pi_n'|^{\lambda_2} \sum_{j_1=1}^{n_1^{(n)}} (x_{j_1+1}^{1,n} - x_{j_1}^{1,n})^{1-\beta_1} \to 0, \quad n \to \infty.
\end{aligned}
$$

$G_3^n$ and $G_4^n$ can be estimated in a similar way. □

**Definition 2.2.11.** We say that the two-parameter left Riemann–Stieltjes integral $l\text{-}\int_{\mathcal{P}} f dg$ exists if the sums $S_n$ have the limit for all sequences of uniform partitions of $\mathcal{P}$ with vanishing diameter.

**Theorem 2.2.12.** *Let $f \in C^{\lambda_1\lambda_2}(\mathcal{P})$, $g \in C^{\mu_1\mu_2}(\mathcal{P})$ and $\lambda_i + \mu_i > 1, i = 1, 2$. Then the generalized two-parameter Lebesgue–Stieltjes integrals $\int_{\mathcal{P}} f dg$ and $l$-$\int_{\mathcal{P}} f dg$ exist and coincide.*

*Proof.* It is sufficient to prove that $S_n \to \int_P f dg$. But the sums $S_n$ equal $S_n = \int_P f_{\pi_n} dg$. Denote $f^{(n)} := f_{\pi_n}$. Then

$$\int_{\mathcal{P}} f^{(n)} dg = \int_{\mathcal{P}} D_{a+}^{\beta_1\beta_2} f^{(n)}(x) D_{b-}^{1-\beta_1 1-\beta_2} g_{b-}(x) dx$$

for any $1-\mu_i < \beta_i < \lambda_i$. According to previous theorem, $D_{a+}^{\beta_1\beta_2} f^{(n)} \to D_{a+}^{\beta_1\beta_2} f$ in $L_1(\mathcal{P})$, whence the proof follows. □

*Remark* 2.2.13. We can use the Hölder properties of $f$ in order to establish that $\int_{\mathcal{P}} f dg = \lim \widetilde{S}_n$, where

$$\widetilde{S}_n = \sum_{j_1 j_2} (f(x_{j_1}^{1,n}, \xi_{j_2}^{2,n}) + f(\xi_{j_1}^{1,n}, x_{j_2}^{2,n}) - f(\xi_j^n)) \Delta_{x_j^n} g(x_{j+1}^n)$$

and $\xi_j^n$ is any point of $\mathcal{P}_j^n$.

### 2.2.3 Generalized Integrals of Two-parameter fBm in the Case of the Integrand Depending on fBm

Since the trajectories of two-parameter fBm $B^{H_1H_2}$ a.s. belong to $C^{H_1-\varepsilon_1 H_2-\varepsilon_2}(\mathcal{P})$ for any rectangle $\mathcal{P} \subset \mathbb{R}_+^2$ and any $0 < \varepsilon_i < H_i$, the next result is a direct consequence of Theorem 2.2.12.

**Theorem 2.2.14.** *Let $B^{H_1H_2}$ be a two-parameter fBm with $H_i \in (1/2, 1)$, and the function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $F \in C^1(\mathbb{R}_+ \times \mathbb{R})$. Then there exists the generalized two-parameter Lebesgue–Stieltjes integral $\int_{\mathcal{P}} F(\cdot, B^{H_1H_2}) dB^{H_1H_2}$ which coincides with the left Riemann–Stieltjes integral $l$-$\int_P F(\cdot, B^{H_1H_2}) dB^{H_1H_2}$.*

*Remark* 2.2.15. Theorem 2.2.14 holds if we replace $F(\cdot, B^{H_1H_2})$ with any Hölder field $f \in C^{\lambda_1\lambda_2}(\mathcal{P})$, such that $\lambda_i + H_i > 1$. It means that for such an $f$, we can consider the integral $\int_P f dB^{H_1H_2}$ for any $\omega \in \Omega', P(\Omega') = 1$ as the limit of corresponding integral sums.

### 2.2.4 Pathwise Integration in Two-parameter Besov Spaces

According to the form of two-parameter forward and backward fractional Marchaud derivatives (Definition 1.20.8), the Besov type spaces in this case receive the following form.

Let $\mathcal{P}_t := [0, t] = \prod_{i=1,2} [0, t_i]$,
$\varphi_1^{\beta_1}(f)(t) := \int_0^{t_1} |f(t) - f(s_1, t_2)| (t_1 - s_1)^{-\beta_1 - 1} ds_1$,
$\varphi_2^{\beta_2}(f)(t) := \int_0^{t_2} |f(t) - f(t_1, s_2)| (t_2 - s_2)^{-\beta_2 - 1} ds_2$,

$\varphi_3^{\beta_1\beta_2}(f)(t) := \int_{\mathcal{P}_t} |\Delta_s f(t)|(\varphi(t,s,1+\beta))^{-1}ds$, $0<\beta_i<1$,
and $\varphi_f^{\beta_1\beta_2}(t) := |f(t)| + \sum\limits_{i=1,2} \varphi_i^{\beta_i}(f)(t) + \varphi_3^{\beta_1\beta_2}(f)(t)$.

Denote by $W_0^{\beta_1,\beta_2}(\mathcal{P}_T)$ the Banach space of measurable functions $f:\mathcal{P}_T\to\mathbb{R}$, such that

$$\|f\|_{0,\beta_1,\beta_2} := \sup_{t\in\mathcal{P}_T} \varphi_f^{\beta_1\beta_2}(t) < \infty,$$

$W_1^{\beta_1,\beta_2}(\mathcal{P}_T)$ the Banach space of measurable functions $f:\mathcal{P}_T\to\mathbb{R}$, such that

$$\begin{aligned}\|f\|_{1,\beta_1,\beta_2} &:= \sup\nolimits_{0<s\le t<T}\Big(|\Delta_s f(t)| \prod_{i=1,2}(t_i-s_i)^{-\beta_i}\\ &+ (t_2-s_2)^{-\beta_2}\int_{s_1}^{t_1}|f_{t-}(u,s_2)-f_{t-}(s)|(u-s_1)^{-1-\beta_1}du\\ &+ (t_1-s_1)^{-\beta_1}\int_{s_2}^{t_2}|f_{t-}(s_1,v)-f_{t-}(s)|(v-s_2)^{-1-\beta_2}dv\\ &+ \int_{[s,t]}|\Delta_s f(r)|(\varphi(r,s,1+\beta))^{-1}dr\Big) < \infty\end{aligned}$$

(for the notation of $\varphi(r,s,\beta)$ see Definition 1.20.3) and $W_2^{\beta_1,\beta_2}(\mathcal{P}_T)$ the Banach space of measurable functions $f:\mathcal{P}_T\to\mathbb{R}$, such that

$$\begin{aligned}\|f\|_{2,\beta_1,\beta_2} &:= \int_{\mathcal{P}_T}\Big(|f(s)|\prod_{i=1,2}s_i^{-\beta_i} + s_2^{-\beta_2}\varphi_1^{\beta_1}(f)(s)\\ &+ s_1^{-\beta_1}\varphi_2^{\beta_2}(f)(s) + \varphi_3^{\beta_1\beta_2}(f)(s)\Big)ds < \infty.\end{aligned}$$

Similarly to Lemmas 2.1.9 and 2.1.10, the following bounds can be established. Let $0<\beta_i<1/2, i=1,2$, $G_t(f)=\int_{\mathcal{P}_t} f dg$, $F_t(f)=\int_{\mathcal{P}_t} f ds$.

**Lemma 2.2.16.** *1. Let $f\in W_2^{\beta_1\beta_2}(\mathcal{P}_T)$, $g\in W_1^{1-\beta_1,1-\beta_2}(\mathcal{P}_T)$. Then*

$$\varphi_{G.(f)}^{\beta_1\beta_2}(t) \le C^1_{\beta_1,\beta_2,T}\Lambda_{1-\beta_1 1-\beta_2}(g)\int_{\mathcal{P}_t}\prod_{i=1,2}(r_i^{-\beta_i}+(t_i-r_i)^{-2\beta_i})\varphi_f^{\beta_1\beta_2}(r)dr.$$

*2. Let $f\in W_0^{\beta_1\beta_2}(\mathcal{P}_T)$, $g\in W_1^{1-\beta_1,1-\beta_2}(\mathcal{P}_T)$. Then $G.(f)\in C^{1-\beta_1 1-\beta_2}(\mathcal{P}_T)$ and*

$$\|G(f)\|_{1-\beta_1,1-\beta_2} \le C^2_{\beta_1,\beta_2,T}\Lambda_{1-\beta_1 1-\beta_2}(g)\|f\|_{0,\beta_1,\beta_2}.$$

*3. Let $0<\beta_i<1$ and $f_T^* := \sup_{t\in\mathcal{P}_T}|f(t)|<\infty$. Then $F.(f)\in W_0^{\beta_1\beta_2}(\mathcal{P}_T)\cap C^2(\mathcal{P}_T)$ and*

$$\|F(f)\|_{0,\beta_1,\beta_2} \le C^3_{\beta_1,\beta_2,T} f_T^*.$$

### 2.2.5 The Existence of the Integrals of the Second Kind of a Two-parameter fBm

We fix the rectangle $\mathcal{P}=[0,T]\subset\mathbb{R}_+^2$ and consider the sequence of uniform partitions

$$\pi_n = \{t_j^n = (T_1 j_1 \cdot 2^{-n}, T_2 j_2 \cdot 2^{-n}), 0 \le j_i \le 2^n\}.$$

Let the functions $f, g : \mathcal{P} \to \mathbb{R}$, $f|_{\partial\mathbb{R}_+^2} = f_0 \in \mathbb{R}$, $g|_{\partial\mathbb{R}_+^2} = g_0 \in \mathbb{R}$, $f \in C^{\lambda_1\lambda_2}(\mathcal{P})$ and $g \in C^{\mu_1\mu_2}(\mathcal{P})$.

Consider the sequence of integral sums of the second kind, i.e.

$$\widetilde{S}_n := \sum_{j_1,j_2=0}^{2^n-1} f(t_j^n)\Delta_j^1 g \Delta_j^2 g,$$

where $\Delta_j^1 g = g(t_{j_1+1j_2}^n) - g(t_j^n)$, $\Delta_j^2 g = g(t_{j_1j_2+1}^n) - g(t_j^n)$.

**Theorem 2.2.17.** *Let $\lambda_i, \mu_i > \frac{1}{2}$, $\lambda_i + \mu_1 + \mu_2 > 2$, $i = 1, 2$. Then there exists $\lim_{n\to\infty} \widetilde{S}_n =: \widetilde{S}$. This limit will be called the integral of the second kind of $f$ w.r.t. $g$ and denoted as $\widetilde{S} = \int_P f d_1 g d_2 g$.*

*Proof.* Let, for technical simplicity, $T_1 = T_2 = 1$. Also, let $m > n$. Consider the difference $S_n - S_m = S_n - S_{mn} + S_{mn} - S_m$, where

$$\begin{aligned} S_{mn} = &\sum_{j_1,j_2=0}^{2^n-1} \sum_{r\in A_{j_1}} f(r2^{-m}, j_2 2^{-n})(g((r+1)2^{-m}, j_2 2^{-n}) - g(r2^{-m}, j_2 2^{-n})) \\ &\times (g(r2^{-m}, (j_2+1)2^{-n}) - g(r2^{-m}, j_2 2^{-n})), \\ A_{j_1} = &\{r : j_1 2^{m-n} \le r < (j_1+1)2^{m-n}\}. \end{aligned}$$

It is sufficient to estimate only $S_n - S_{mn}$, because $S_{mn} - S_m$ can be estimated similarly. We have that

$$|S_n - S_{mn}| \le |\Delta_{mn}^1| + |\Delta_{mn}^2|,$$

where

$$\begin{aligned} &\Delta_{mn}^1 = \sum_{j_1,j_2=0}^{2^n-1} \sum_{r\in A_{j_1}} f(t_j^n)\Delta_{jr} g \Delta_{j_2 r}^1 g, \Delta_{mn}^2 = \sum_{j_1,j_2=0}^{2^n-1} \sum_{r\in A_{j_1}} \Delta_{jr}^1 f \Delta_{j_2 r}^1 g \Delta_{j_2 r}^2 g, \\ &\Delta_{jr} g = \Delta_{t_j^n} g(r2^{-m}, (j_2+1)2^{-n}), \\ &\Delta_{j_2 r}^1 g = \Delta_{(r2^{-m}, j_2 2^{-n})}^1 g((r+1)2^{-m}, (j_2+1)2^{-n}), \\ &\Delta_{jr}^1 f = \Delta_{t_j^n}^1 f(r2^{-m}, j_2 2^{-n}), (j_2+1)2^{-n}), \\ &\Delta_{j_2 r}^2 g = \Delta_{(r2^{-m}, j_2 2^{-n})}^2 g(r2^{-m}, (j_2+1)2^{-n}). \end{aligned}$$

Transform $\Delta_{mn}^1$ into the sum

$$\Delta_{mn}^1 = \sum_{j_1,j_2=0}^{2^n-1} \sum_{r\in A_{j_1}} f(t_j^n)\Delta_{j_2 r} g \Delta_{jr}^1 g,$$

where $\Delta_{j_2 r} g = \Delta_{(r2^{-m}, j_2 2^{-n})}(g((r+1)2^{-m}(j_2+1)2^{-n}))$, and $\Delta_{jr}^1 g = \Delta_{(r2^{-m}, j_2 2^{-n})}^1 g(t_{j_1+1j_2}^n)$. The increments $\Delta_{j_2 r} g$ correspond to the rectangles $\Delta_{j_2r} = (\, r2^{-m}, (r+1)2^{-m}\,] \times (\, j_2 2^{-n}, (j_2+1)2^{-n}\,]$, that do not intersect, and $\cup \Delta_{j_2r} = (\,0,1\,]^2$. Therefore the sum $\Delta^1_{n,m}$ can be presented as a two-parameter generalized Lebesgue–Stieltjes integral $\int_{\mathcal{P}} \widetilde{f}_{mn} dg$, where

$$\widetilde{f}_{mn}(s) = f(t^n_j)\Delta^1_{jr} g \cdot \mathbf{1}_{\{s\in\Delta_{j_2r}\}}.$$

In turn,

$$\int_{\mathcal{P}} \widetilde{f}_{mn} dg = \int_{\mathcal{P}} (D^{\beta_1\beta_2}_{0+}\widetilde{f}_{mn})(s)(D^{1-\beta_1 1-\beta_2}_{1-} g_{1-})(s)ds,$$

where $1=(1,1), 0=(0,0)$, $1-\mu_i<\beta_i<\lambda_i, i=1,2$. With such a choice of $\beta_i$ $D^{1-\beta_1 1-\beta_2}_{1-} g_{1-} \in C^{\mu_1+\beta_1-1\mu_2+\beta_2-1}(\mathcal{P})$, in particular, there exists such a $C>0$ that $|(D^{1-\beta_1 1-\beta_2}_{1-} g_{1-})(s)| \le C$, $s\in\mathcal{P}$. Therefore, it is sufficient to prove that $\int_{\mathcal{P}} |(D^{\beta_1\beta_2}_{0+}\widetilde{f}_{mn})(s)|ds \to 0$, $n,m\to\infty$. Since $D^{\beta_1\beta_2}_{0+}\widetilde{f}_{mn}$ consists of four terms, we must consider them separately. Estimate only $\int_{\mathcal{P}} |\varphi_{mni}(s)|ds$, where

$\varphi_{mn1}(s) = s_2^{-\beta_2}\int_0^{s_1}(\widetilde{f}_{mn}(s)-\widetilde{f}_{mn}(u_1,s_2))(s_1-u_1)^{-1-\beta_1}du_1$,

and

$\varphi_{mn2}(s) = \int_{[0,s]} \Delta_u\widetilde{f}_{mn}(s)\prod_{i=1,2}(s_i-u_i)^{-1-\beta_i}du_1$;

the other two terms can be considered similarly.

Let $s\in\Delta_{j_2r}$. Then, taking into account that $|f(s)|\le C$ for some $C>0$, we obtain that

$$\begin{aligned}
&|\varphi_{mn1}(s)| \le s_2^{-\beta_2}\Big(\int_0^{j_1 2^{-n}} + \int_{j_1 2^{-n}}^{r2^{-m}}\Big)|\widetilde{f}_{mn}(s)-\widetilde{f}_{mn}(u_1,s_2)|(s_1-u_1)^{-1-\beta_1}du_1\\
&\le s_2^{-\beta_2}\int_0^{j_1 2^{-n}}(|\widetilde{f}_{mn}(s)|+|\widetilde{f}_{mn}(u_1,s_2)|)(s_1-u_1)^{-1-\beta_1}du_1\\
&\quad + Cs_2^{-\beta_2}\int_{j_1 2^{-n}}^{r2^{-m}}|f(t^n_j)|(s_1-u_1+2^{-m})^{\mu_1}(s_1-u_1)^{-1-\beta_1}du_1 \le Cs_2^{-\beta_2}\\
&\times(2^{-n\mu_1}(s_1-j_1 2^{-n})^{-\beta_1}+(s_1-r2^{-m})^{\mu_1-\beta_1}+2^{-m\mu_1}(s_1-r2^{-m})^{-\beta_1}),
\end{aligned}$$

whence

$$\begin{aligned}
&\int_{\mathcal{P}}|\varphi_{mn1}(s)|ds \le C\sum_{j_1,j_2=0}^{2^n-1}\sum_{r\in A_{j_1}}\Big(2^{-n\mu_1}\int_{\Delta_{j_2r}} s_2^{-\beta_2}(s_1-j_1 2^{-n})ds\\
&+\int_{\Delta_{j_2r}} s_2^{-\beta_2}(s_1-r2^{-m})^{\mu_1-\beta_1}ds + 2^{-m\mu_1}\int_{\Delta_{j_2r}} s_2^{-\beta_2}(s_1-r2^{-m})^{-\beta_1}ds\Big)\\
&\le C(1-\beta_2)^{-1}(2^{n(\beta_1-\mu_1)}+2^{m(\beta_1-\mu_1)}) \to 0, \quad m,n\to\infty.
\end{aligned}$$

Further, from Hölder properties of $f$ and $g$, it follows that for $u \le (j_1 2^{-n}, j_2 2^{-n})$ we have the estimate $|\Delta_u\widetilde{f}_{mn}(s)| \le 2(s_2-u_2+2^{-n})^{\lambda_2}2^{-n\mu_1}+C(s_2-u_2+2^{-n})^{\mu_2}(s_1-u_1)^{-n\mu_1}$, for $u\in(j_1 2^{-n}, r2^{-m})\times(0, j_2 2^{-n})$ the estimate is $|\Delta_u\widetilde{f}_{mn}(s)| \le 2(s_2-u_2+2^{-n})^{\lambda_2}(s_1-u_1+2^{-m})^{\mu_1}+C2^{-m\mu_1}(s_2-u_2+2^{-n})^{\mu_2}$, and $\Delta_u\widetilde{f}_{mn}(s)=0$ otherwise. Hence,

$$\begin{aligned}
|\varphi_{mn2}(s)| &\le C2^{-n\mu_1}(s_1-j_2 2^{-n})^{-\beta_1}(s_2-(j_1-1)2^{-n})^{\lambda_2\wedge\mu_2-\beta_2}\\
&+C(s_1+j_2 2^{-n}+2^{-m})^{\mu_1-\beta_1}(s_2-j_2 2^{-m}+2^{-n})^{\mu_2\wedge\mu_2-\beta_2},
\end{aligned}$$

and $\int_{\mathcal{P}} |\varphi_{mn2}(s)|ds \le C2^{n(\beta_1+\beta_2-\mu_1-\mu_2\wedge\lambda_2)} \to 0, m, n \to \infty$. So, $|\Delta^1_{mn}| \to 0, m, n \to \infty$. Now we want to prove that $|\Delta^2_{mn}| \to 0, m, n \to \infty$. We can present $\Delta^2_{mn}$ as

$$\Delta^2_{mn} = \sum_{j_2=0}^{2^n-1} \Delta^{2,j_2}_{mn},$$

where

$$\Delta^{2,j_2}_{mn} = \sum_{j_1=0}^{2^n-1} \sum_{r\in A_{j_1}} \Delta^1_{jr} f \Delta^1_{j_2 r} g \Delta^2_{j_2 r} g.$$

Moreover, $\Delta^{2,j_2}_{mn}$ can be presented as one-parameter generalized Lebesgue–Stieltjes integral $\int_0^1 \psi_{j_2}(u) d_1 g(u, j_2 2^{-n})$, where $\psi_{j_2}(u) = \Delta^1_{jr} f \Delta^2_{j_2 r} g \mathbf{1}_{\{r2^{-m} \le u < (r+1)2^{-m}\}}$, $\psi(0) = 0$. Then $\int_0^1 \psi_{j_2}(u) d_1 g(u, j_2 2^{-n}) = \int_0^1 (D^\beta_{0+}\psi_{j_2})(u)(D^{1-\beta}_{1-} g_{1-})(u, j_2 2^{-n}) du$, where $1-\mu_1 < \beta < 1/2$. Evidently, $|(D^{1-\beta}_{1-} g_{1-})(u, j_2 2^{-n})| \le C$, therefore, it is sufficient to prove that

$$\sum_{j_2=0}^{2^n-1} \int_0^1 |(D^\beta_{0+}\psi_{j_2})(u)| du \to 0, m, n \to \infty.$$

Note that

$$\begin{aligned}(D^\beta_{0+}\psi_{j_2})(u) = (\Gamma(1-\beta))^{-1}\Big(&\psi_{j_2}(u) u^{-\beta} \\ &+ \beta \int_0^u (\psi_{j_2}(u) - \psi_{j_2}(z))(u-z)^{-1-\beta} dz\Big),\end{aligned}$$

and $|\psi_{j_2}(u)| \le C 2^{-n(\lambda_1+\mu_2)}$, whence

$$\sum_{j_2=0}^{2^n-1} \int_0^1 |\psi_{j_2}(u)| u^{-\beta} du \le C \int_0^1 u^{-\beta} du \cdot 2^{n(1-\lambda_1-\mu_2)} \to 0, n \to \infty.$$

Further, for $j_1 2^{-n} \le r2^{-m} \le u < (r+1)2^{-m} \le (j_1+1)2^{-n}$,

$$\int_0^u (\psi_{j_2}(u) - \psi_{j_2}(z))(u-z)^{-1-\beta} dz = \int_0^{j_1 2^{-n}} + \int_{j_1 2^{-n}}^{r2^{-m}},$$

and

$$|\psi_{j_2}(u) - \psi_{j_2}(z)| \le |\psi_{j_2}(u)| + |\psi_{j_2}(z)| \le C 2^{-n(\lambda_1+\mu_2)}.$$

From here,

$$\begin{aligned}&\sum_{j_2=0}^{2^n-1} \int_0^1 \Big|\int_0^{j_1 2^{-n}} (\psi_{j_2}(u) - \psi_{j_2}(z))(u-z)^{-1-\beta} dz\Big| du \\ &\le C 2^{-n(\lambda_1+\mu_2)} \sum_{j_1,j_2=0}^{2^n-1} \sum_{r\in A_{j_1}} \int_{r2^{-m}}^{(r+1)2^{-m}} \Big|\int_0^{j_1 2^{-n}} (u-z)^{-1-\beta} dz\Big| du \\ &\le C 2^{n(1+\beta-\lambda_1-\mu_2)} \to 0, n \to \infty,\end{aligned}$$

since under assumption $\lambda_1 + \mu_1 + \mu_2 > 2$ we can choose $\frac{1}{2} > \beta > 1 - \mu_1$ in such a way that $1 + \beta - \lambda_1 - \mu_2 < 0$. Finally, for $j_1 2^{-n} \le z \le u \le (r+1)2^{-m}$

$$|\psi_{j_2}(u) - \psi_{j_2}(z)| \le 2^{-n\mu_2}(u - z + 2^{-m})^{\lambda_1},$$

and

$$\begin{aligned}&\sum_{j_2=0}^{2^n-1} \int_0^1 |\int_{j_1 r^{-n}}^{r2^{-m}} (\psi_{j_2}(u) - \psi_{j_2}(z))(u-z)^{-1-\beta} dz| du \\ &\le C 2^{m(1+\beta_1-\lambda_1-\mu_2)} \to 0, m \to \infty.\end{aligned}$$

□

*Remark* 2.2.18. For $f(s) = C$ $\Delta^2_{mn} = 0$, and it is easy to see from the bounds of $\Delta^1_{mn}$ that the theorem will hold under the assumption $\lambda_i, \mu_i > \frac{1}{2}, i = 1, 2$.

*Remark* 2.2.19. Multiple stochastic fractional integral with Hurst parameter less than 1/2 was considered in (BJ06).

## 2.3 Wick Integration with Respect to fBm with $H \in [1/2, 1)$ as $S^*$-integration

### 2.3.1 Wick Products and $S^*$-integration

Recall (see Sections 1.4–1.5), that the random variable $F$ on the probability space $S'(R)$ belongs to $S^*$ if F admits the formal expansion (1.5.1) with finite negative norm

$$\|F\|^2_{-q} = \sum_{\alpha \in \mathcal{I}} \alpha!\, c^2_\alpha (2\mathbb{N})^{-q\alpha} < \infty$$

for at least one $q \in \mathbb{N}$. Introduce the following notations:

(i) Let the function $Z : \mathbb{R} \to S^*$, and for any $F \in S$ we have that $\langle\langle Z(t), F \rangle\rangle \in L_1(\mathbb{R})$ as a function of $t \in \mathbb{R}$.

(ii) In this case, define $\int_{\mathbb{R}} Z(t) dt$ as the unique element of $S^*$ such that

$$\left\langle\left\langle \int_{\mathbb{R}} Z(t) dt, F \right\rangle\right\rangle = \int_{\mathbb{R}} \langle\langle Z(t), F \rangle\rangle \, dt,$$

and say that $Z$ is integrable in $S^*$.

(iii) Define the Wick products: for $F(\omega) = \sum_\alpha c_\alpha \mathcal{H}_\alpha(\omega)$, and $G(\omega) = \sum_\beta d_\beta \mathcal{H}_\beta(\omega)$, put $(F \diamondsuit G)(\omega) = \sum_{\alpha,\beta} c_\alpha d_\beta \mathcal{H}_{\alpha+\beta}(\omega)$.

According to the (HOUZ96), for $F, G, H \in S$ it holds that

(iv) $F \diamondsuit G = G \diamondsuit F$;

(v) $(F \diamondsuit G) \diamondsuit H = F \diamondsuit (G \diamondsuit H)$;

(vi) $H \diamondsuit (F + G) = H \diamondsuit F + H \diamondsuit G$;

(vii) $F \diamondsuit G \in S$ if $F, G \in S$; $F \diamondsuit G \in S^*$ if $F, G \in S^*$.

In this section we consider only the case $H \in [1/2, 1)$.

**Theorem 2.3.1.** *Let the process $Y(t) \in S^*$ and admit an expansion $Y(t) = \sum_\alpha c_\alpha(t)\mathcal{H}_\alpha(\omega)$, $t \in \mathbb{R}$, with the coefficients, satisfying the inequality*

$$K := \sup_\alpha \{\alpha! \left\| c_\alpha \right\|^2_{L_1(\mathbb{R})} (2\mathbb{N})^{-q\alpha}\} < \infty$$

*for some $q > 0$.*

*Then the Wick product $Y(t) \diamondsuit \dot{B}^M_t$ is $S^*$-integrable, and, moreover,*

$$\int_{\mathbb{R}} Y(t) \diamondsuit \dot{B}^M_t dt = \sum_{\alpha,k} \int_{\mathbb{R}} c_\alpha(t) M_+ \widetilde{h}_k(t) dt \cdot \mathcal{H}_{\alpha+\varepsilon_k}(\omega). \tag{2.3.1}$$

*Proof.* Consider only $\dot{B}^H_t$, and for arbitrary $\dot{B}^M_t$ the proof is the same. Since $\langle \widetilde{h}_k, \omega \rangle = \mathcal{H}_{\varepsilon_k}(\omega)$, we have that the Wick product $Y(t) \diamondsuit \dot{B}^H_t \in S^*$ and equals $\sum_{\alpha,k} c_\alpha(t) M^H_+ \widetilde{h}_k(t) \mathcal{H}_{\alpha+\varepsilon_k}(\omega)$. According to (HOUZ96, Lemmas 2.5.6 and 2.5.7), the $S^*$-integrability of $Y(t) \diamondsuit \dot{B}^H_t$ follows from the inequality

$$\sum_{\beta \in \mathcal{I}} \beta! \left\| \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} c_\alpha(t) M^H_+ \widetilde{h}_k(t) \right\|^2_{L_1(\mathbb{R})} (2\mathbb{N})^{-p\beta} < \infty$$

for some $p > 0$. According to estimate (1.5.3),
$\left| M^H_+ \widetilde{h}_k(t) \right| \le C k^{2/3-H/2} < C k^{5/12}$ for any $k \ge 1$ and some $C > 0$.

Therefore,

$$\int_{\mathbb{R}} \left| c_\alpha(t) M^H_+ \widetilde{h}_k(t) \right| dt \le C k^{5/12} \left\| c_\alpha \right\|_{L_1(\mathbb{R})},$$

and

$$\begin{aligned} \left\| \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} c_\alpha(t) M^H_+ \widetilde{h}_k(t) \right\|^2_{L_1(\mathbb{R})} &\le \left( \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \left\| c_\alpha(t) M^H_+ \widetilde{h}_k(t) \right\|_{L_1(\mathbb{R})} \right)^2 \\ &\le C \left( \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} k^{5/12} \left\| c_\alpha \right\|_{L_1(\mathbb{R})} \right)^2 . \end{aligned}$$

Consider the sum

$$\begin{aligned} S &:= \sum_{\beta \in \mathcal{I}} \beta! \left( \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} k^{5/12} \left\| c_\alpha \right\|_{L_1(\mathbb{R})} \right)^2 (2\mathbb{N})^{-p\beta} \\ &\le \sum_{\beta \in \mathcal{I}} \beta! (l(\beta))^{5/6} \left( \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \left\| c_\alpha \right\|_{L_1(\mathbb{R})} \right)^2 (2\mathbb{N})^{-p\beta}, \end{aligned}$$

where $l(\beta)$ equals the number of the last nonzero element in the index $\beta$ (the length of the index $\beta$). Further, for any $\alpha,\beta$ there exists no more than one $k$, such that $\alpha+\varepsilon_k=\beta$. Therefore,

$$\left(\sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \|c_\alpha\|_{L_1(\mathbb{R})}\right)^2 \le l^2(\beta) \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \|c_\alpha\|^2_{L_1(\mathbb{R})}.$$

It means that

$$\begin{aligned} S &\le \sum_{\alpha,k} (\alpha+\varepsilon_k)!(l(\alpha+\varepsilon_k))^{17/6}\|c_\alpha\|^2_{L_1(\mathbb{R})}(2\mathbb{N})^{-p\alpha-p\varepsilon_k} \\ &\le K\sum_{\alpha,k}\frac{(\alpha+\varepsilon_k)!}{\alpha!}(l(\alpha+\varepsilon_k))^3(2\mathbb{N})^{-(p-q)\alpha-p\varepsilon_k} \\ &\le K\sum_{\alpha,k}(|\alpha|+1)^4 2^{-|\alpha|(p-q)}k^{-p}<\infty, \end{aligned}$$

for $p>q+1$, and we have established the $S^*$-integrability of $Y(t)\diamondsuit \dot{B}^H_t$. Now, for any $F=\sum_{\beta,k} d_{\beta,k}\mathcal{H}_{\beta+\varepsilon_k}(\omega)\in S$, we have from the definition of the $S^*$-integral and of Wick product, that

$$\begin{aligned} \left\langle\!\!\left\langle \int_{\mathbb{R}} Y(t)\diamondsuit \dot{B}^H_t dt, F\right\rangle\!\!\right\rangle &= \int_{\mathbb{R}} \left\langle\!\!\left\langle \sum_{\alpha,k} c_\alpha(t) M^H_+ \widetilde{h}_k(t)\mathcal{H}_{\alpha+\varepsilon_k}(\omega), F\right\rangle\!\!\right\rangle dt \\ &= \int_{\mathbb{R}} \sum_{\alpha,k}(\alpha+\varepsilon_k)! c_\alpha(t) d_{\alpha,k} M^H_+\widetilde{h}_k(t)(\omega)dt. \end{aligned} \tag{2.3.2}$$

Note that

$$\sum_{\alpha,k}(\alpha+\varepsilon_k)!\,|d_{\alpha,k}|^2\,(2\mathbb{N})^{2q(\alpha+\varepsilon_k)} =: C_q<\infty$$

for any $q\in\mathbb{N}$. Therefore

$$\begin{aligned} \sum_{\alpha,k}\int_{\mathbb{R}}(\alpha+\varepsilon_k)!\,|c_\alpha(t)|\,|d_{\alpha,k}|\left|M^H_+\widetilde{h}_k(t)\right|dt &\le \sum_{\alpha,k}(\alpha+\varepsilon_k)!\,|d_{\alpha,k}|\,k^{5/12}\,\|c_\alpha\|_{L_1(\mathbb{R})} \\ &\le \left(\sum_{\alpha,k}\beta_k!\,|d_{\alpha,k}|^2\,(2\mathbb{N})^{2q\beta_k}\sum_{\alpha,k}k^{5/6}\,\|c_\alpha\|^2_{L_1(\mathbb{R})}\,\beta_k!(2\mathbb{N})^{-2q(\alpha+\varepsilon_k)}\right)^{1/2} \\ &\le \left(C_qK\sum_{\alpha,k}k^{5/6}\frac{\beta_k!}{\alpha!}(2\mathbb{N})^{-q|\alpha|}k^{-2q}\right)^{1/2}<\infty \end{aligned}$$

for $q > 11/12$, $\beta_k = \alpha + \varepsilon_k$, because $\sum_\alpha \frac{\beta_k!}{\alpha!}(2\mathbb{N})^{-q|\alpha|} \le \sum_\alpha (|\alpha|+1)2^{-q|\alpha|} < \infty$. So, we can change the signs of sum and integral in (2.3.2) and obtain

$$\left\langle\!\!\left\langle \int_{\mathbb{R}} Y(t) \diamondsuit \dot{B}_t^H dt, F \right\rangle\!\!\right\rangle = \sum_{\alpha,k} (\alpha+\varepsilon_k)! d_{\alpha,k} \int_{\mathbb{R}} c_\alpha(t) M_+^H \widetilde{h}_k(t)(\omega) dt$$
$$= \left\langle\!\!\left\langle \sum_{\alpha,k} \int_{\mathbb{R}} c_\alpha(t) M_+^H \widetilde{h}_k(t)(\omega) dt, F \right\rangle\!\!\right\rangle,$$

whence (2.3.1) follows. □

**Corollary 2.3.2.** *Let $Y(t) = \sum_\alpha c_\alpha(t)\mathcal{H}_\alpha(\omega) \in S^*$ be a process such that $\int_0^T EY^2(t)dt < \infty$ for some $T > 0$. Then $\sum_\alpha \alpha! \int_0^T c_\alpha^2(t)dt = \int_0^T EY^2(t)dt < \infty$, whence $K := \sup_\alpha\{\alpha! \left\|\overline{c}_\alpha\right\|^2_{L_1(\mathbb{R})} (2\mathbb{N})^{-q\alpha}\} < \infty$ for any $q > 0$ (hereafter we put $\overline{c}_\alpha(t) := c_\alpha(t)\mathbf{1}_{[0,T]}(t)$).*

So, we can use Theorem 2.3.1 and conclude that $Y(t) \diamondsuit \dot{B}_t^M$ is $S^*$-integrable, and, moreover, equality (2.3.1) holds.

**Corollary 2.3.3.** *Let $Y(t) \equiv 1$. Then the previous corollary holds with $c_0(t) = 1, c_\alpha(t) = 0$ for $\alpha \ne 0$, whence*

$$\int_0^T \dot{B}_t^M dt = \sum_k \int_0^T M_+ \widetilde{h}_k(t)dt \cdot \mathcal{H}_{\varepsilon_k}(\omega) = B_T^M.$$

*In this connection, we can say that the fractional noise is the $S^*$-derivative of fBm.*

As a consequence, we can define $\int_{\mathbb{R}} Y_t \diamondsuit dB_t^M := \int_{\mathbb{R}} Y_t \diamondsuit \dot{B}_t^M dt$ for the process $Y_t$, satisfying the conditions of Theorem 2.3.1.

Now, let $Y \in L_2[0,T]$ be some nonrandom function, $H \in (1/2, 1)$.

Then $c_\alpha(t) = Y(t) = \overline{c}_\alpha(t)$, for $\alpha = 0$ and $c_\alpha \equiv 0$ for other $\alpha$, so, by using Theorem 2.3.1, we obtain that

$$\int_0^T Y(t) \diamondsuit \dot{B}_t^H dt = \sum_k \int_0^T Y(t) M_+^H \widetilde{h}_k(t)dt \cdot \langle \widetilde{h}_k, \omega\rangle.$$

Further, even for $Y \in L_1[0,T]$ we can replace the operator $M_+^H$ and obtain $\int_0^T Y(t) M_+^H \widetilde{h}_k(t)dt = \int_0^T M_-^H Y(t)\widetilde{h}_k(t)dt$, whence

$$\begin{aligned} \int_0^T Y(t) \diamondsuit \dot{B}_t^H dt &= \sum_k \int_{\mathbb{R}} M_-^H \overline{Y}(t)\widetilde{h}_k(t)dt \cdot \langle \widetilde{h}_k, \omega\rangle \\ &= \sum_k \int_{\mathbb{R}} M_-^H \overline{Y}(t)\widetilde{h}_k(t)dt \cdot \mathcal{H}_{\varepsilon_k}(\omega), \end{aligned} \tag{2.3.3}$$

where $\overline{Y}(t) = Y(t)\mathbf{1}_{[0,T]}(t)$. The right-hand side of (2.3.3) corresponds to (HOUZ96, representation (2.5.22)) of the integral $\int_0^T M_-^H Y(t) \diamondsuit \dot{B}_t dt$, where $\dot{B}_t = \dot{B}_t^{1/2}$ is a white noise:

$$\int_0^T M_-^H Y(t) \diamondsuit \dot{B}_t dt = \sum_{\alpha,k} \int_0^T c_\alpha(t) \widetilde{h}_k(t) dt \cdot \mathcal{H}_{\alpha+\varepsilon_k}(\omega).$$

Therefore, for $Y \in L_2^H[0,T]$

$$\int_0^T Y(t) \diamondsuit \dot{B}_t^M dt = \int_{\mathbb{R}} M_- \overline{Y}(t) \diamondsuit \dot{B}_t dt = \int_{\mathbb{R}} M_- \overline{Y}(t) \cdot \dot{B}_t dt. \qquad (2.3.4)$$

### 2.3.2 Comparison of Wick and Pathwise Integrals for "Markov" Integrands

In this subsection we can, without losing generality, consider instead of $S'(\mathbb{R})$ the probability space $\Omega = C_0(\mathbb{R}_+, \mathbb{R})$ of real-valued continuous functions on $\mathbb{R}_+$ with the initial value zero and the topology of local uniform convergence. There exists a probability measure $P$ on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the Borel $\sigma$-field, such that on the probability space $(\Omega, \mathcal{F}, P)$ the coordinate process $B : \Omega \to \mathbb{R}$ defined as,

$$B_t(\omega) = \omega(t), \ \omega \in \Omega$$

is the Wiener process.

(i) Recall the notion of a stochastic derivative. Let $F$ be a square-integrable random variable, and suppose that the limit

$$\lim_{\beta \to 0} \frac{1}{\beta} \left( F(\omega. + \beta \int_0^{\cdot} h(s) ds) - F(\omega.) \right) \quad \text{exists in} \quad L_2(P)$$

for any $h \in L_2(\mathbb{R})$. Then this limit is called the directional derivative $D_h F$.

(ii) If the directional derivative $D_h F$, $h \in L_2(\mathbb{R})$, is absolutely continuous w.r.t. the measure $h(x)dx$, i.e.

$$D_h(F) = \int_{\mathbb{R}} \frac{dD_h(F)}{dh}(x) \cdot h(x) dx,$$

and $(dD_h(F))/(dh)$ does not depend on $h$, then the Radon–Nikodym derivative $(dD_h(F))/(dh)$ is called the stochastic derivative of $F$ and is denoted by $D_x F$.

(iii) We have a chain rule for the stochastic derivative: if $D_x F$ exists and $\varphi \in C^1(\mathbb{R})$, then $D_x \varphi(F)$ has the stochastic derivative

$$D_x \varphi(F) = \varphi'(F) D_x F.$$

(iv) Let $u \in L_2(\mathbb{R})$ be a nonrandom function. Then it follows from (NP95, Proposition 5.5), that

$$D_x \int_{\mathbb{R}} u_s dB_s = u_x \quad \text{a.e.}$$

(v) Recall the notion of the class $\mathbb{D}_{1,2}$. This is the Banach space, obtained as a completion of the set $\mathcal{P}_0$ of smooth functionals $F = f(B_{t_1}, \dots, B_{t_i})$, w.r.t. the norm $\|F\|_{1,2} := \|F\|_{L_2(P)} + \big\| \|D_x F\|_{HS} \big\|_{L_1(P)}$, where $F \in \mathcal{P}_0$, and $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm.

Denote $L_2^M(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} |M_- f(x)|^2 \, dx < \infty\}$.

**Lemma 2.3.4.** *Let $F \in \mathbb{D}_{1,2},\ f \in L_2^M(\mathbb{R})$. Suppose that the integrals*

$$\int_{\mathbb{R}} (M_- f)(s) \cdot D_s F ds \ \text{and}\ F \cdot \int_{\mathbb{R}} (M_- f)(s) dB_s = F \cdot \int_{\mathbb{R}} f(s) dB_s^M$$

*belong to $L_2(P)$. Then $F \diamondsuit \int_{\mathbb{R}} f(s) dB_s^M$ exists and*

$$F \diamondsuit \int_{\mathbb{R}} f(s) dB_s^M = \int_{\mathbb{R}} (F \cdot M_- f)(s) \delta B_s$$

$$= F \cdot \int_{\mathbb{R}} f(s) dB_s^M - \int_{\mathbb{R}} (M_- f)(s) \cdot D_s F ds. \tag{2.3.5}$$

*Proof.* By using (HOUZ96, Corollary 2.5.12) and (NP95, Theorem 3.2), we obtain for nonrandom $f$ that

$$\begin{aligned} F \diamondsuit \int_{\mathbb{R}} f(s) dB_s^M &= F \diamondsuit \int_{\mathbb{R}} (M_- f)(s) dB_s \\ &= \int_{\mathbb{R}} (F \diamondsuit M_- f)(s) \delta B_s = \int_{\mathbb{R}} (F \cdot M_- f)(s) \delta B_s \\ &= F \cdot \int_{\mathbb{R}} (M_- f)(s) dB_s - \int_{\mathbb{R}} (M_- f)(s) \cdot D_s F ds \\ &= F \cdot \int_{\mathbb{R}} f(s) dB_s^M - \int_{\mathbb{R}} (M_- f)(s) \cdot D_s F ds. \end{aligned}$$

(Note that according to (NP95, Theorem 3.2), the Skorohod integral $\int_{\mathbb{R}} F \cdot (M_- f)(s) \delta B_s$ exists if and only if the difference $F \cdot \int_{\mathbb{R}} (M_- f)(s) dB_s - \int_{\mathbb{R}} (M_- f)(s) \cdot D_s F ds$ belongs to $L_2(P)$). □

Using this result, we can compare the Wick integral and the pathwise integral w.r.t. fBm $B_t^H$, $H \in (1/2, 1)$(the latter integral coincides with Stratonovich integral). Therefore, now $M_\pm = M_\pm^H$.

**Lemma 2.3.5.** *Let $\varphi \in C^1(\mathbb{R})$, $F_t = \varphi(B_t^H)$, $f(s) = \mathbf{1}_{[t,t+h]}(s)$, $t, h > 0$. If $\varphi'(B_t^H)$ and $F_t \cdot (B_{t+h}^H - B_t^H)$ belong to $L_2(P)$, then*

$$F_t \diamondsuit (B_{t+h}^H - B_t^H) = F \cdot (B_{t+h}^H - B_t^H) \\ - H\varphi'(B_t^H) t^{2\alpha} h + c(\omega)(t^{2\alpha-1}h^2 + h^{2H}),$$

*where $c(\omega)$ is a.s. finite and independent of $t$ and $h$.*

*Proof.* According to equation (2.3.5), we can rewrite formally the left-hand side of the previous equality:

$$F_t \diamondsuit (B_{t+h}^H - B_t^H) = F_t \cdot (B_{t+h}^H - B_t^H) \\ - \int_{\mathbb{R}} \left(M_-^H \mathbf{1}_{[t,t+h]}\right)(s) D_s \varphi(B_t^H) ds. \quad (2.3.6)$$

Further, according to the chain rule (iii), it holds that

$$D_s \varphi(B_t^H) = \varphi'(B_t^H) D_s B_t^H,$$

and

$$D_s B_t^H = D_s \int_{\mathbb{R}} \left(M_-^H \mathbf{1}_{[0,t]}\right)(u) dB_u = (M_-^H \mathbf{1}_{[0,t]})(s).$$

Therefore,

$$F_t \diamondsuit (B_{t+h}^H - B_t^H) = F_t \cdot (B_{t+h}^H - B_t^H) \\ - \varphi'(B_t^H) \int_{\mathbb{R}} \left(M_-^H \mathbf{1}_{[t,t+h]}\right)(s) \left(M_-^H \mathbf{1}_{[0,t]}\right)(s) ds,$$

and under the conditions of the lemma the right-hand side of equation (2.3.6) is well-defined. Finally,

$$\int_{\mathbb{R}} (M_-^H \mathbf{1}_{[t,t+h]})(s)(M_-^H \mathbf{1}_{[0,t]})(s) ds = E(B_{t+h}^H - B_t^H) B_t^H \\ = \frac{1}{2}((t+h)^{2H} - t^{2H} - h^{2H}) = Ht^{2\alpha}h + 2H\alpha\theta^{2\alpha-1}h^2 - h^{2H},$$

where $\theta \in (t, t+h)$. The lemma is proved. □

*Remark* 2.3.6. Evidently, the assumption $E\big(\varphi(B_t^H)\big)^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$ is sufficient for $F_t(B_{t+h}^H - B_t^H)$ to belong to $L_2(P)$.

Now, fix some $T > 0$ and consider the sequence $\pi_n = \{0 = t_0^n < \cdots < t_n^n = T\}$ of partitions of $[0, T]$, such that $\pi_n \subset \pi_{n+1}$ and $|\pi_n| \to 0$ as $n \to \infty$. Suppose that

$$\varphi'(B_t^H) \in L_2(P),\ \varphi(B_t^H) \in L_{2+\varepsilon}(P),\ t \in [0, T] \quad (2.3.7)$$

for some $\varepsilon > 0$.

According to Lemma 2.3.5, we can write

$$\sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \diamond \Delta B^H_{i,n} = \sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \Delta B^H_{i,n} \\ - H \sum_{i=1}^{n} \varphi'(B^H_{t^n_{i-1}})(t^n_{i-1})^{2\alpha} \Delta t_{i,n} + R_n(T),$$

where $\Delta t_{i,n} = t^n_i - t^n_{i-1}$, $\Delta B^H_{i,n} = B^H_{t^n_i} - B^H_{t^n_{i-1}}$. Here $R_n(T)$ is a remainder term and $R_n(T) \to 0$ a.s. as $n \to \infty$. Furthermore, the process $C_t := \varphi(B^H_t)$ is Hölder continuous up to order $H$. Also, by Theorem 2.1.7, part 2), the sum $\sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \Delta B^H_{i,n}$ converges a.s. as $n \to \infty$ to the pathwise integral $\int_0^T \varphi(B^H_s) dB^H_s$. Clearly,

$$\sum_{i=1}^{n} \varphi'(B^H_{t^n_{i-1}})(t^n_{i-1})^{2\alpha} \Delta t_{i,n} \to \int_0^T \varphi'(B^H_s) s^{2\alpha} ds \quad \text{a.s.}$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \diamond \Delta B^H_{i,n} = \int_0^T \varphi(B^H_s) dB^H_s - H \int_0^T \varphi'(B^H_s) s^{2\alpha} ds \quad \text{a.s.}$$

Moreover, under assumption (2.3.7) and

$$E \int_0^T \left(\varphi(B^H_s)\right)^2 ds < \infty, \tag{2.3.8}$$

there exists the Wick integral $\int_0^T \varphi(B^H_s) \diamond dB^H_s$. Now we are in a position to prove that

$$\int_0^T \varphi(B^H_s) \diamond dB^H_s = \lim_{n\to\infty} \sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \diamond \Delta B^H_{i,n}. \tag{2.3.9}$$

**Theorem 2.3.7.** *Under conditions* (2.3.7) *and*

$$E \sup_{s\le T} \left(\varphi(B^H_s)\right)^2 + E \sup_{s\le T} (\varphi'(B^H_s))^2 < \infty \tag{2.3.10}$$

*equality* (2.3.8) *and* (2.3.9), *consequently, the equality*

$$\int_0^T \varphi(B^H_s) \diamond dB^H_s = \int_0^T \varphi(B^H_s) dB^H_s - H \int_0^T \varphi'(B^H_s) s^{2\alpha} ds$$

*holds a.s.*

*Proof.* Let the random variables $F, G \in \mathbb{D}_{1,2}$. According to equality (2.3.5) and (NP95, Theorem 3.2), for $i \le k$

$$
\begin{aligned}
&E\left[F \diamondsuit \Delta B^H_{i,n} \cdot G \diamondsuit \Delta B^H_{k,n}\right] \\
&= E\left[\int_{\mathbb{R}} F M^H_- \mathbf{1}_{[t^n_{i-1}, t^n_i]}(s)\delta B_s \cdot \int_{\mathbb{R}} G M^H_- \mathbf{1}_{[t^n_{k-1}, t^n_k]}(s)\delta B_s\right] \\
&= E\left[\int_{\mathbb{R}} FG M^H_- \mathbf{1}_{[t^n_{i-1}, t^n_i]}(s) M^H_- \mathbf{1}_{[t^n_{k-1}, t^n_k]}(s) ds\right] \\
&\quad + E\left[\int_{\mathbb{R}\times\mathbb{R}} D_t F D_s G M^H_- \mathbf{1}_{[t^n_{i-1}, t^n_i]}(t) M^H_- \mathbf{1}_{[t^n_{k-1}, t^n_k]}(s) ds\, dt\right] \\
&= \frac{1}{2} E[FG r_{ik}] \\
&\quad + E\left[\int_{\mathbb{R}} D_t F M^H_- \mathbf{1}_{[t^n_{i-1}, t^n_i]}(t) dt \cdot \int_{\mathbb{R}} D_s G M^H_- \mathbf{1}_{[t^n_{k-1}, t^n_k]}(s) ds\right],
\end{aligned}
\tag{2.3.11}
$$

where

$$
r_{ik} = \left|t^n_{k-1} - t^n_i\right|^{2H} + (t^n_k - t^n_{i-1})^{2H} - (t^n_k - t^n_i)^{2H} - (t^n_{k-1} - t^n_{i-1})^{2H}.
$$

Put in (2.3.11) $F = \varphi(B^H_{t^n_{i-1}})$, $G = \varphi(B^H_{t^n_{k-1}})$ and take the sum over $1 \le i \le k \le n$. We obtain that

$$
E\left(\sum_{i=1}^n \varphi(B^H_{t^n_{i-1}}) \diamondsuit \Delta B^H_{i,n}\right)^2 = S^n_1 + S^n_2,
$$

where

$$
S^n_1 = \sum_{1 \le i \le k \le n} E\varphi(B^H_{t^n_{i-1}})\varphi(B^H_{t^n_{k-1}}) r_{ik},
$$

and

$$
\begin{aligned}
S^n_2 = &\sum_{1 \le i \le k \le n} E \int_{\mathbb{R}} \varphi'(B^H_{t^n_{i-1}}) M^H_- \mathbf{1}_{[t^n_{i-1}, t^n_i]}(t) M^H_- \mathbf{1}_{[0, t^n_{i-1}]}(t) dt \\
&\times \int_{\mathbb{R}} \varphi'(B^H_{t^n_{k-1}}) M^H_- \mathbf{1}_{[t^n_{k-1}, t^n_k]}(s) M^H_- \mathbf{1}_{[0, t^n_{k-1}]}(s) ds \\
= &\frac{1}{4} \sum_{1 \le i \le k \le n} E\varphi'(B^H_{t^n_{i-1}})\varphi'(B^H_{t^n_{k-1}}) \left((t^n_k)^{2H} - (t^n_{k-1})^{2H} - (\Delta t^n_k)^{2H}\right) \\
&\times \left((t^n_i)^{2H} - (t^n_{i-1})^{2H} - (\Delta t^n_i)^{2H}\right).
\end{aligned}
$$

Evidently,

$$
|S^n_2| \le H^2 E\left(\sum_{i=1}^n \left|\varphi'(B^H_{t^n_{i-1}})\right| t^{2\alpha}_i \cdot \Delta t^n_i\right)^2. \tag{2.3.12}
$$

If the partition $\pi_n$ is uniform, i.e. $t_i^n = \frac{iT}{n}$, then for some $C_H > 0$

$$S_1^n \le 2 \sum_{1\le i\le n} E\left|\varphi(B^H_{t^n_{i-1}})\right|^2 \left(\frac{iT}{n}\right)^{2H} + \left(\frac{T}{n}\right)^{2H} C_H \sum_{1\le i\le k\le n} \left|\varphi(B^H_{t^n_{i-1}})\varphi(B^H_{t^n_{k-1}})\right| \cdot \int_{i-1}^{i}\int_{k-1}^{k} (u-v)^{2\alpha-1} du\, dv. \tag{2.3.13}$$

Now it is very easy to conclude from (2.3.10)–(2.3.13), that the sums

$$S_n := \sum_{k=1}^{n} \varphi(B^H_{t^n_k}) \diamondsuit \Delta B^H_{k,n}$$

form a Cauchy sequence in $L_2(P)$, at least, for uniform $\pi_n$. From the estimate

$$|\langle\langle F, g\rangle\rangle| \le \|F\|_{L_2(P)} \|g\|_{L_2(P)}, \; F \in L_2(P), \; g \in S,$$

we obtain that $\langle\langle S_n - S_m, g\rangle\rangle \to 0, \; n, m \to \infty$ for any $g \in S$. This means that $\{S_n\}$ is a Cauchy sequence in the weak sense. If we establish the weak convergence $S_n \to \widetilde{S} := \int_0^T \varphi(B^H_s) \diamondsuit dB^H_s$, then the theorem will be proved, since the convergence will be in $L_2(P)$, as well. According to (2.3.1) and Corollary 2.3.2, we have that

$$\widetilde{S} = \int_0^T \varphi(B^H_t) \diamondsuit \dot{B}^H_t dt = \sum_{\alpha,k} \int_0^T c_\alpha(t) M^H_+ \widetilde{h}_k(t) dt \cdot \mathcal{H}_{\alpha+\varepsilon_k}(\omega),$$
$$S_n = \int_0^T \varphi_n(t) \diamondsuit \dot{B}^H_t dt = \sum_{\alpha,k} \int_0^T c^n_\alpha(t) M^H_+ \widetilde{h}_k(t) dt \cdot \mathcal{H}_{\alpha+\varepsilon_k}(\omega),$$

where

$$\varphi_n(t) = \sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \mathbf{1}_{[t^n_{i-1}, t^n_i)}(t),$$
$$\varphi(B^H_t) = \sum_\alpha c_\alpha(t) \mathcal{H}_\alpha(\omega), \; c^n_\alpha(t) = \sum_{i=1}^{n} c_\alpha(t^n_{i-1}) \mathbf{1}_{[t^n_{i-1}, t^n_i)}(t).$$

Denote $d^n_\alpha := c_\alpha - c^n_\alpha$. Then

$$S - S_n = \sum_\beta \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \int_0^T d^n_\alpha(t) M^H_+ \widetilde{h}_k(t) dt \cdot \mathcal{H}_\beta(\omega).$$

Furthermore, for any $g = \sum_\beta g_\beta \mathcal{H}_\beta(\omega) \in S$ and any $q > 0$

$$\begin{aligned}\left|\left\langle\!\left\langle \widetilde{S} - S_n, g\right\rangle\!\right\rangle\right| &\le \sum_\beta \beta! \left| g_\beta \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \int_0^T d_\alpha^n(t) M_+^H \widetilde{h}_k(t)dt \right| \\ &\le \left(\sum_\beta \beta! (g_\beta)^2 (2\mathbb{N})^{\beta q}\right)^{1/2} \\ &\quad \times \left(\sum_\beta \beta! \left\| \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \left| d_\alpha^n M_+^H \widetilde{h}_k \right| \right\|_{L_1[0,T]}^2 (2\mathbb{N})^{-\beta q}\right)^{1/2} .\end{aligned}$$

We estimate only the second multiplicand. According to (1.5.3), for $H \in (1/2, 1)$ $\left|M_+^H \widetilde{h}_k(t)\right| \le C k^{5/12}$ with constant $C$ independent of $t, k$. So,

$$\begin{aligned}\left\| \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \left| d_\alpha^n M_+^H \widetilde{h}_k \right| \right\|_{L_1[0,T]}^2 &\le C \left( \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} k^{5/12} \left\|d_\alpha^n\right\|_{L_1[0,T]} \right)^2 \\ &\le C (l(\beta))^{5/6} \left( \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \left\|d_\alpha^n\right\|_{L_1[0,T]} \right)^2 ,\end{aligned}$$

where $l(\beta)$ equals the number of nonzero entries in $\beta$. Further,

$$\begin{aligned}&\sum_\beta \beta!\, (2\mathbb{N})^{-\beta q} \left\| \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \left| d_\alpha^n M_+^H \widetilde{h}_k \right| \right\|_{L_1[0,T]}^2 \\ &\le \sum_\beta \beta!\, (2\mathbb{N})^{-\beta q} l(\beta)^{5/6} \left( \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \left\|d_\alpha^n\right\|_{L_1[0,T]} \right)^2 \\ &\le \sum_\beta \beta!\, l(\beta)^{17/6} \sum_{\alpha:\exists k,\alpha+\varepsilon_k=\beta} \left\|d_\alpha^n\right\|_{L_1[0,T]} (2\mathbb{N})^{-\beta q} \\ &\le \sum_{\alpha,k} (\alpha+\varepsilon_k)! (l(\alpha+\varepsilon_k))^{17/6} \left\|d_\alpha^n\right\|_{L_1[0,T]}^2 (2\mathbb{N})^{-q(\alpha+\varepsilon_k)} \\ &\le \sup_\alpha \left\{ \alpha!\ \left\|d_\alpha^n\right\|_{L_1[0,T]}^2 \right\} \sum_{\alpha,k} \frac{(\alpha+\varepsilon_k)!}{\alpha!} (l(\alpha+\varepsilon_k))^{17/6} (2\mathbb{N})^{-q\alpha} (2\mathbb{N})^{-q\varepsilon_k} \\ &\le \sup_\alpha \left\{ \alpha!\, \left\|d_\alpha^n\right\|_{L_1[0,T]}^2 \right\} \sum_{\alpha,k} (|\alpha|+1)^{23/6} 2^{-|\alpha| q} k^{-q}.\end{aligned}$$

The last series converges for $q > 1$, and it follows from the continuity of $\varphi$ and condition (2.3.10), that

$$\sup_\alpha \left\{ \alpha! \; \|d_\alpha^n\|^2_{L_1[0,T]} \right\} \le \sum_\alpha \alpha! \; \|d_\alpha^n\|_{L_2[0,T]} \cdot T$$
$$= T \left\| \varphi(B_\cdot^H) - \varphi_n(\cdot) \right\|_{L_2[0,T]} \to 0, \; n \to \infty.$$

□

Theorem 2.3.7 can be generalized to the processes of the form

$$B_t^M := \sum_{k=1}^m \sigma_k B_t^{H_k}.$$

Suppose that $H_1 = \frac{1}{2}$ and $H_k \in (1/2, 1), \; 2 \le k \le m$.

**Theorem 2.3.8.** *Assume that conditions* (2.3.7), (2.3.8) *and* (2.3.10) *hold with* $B_t^H$ *replaced by* $B_t^M$. *Then*

$$\int_0^T \varphi(B_t^M) \diamondsuit \, dB_t^M = \int_0^T \varphi(B_t^M) dB_t^M$$
$$- \sum_{i,k=1}^n \sigma_i \sigma_k \widetilde{C}_{H_i H_k} (H_i + H_k) \int_0^T \varphi'(B_s^M) s^{H_i + H_k - 1} ds + \frac{1}{2} \sigma_1^2 \int_0^T \varphi'(B_s^M) ds,$$

*where*

$$\widetilde{C}_{H_i H_k} = \begin{cases} \dfrac{C_{H_i}^{(3)} C_{H_k}^{(3)} B(H_i - 1/2, 2 - H_i - H_k)}{(H_i + H_k)(H_i + H_k - 1)\Gamma(H_i - 1/2)\Gamma(H_k - 1/2)}, \\ H_i, H_k \in (1/2, 1), \\ \dfrac{C_{H_k}^{(3)}}{\Gamma(H_k + 3/2)}, \; H_i = 1/2, H_k \in (1/2, 1), \\ 0, H_i \in (1/2, 1), H_k = 1/2, \\ \dfrac{1}{2}, H_i = H_k = 1/2. \end{cases}$$

*Proof.* We start with (2.3.5) and conclude that

$$\varphi(B_t^M) \diamondsuit (B_{t+h}^M - B_t^M) = \varphi(B_t^M) \cdot (B_{t+h}^M - B_t^M)$$
$$- \varphi'(B_t^M) \sum_{i,k=1}^m \sigma_i \sigma_k \int_{\mathbb{R}} M_-^{H_i} \mathbf{1}_{[t,t+h]}(s) M_-^{H_k} \mathbf{1}_{[0,t]}(s) ds.$$

Further, for $f \in L_2^{H_i}(\mathbb{R}), g \in L_2^{H_k}(\mathbb{R}), H_i, H_k \in (1/2, 1)$

$$\int_{\mathbb{R}} M_-^{H_i} f(s) M_-^{H_k} g(s) ds = C_{i,k,H}^{(1)} \int_{\mathbb{R}} \int_s^\infty (x - s)^{H_i - 3/2} f(x) dx$$
$$\times \int_s^\infty (y - s)^{H_k - 3/2} g(y) dy \, ds = C_{i,k,H}^{(1)} \int_{\mathbb{R}^2} f(x) g(y) dx \, dy$$
$$\times \int_{-\infty}^{x \wedge y} (x - s)^{H_i - 3/2} (y - s)^{H_k - 3/2} ds,$$

where $C^{(1)}_{i,k,H} = \frac{C^{(3)}_{H_i} C^{(3)}_{H_k}}{\Gamma(H_i-1/2)\Gamma(H_k-1/2)}$. Evidently,

$$\int_{-\infty}^{x\wedge y} (x-s)^{H_i-3/2}(y-s)^{H_k-3/2} ds$$
$$= |y-x|^{H_i+H_k-2}\left(C^{(2)}_{i,k,H}\mathbf{1}\{y>x\} + C^{(2)}_{k,i,H}\mathbf{1}\{y\le x\}\right).$$

with $C^{(2)}_{i,k,H} = \int_0^\infty z^{H_i-3/2}(1+z)^{H_k-3/2}dz = B(H_i-1/2, 2-H_i-H_k)$. Therefore,

$$\int_{\mathbb{R}} M_-^{H_i} f(s) M_-^{H_k} g(s) ds = C^{(1)}_{i,k,H}\int_{\mathbb{R}} f(x)\,|y-x|^{H_i+H_k-2}$$
$$\cdot\left(C^{(2)}_{i,k,H}\mathbf{1}\{x<y\} + C^{(2)}_{k,i,H}\mathbf{1}\{y<x\}\right) dx\,dy.$$

Let $f(x) = \mathbf{1}_{[t,t+h]}(x),\ g(y) = \mathbf{1}_{[0,t]}(y)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} M_-^{H_i}\mathbf{1}_{[t,t+h]}(s) M_-^{H_k}\mathbf{1}_{[0,t]}(s) ds \\
&= \prod_{j=1,2} C^{(j)}_{k,i,H}\int_0^t\int_t^{t+h}(y-x)^{H_i+H_k-2}dy\,dx \\
&= \prod_{j=1,2} C^{(j)}_{k,i,H}((H_i+H_k)(H_i+H_k-1))^{-1} \\
&\quad\times\left[(t+h)^{H_i+H_k} - t^{H_i+H_k} - h^{H_i+H_k}\right] \\
&=: \widetilde{C}_{H_iH_k}\left[(t+h)^{H_i+H_k} - t^{H_i+H_k} - h^{H_i+H_k}\right] \\
&= \widetilde{C}_{H_iH_k}\left[(H_i+H_k)t^{H_i+H_k-1}h + (H_i+H_k)(H_i+H_k-1)\theta^{H_i+H_k-1}h^2\right. \\
&\quad\left. - h^{H_i+H_k}\right],\ \theta\in(t,t+h).
\end{aligned} \tag{2.3.14}$$

For $H_i = 1/2$ and $H_k \in (1/2,1)$ we have that $M_-^{1/2} = I$ is identity operator, and

$$\int_{\mathbb{R}} M_-^{1/2} f(s) M_-^{H_k} g(s) ds = \frac{C^{(3)}_{H_k}}{\Gamma(H_k-1/2)}\int_{\mathbb{R}} f(s)\int_s^\infty g(y)(y-s)^{H_k-3/2}dy\,ds.$$

For $f$ and $g$ as above, the last integral equals

$$\begin{aligned}
&\frac{C^{(3)}_{H_k}}{\Gamma(H_k-1/2)}\int_0^t\int_t^{t+h}(y-s)^{H_k-3/2}dy\,ds \\
&= \frac{C^{(3)}_{H_k}}{\Gamma(H_k+3/2)}\left[(t+h)^{H_k+1/2} - t^{H_k+1/2} - h^{H_k+1/2}\right] \\
&=: \widetilde{C}_{\frac{1}{2}H_k}\left[(H_k+1/2)t^{H_k-1/2}h\right. \\
&\quad\left.+ (H_k+1/2)(H_k-1/2)t^{H_k-2}h^2 - h^{H_k+1/2}\right].
\end{aligned} \tag{2.3.15}$$

At last, for $H_i = H_k = 1/2$

$$\int_{\mathbb{R}} M_-^{1/2} \mathbf{1}_{[0,t]}(s) M_-^{1/2} \mathbf{1}_{[t,t+h]}(s) ds = 0. \tag{2.3.16}$$

Now we can proceed as in Lemma 2.3.5 and Theorem 2.3.7, put $\widetilde{C}_{\frac{1}{2}\,\frac{1}{2}} := \frac{1}{2}$, take into account (2.3.14)–(2.3.16) and obtain the proof. □

### 2.3.3 Comparison of Wick and Stratonovich Integrals for "General" Integrands

Now we consider the general process $F_t$ instead of $\varphi(B_t^M)$. Suppose that fBm $\{B_t^H, t \geq 0\}$ is "one-sided", $H \in (\frac{1}{2}, 1)$.

**Theorem 2.3.9.** *Let $\{F_t, \mathcal{F}_t, t \in [0,T]\}$ be the stochastic process satisfying the conditions*

*(i) $F_t \in \mathbb{D}_{1,2}$ for any $t \in [0,T]$, $E\,|F_t|^{2+\varepsilon} < \infty$ for any $t \in [0,T]$ and some $\varepsilon > 0$, $\sup_{s,t\in[0,T]} |D_s F_t|$ is bounded in probability;*

*(ii) $\lim_{h\downarrow 0} \sup_{t\in[0,T]} |D_t F_s - D_t F_{s+h}| = 0$ in probability;*

*(iii) $F_t$ is a.s. Hölder continuous of order $\alpha > 1 - H$ (this condition implies the existence of the Stratonovich integral $\int_0^T F_t dB_t^H$, $H \in (1/2, 1)$);*

*(iv) $E \int_0^T F_t^2 dt < \infty$ (this condition implies the existence of the Wick integral $\int_0^T F_t \diamondsuit dB_t^H$, according to Corollary 2.3.2);*

*(v) there exists a sequence of partitions $\{\pi_n, n \geq 1\}$ with $|\pi_n| \to 0$ as $n \to \infty$ such that the integral sums $\sum_{k=1}^n F_{t_{k-1}^n} \diamondsuit \Delta B_{k,n}^H$ converge to $\int_0^T F_t \diamondsuit dB_t^H$ in probability.*

*Then*

$$\int_0^T F_s \diamondsuit dB_s^H = \int_0^T F_s dB_s^H - C_H^{(3)} \int_0^T \left( \int_0^s (s-t)^{\alpha-1} D_s F_t dt \right) ds.$$

*Proof.* Consider for any $0 \leq t < t+h \leq T$ the function $f(u) = \mathbf{1}_{[t,t+h]}(u)$. Then we take into account that $D_s F_t = 0$ for $s > t$ and $s < 0$ (since $F_t$ is $\mathcal{F}_t$-adapted) and obtain that $\int_{\mathbb{R}} M_-^H f D_s F_t ds = C_H^{(3)} \int_0^t \int_t^{t+h} (u-s)^{\alpha-1} du D_s F_t ds$, where $\int_t^{t+h} (u-s)^{\alpha-1} du \leq \frac{h^\alpha}{\alpha}$. Hence,

$$E \left( \int_{\mathbb{R}} M_-^H f D_s F_t ds \right)^2 \leq \frac{\left(C_H^{(3)}\right)^2}{\alpha^2} h^{2\alpha} t E \int_0^t |D_s F_t|^2 ds < \infty.$$

Further, $F_t \cdot \int_{\mathbb{R}} M_-^H f dB_s = F_t \cdot \left(B_{t+h}^H - B_t^H\right)$, and, according to (i),

$$E\left|F_t \cdot \left(B_{t+h}^H - B_t^H\right)\right|^2 \leq \left(E\left|F_t\right|^{2+\varepsilon}\right)^{\frac{2}{2+\varepsilon}} \left(E\left|B_{t+h}^H - B_t^H\right|^{\frac{2(2+\varepsilon)}{\varepsilon}}\right)^{\frac{\varepsilon}{2+\varepsilon}} < \infty.$$

Therefore, $\int_{\mathbb{R}} M_-^H f \cdot D_s F_t ds$ and $F_t \cdot \int_{\mathbb{R}} M_-^H f dB_s$ belong to $L_2(P)$ and it follows from Lemma 2.3.4 that the integral sums $\sum_{k=1}^n F_{t_{k-1}^n} \diamondsuit \Delta B_{k,n}^H$ exist. Moreover,

$$\begin{aligned} F_{t_{k-1}^n} \diamondsuit \Delta B_{k,n}^H &= F_{t_{k-1}^n} \cdot \Delta B_{k,n}^H - \int_{\mathbb{R}} M_-^H \mathbf{1}_{[t_{k-1}^n, t_k^n]}(s) D_s F_{t_{k-1}^n} ds \\ &= F_{t_{k-1}^n} \cdot \Delta B_{k,n}^H - \int_{\mathbb{R}} \mathbf{1}_{[t_{k-1}^n, t_k^n]}(s) \big(M_+^H (D_. F_{t_{k-1}^n})\big)(s) ds \\ &= F_{t_{k-1}^n} \cdot \Delta B_{k,n}^H - C_H^{(3)} \int_{t_{k-1}^n}^{t_k^n} \int_0^{t_{k-1}^n} (s-u)^{\alpha-1} D_u F_{t_{k-1}^n} du\, ds. \end{aligned} \tag{2.3.17}$$

Consider the difference,

$$\begin{aligned} &\Bigg|\sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \int_0^{t_{k-1}^n} (s-u)^{\alpha-1} D_u F_{t_{k-1}^n} du\, ds \\ &\qquad - \int_0^T \int_0^s (s-u)^{\alpha-1} D_u F_{t_{k-1}^n} \mathbf{1}_{[t_{k-1}^n, t_k^n)}(s) du\, ds\Bigg| \\ &\qquad\qquad \leq C \cdot \sup_{0 \leq u \leq t \leq T} |D_u F_t| \cdot |\pi_n|^\alpha \cdot T \to 0, \end{aligned} \tag{2.3.18}$$

as $n \to \infty$ in probability, according to (i). Further, according to (i) and (ii),

$$\begin{aligned} &\Bigg|\int_0^T \int_0^s (s-u)^{\alpha-1} D_u F_{t_{k-1}^n} \mathbf{1}_{[t_{k-1}^n, t_k^n)}(s) du\, ds \\ &\qquad - \int_0^T \int_0^s (s-u)^{\alpha-1} D_u F_s du\, ds\Bigg| \to 0 \end{aligned} \tag{2.3.19}$$

in probability. Now, the proof follows from (v) and (2.3.17)–(2.3.19). □

Now consider one sufficient condition for (v) (condition (v) seems to be the most artificial among other conditions (i)–(iv)). To this end, consider the middle part of (2.3.11), from which we obtain that for any step processes $F_n(t) = \sum_{k=1}^n F_{k,n} \mathbf{1}_{[t_{k-1}^n, t_{k-1}^n)}(t)$ and $G_n(t) = \sum_{k=1}^n G_{k,n} \mathbf{1}_{[t_{k-1}^n, t_{k-1}^n)}(t)$

$$\begin{aligned} &E\left[\sum_{k=1}^n F_n(t) \diamondsuit dB_t^H \cdot \sum_{k=1}^n G_n(t) \diamondsuit dB_t^H\right] \\ &\quad = E \int_{\mathbb{R}} M_-^H F_n(t) M_-^H G_n(t) dt + E \int_{\mathbb{R}^2} M_-^H D_s F_n(t) M_-^H D_t G_n(s) ds\, dt. \end{aligned} \tag{2.3.20}$$

The next result was motivated by (Ben03a, Theorem 2.2.8).

**Theorem 2.3.10.** *Let the stochastic process* $\{F_t, \mathcal{F}_t, t \in [0,T]\}$ *satisfy the assumptions (i)–(iv) and*

*(vi)* $E\int_0^T F_t^2 dt < \infty$;
*(vii) the operator* $F_t : [0,T] \to \mathbb{D}_{1,2}$ *is continuous in* $L_2([0,T]\times P)$.

*Then the integral sums* $\sum_{k=1}^n F_{t_{k-1}^n} \diamondsuit \Delta B_{k,n}^H$ *exist, the integral* $\int_0^T F_s \diamondsuit dB_s^H$ *exists and*

$$\int_0^T F_s \diamondsuit dB_s^H = \lim_{n\to\infty} \sum_{k=1}^n F_{t_{k-1}^n} \diamondsuit \Delta B_{k,n}^H \quad in \quad L_2(P)$$

*for any sequence of increasing partitions* $\pi_n$ *with* $|\pi_n| \to 0$ *as* $n \to \infty$.

*Proof.* Under condition (vi), the existence of sums $\sum_{k=1}^n F_{t_{k-1}^n} \diamondsuit \Delta B_{k,n}^H$ and the integral $\int_0^T F_s \diamondsuit dB_s^H$ was established in Theorem 2.3.9. Further, using (2.3.20) and (vii), we obtain that

$$\begin{aligned} E\left|\int_0^T F_t \diamondsuit dB_t^H - \sum_{k=1}^n F_{t_{k-1}^n} \diamondsuit \Delta B_{k,n}^H\right|^2 & \\ &= E\int_{\mathbb{R}} \left[M_-^H(F_\cdot - F_\cdot^n)(t)\right]^2 dt \\ &+ \int_{\mathbb{R}^2} E\left[M_-^H(D_t F_\cdot - D_t F_\cdot^n)(s)\right]^2 ds dt =: E_n < \infty \end{aligned}$$

From the Hardy–Littlewood theorem (Theorem 1.1.1) with $q = 2$, $\alpha = H - 1/2$ and $p = \frac{1}{H}$

$$\int_{\mathbb{R}} \left[M_-^H(F_\cdot - F_\cdot^n)(t)\right]^2 dt \le C_H \left\|F_\cdot - F_\cdot^n\right\|_{L_{\frac{1}{H}}[0,T]}^2$$

and from condition (vii) it follows that

$$\int_{\mathbb{R}} \left[M_-^H(D_t F_\cdot - D_t F_\cdot^n)(s)\right]^2 ds \le C_H \left\|D_t F_\cdot - D_t F_\cdot^n\right\|_{L_{\frac{1}{H}}[0,T]}^2$$

whence from (vii) and (iv) we obtain that

$$\begin{aligned} E_n &\le C_H E\left(\left\|F_\cdot - F_\cdot^n\right\|_{L_{\frac{1}{H}}[0,T]}^2 + \int_0^T E\left\|D_t F_\cdot - D_t F_\cdot^n\right\|_{L_{\frac{1}{H}}[0,T]}^2 dt\right) \\ &\le C_H T^{2\alpha} E\left(\left\|F_\cdot - F_\cdot^n\right\|_{L_2[0,T]}^2 + \int_0^T \left\|D_t F_\cdot - D_t F_\cdot^n\right\|_{L_2[0,T]}^2 dt\right) \\ &\le C_H T^{2\alpha} \int_0^T E\left\|F_\cdot - F_\cdot^n\right\|_{1,2}^2 dt \\ &\le C_{H,1} T^{2\alpha} \left\|F - F^{(n)}\right\|_{L_2([0,T]\times P)} \to 0,\ n \to \infty. \end{aligned}$$

□

### 2.3.4 Reduction of Wick Integration w.r.t. Fractional Noise to the Integration w.r.t. White Noise

Recall that for nonrandom integrands $f \in L_2^H(\mathbb{R})$

$$\int_{\mathbb{R}} f(t)dB_t^H := \int_{\mathbb{R}} (M_-^H f)(t)dB_t.$$

In this subsection we reduce $\int_{\mathbb{R}} X_t \diamondsuit \dot{B}_t^H dt$ to the corresponding integral $\int_{\mathbb{R}}(M_-^H X)(t) \diamondsuit \dot{B}_t dt$ w.r.t. white noise.

**Theorem 2.3.11.** *Let the following conditions hold:*

$$E\int_{\mathbb{R}} |X_t|^2 dt < \infty \quad \textit{and} \quad E\int_{\mathbb{R}} ((M_-^H|X_t|)(t))^2 dt < \infty.$$

*Then*

$$\int_{\mathbb{R}} X_t \diamondsuit \dot{B}_t^H dt = \int_{\mathbb{R}} (M_-^H X)(t) \diamondsuit \dot{B}_t dt \quad \textit{a.s.}$$

*Proof.* According to Theorem 2.3.1 and Corollary 2.3.2, the condition $E\int_{\mathbb{R}} |X_t|^2 dt < \infty$ supplies the equality

$$\int_{\mathbb{R}} X_t \diamondsuit \dot{B}_t^H dt = \sum_{\alpha,k} \int_{\mathbb{R}} c_\alpha(t) M_+^H \widetilde{h}_k(t)dt \cdot \mathcal{H}_{\alpha+\varepsilon_k}(\omega). \tag{2.3.21}$$

First, replace the operator $M_+^H$ in the last equality. Evidently,

$$\int_{\mathbb{R}} f(t) M_+^H g(t)dt = \int_{\mathbb{R}} M_-^H f(t)g(t)dt \tag{2.3.22}$$

for $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$ with $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha = H + 1/2$. Moreover, $\widetilde{h}_k \in L_q(\mathbb{R})$ for any $q > 1$. Since $E\int_{\mathbb{R}} |X_t|^2 dt$ $= \sum\limits_{\alpha} \alpha! \int_{\mathbb{R}} c_\alpha^2(t)dt < \infty$, we can take $p = 2$, $q = \frac{1}{H}$ and obtain from (2.3.22) that

$$\int_{\mathbb{R}} c_\alpha(t) M_+^H \widetilde{h}_k(t)dt = \int_{\mathbb{R}} (M_-^H c_\alpha)(t)\widetilde{h}_k(t)dt. \tag{2.3.23}$$

Further, consider the formal expansion $Y_t := \sum\limits_{\alpha}(M_-^H c_\alpha)(t)\mathcal{H}_\alpha(\omega)$. Again, from Corollary 2.3.2, the condition

$$E\int_{\mathbb{R}} Y_t^2 dt = \sum_{\alpha} \alpha! \int_{\mathbb{R}} |(M_-^H c_\alpha)(t)|^2 dt < \infty \tag{2.3.24}$$

ensures the equality

$$\int_{\mathbb{R}} Y_t \diamondsuit \dot{B}_t dt = \sum_{\alpha,k} \int_{\mathbb{R}} (M_-^H c_\alpha)(t)\widetilde{h}_k(t)dt\,\mathcal{H}_{\alpha+\varepsilon_k}(\omega). \tag{2.3.25}$$

So, we want to know when (2.3.24) holds and we need the equality $Y_t = (M_-^H X)(t)$. This follows from the equalities

$$((M_-^H X)(t), \mathcal{H}_\alpha(\omega))_{L_2(P)} = (M_-^H c_\alpha)(t) = M_-^H (X_t, \mathcal{H}_\alpha(\omega))_{L_2(P)}, \tag{2.3.26}$$

if they hold for any $\alpha \in \mathcal{I}$. Equalities (2.3.26) can be reduced to

$$\int_\Omega \Big( \int_t^\infty (x-t)^{\alpha-1} X_x(\omega)dx \Big) \mathcal{H}_\alpha(\omega)dP \\ = \int_t^\infty (x-t)^{\alpha-1} \Big( \int_\Omega X_x(\omega)\mathcal{H}_\alpha(\omega)dP \Big) dx \tag{2.3.27}$$

for a.a. $t \in \mathbb{R}$. In turn, the Fubini theorem can be applied to (2.3.27) in the case when

$$E\Big( \int_t^\infty (x-t)^{\alpha-1} |X_x(\omega)| dx \Big)^2 < \infty \quad \text{for a.a. } t \in \mathbb{R} \tag{2.3.28}$$

because $E\mathcal{H}_\alpha^2(\omega) = \alpha! < \infty$. Evidently, the condition $E\int_{\mathbb{R}}((M_-^H|X|)(t))^2 dt < \infty$ ensures both (2.3.24) and (2.3.28). The proof now follows from (2.3.21), (2.3.23), (2.3.25) and (2.3.26). □

## 2.4 Skorohod, Forward, Backward and Symmetric Integration w.r.t. fBm. Two Approaches to Skorohod Integration

Taking into account the definition of the integral for nonrandom function w.r.t. fBm: $\int_{\mathbb{R}} f(t)dB_t^H := \int_{\mathbb{R}} (M_-^H f)(t)dB_t$, and Theorem 2.3.11, it is desirable to define the integral $\int_{\mathbb{R}} f(t)dB_t^H$ for stochastic integrands in a similar way. Evidently, in this case, even for very simple and natural integrands, such as $f(t) = B_t^H$, we have that $(M_-^H B^H)(t) = C_H^{(3)} \int_t^\infty (x-t)^{\alpha-1} B_x^H dx$ is not adapted. So, we must in this case address the theory of integration of non-adapted processes. To this end, recall the definition of the Skorohod integral (see also the pioneer paper (Sko75)).

Let the stochastic process $X_t = X_t(\omega)$ be such that

$$EX_t^2 < \infty \quad \text{for all } t \in \mathbb{R}.$$

Then $X_t$ admits a Wiener–Itô chaos expansion

$$X_t = \sum_{n=0}^\infty \int_{\mathbb{R}^n} f_n(s_1, \ldots, s_n, t) dB^{\otimes n}(s_1, \ldots, s_n),$$

where the functions $f_n(\cdot) \in L_2(\mathbb{R}^n)$ and are symmetric in variables $(s_1, \ldots, s_n)$, for $n = 0, 1, 2, \ldots$ and for each $t \in \mathbb{R}$. See, for example, (HOUZ96, Theorem 2.2.5). Let $\widehat{f}_n(s_1, \ldots, s_n, s_{n+1})$ be the symmetrization of $f_n(s_1, \ldots, s_n, s_{n+1})$ with respect to $(n+1)$ variables $s_1, \ldots, s_n, s_{n+1}$.

**Definition 2.4.1.** Assume that

$$\sum_{n=0}^{\infty}(n+1)! \left\| \widehat{f}_n \right\|_{L_2(\mathbb{R}^{n+1})} < \infty.$$

Then we say that the process $X$ is Skorohod integrable, write $X \in Dom(\delta)$, denote the Skorohod integral as $\int_{\mathbb{R}} X_t \delta B_t$, and define it as $\int_{\mathbb{R}} X_t \delta B_t := \sum\limits_{n=0}^{\infty} \int_{\mathbb{R}^{n+1}} \widehat{f}_n(s_1, \ldots, s_{n+1}) dB^{\otimes(n+1)}(s_1, \ldots, s_{n+1})$. The Skorohod integral belongs to $L_2(P)$,

$$E \int_{\mathbb{R}} X_t \delta B_t = 0, \text{ and } E| \int_{\mathbb{R}} X_t \delta B_t|^2 = \sum_{n=0}^{\infty}(n+1)! \left\| \widehat{f}_n \right\|_{L_2(\mathbb{R}^{n+1})}.$$

*Remark* 2.4.2 ((NP95)). Define by $\mathbb{L}_{1,2}$ the class of stochastic processes $X \in L_2(\mathbb{R} \times \Omega)$ such that $X \in \mathbb{D}_{1,2}$ for almost all $t$, and there exists a measurable version of two-parameter process $D_s X_t$ satisfying the relation $E \int_{\mathbb{R}^2} (D_s X_t)^2 ds\, dt < \infty$. Then $\mathbb{L}_{1,2} \subset Dom(\delta)$.

**Definition 2.4.3** ((Ben03a))**.** Let the stochastic process $X_t = X_t(\omega)$ be such that $(M_-^H X)(t)$ exists and belongs to $Dom(\delta)$. Then we define the Skorohod integral with respect to fBm $B^H$ as

$$\int_{\mathbb{R}} X_t \delta B_t^H := \int_{\mathbb{R}} (M_-^H X)(t) \delta B_t$$

for the underlying Wiener process $B$.

Evidently, $E \int_{\mathbb{R}} X_t \delta B_t^H = 0$. Of course, we can define in the usual way the Skorohod integral with finite limits and indefinite integral $\int_0^t X_t \delta B_t^H$, $t \in [0, T]$. It is easy to compare now the Skorohod and Wick integral w.r.t. fBm.

**Theorem 2.4.4.** *Let* $M_-^H X \in Dom(\delta)$, $E \int_{\mathbb{R}} |X_t|^2 dt < \infty$ *and* $E \int_{\mathbb{R}} ((M_-^H |X|)(t))^2 dt < \infty$. *Then*

$$\int_{\mathbb{R}} X_t \delta B_t^H = \int_{\mathbb{R}} X_t \diamondsuit \dot{B}_t^H dt.$$

*Proof.* According to (HOUZ96, Theorem 2.5.9), the condition $M_-^H X \in Dom(\delta)$ ensures the existence of $\int_{\mathbb{R}} (M_-^H X)(t) \diamondsuit \dot{B}_t dt$ and the equalities:

$$\int_{\mathbb{R}} (M_-^H X)(t) \diamondsuit \dot{B}_t dt = \int_{\mathbb{R}} (M_-^H X)(t) \delta B_t = \int_{\mathbb{R}} X_t \delta B_t^H.$$

Further, according to Theorem 2.3.11, in our case

$$\int_{\mathbb{R}} (M_-^H X)(t) \diamondsuit \dot{B}_t dt = \int_{\mathbb{R}} X_t \diamondsuit \dot{B}_t^H dt,$$

whence the proof follows. □

*Remark* 2.4.5. Let $Y \in L_2^H[0,T]$. Then $Y$ is a Skorohod integrable adapted stochastic process. Indeed, it is nonrandom thus adapted. From (2.3.4) and (HOUZ96, Theorem 2.5.9), $Y(t) \diamondsuit \dot{B}_t^M$ is $S^*$-integrable, and

$$\begin{aligned}\int_0^T Y(t) \diamondsuit \dot{B}_t^M dt &= \int_{\mathbb{R}} M_- \overline{Y}(t) \cdot \dot{B}_t dt \\ &= \int_0^T M_- \overline{Y}(t) \delta B_t = \int_0^T M_- \overline{Y}(t) dB_t,\end{aligned}$$

where $\delta$ means Skorohod integration, and the last integral is the Itô, and even the Wiener, integral. Note that, according to Corollary 1.9.4 (for $H > 1/2$, or $1/H < 2$) $L_2[0,T] \subset L_2^H[0,T]$. We obtain that the $S^*$-integral for nonrandom functions from $L_2[0,T]$ coincides with the Wiener integral $\int_0^T Y(t) dB_t^H$ from Definition 1.6.1.

Another approach to Skorohod integration w.r.t. fBm was developed in the papers (AN02), (Nua03), (Nua06). The main idea is to use the basic tools of a stochastic calculus of variations (Malliavin calculus) with respect to $B^H$. Recall some of these notions for $H \in (1/2, 1)$. (For $H \in (0, 1/2)$ see, for example, (AMN00).)

Let $\mathcal{S}$ be a family of smooth random variables of the form

$$F = f(B_{t_1}^H, \dots, B_{t_n}^H)$$

with $f \in C_b^\infty(\mathbb{R}^n)$ and $t_i \in [0,T], 1 \le i \le n$. Let $\mathcal{H}$ be a closure of the linear space of step functions defined on $[0,T]$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} := 2\alpha H \int_0^t \int_0^s |r-u|^{2\alpha-1} du\, dr.$$

Then the derivative operator $D : \mathcal{S} \to L_p(\Omega, \mathcal{H})$ for $p \ge 1$ is defined as

$$D_H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (B_{t_1}^H, B_{t_2}^H, \dots, B_{t_n}^H) 1_{[0,t_i]}.$$

Let $D_{k,p}(\mathcal{H})$ be the Sobolev space, the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^k E(\|D^\nu F\|_{\mathcal{H}^{\otimes j}}^p),$$

where $D^j$ is the $j$th iteration of $D$. The Skorohod integral (divergence operator) $\delta_H$ is defined as the adjoint of $D_H : \mathbb{D}_{1,2}(\mathcal{H}) \subset L_2(\Omega) \to L_2(\Omega, \mathcal{H})$, defined by the means of the duality relationship

$$E(G\delta_H(u)) = E\langle D_H G, u\rangle_{\mathcal{H}}, u \in L_2(\Omega, \mathcal{H}), G \in S.$$

Its domain is denoted by $Dom(\delta_H)$.

Introduce the Banach space $|\mathcal{H}| \otimes |\mathcal{H}|$ as the class of all the measurable functions $\varphi : [0,T]^2 \to \mathbb{R}$ such that

$$\|\varphi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} := (2\alpha H)^2 \int_{[0,T]^4} |\varphi_{u,v}||\varphi_{s,t}||s-u|^{2\alpha-1}|t-v|^{2\alpha-1} du\, dv\, ds\, dt < \infty,$$

and denote $|\mathcal{H}| := |R_H|$ with the norm $\|\cdot\|_{|R_H|,2}$ (see (1.6.7)). Denote also $\mathcal{S}_{|\mathcal{H}|}$ the family of $|\mathcal{H}|$-valued random variables of the form $F = \sum_{i=1}^{n} F_i h_i$, where $F_i \in S$ and $h_i \in |\mathcal{H}|$. Put $D^k F := \sum_{i=1}^{n} D^k F_i \otimes h_i$, and define the space $\mathbb{D}_{k,p}(|\mathcal{H}|)$ as the completion of $\mathcal{S}_{|\mathcal{H}|}$ with respect to the norm

$$\|F\|^p_{k,p,|\mathcal{H}|} = E(\|F\|^p_{|\mathcal{H}|}) + \sum_{i=1}^{k} E(\|D^i F\|^p_{\mathcal{H}^{\otimes i}\otimes|\mathcal{H}|}).$$

Then $\mathbb{D}_{1,2}(|\mathcal{H}|) \subset Dom(\delta_H)$. The basic property of the divergence operator is that for every $u \in \mathbb{D}_{1,2}(|\mathcal{H}|)$ we have

$$E(|\delta(u)|^2) \le \|u\|^2_{\mathbb{D}_{1,2}(|\mathcal{H}|)}.$$

Consider the forward integral w.r.t. fBm ((AN02), (LT02)). It is defined as

$$\int_0^t u_s dB_s^{H,-} := P - \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_0^t u_s (B^H_{(s+\varepsilon)\wedge t} - B^H_s) ds. \tag{2.4.1}$$

(Note that in a similar way the symmetric Stratonovich integral can be defined: $\int_0^t u_s dB_s^{H,-} := P - \lim_{\varepsilon\to 0}(2\varepsilon)^{-1} \int_0^t u_s (B^H_{(s+\varepsilon)\wedge t} - B^H_{(s-\varepsilon)\wedge t}) ds$, and also backward integral can be defined.) In (LT02) the ucp-limit is considered instead of the $P$-limit, where ucp-convergence is uniform convergence in probability on $[0,T]$. Moreover, it is mentioned in (AN02) that forward, backward and symmetric integrals with integrand $u$ and w.r.t. fBm coincide with each other under the following suppositions: $u \in \mathbb{D}_{1,2}(|\mathcal{H}|)$ with $\int_0^t \int_0^t |D_s u_r||r-s|^{2\alpha-1} ds\, dr < \infty$ a.s.). Also, it was proved that for processes $u \in \mathbb{D}_{1,2}(|\mathcal{H}|)$ with $\int_0^t \int_0^t |D_s u_r||r-s|^{2\alpha-1} ds\, dr < \infty$ a.s. we have the equality

$$\int_0^t u_s dB_s^{H,s} = \delta_H(u) + 2\alpha H \int_0^t \int_0^t |D_s u_r||r-s|^{2\alpha-1} dr ds. \tag{2.4.2}$$

Evidently, for $u \in C^{\beta}[0,T]$ with $\beta + H > 1$ all the integrals, symmetric, forward, backward, and pathwise, coincide. We use this fact in order to establish the conditions of coincidence of Skorohod integrals introduced in (Ben03a) and in (AN02).

**Theorem 2.4.6.** *Fix a time interval $[0,T]$. Let $\phi \in C^1(\mathbb{R})$ and satisfy, together with its derivative $\phi'$, the growth condition $|\phi(x)| \le C\exp(\lambda x^b)$ for some $\lambda > 0$ and $0 < b < 2$. Then the integrals $\delta_H(\phi(B^H))$ and $\int_0^t \phi(B_s^H)\delta B_s^H$ coincide on $[0,T]$ a.s.*

*Proof.* According to Proposition 3.3 (Nua06), under the condition of the theorem (even under the less restrictive condition $|\phi(x)| \le C\exp(\lambda x^2)$ for $\lambda < (4T^{2H})^{-1}$), the divergence operator $\delta_H(\phi(B^H))$ exists on $[0,T]$ and satisfies the relation

$$\delta_H(\phi(B^H)) = \int_0^T \phi(B_s^H)dB_s^H - H\int_0^T \phi'(B_s^H)s^{2\alpha}ds \quad \text{a.s.},$$

where $\int_0^T \phi(B_s^H)dB_s^H$ is the pathwise integral. According to Theorem 2.3.7, under conditions (2.3.10), which evidently hold now, the same equality is valid for the integral $\int_0^T \phi(B_s^H)\diamondsuit dB_s^H$. Therefore, $\delta_H(\phi(B^H))$ and $\int_0^T \phi(B_s^H)\diamondsuit dB_s^H$ coincide a.s. on $[0,T]$. Further, the conditions of Theorem 2.4.4 also hold now. Indeed, for example, $E\int_{\mathbb{R}}((M_-^H|X|)(t))^2dt$ can be bounded in our case by $C\int_0^T |\phi(B_s^H)|^2ds$. Therefore, $\int_0^T \phi(B_t^H)\delta B_t^H$ exists and equals $\int_0^T \phi(B_t^H)\diamondsuit \dot{B}_t^H dt$. Finally, we use Theorem 2.3.1 and Corollary 2.3.2 and obtain the proof.

□

*Remark* 2.4.7. A general $S$-transform approach to the stochastic fractional integration is presented in (Ben03b); see also (CC00) and (Cou07).

## 2.5 Isometric Approach to Stochastic Integration with Respect to fBm

### 2.5.1 The Basic Idea

Some special approach to stochastic integration w.r.t. fBm was considered in (MV00). We will work with a continuous stochastic process $\{X_t, 0 \le t \le T\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t := \mathcal{F}_t^X$ be the sigma-field generated by $X$ on $[0,t]$. We assume that $X_0 = 0$. Given a partition $\pi_n := \{t_i : 0 = t_0 < t_1 < \cdots < t_n = T\}$ and $X$ a stochastic process, define $\Delta X_i$ by $\Delta X_i := X_{t_i} - X_{t_{i-1}}$ for $1 \le i \le n$. Assume first that the integrand $f$ is a simple predictable process: $f_t = \sum\limits_i f_i \mathbf{1}_{[t_{i-1},t_i)}(t)$, where the random variables $f_i$ are assumed to be $\mathcal{F}_{t_{i-1}}$ measurable and $t_i \in \pi_n$; denote the class of simple predictable processes by $L^s$. With such an $f \in L^s$ and any (continuous) process $X$, define the stochastic integral of $f$ with respect to $X$ by

$$(f, X) := \sum_i f_i \Delta X_i.$$

Assume now that $|\pi_n| \to 0$ as $n \to \infty$. If the process $X$ is the standard Brownian motion $B$, $f := L_2(P \otimes \lambda)\text{-}\lim f^n$, where $\lambda$ is the Lebesgue measure on $[0, T]$, one can define the integral $(f, B)$ as the $L_2$-limit of the simple stochastic integrals $(f^{(n)}, B)$ using the classical Itô isometry

$$E(f^{(n)}, B)^2 = E \int_0^T (f_s^{(n)})^2 ds. \tag{2.5.1}$$

Assume now that the process $X$ is any continuous stochastic process and $f$ is a simple predictable process. Define now a semi-norm for $(f, X)$ using (2.5.1). Note that such a semi-norm does not depend on the process $X$. It is the main feature of this approach. If the process $X$ is the standard Brownian motion, then the semi-norm is a norm and the integrals of simple function converge to the classical stochastic integral defined by Itô. For an arbitrary integrator $X$, even if the semi-norm is a norm, it may happen that the integrals of simple functions of processes have no limit. However, they have a limit in the completion of the space integral sums with respect to this norm. In this sense we generalize the Itô construction of stochastic integrals.

In particular, we show that if $X$ is a fractional Brownian motion $B^H$, then we can define a norm by putting

$$\|(f, B^H)\|_G := \left(E \int_0^T f_s^2 ds\right)^{1/2}$$

in the space $G$ of random variables of the form $\{g \in G : G = (f, B^H), f \in L^s\}$.

Even more turns out to be true: for any $k \geq 2$ define random variables $(f, X^{(k)})$ by the formula

$$(f, X^{(k)}) := \sum_i f_i (\Delta X_i)^k$$

and define again a semi-norm for such random variables by putting

$$\left\|(f, X^{(k)})\right\|_{G^k} := \left(E \int_0^T f_s^2 ds\right)^{1/2}.$$

Again, if the process $X$ is a fractional Brownian motion $B^H$, then $\left\|(f, (B^H)^{(k)})\right\|_{G^k}$ is a norm. Denote by $L_2^{pr}(P \otimes \lambda)$ the space of predictable process $f$ with the property $E \int_0^T f_s^2 ds < \infty$. Now, let $f \in L_2^{pr}(P \otimes \lambda)$ be a predictable process and $f^{(n)}$ a sequence of simple predictable processes such that

$$\left\| f^{(n)} - f \right\|_{L^2(P\otimes\lambda)} \to 0$$

as $n \to \infty$. Define the higher-order generalized integral $(f, (B^H)^{(k)})$ as a limit in the Banach space $(\mathcal{J}^k, \|\cdot\|_{G^k})$, which is the space of some kind of extended random variables $g$, which are limits of the sequences of the form $(f, (B^H)^{(k)})$ with respect the norm $\|\cdot\|_{G^k}$.

### 2.5.2 First- and Higher-order Integrals with Respect to $X$

#### Wiener Integrals

Further, if $(Y, \|\cdot\|_Y)$ is a complete metric space, then the $Y$-lim stands for the limit on the space $Y$ with respect to the norm $\|\cdot\|_Y$. Assume that $f$ is a simple deterministic process, $f_t = \sum_{i=1}^m f_i \mathbf{1}_{[t_{i-1},t_i)}(t)$. Then $\|\cdot\|_G$ is a norm if and only if

$$(f, X) = \sum_{i=1}^{m} f_i \Delta X_i = 0 \Longleftrightarrow f_i = 0, 1 \le i \le m. \tag{2.5.2}$$

Let $X = (X_t)_{t\in[0,T]}$ be a square integrable process with $EX_t = 0, X_0 = 0$, and write $R(t,s)$ for the covariance function, $R(t,s) = E\, X_t X_s$. Consider the quadratic forms

$$B_m = E((f,X))^2$$

where $f \in L^s$ has deterministic coefficients $f_i, 1 \le i \le m$. Then condition (2.5.2) is equivalent to the following:

The quadratic form $B_m$ is positive definite for each $m \ge 1$. (2.5.3)

We can write $B_m$ in terms of the correlation function $R$:

$$\begin{aligned} B_m = &\sum_{i=1}^{m} [f_i^2 (R(t_i,t_i) - 2R(t_{i-1},t_i) + R(t_{i-1},t_{i-1}))] \\ &+ 2 \sum_{i\ne j, i,j\le m} f_i f_j [R(t_i,t_j) - R(t_{i-1},t_j) - R(t_i,t_{j-1}) + R(t_{i-1},t_{j-1})]. \end{aligned} \tag{2.5.4}$$

Put

$$\delta_{ii} := R(t_i,t_i) - 2R(t_{i-1},t_i) + R(t_{i-1},t_{i-1})$$

and

$$\delta_{ij} := R(t_i,t_j) - R(t_{i-1},t_j) - R(t_i,t_{j-1}) + R(t_{i-1},t_{j-1}).$$

Then condition (2.5.3) is equivalent to the property that the matrix $(\delta_{ij})_{i,j\le m}$ is positive definite for each $m \ge 1$. Assume that condition (2.5.2) is valid for the process $X$ and assume that $f \in L_2[0,T]$. Then there exists $f^n \in L^s$ such that $\|f^n - f\|_{L_2[0,T]} \to 0$ as $n \to \infty$. Moreover, the sequence $(f^n, X)$

is a Cauchy sequence in the space $(E^s, \|\cdot\|_{E^s})$, where $E^s$ is the subspace of $L^s$ consisting of deterministic simple functions $f$. Complete $E^s$ with respect the norm $\|\cdot\|_{E^s}$ and denote this Banach space by $\overline{E}$. Define now the integral $\int_0^T f_s dX_s$ as the limit of $(f^n, X)$ in the space $\overline{E}$. We say that $\int_0^T f_s dX_s$ is the generalized Wiener integral with respect to process $X$. Note that $L^s$ in dense in $L_2[0,T]$ and hence also $E^s$ is dense in $\overline{E}$, by using the isometry.

We clarify the connection between random variables and Wiener integrals defined above. Let $\zeta^n$ be a sequence of random variables of the form

$$\zeta^n := (f^n, X)$$

with some $f^n \in L^s$. Assume now that $\zeta = P\text{-}\lim_n \zeta^n$ and $\|f - f^n\|_{L_2[0,T]} \to 0$, $n \to \infty$. We show later that it may happen that $P\{|\zeta| < \infty\} < 1$ or even $P\{|\zeta| < \infty\} = 0$. But even in the above situation the limit

$$\int_0^T f_s dX_s = \overline{E}\text{-}\lim_n (f^n, X)$$

defines the generalized Wiener integral. In this kind of situation we say that the random variable $\zeta$ is one of the representatives of $\int_0^T f_s dX_s$ in the space of random variables and $\int_0^T f_s dX_s$ is one of the representatives of the random variable $\zeta$ in the space $\overline{E}$: write this as $\zeta \leftrightarrow \int_0^T f_s dX_s$. It is easy to check that if $X$ is a process with non-correlated increments and with the property

$$E\,X_t^2 > E\,X_s^2 \tag{2.5.5}$$

where $s < t$, then condition (2.5.2) is satisfied. Note first that condition (2.5.5) is equivalent to the condition $E(X_t - X_s)^2 > 0$ for $s < t$. Since the process $X$ has non-correlated increments, we have that

$$E\Big(\sum_{i=1}^m f_i \Delta X_i\Big)^2 = \sum_{i=1}^m f_i^2 E(\Delta X_i)^2 = 0$$

if and only if $f_i = 0, i \le m$. Note that if $X$ is a square integrable martingale and $EX_t^2 > EX_s^2, s < t$, then (2.5.2) is satisfied.

Similarly, if $X$ is a stationary process with so-called orthogonal vector measure $\varphi(d\lambda)$ such that the spectral measure $F(d\lambda) := E|\varphi(d\lambda)|^2$ is equivalent to the Lebesgue measure, then condition (2.5.2) is satisfied.

If the process $X$ is the standard Brownian motion $B$, then

$$\|(f, B)\|_{E^s} = E(f, B)^2 = \|f\|_{L_2[0,T]}$$

and then the limits of simple integrals $(f^{(n)}, B)$ in the space $\overline{E}$ and in $L_2(P)$ are the same. Similarly, if the process $X$ is a continuous square integrable martingale $M$ with the angle bracket $\langle M\rangle_t = \int_0^t a_s ds$, where $1/K \le Ea_s \le K$, the limits in the space $\overline{E}$ and $L_2(P)$ are the same.

**First-order Stochastic Integrals with Respect to $X$**

Let $\mathcal{F} := \{\mathcal{F}_t, t \in [0,T]\}$ be a filtration on $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions of right continuity and completeness.

The notation $X \in \mathcal{F}$ means that $X_t$ is $\mathcal{F}_t$ measurable. So, let $X \in \mathcal{F}$ be a process and introduce the space $G^s$ of random variables $\xi$:

$$\xi = \sum_{i=1}^{m} f_i \Delta X_i$$

where $f_i \in \mathcal{F}_{t_{i-1}}$ and $f_i \in L_2(P), 1 \le i \le m, m \ge 1$. Let $f$ be as above, i.e., $f \in L^s$ and the coefficients $f_i, 1 \le i \le m$ satisfy $f_i \in \mathcal{F}_{t_{i-1}}$ and $f_i \in L_2(P)$. Then we can define a surjection $\mathcal{I}$ from $L^s \to G^s$ by

$$\mathcal{I}(f) := (f, X) = \sum_{i=1}^{m} f_i \Delta X_i.$$

Introduce the following semi-norm on $G^s$:

$$\|(f,X)\|_{G^s} := \Big(E \sum_{i=1}^{m} f_i^2 (t_i - t_{i-1})\Big)^{1/2}. \tag{2.5.6}$$

It is easy to check that the condition

$$(f,X) = 0 \ P\text{-a.s. if and only if } f_i = 0 \ P\text{-a.s. for } 1 \le i \le m \tag{2.5.7}$$

is a necessary and a sufficient condition for $\mathcal{I}$ to be a bijection and $\|\cdot\|_{G^s}$ to be a norm.

Let $X$ be a square integrable process, which satisfies (2.5.7). Now let $f$ be a predictable process with $E\int_0^T f_s^2 ds < \infty$. Then there exist processes $f^n \in L^s$ such that

$$E\int_0^T (f_s - f_s^n)^2 ds \to 0$$

as $n \to \infty$. Now $L^s$ is the space of elementary "predictable" processes $g$, where $g_t := \sum_{i=1}^{m} f_i \mathbf{1}_{[t_{i-1}, t_i)}(t)$, and $f_i \in \mathcal{F}_{t_{i-1}}, 1 \le i \le m$. Complete again the space $G^s$ with respect to the norm $\|\cdot\|_{G^s}$. The integral $\int_0^T f_s dX_s =: \mathcal{I}(f)$ is defined using the extension of the isometry $\mathcal{I}$ on the completed Banach space $\overline{G}$. The sequence $f^n$ is a Cauchy sequence with respect the norm $\|\cdot\|_{\overline{G}}$ and the integral $\int_0^T f_s dX_s$ is the limit of the elementary integrals $(f^n, X)$ in the space $(\overline{G}, \|\cdot\|_{\overline{G}})$. We say that the integral $\int_0^T f_s dX_s$ defined for predictable $f \in L_2^{pr}(P \otimes \lambda)$ is the first order generalized stochastic integral with respect to the process $X$. Later we will use the notation $\int_0^T f_s dX_s^{(1)}$ for this integral. If $\zeta^n$ be a sequence of random variables of the form

$$\zeta^n := (f^n, X)$$

with some $f^n \in L^s$ and assume that $\zeta = P\text{-}\lim_n \zeta^n$ and $\|f - f^n\|_{L_2^{pr}(P\otimes\lambda)} \to 0$, $n \to \infty$. Hence also

$$\int_0^T f_s dX_s = \overline{G}\text{-}\lim_n (f^n, X).$$

It may happen that $P\{|\zeta| < \infty\} < 1$ or even $P\{|\zeta| < \infty\} = 0$. Again the random variable $\zeta$ is one of the representatives of the integral $\int_0^T f_s dX_s^{(1)}$ in the space of random variables and $\int_0^T f_s dX_s^{(1)}$ is one of the representatives of the random variable $\zeta$ in the space $\overline{G}$: write this again as $\zeta \leftrightarrow \int_0^T f_s dX_s^{(1)}$. The first-order integral is linear: $(af + bg, X) = a(f, X) + b(g, X)$.

### Higher-order Stochastic Integrals with Respect to *X*

Let $(X, \mathcal{F})$ be again a stochastic process defined on $(\Omega, \mathcal{F}, P)$. Introduce the space $G^{s,k}$ of the random variables $\xi$:

$$\xi := \sum_{i=1}^m f_i (\Delta X_i)^k$$

where $k > 1, f_i \in \mathcal{F}_{t_{i-1}}, f_i \in L_2(P), 1 \le i \le m$. If $f \in L^s$ is a predictable step function, define a surjection $\mathcal{I}^k$ from $L^s$ to $G^{s,k}$ by putting

$$\mathcal{I}^k(f) := (f, X^{(k)}) := \sum_{i=1}^m f_i (\Delta X_i)^k.$$

We suppose that any simple function has different values on the adjoining segments of the partition. With this assumption only one partition corresponds to a simple function, we have only one zero function and $\mathcal{I}^k$ is a surjection.

Introduce the following semi-norm on $G^{s,k}$:

$$\left\|(f, X^{(k)})\right\|_{G^{s,k}} := \Big(E \sum_{i=1}^m f_i^2 (t_i - t_{i-1})\Big)^{1/2} = \|f\|_{L_2(P\otimes\lambda)}.$$

Let $f$ and $g$ be simple predictable processes, defined with respect to different partitions $\pi_f$ and $\pi_g$. Consider $f + g$ on the partition $\pi := \pi_f \cup \pi_g$, put $(f, X^{(k)}) + (g, X^{(k)}) := (f + g, X^{(k)})$ and see that

$$\left\|(f, X^{(k)}) + (g, X^{(k)})\right\|_{G^{s,k}} \le \left\|(f, X^{(k)})\right\|_{G^{s,k}} + \left\|(g, X^{(k)})\right\|_{G^{s,k}}. \tag{2.5.8}$$

Again it is easy to check that the condition

$$(f, X^{(k)}) = 0 \ P\text{-a.s. if and only if } f_i = 0 \text{ for } 1 \le i \le m,$$
$$\text{when } f \in L^s,\ f = \sum_{i=1}^m f_i \mathbf{1}_{[t_{i-1}, t_i)}(\cdot) \tag{2.5.9}$$

is a necessary and sufficient condition for $\mathcal{I}^k$ to be a bijection, for $G^{s,k}$ to be a linear space and for $\|\cdot\|_{G^{s,k}}$ to be a norm.

If $f$ is a predictable process from $L_2^{pr}(P \otimes \lambda)$, take $f^n \in L^s$ such that $\|f - f^n\|_{L_2(P\otimes\lambda)} \to 0$. Assume that property (2.5.9) holds for the process $X$ with some $k > 1$. Define the integral $\int_0^T f_s dX_s^{(k)} := \mathcal{I}^k(f)$ as the limit of $(f^n, X^{(k)})$ in the completed Banach space $(\overline{G}^k, \|\cdot\|_{\overline{G}^k})$, where $\overline{G}^k$ is the completion of $G^{s,k}$ with respect to norm $\|\cdot\|_{G^{s,k}}$. We say that such an integral $\int_0^T f_s dX_s^{(k)}$ is the $k$th order generalized stochastic integral of $f$ with respect to the process $X$.

Assume now that property (2.5.9) holds for all $k \le N$. Define the Banach space $G^N$ by

$$G^N := \overline{G}^1 \times \overline{G}^2 \times \cdots \times \overline{G}^N$$

and define the norm in $G^N$ by

$$\|\cdot\|_{G^N} := \sum_{k=1}^{N} \|\cdot\|_{\overline{G}^k} .$$

In view of (2.5.8), $\|\cdot\|_{G^N}$ satisfies the triangle inequality and hence it is really a norm.

The elements $g \in \overline{G}^N$ have the form

$$g = \sum_{k=1}^{N} \int_0^T f_k(s) dX_s^{(k)}$$

where $f_k$ is a predictable process from $L_2(P \otimes \lambda)$. Note also that there is a bijection between such a $g$ from $\overline{G}^N$ and $(f_1, \dots, f_N) \in \otimes_{k=1}^N L_2^{pr}(P \otimes \lambda)$ equipped with the norm $\sum\limits_{k=1}^{N} \|f_k\|_{L_2(P\otimes\lambda)}$.

The following examples clarify the definition of the generalized integrals of higher order. We assume that the process $X$ satisfies property (2.5.9) for each $1 \le k \le N$ below.

*Processes with bounded variation.* Assume that the process $X$ is a continuous process with bounded variation and consider the random variables $X_T^m$, where

$$X_T^m := \sum_{l=1}^{N} \sum_{k=1}^{m} (\Delta X_k)^l.$$

When $|\pi| \to 0$ we have that $X_T^m \xrightarrow{P} X_T$ and the right-hand side converges in the space $\overline{G}^N$ towards the element

$$\sum_{l=1}^{N} \int_0^T dX_s^{(l)}.$$

Here the random variable $X_T$ is a representative of the integral $\int_0^T dX_s^{(1)}$ and zero is a representative of the sum $\sum\limits_{l=2}^{N} \int_0^T dX_s^{(l)}$.

*Standard Brownian Motion.* Assume that $X$ is a standard Brownian motion, $X = B$. Define again the random variable $X_T^m$ by

$$X_T^m := \sum_{l=1}^{N} \sum_{k=1}^{m} (\Delta B_k)^l.$$

Now, when $|\pi| \to 0$, $X_T^m \xrightarrow{P} B_T + T$, so the constant $T$ is a representative of the integral $\int_0^T dB_s^{(2)}$ and zero is a representative of the sum $\sum\limits_{l=3}^{N} \int_0^T dB_s^{(l)}$.

### 2.5.3 Generalized Integrals with Respect to fBm

#### Fractional Brownian Motion and Property (2.5.7)

**Theorem 2.5.1.** *Property* (2.5.7) *holds for fBm* $B^H, H \in (0,1)$.

*Proof.* Assume that $\sum\limits_{i \le m} f_i \Delta B_i^H = 0$ almost surely. Assume that $m_0$ is the largest index for which $P\{f_{m_0} \ne 0\} > 0$. Then from presentations (1.8.17)–(1.8.18) we have

$$\begin{aligned}
\Delta B_{m_0}^H &= \int_{t_{m_0-1}}^{t_{m_0}} m_H(t_{m_0}, s) dW_s + \int_0^{t_{m_0-1}} (m_H(t_{m_0}, s) - m_H(t_{m_0-1}, s)) dW_s \\
&= A_{m_0} + B_{m_0},
\end{aligned}$$

For the term $B_{m_0}$ we have $B_{m_0} \in \mathcal{F}_{t_{m_0-1}}$. Put $\Omega_c := \{\omega : |f_i| \le c, i \le m_0\}$. Then $\Omega_c \in \mathcal{F}_{t_{m_0-1}}$ and

$$\sum_{i=1}^{m_0} \mathbf{1}_{\Omega_c} f_i \Delta B_i^H = \sum_{i=1}^{m} \mathbf{1}_{\Omega_c} f_i \Delta B_i^H = 0.$$

Hence we can conclude the following:

$$\begin{aligned}
0 &= E\Big( \sum_{i=1}^{m_0} \mathbf{1}_{\Omega_c} f_i \Delta B_i^H \Big)^2 \\
&= E\bigg( \Big( \sum_{i \le m_0 - 1} \mathbf{1}_{\Omega_c} f_i \Delta B_i^H \Big) + f_{m_0} \mathbf{1}_{\Omega_c} B_{m_0-1} + f_{m_0} A_{m_0} \bigg)^2 .
\end{aligned} \tag{2.5.10}$$

The right-hand side of (2.5.10) is equal to

$$E\Big(\sum_{i\le m_0-1}(f_i\Delta B_i^H\mathbf{1}_{\Omega_c})+f_{m_0}\mathbf{1}_{\Omega_c}B_{m_0-1}\Big)^2$$
$$+E\Big(f_{m_0}^2\mathbf{1}_{\Omega_c}\int_{t_{m_0-1}}^{t_{m_0}}(B^H(t_{m_0},s))^2ds\Big).$$

Hence, from (2.5.10), since

$$\int_{t_{m_0-1}}^{t_{m_0}}(B^H(t_{m_0},s))^2ds>0$$

we have that $f_{m_0}\mathbf{1}_{\Omega_c}=0$ almost surely for any $c>0$ and so $f_{m_0}=0$ $P$-a.s. This shows that condition (2.5.7) is fulfilled. Hence $f_i=0$ for all $i\le m$. □

## Fractional Brownian Motions and Property (2.5.9)

**Theorem 2.5.2.** *Property* (2.5.9) *holds for fBm* $B^H, H\in(0,1)$.

*Proof.* We know from Theorem 2.5.1 that the claim holds for $k=1$. Assume now that $k>1$ and let $m_0, A_{m_0}, B_{m_0}$ and $W$ be as in the proof of Theorem 2.5.1. Put $f_i^c:=\mathbf{1}_{\Omega_c}f_i$. Note that $f_i^c\in\mathcal{F}_{t_{m_0-1}}$ for $i\le m_0$. Denote by $\chi$ the random variable

$$\chi:=\sum_{i=1}^{m_0-1}f_i^c(\Delta B_i^H)^k.$$

For the random variable $\chi$ we have that $\chi\in\mathcal{F}_{t_{m_0-1}}$, and this fact is used below. Assume that $\sum_{i\le m}f_i(\Delta B_i^H)^k=0$. With the above notation we have from this assumption that also

$$\chi+f_{m_0}^c\sum_{r=0}^{k}\binom{k}{r}(B_{m_0})^{k-r}(A_{m_0})^r=0. \tag{2.5.11}$$

Write the expression in (2.5.11) as

$$\begin{aligned}&\left(\chi+f_{m_0}^c\sum_{0\le r\le k,\ r\text{ even}}\binom{k}{r}(B_{m_0})^{k-r}(A_{m_0})^r\right)\\&+\left(f_{m_0}^c\sum_{0\le r\le k,\ r\text{ odd}}\binom{k}{r}(B_{m_0})^{k-r}(A_{m_0})^r\right)=:\chi_1+\chi_2.\end{aligned} \tag{2.5.12}$$

The random variable $A_{m_0}$ is a Gaussian random variable with zero expectation and hence for odd $r$ $E(A_{m_0})^r=0$ and by conditioning on $\mathcal{F}_{t_{m_0-1}}$ in (2.5.12) it is easy to see that $E(\chi_1\chi_2)=0$. So from this we can conclude that $E\chi_2^2=0$, using also (2.5.11) and (2.5.12). But

$$\chi_2^2 = f_{m_0}^2(\gamma_1 + \gamma_2)$$

with

$$\gamma_1 := \sum_{0 \le r \le k,\ r \text{ odd}} \left( \binom{k}{r} (B_{m_0})^{k-r} (A_{m_0})^r \right)^2 \tag{2.5.13}$$

and

$$\gamma_2 := \sum_{r \neq q,\ r,q \text{ odd}} \binom{k}{r} \binom{k}{q} (B_{m_0})^{2k-r-q} (A_{m_0})^{r+q}. \tag{2.5.14}$$

All the terms in (2.5.13) are nonnegative and since $r + q$ is even, the same holds for the expression (2.5.14), too. Note also that if $r = 1$, then

$$k^2 (B_{m_0})^{2k-2} (A_{m_0})^2 > 0$$

almost surely. But at the same time $E(f_{m_0}^2(\gamma_1 + \gamma_2)) = 0$. Hence $f_{m_0} = 0$ almost surely. From this follows that $f_i = 0$ almost surely for all $i \le m$. We have shown that fBm $B^H$ satisfies property (2.5.9) for all $k \ge 1$. $\square$

### Some Properties of the Generalized Integrals

In this subsection we discuss some of the properties of the generalized integrals. At this stage we have results mostly on Wiener integrals.

Assume that $B^H$ is again an fBm with index $H$. Take

$$f_s^n := n^\gamma \mathbf{1}_{(T/2-1/2n, T/2+1/2n]}(s).$$

Then $\|f^n\|^2_{L_2[0,T]} = n^{2\gamma-1}$. If $H \in (1/2, 1), 1/2 < \gamma < H$, then $\|f^n\|_{L_2[0,T]} \to \infty$ and the generalized integral does not exist, but $E((f^n, B^H))^2 = n^{2\gamma-2H} \to 0$, and the limit exists in $L_2(P)$. If $H < \gamma < 1/2$, then $E((f^n, B^H))^2 \to \infty$, but $\|f^n\|_{L_2[0,T]} \to 0$. Hence the integral exists in $\overline{G}$ and it is $= 0$, but the limit does not exist in $L_2(P)$. Note also that here we have that $|(f^n, B^H)| \xrightarrow{P} \infty$.

*$L_2$-integrals and Wiener integrals, $H \in (1/2, 1)$.* If $B^H$ is an fBm with Hurst index $H \in (1/2, 1)$, then according to (1.9.2) we have the following estimate for $L_2$-integral, valid for any $p > 0$:

$$E\left|\int_0^T f_s dB_s^H\right|^p \le c_{H,p} \|f\|^p_{L_{\frac{1}{H}}[0,T]}. \tag{2.5.15}$$

Hence, if $(f^{(n)}, B^H)$ converges in $\overline{G}$, it also converges in $L_2(P)$.

*$L_2$-integrals and Wiener integrals, $H \in (0, 1/2)$.* Before the continuation, we prove the following theorem, which is the opposite to (2.5.15).

**Theorem 2.5.3.** *Let $f \in L^s$ and $B^H$ is an fBm with Hurst index $H \in (0, 1/2)$. Then*

$$E\left|\int_0^T f_s dB_s^H\right|^2 \ge C \|f\|^2_{L_2[0,T]}. \tag{2.5.16}$$

*Proof.* If $f \in L^s$ and $(f, B^H) = \sum_i f_i \Delta B_i^H$, then

$$E(f, B^H)^2 = \sum_i (f_i^2 E\Delta B_i^H)^2 + \sum_{i \neq k} f_i f_k E(\Delta B_i^H \Delta B_k^H). \tag{2.5.17}$$

But $E(\Delta B_i^H \Delta B_k^H) < 0$ and hence

$$f_i f_k E(\Delta B_i^H \Delta B_k^H) \geq |f_i||f_k| E(\Delta B_i^H \Delta B_k^H).$$

Use this in (2.5.17) to obtain the inequality

$$E(f, B^H)^2 \geq E\left(\sum_i |f_i| \Delta B_i^H\right)^2.$$

Hence we can assume that $f_i \geq 0$ for all $i \leq n$ in proving (2.5.16).

Denote by $\mathcal{D}(\mathbb{R})$ the space of functions $f$ with the two properties: $f \in C^\infty(\mathbb{R})$ and $f$ has compact support.

Let $\phi \in \mathcal{D}(\mathbb{R})$. Then the Fourier transform $\widehat{\phi}$ of $\phi$ belongs to $S(\mathbb{R}) \subset \mathcal{F}_H \subset L_2^H(\mathbb{R})$ (see Lemma 1.6.8), and moreover,

$$E\left|\int_{\mathbb{R}} \phi_t dB_t^H\right|^2 = E\left|\int_{\mathbb{R}} \phi'(t) B_t^H dt\right|^2 = c_H \int_{\mathbb{R}} |\widehat{\phi}(\lambda)||\lambda|^{-2\alpha} d\lambda, \tag{2.5.18}$$

where $c_H$ is some constant.

We want to prove that there exists a sequence $(\phi^n)_{n \geq 1}, \phi^n \in \mathcal{D}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} (\phi^n)'(t) B_t^H dt \stackrel{L_2(P)}{\to} (f, B^H). \tag{2.5.19}$$

To prove (2.5.19) it is sufficient to prove it for $f \in L^s$, $f_u = a\mathbf{1}_{[s,t)}(u)$, $s < t \leq T$ and $a > 0$. Take $\phi^n \in \mathcal{D}(\mathbb{R})$ such that $\text{supp}(\phi^n) \subset [s - 1/n, t + 1/n]$ and $\phi^n = a$ on $[s + 1/n, t - 1/n]$. Then

$$\int_{\mathbb{R}} (\phi^n)'(u) B_u^H du = \int_{t-1/n}^{t+1/n} (\phi^n)'(u) B_u^H du + \int_{s-1/n}^{s+1/n} (\phi^n)'(u) B_u^H du$$

and, for example,

$$\left|aB_{t+1/n}^H - \int_{t-1/n}^{t+1/n} (\phi^n)'(u) B_u^H du\right| \leq \left|\int_{t-1/n}^{t+1/n} (\phi^n)'(u)(B_{t+1/n}^H - B_u^H) du\right|$$
$$\leq a \sup_{u \in [t-1/n, t+1/n]} |B_{t+1/n}^H - B_u^H|.$$

From self-similarity of $B^H$ and Remark 1.10.7 with $f = 1, T = 2/n$

$$\sup_{u \in [t-1/n, t+1/n]} |B_{t+1/n}^H - B_u^H| \xrightarrow{L_2(P)} 0$$

and so

$$E(f, B^H)^2 = \lim_n \int_{\mathbb{R}} |\widehat{\phi}^n(\lambda)||\lambda|^{-2\alpha} d\lambda.$$

Since for any $\lambda \in \mathbb{R}$ $\widehat{f}(\lambda) = \lim_{n\to\infty} \widehat{\phi}^n(\lambda)$, we have, using the Fatou lemma and relation (2.5.18),

$$\int_{-\infty}^{\infty} |\widehat{f}(\lambda)|^2 |\lambda|^{-2\alpha} d\lambda \le \liminf_{n\to\infty} \int_{-\infty}^{\infty} |\widehat{\phi}^n(\lambda)|^2 |\lambda|^{-2\alpha} d\lambda = E\left|\sum_i f_i \Delta B_i^H\right|^2 .$$

We have that

$$\begin{aligned} &\int_{-\infty}^{\infty} |\widehat{f}(\lambda)|^2 |\lambda|^{-2\alpha} d\lambda \\ &\ge \varepsilon^{-2\alpha} \int_{|\lambda|>\varepsilon} |\widehat{f}(\lambda)|^2 d\lambda + \int_{|\lambda|\le\varepsilon} |\widehat{f}(\lambda)|^2 |\lambda|^{-2\alpha} d\lambda. \end{aligned} \tag{2.5.20}$$

Put $\rho(\lambda) := |\lambda|^{-\alpha} \mathbf{1}_{[-\varepsilon,\varepsilon]}(\lambda)$. Since $H \in (0, 1/2)$, we have that $\rho \in L_1(\mathbb{R})$. Also,

$$\widehat{\rho}(t) := \int_{-\infty}^{\infty} e^{it\lambda} \rho(\lambda) d\lambda = \int_{-\varepsilon}^{\varepsilon} \cos(t\lambda)|\lambda|^{-\alpha} d\lambda.$$

This integral is finite and hence $\rho(\cdot)$ is the Fourier transform of $\widehat{\rho}(\cdot)$. Use the Parceval identity to obtain

$$\begin{aligned} &\int_{|\lambda|<\varepsilon} |\widehat{f}(\lambda)|^2 |\lambda|^{-2\alpha} d\lambda \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(s) \left( \int_{-\varepsilon}^{\varepsilon} \cos((t-s)\lambda)|\lambda|^{-\alpha} d\lambda \right) ds \right|^2 dt. \end{aligned} \tag{2.5.21}$$

Estimate the right-hand side of (2.5.21) from below by

$$\int_{-1}^{1} \left| \int_{0}^{T} f(s) \left( \int_{-\varepsilon}^{\varepsilon} \cos((t-s)\lambda)|\lambda|^{-\alpha} d\lambda \right) ds \right|^2 dt. \tag{2.5.22}$$

Take in (2.5.22) such an $\varepsilon$ that $\varepsilon(T+1) \le \pi/3$. Then $\cos((t-s)\lambda) \ge 1/2$ and the left-hand side of inequality (2.5.21) can be estimated from below, using the estimate (2.5.22) and the chosen $\varepsilon$ by the expression

$$\frac{1}{2} \left| \int_0^T f(s) ds \right|^2 \left( \int_{-\varepsilon}^{\varepsilon} |\lambda|^{-\alpha} d\lambda \right)^2 = \frac{2\varepsilon^{2-2\alpha}}{(1-\alpha)^2} |\widehat{f}(0)|^2,$$

but since $f$ is nonnegative, we also have the estimate $|\widehat{f}(0)| \ge |\widehat{f}(\lambda)|$. Therefore, from the above estimates we obtain

$$\textstyle\int_{|\lambda|\le\varepsilon} |\widehat{f}(\lambda)|^2 |\lambda|^{-2\alpha} d\lambda \ge \frac{\varepsilon^{1-2\alpha}}{(1-\alpha)^2} \int_{-\varepsilon}^{\varepsilon} |\widehat{f}(\lambda)|^2 d\lambda. \tag{2.5.23}$$

Take $C = \min\{\varepsilon^{-2\alpha}, \varepsilon^{1-2\alpha}/(1-\alpha)^2\}$ and use (2.5.23) in (2.5.20) to obtain

$$\int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 |\lambda|^{1-2H} d\lambda \geq C \int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 d\lambda = C_1 \|f\|^2_{L_2[0,T]} \,.$$

□

*Random variables and the corresponding integrals.* Assume first that $H \in (1/2, 1)$. Let $f^n \in L^s$ be such that $f = L_2(P)\text{-}\lim f^n$. Put $\zeta^n := (f^n, B^H)$ and assume that $\zeta := L_2(P)\text{-}\lim \zeta^n$. Let $g^n \in L^s$ be another sequence such that $\zeta = L_2(P)\text{-}\lim(g^n, B^H)$. Use the beginning of this subsection to conclude that the corresponding integral may not exist, and hence the representative of the random variable $\zeta$ need not to be unique in the space $\overline{E}$. On the other hand, it follows from inequality (2.5.15) that the integral $\int_0^T f_s dB_s^H$ has only one random variable as a representative.

If $H \in (0, 1/2)$ then the picture is the opposite. Namely, a random variable $\zeta$ can represent only one Wiener integral; this follows from Theorem 2.5.3. On the other hand, the zero Wiener integral has at least two representatives as extended random variables, namely $\zeta = 0$ and $\zeta = \infty$; this follows again from the beginning of this subsection.

## 2.6 Stochastic Fubini Theorem for Stochastic Integrals w.r.t. Fractional Brownian Motion

In this section we prove the generalization of stochastic Fubini theorem for the Wiener integrals with respect to fBm (Theorem 1.13.1). First, we consider pathwise integrals and the result is for the most part based on Hölder properties of fBm and of corresponding integrals. Then, the extension to Wick and Skorohod integration is more or less evident, due to comparison results of Sections 2.3 and 2.4.

**Definition 2.6.1.** The nonrandom function $f : \mathbb{R} \to \mathbb{R}$ is called piecewise Hölder of order $\alpha$ on the interval $[T_1, T_2] \subset \mathbb{R}$ ($f \in C^\alpha_{pw}[T_1, T_2]$), if there exists a finite set of disjoint subintervals $\{[a_i, b_i), 1 \leq i \leq N \mid \bigcup_{i=1}^N [a_i, b_i] \cup T_2 = [T_1, T_2]\}$ and the function $f \in C^\alpha[a_i, b_i)$ for $1 \leq i \leq N$.

As before, we denote

$$\|f\|_{C^\alpha[a_i,b_i)} := \sup_{a_i \leq t < b_i} |f(t)| + \sup_{a_i \leq s < t < b_i} \frac{|f(t) - f(s)|}{|t-s|^\alpha}.$$

**Definition 2.6.2.** For $f \in C^\alpha_{pw}[T_1, T_2]$, let

$$\|f\|_{C^\alpha_{pw}[T_1,T_2]} = \max_{1 \leq i \leq N} \|f\|_{C^\alpha[a_i,b_i)} \,.$$

Let $f \in C^\alpha[a,b]$, $g \in C^\beta[a,b]$ with $\alpha+\beta>1$. Then we know that the Riemann–Stieltjes integral exists,

$$\int_a^b f(t)dg(t) := \lim_{|\pi_n|\to 0} \sum_{k=0}^{k_n-1} f(t_k^n)\Delta g(t_k^n), \tag{2.6.1}$$

where $\pi_n = \{a = t_k^0 < t_k^1 < \cdots < t_k^{k_n} = b\}, \Delta g(t_k^n) = g(t_{k+1}^n) - g(t_k^n)$, $\pi_n \subset \pi_{n+1}$.

Moreover, according to (FdP01, Theorem 2.1), there exist the sequences $\{f_n, g_n\} \subset C^{(1)}[a,b]$ such that $\|f_n - f\|_{C^\alpha[a,b]} \to 0,\ n \to \infty$, $\|g_n - g\|_{C^\beta[a,b]} \to 0,\ n \to \infty$.

We shall use some bounds for integrals involving Hölder functions. They are proved in the next lemma.

**Lemma 2.6.3.** *Let $f \in C^\alpha[a,b]$, $g \in C^\beta[a,b]$, $\alpha+\beta>1$, $f_m, g_m \in C^1[a,b], m \geq 1$ and $\|f_m - f\|_{C^\alpha[a,b]} \to 0$, $\|g_m - g\|_{C^\beta[a,b]} \to 0$, as $m \to \infty$.*

*Then 1)$\int_a^b f(t)dg(t) = \lim_{m\to\infty} \int_a^b f_m(t)g_m'(t)dt$;*

*2) the following estimate holds:*

$$\left|\int_a^b f(t)dg(t)\right| \leq C\,\|f\|_{C^\alpha[a,b]} \cdot \|g\|_{C^\beta[a,b]} \cdot ((b-a)^{1+\varepsilon} \vee (b-a)^\beta);$$

*3) if $f(a)=0$, then*

$$\left|\int_a^b f(t)dg(t)\right| \leq C\,\|f\|_{C^\alpha[a,b]} \cdot \|g\|_{C^\beta[a,b]} \cdot (b-a)^{1+\varepsilon}, \tag{2.6.2}$$

*where $0<\varepsilon<\alpha+\beta-1$, $C>0$ is a constant not depending on $\alpha$ and $\beta$.*

*Proof.* 1) Evidently,

$$\begin{aligned}\left|\int_a^b f(t)dg(t) - \int_a^b f_m(t)g_m'(t)dt\right| &\leq \left|\int_a^b f(t)dg(t) - \sum_{k=1}^{k_n} f(t_k^n)\Delta g(t_k^n)\right| \\ &+ \left|\int_a^b f_m(t)g_m'(t)dt - \sum_{k=1}^{k_n} f_m(t_k^n)\Delta g_m(t_k^n)\right| \\ &+ \left|\sum_{k=1}^{k_n} f(t_k^n)\Delta g(t_k^n) - \sum_{k=1}^{k_n} f_m(t_k^n)\Delta g_m(t_k^n)\right|.\end{aligned}$$

According to (2.6.1), for any fixed $\delta>0$ we can choose $\pi_n$ in such a way that

$$\left|\int_a^b f(t)dg(t) - \sum_{k=1}^{k_n} f(t_k^n)\Delta g(t_k^n)\right| < \delta. \tag{2.6.3}$$

Further, according to (FdP01, Corollary 20),

$$\left|\int_a^b f_m(t)g_m'(t)dt - \sum_{k=1}^{k_n} f_m(t_k^n)\Delta g_m(t_k^n)\right| \le C\,|\pi_n|^{\varepsilon} \cdot \|f_m\|_{C^{\alpha'}[a,b]} \cdot \|g_m\|_{C^{\beta'}[a,b]}\,, \tag{2.6.4}$$

where $0 < \alpha' < \alpha$, $0 < \beta' < \beta$, and $\alpha' + \beta' = 1 + \varepsilon$. If $\|f_n - f\|_{C^{\alpha}[a,b]} \to 0$, $m \to \infty$, then $\|f_m - f\|_{C^{\alpha'}[a,b]} \to 0$, $m \to \infty$ for $0 < \alpha' < \alpha$, and $\|f_m\|_{C^{\alpha'}[a,b]} \le C_1$, where $C_1$ does not depend on $m \ge 1$. Similarly, $\|g_m\|_{C^{\beta'}[a,b]} \le C_2$. From these bounds and from (2.6.4) we obtain that

$$\left|\int_a^b f_m(t)g_m'(t)dt - \sum_{k=1}^{k_n} f_m(t_k^n)\Delta g_m(t_k^n)\right| \le C_3\,|\pi_n|^{\varepsilon}\,. \tag{2.6.5}$$

Choose such $n$ that (2.6.3) holds and also $C_3\,|\pi_n|^{\varepsilon} < \delta$; then for such fixed $n$ we can choose such $m$ that

$$\left|\sum_{k=1}^{k_n} f(t_k^n)\Delta g(t_k^n) - \sum_{k=1}^{k_n} f_m(t_k^n)\Delta g_m(t_k^n)\right| < \delta. \tag{2.6.6}$$

It is possible since $\sup_{t\in[a,b]} |g_m(t) - g(t)| \le \|g_m - g\|_{C^{\beta'}[a,b]} \to 0$, and the same is true for $f_m$.

The proof of the first statement follows now from (2.6.3)–(2.6.6).

The third statement follows from 1) and (FdP01, Lemma 19), which states that the bound (2.6.2) holds for any $f \in C_0^{(1)}[a,b]$ (it means that $f \in C^{(1)}[a,b]$ and $f(a) = 0$) and $g \in C^{(1)}[a,b]$.

The second statement follows from 1) and (FdP01, Theorem 22). Indeed, according to 3)

$$\left|\int_a^b \Big(f(t) - f(0)\Big)dg(t)\right| \le C\,\|f\|_{C^{\alpha}[a,b]} \cdot \|g\|_{C^{\beta}[a,b]} \cdot (b-a)^{1+\varepsilon},$$

whence

$$\left|\int_a^b f(t)dg(t)\right| \le C\,\|f\|_{C^{\alpha}[a,b]} \cdot \|g\|_{C^{\beta}[a,b]} \cdot ((b-a)^{1+\varepsilon} \vee (b-a)^{\beta}).$$

□

Further we consider $H \in (\frac{1}{2}, 1)$. Let $f \in C_{pw}^{\beta}[a,b]$ with $\beta > 1 - H$. In this case the sum $\sum_{i=1}^{N} \int_{a_i}^{b_i} f(t)dB_t^H$ exists. The next result means that this sum can be represented as a unique integral.

**Lemma 2.6.4.** *Let $f$ be piecewise Hölder of order $\beta > 1 - H$ on the interval $[a,b]$. Then there exists the Riemann–Stieltjes integral*

$$\int_a^b f(u)dB_u^H = \sum_{i=1}^N \int_{a_i}^{b_i} f(u)dB_u^H$$

*and for an arbitrary sequence* $\pi_n$ *of partitions of* $[a,b]$ *it can be represented as a limit*

$$\int_a^b f(u)dB_u^H = \lim_{|\pi_n|\to 0} \sum_{k=1}^{k_n} f(u_k^n)\Delta B_{u_k^n}^H.$$

(*We suppose that* $\bigcup_{i=1}^N [a_i,b_i) = [a,b)$, $[a_i,b_i)$ *are disjoint and* $f \in C^\alpha[a_i,b_i)$).

*Proof.* Put $\pi_n^i := [a_i,b_i) \cap \pi_n$. Evidently, $\left|\pi_n^i\right| \le |\pi_n|$. It follows from boundedness of $f$ and continuity of $B^H$ that

$$\sum_{j:u_j^n \in \pi_n^i} f(u_j^n)\Delta B_{u_j^n}^H \to \int_{a_i}^{b_i} f(u)dB_u^H,$$

even in the case when $\pi_n^i$ does not contain $a_i$ or(and) $b_i$.

Therefore, $\sum_{k:u_k^n\in\pi_n} f(u_k^n)\Delta B_{u_k^n}^H = \sum_{i=1}^N \sum_{k:u_k^n\in\pi_n^i} f(u_k^n)\Delta B_{u_k^n}^H$ $\to \sum_{i=1}^N \int_{a_i}^{b_i} f(u)dB_u^H = \int_a^b f(u)dB_u^H$, as $|\pi_n| \to 0$. □

Let $0 < T_1 < T_2$, $\Phi = \Phi(t,u,\omega) : \mathcal{P}_T := [T_1,T_2]^2 \times \Omega \to \mathbb{R}$ be the random function measurable in all the variables.

**Theorem 2.6.5.** *Let there exist the set* $\Omega' \subset \Omega$ *such that* $P(\Omega') = 1$ *and let for any* $\omega \in \Omega'$ *the function* $\Phi(s,u,\omega)$ *satisfy the conditions:*

*1)* $\forall s \in (T_1,T_2)$ $\Phi(t,\cdot,\omega)$ *is piecewise Hölder of order* $\beta > 1-H$ *in* $u \in [T_1,T_2]$*, and there exists* $C = C(\omega) > 0$ *such that* $\|\Phi(t,\cdot,\omega)\|_{C_{pw}^\beta[T_1,T_2]} \le C$*;*

*2) the function* $\int_{T_1}^{T_2} \Phi(t,u,\omega)dB_u^H$ *is Riemann integrable in the interval* $[T_1,T_2]$.

*Then there exist the repeated integrals*

$$I_1 := \int_{T_1}^{T_2} \Big(\int_{T_1}^{T_2} \Phi(t,u,\omega)dB_u^H\Big)dt \quad \text{and} \quad I_2 := \int_{T_1}^{T_2} \Big(\int_{T_1}^{T_2} \Phi(t,u,\omega)dt\Big)dB_u^H,$$

*and* $I_1 = I_2$ $P$*-a.s.*

*Proof.* We fix $\omega \in \Omega'$ and omit $\omega$ throughout the proof. The integral $\int_{T_1}^{T_2} \Phi(t,u)dB_u^H$ exists according to Lemma 2.6.4 and condition 1); the repeated integral $I_1$ exists according to condition 2). Since $\Phi(t,\cdot)$ is piecewise Hölder, then from the evident bound $\int_{T_1}^{T_2} |\Phi(t,u_1) - \Phi(t,u_2)|\, ds \le C(T_2 - T_1)\,|u_1-u_2|^\alpha$ we obtain that $\int_{T_1}^{T_2} \Phi(t,u)ds$ is piecewise Hölder of order $\alpha$ in $u \in [T_1,T_2]$. Further, since $B^H$ is Hölder up to order $H > \frac{1}{2}$ and $\alpha + H > 1$, the integral $I_2$ also exists. The integral $I_1$ can be presented as a limit of integral sums,

$$I_1 = \lim_{|\pi_n|\to 0} \sum_{k=0}^{k_n-1} \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H \Delta t_k^n. \tag{2.6.7}$$

For any point $t_k^n \in \pi_n$, according to condition 1), there exists a finite number of points $\{u_{1,k} < u_{2,k} < \cdots < u_{l(k),k}\}$ such that $\Phi(\cdot, u)$ is Hölder between them. Denote

$$\begin{aligned}\{T_1 = u_0 < u_1 < u_2 < \cdots < u_{L(n)} = T_2\} \\ := \bigcup_{k=1}^{k_n} \{u_{1,k} < u_{2,k} < \cdots < u_{l(k),k}\} \cup \{T_1, T_2\}.\end{aligned}$$

For any interval $[u_i, u_{i+1}]$ we consider the sequence of partitions $\pi_{i,r}, r \geq 1$ of the form

$$\pi_{i,r} := \{u_i = u_{i,r}^{(0)} < u_{i,r}^{(1)} < \cdots < u_{i,r}^{(m_r)} = u_{i+1}\}, |\pi_{i,r}| \to 0, r \to \infty.$$

Then $\tilde{\pi}_r := \bigcup_{i=0}^{L(n)-1} \pi_{i,r} \cup \{T_1, T_2\} := \{T_1 = u_r^{(0)} < \cdots < u_r^{(N_r)} = T_2\}$ is a partition of interval $[T_1, T_2]$ w.r.t. argument $u$, its diameter $|\tilde{\pi}_r| = \max_{1 \leq i \leq L(n)-1} |\pi|_{i,r}$, and $|\tilde{\pi}_r| \to 0, r \to \infty$.

Estimate the difference $|I_1 - I_2|$:

$$\begin{aligned}|I_1 - I_2| \leq &\left| I_1 - \sum_{k=0}^{k_n-1} \sum_{j=0}^{N_r-1} \Phi(t_k^n, u_r^{(j)}) \Delta B_{u_r^{(j)}}^H \Delta t_k^n \right| \\ &+ \left| I_2 - \sum_{j=0}^{N_r-1} \sum_{k=0}^{k_n-1} \Phi(t_k^n, u_r^{(j)}) \Delta t_k^n \Delta B_{u_r^{(j)}}^H \right| =: \Delta_1^{n,r} + \Delta_2^{n,r}.\end{aligned} \tag{2.6.8}$$

Further,

$$\begin{aligned}\Delta_1^{n,r} \leq &\left| I_1 - \sum_{k=0}^{k_n-1} \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H \cdot \Delta t_k^n \right| \\ &+ \sum_{k=0}^{k_n-1} \left| \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H - \sum_{j=0}^{N_r-1} \Phi(t_k^n, u_r^{(j)}) \Delta B_{u_r^{(j)}}^H \right| \Delta t_k^n.\end{aligned}$$

Since $\Phi$ is piecewise Hölder, then, according to Lemma 2.6.4,

$$\left| \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H - \sum_{j=0}^{N_r-1} \Phi(t_k^n, u_r^{(j)}) \Delta B_{u_r^{(j)}}^H \right| \to 0, r \to \infty.$$

According to (2.6.7), $\left| I_1 - \sum_{k=0}^{k_n-1} \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H \cdot \Delta t_k^n \right| \to 0, n \to \infty$.

Therefore,

$$\lim_{n\to\infty}\lim_{r\to\infty}\Delta_1^{n,r}=0. \tag{2.6.9}$$

Further,

$$\begin{aligned}\Delta_2^{n,r} &\le \left| I_2-\sum_{j=0}^{N_r-1}\int_{T_1}^{T_2}\Phi(t,u_r^{(j)})dt\cdot\Delta B^H_{u_r^{(j)}}\right| \\ &\quad+\left|\sum_{j=0}^{N_r-1}\sum_{k=0}^{k_n-1}\int_{t_k^n}^{t_{k+1}^n}\Big(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)})\Big)dt\cdot\Delta B^H_{u_r^{(j)}}\right|.\end{aligned} \tag{2.6.10}$$

The second term can be expanded as

$$\begin{aligned}&\left|\sum_{k=0}^{k_n-1}\int_{t_k^n}^{t_{k+1}^n}\sum_{j=0}^{N_r-1}\Big(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)})\Big)\Delta B^H_{u_r^{(j)}}dt\right| \qquad (2.6.11)\\ &=\left|\sum_{k=0}^{k_n-1}\sum_{i=0}^{L(N)-1}\int_{t_k^n}^{t_{k+1}^n}\sum_{u_r^{(j)}\in\pi_{i,r}}\Big(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)})\Big)\Delta B^H_{u_r^{(j)}}dt\right|.\end{aligned}$$

Since the function $\Phi(s,u)-\Phi(t_k^n,u)$ is Hölder on any interval $[u_i,u_{i+1})$, we have that

$$\begin{aligned}&\lim_{|\pi_{i,r}|\to 0}\sum_{u_r^{(j)}\in\pi_{i,r}}\Big(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)})\Big)\Delta B^H_{u_r^{(j)}}\\ &\qquad=\int_{u_i}^{u_{i+1}}\Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H.\end{aligned} \tag{2.6.12}$$

Moreover, $\forall\ 0\le i\le L(n)-1$ the sequence $f_i^r(t,t_k^n):=\sum_{u_r^{(j)}\in\pi_{i,r}}\Big(\Phi(t,u_r^{(j)})$ $-\ \Phi(t_k^n,u_r^{(j)})\Big)\Delta B^H_{u_r^{(j)}}$ has the integrable dominant. Indeed, we can use the bounds from (FdP01, Corollary 20), Lemma 2.6.3, and the boundedness of Hölder norms, and obtain that

$$\begin{aligned}|f_i^r(t,t_k^n)| &\le \left|f_i^r(t,t_k^n)-\int_{u_r^{(j)}}^{u_{r+1}^{(j)}}\Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H\right|\\ &\quad+\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}}\Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H\right|\\ &\le C\,|\pi_{i,r}|^{\varepsilon}\cdot\|\Phi(t,\cdot)-\Phi(t_k^n,\cdot)\|_{C[u_r^{(j)},u_{r+1}^{(j)}]^{\beta'}}\cdot\left\|B^H\right\|_{C[u_r^{(j)},u_{r+1}^{(j)}]^{H'}}\end{aligned}$$

$$\begin{aligned} &+\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}} \Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H\right| \\ &\le C+\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}} \Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H\right|, \end{aligned} \tag{2.6.13}$$

where $\beta' < \beta$, $H' < H$ and $\beta' + H' > 1$.

Using the second statement of Lemma 2.6.3 and condition 1) of this theorem, we obtain the bound

$$\begin{aligned} &\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}} \Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H\right| \\ &\qquad \le C\,\|\Phi(t,\cdot)-\Phi(t_k^n,\cdot)\|_{C_{pw}^{\alpha'}[T_1,T_2]}\cdot\left\|B^H\right\|_{C^{H'}[T_1,T_2]} \le C. \end{aligned} \tag{2.6.14}$$

Estimates (2.6.13) and (2.6.14) mean that we can use the Lebesgue dominant convergence theorem and obtain that

$$\lim_{r\to\infty}\int_{t_k^n}^{t_{k+1}^n} f_i^r(t,t_k^n)dt = \int_{t_k^n}^{t_{k+1}^n}\int_{u_i}^{u_{i+1}}\Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H dt,$$

where the integrand $\int_{u_i}^{u_{i+1}}\Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H$ is measurable and bounded in $t$.

Therefore,

$$\begin{aligned} &\lim_{r\to\infty}\sum_{k=0}^{k_n-1}\sum_{i=0}^{L(n)-1}\int_{t_k^n}^{t_{k+1}^n}\sum_{u_r^{(j)}\in\,\pi_{i,r}}\Big(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)})\Big)\Delta B_{u_r^{(j)}}^H dt \\ &\qquad = \sum_{k=0}^{k_n-1}\int_{t_k^n}^{t_{k+1}^n}\int_{T_1}^{T_2}\Big(\Phi(t,u)-\Phi(t_k^n,u)\Big)dB_u^H dt \\ &= \int_{T_1}^{T_2}\Big(\int_{T_1}^{T_2}\Phi(t,u)dB_u^H\Big)dt - \sum_{k=0}^{k_n-1}\int_{T_1}^{T_2}\Phi(t_k^n,u)dB_u^H\,\Delta t_k^n. \end{aligned} \tag{2.6.15}$$

According to condition 2) of this theorem, the integral $\int_{T_1}^{T_2}\Phi(t,u)dB_u^H$ is Riemann integrable in $t$, therefore

$$\lim_{n\to\infty}\sum_{k=0}^{k_n-1}\int_{T_1}^{T_2}\Phi(t_k^n,u)dB_u^H\,\Delta t_k^n = \int_{T_1}^{T_2}\Big(\int_{T_1}^{T_2}\Phi(t,u)dB_u^H\Big)dt. \tag{2.6.16}$$

From Lemma 2.6.4,

$$\left|I_2 - \sum_{r=0}^{L(n)-1}\int_{T_1}^{T_2}\Phi(t,u_j^{(r)})dt\cdot\Delta B_{u_j^{(r)}}^H\right| \to 0, \text{as } n\to\infty. \tag{2.6.17}$$

Now the proof follows from (2.6.8)–(2.6.17). □

Let $I(t) = \int_0^t f(s)dB_s^H$ for some stochastic process $f$ with trajectories from $C^\beta[0,T]$ with $\beta + H > 1$. Consider the integral $\big(H \in (\frac{1}{2}, 1)\big)$ $J_1(t) = \int_0^t l_H(t,s)I(s)ds$ that will appear in connection with the Girsanov theorem and stochastic differential equations in subsections 2.8.2 and 3.2.3, and also, let $J_2(t) = \int_0^t f(u)\Big(\int_u^t l_H(t,s)ds\Big)dB_u^H$.

**Lemma 2.6.6.** *Both the integrals, $J_1$ and $J_2$, exist and $J_1 = J_2$ $P$-a.s.*

*Proof.* It follows from (FdP01) that the trajectories of $I(t)$, $t \in [0,T]$ are Hölder of order $H - \varepsilon$ for any $0 < \varepsilon < H$, whence the existence of $J_1(t)$ follows. Further, elementary calculations

$$\int_{u_1}^{u_2} (t-s)^{-\alpha} s^{-\alpha} ds \le \frac{1}{2}\left[\int_{u_1}^{u_2} (t-s)^{-2\alpha} ds + \int_{u_1}^{u_2} s^{-2\alpha} ds\right] \le (u_2 - u_1)^{1-2\alpha}$$

demonstrate that the function $f(u) \cdot \int_u^t l_H(t,s)ds$ is Hölder up to order $\beta \wedge (1 - 2\alpha) > 1 - H$, and $J_2(t)$ exists. We can present these integrals in the following way:

$$J_1 = \int_0^t \Big(\int_0^t \Phi(s,u)dB_u^H\Big)ds, \ J_2 = \int_0^t \Big(\int_0^t \Phi(s,u)ds\Big)dB_u^H,$$

where $\Phi(s,u) = l_H(t,s)f(u)\mathbf{1}_{\{0 \le u \le s\}}$.

The function $\Phi$ will satisfy both the conditions of Theorem 2.6.5, if we put $T_1 = \delta$ and $T_2 = t - \delta$ for any $0 < \delta < \frac{t}{2}$. In particular, $\Phi(s,\cdot)$ is piecewise Hölder of order $\beta$ on $[\delta, t-\delta]$ with one point $u = s$ of Hölder discontinuity for any $s \in [\delta, t-\delta]$.

Therefore, the following equality holds a.s.:

$$\int_\delta^{t-\delta} l_H(t,s) \int_\delta^s f(u)dB_u^H ds = \int_\delta^{t-\delta} f(u) \int_u^{t-\delta} l_H(t,s)ds dB_u^H.$$

The last equality can be rewritten as

$$J_1 - R_1 = J_2 - R_2, \tag{2.6.18}$$

where

$$R_1 = \int_0^\delta l_H(t,s)\Big(\int_0^s f(u)dB_u^H\Big)ds + \int_\delta^{t-\delta} l_H(t,s)\Big(\int_0^\delta f(u)dB_u^H\Big)ds$$
$$+ \int_{t-\delta}^t l_H(t,s)\Big(\int_0^s f(u)dB_u^H\Big)ds =: R_{11} + R_{12} + R_{13};$$

$$R_1 = \int_0^\delta f(u)\Big(\int_u^t l_H(t,s)ds\Big)dB_u^H + \int_\delta^{t-\delta} f(u)\Big(\int_{t-\delta}^t l_H(t,s)ds\Big)dB_u^H$$
$$+ \int_{t-\delta}^t f(u)\Big(\int_u^t l_H(t,s)ds\Big)dB_u^H =: R_{21} + R_{22} + R_{23}.$$

According to (FdP01, Theorem 22), there exists $C > 0$ such that $\left|\int_0^s f(u)dB_u^H\right| \le Cs^{H-\varepsilon}$ for any fixed $0 < \varepsilon < \frac{1}{2}$. Therefore,

$$|R_{11}| \le C\int_0^\delta s^{\frac{1}{2}-\varepsilon}(t-s)^{-\alpha}ds \le Ct^{1-\alpha}(1-\alpha)^{-1}\delta^{\frac{1}{2}-\varepsilon} \to 0 \text{ as } \delta \to 0.$$

Similarly,

$|R_{12}| \le C_1\delta^{H-\varepsilon}\cdot\delta^{-\alpha}\cdot\delta^{1-\alpha} \to 0$ and $|R_{13}| \le C_2 t^{\frac{1}{2}-\varepsilon}\delta^{1-\alpha} \to 0$ as $\delta \to 0$, where $C_1$ and $C_2$ are some constants, possibly depending on $\omega$.

As mentioned above, the process $f(u)\cdot\int_u^t l_H(t,s)ds$ is Hölder of order $\beta\wedge(1-2\alpha) > 1-H$. Therefore, by using again (FdP01, Theorem 22), we obtain the bounds $|R_{21}| \le C\delta^{H-\varepsilon}$, $|R_{22}| \le C_1(t-2\delta)^{H-\varepsilon}$, and $|R_{23}| \le C\delta^{H-\varepsilon}$ with some constants $C$, $C_1$, depending on $\omega$. Taking in (2.6.18) a limit as $\delta \to 0$, we obtain from all these estimates that $J_1 = J_2$ a.s. $\square$

## 2.7 The Itô Formula for Fractional Brownian Motion

### 2.7.1 The Simplest Version

First, we present a very elegant proof of the Itô formula involving fBm from (Shi01).

**Lemma 2.7.1.** *Let $B^H$ be an fBm with $H \in (1/2, 1)$, $F \in C^2(\mathbb{R})$. Then for any $t > 0$*

$$F(B_t^H) = F(0) + \int_0^t F'(B_u^H)dB_u^H.$$

*Proof.* The Taylor formula with the reminder term in the integral form gives us

$$F(x) = F(y) + F'(y)(x-y) + \int_y^x F''(u)(x-u)du.$$

Let the sequence of partitions $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = t\}$, $|\pi_n| \to 0$, $n \to \infty$. Then $F(B_t^H) - F(0) = \sum\limits_{k=1}^{k_n}\left[F(t_k^n) - F(t_{k-1}^n)\right]$

$= \sum\limits_{k=1}^{k_n} F'(B_{t_{k-1}^n}^H)(B_{t_k^n}^H - B_{t_{k-1}^n}^H) + R_t^n$, where $R_t^n = \sum\limits_{k=1}^{k_n}\int_{B_{t_{k-1}^n}^H}^{B_{t_k^n}^H} F''(u)(B_{t_k^n}^H - u)du$. Further, $\sup\limits_{0\leqslant u\leqslant t}\left|F''(B_u^H)\right| < \infty$ a.s. and for $H \in (1/2,1)$, and

$$P\text{-}\lim_{n\to\infty}\sum_{k=1}^{k_n}\left|B_{t_k^n}^H - B_{t_{k-1}^n}^H\right|^2 = 0.$$

Therefore $|R_t^n| \leqslant \frac{1}{2}\sup\limits_{0\leqslant u\leqslant t}\left|F''(B_u^H)\right|\sum\limits_{k=1}^{k_n}\left|B_{t_k^n}^H - B_{t_{k-1}^n}^H\right|^2 \xrightarrow{P} 0$. Even if we do not know that the limit of integral sums $\sum\limits_{k=1}^{k_n} F'(B_{t_{k-1}^n}^H)(B_{t_k^n}^H - B_{t_{k-1}^n}^H)$ exists (but we know it from Theorem 2.1.7), we can obtain this existence now and, moreover,

$$F(B_t^H) - F(0) = \int_0^t F'(B_u^H)dB_u^H.$$

□

### 2.7.2 Itô Formula for Linear Combination of Fractional Brownian Motions with $H_i \in [1/2, 1)$ in Terms of Pathwise Integrals and Itô Integral

Denote $C^{\beta-}[a,b] = \bigcap_{0<\gamma<\beta} C^\gamma[a,b]$.

**Theorem 2.7.2.** *Let the process* $X_t = \sum\limits_{i=1}^m \sigma_i B_t^{H_i}$*, where* $H_1 = 1/2$ *and* $H_i \in (1/2, 1)$ *for* $2 \leqslant i \leqslant m$*. Let the function* $F \in C^2(\mathbb{R})$*. Then for any* $t > 0$

$$F(X_t) = F(0)+\sigma_1 \int_0^t F'(X_s)dW_s+\sum_{i=2}^m \sigma_i \int_0^t F'(X_s)dB_s^{H_i}+\frac{\sigma_1^2}{2}\int_0^t F''(X_s)ds.$$

*Proof.* Note that $\int_0^t |F'(X_s)|^2 ds < \infty$ and $\int_0^t |F''(X_s)| ds < \infty$ a.s., so, the Itô integral $\int_0^t F'(X_s)dW_s$ exists and is a local square-integrable martingale, and the Lebesgue integral $\int_0^t F''(X_s)ds$ also exists. As to integrals $\int_0^t F'(X_s)dB_s^{H_i}$ for $2 \leqslant i \leqslant m$, they exist as pathwise integrals because $X \in C^{1/2-}[0,t]$, $B^{H_i} \in C^{H_i-}[0,t]$ and $H_i + 1/2 > 1$. Further calculations are obvious: we use the Taylor formula and pass to the limit, as usual, taking into account that for any $1 \leqslant i \leqslant m$ and $2 \leqslant j \leqslant m$ $\sum_{k=1}^{k_n}\left(B_{t_k^n}^{H_i} - B_{t_{k-1}^n}^{H_i}\right)\left(B_{t_k^n}^{H_j} - B_{t_{k-1}^n}^{H_j}\right) \xrightarrow{P} 0$ as $n \to \infty$. □

Now, consider the process $Y_t = \sum\limits_{i=1}^m \sigma_i B_t^{H_i}$, where $H_i \in (1/2, 1)$ for any $1 \leqslant i \leqslant m$. We can forecast that in this case the class $C^1(\mathbb{R})$ of functions can be used.

**Theorem 2.7.3.** *Let* $Y_t = \sum\limits_{i=1}^m \sigma_i B_t^{H_i}$*, where* $H_i \in (1/2, 1)$ *for any* $1 \leqslant i \leqslant m$*. Let* $F \in C^1(\mathbb{R})$*, and* $F' \in C^\beta[0,t]$ *with* $(\beta+1)\min H_i > 1$ *for any* $t > 0$*. Then for any* $t > 0$

$$F(Y_t) - F(0) = \sum_{i=1}^m \sigma_i \int_0^t F'(Y_s)dB_s^{H_i}. \tag{2.7.1}$$

*Proof.* Clearly, condition $(\beta + 1)\min H_i > 1$ ensures the existence of $\int_0^t F'(Y_s)dB_s^{H_i}$ as the limit of Riemann sums for any $i > 1$. Consider convolutions $F_n = F * \varphi_n$ with $\varphi_n$ from Lemma 2.1.8. Then $F_n \in C^\infty(\mathbb{R})$, formula (2.7.1) holds for any $F_n$ and for any $1 - \min H_i < \gamma < \beta \cdot \min H_i$ we have that

$D^{\gamma}_{0+}F'_n \to D^{\gamma}_{0+}F'$ in $L_1[a,b]$ as $n\to\infty$ for any $a,b\in\mathbb{R}$, which can be proved similarly to (2.1.10). Therefore,

$$\left|\int_0^t (F'(Y_s)-F'_n(Y_s))dB^{H_i}_s\right|$$
$$\leqslant \sup_{0\leqslant s\leqslant t}\left|D^{1-\gamma}_{t-}B^{H_i}_{t-}(s)\right| \int_{-\sup_{0\leqslant s\leqslant t}|Y_s|}^{\sup_{0\leqslant s\leqslant t}|Y_s|} \left|D^{\gamma}_{0+}F'_n(s)-D^{\gamma}_{0+}F'(s)\right| ds \to 0,$$

whence the proof follows. □

*Remark* 2.7.4. Theorems 2.7.2 and 2.7.3 can be extended to the functions $F$ of several variables, depending also on $t$. The Itô formula has the following form: let $Y^i_t=\int_0^t f_i(s)dB^{H_i}_s$, where $H_1=1/2$, $H_i\in(1/2,1)$, $2\leqslant i\leqslant m-1$, $Y^m_t=\int_0^t g(s)ds$, $\int_0^t f_1^2(s)ds<\infty$ a.s., $f_i\in C^{\beta_i}[0,t]$ a.s. for $\beta_i+H_i>1$, $\int_0^t |g(s)|\,ds<\infty$ a.s., $F=F(t,x):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$, $F\in C^1(\mathbb{R}^+)\times C^2(\mathbb{R})$ $\times\, C^1(\mathbb{R}^{n-1})$, the integrals $\int_0^t \left(\frac{\partial F}{\partial x_1}(Z_s)f_1(s)\right)^2 ds<\infty$, $\int_0^t \left|\frac{\partial F}{\partial t}(Z_s)\right| ds<\infty$, $\int_0^t \left|\frac{\partial^2 F}{\partial x_1^2}(Z_s)\right| f_1^2(s)ds<\infty$, and $\int_0^t \left|\frac{\partial F}{\partial x_1}(Z_s)\right| |g(s)|\,ds<\infty$ a.s, $\frac{\partial F}{\partial x_i}(Z_s)f_i$ $\in C^{\gamma}[0,t]$ a.s. for $\gamma+H_i>1$ and any $t>0$, where $Z_s=(s,Y^1_s,\dots,Y^m_s)$. Then

$$F(t,Y^1_t,\dots,Y^m_t)=F(0)+\int_0^t \frac{\partial F}{\partial t}(Z_s)ds+\sum_{i=1}^{m-1}\int_0^t \frac{\partial F}{\partial x_i}(Z_s)f_i(s)dB^{H_i}_s$$
$$+\int_0^t \frac{\partial F}{\partial x_m}(Z_s)g(s)ds+\frac12\int_0^t \frac{\partial^2 F}{\partial x_1^2}(Z_s)f_1^2(s)ds. \quad (2.7.2)$$

In particular, for the process $Y_t=\int_0^t a(s)dB^H_s+\int_0^t b(s)ds$ we have that

$$F(t,Y_t)=F(0,Y_0)+\int_0^t F'_t(s,Y_s)ds+\int_0^t F'_x(s,Y_s)b(s)ds$$
$$+\int_0^t F'_x(s,Y_s)a(s)dB^H_s,\ H\in(1/2,1). \quad (2.7.3)$$

### 2.7.3 The Itô Formula in Terms of Wick Integrals

The next result is a direct consequence of Theorems 2.3.8 and 2.7.3.

**Theorem 2.7.5.** *Let the function* $F=F(t,x):\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ *be continuously differentiable in* $t$ *and twice continuously differentiable in* $x$. *Let* $Y_t$ *be as in Theorem* 2.7.2, $E\left|\frac{\partial F}{\partial x}(t,Y_t)\right|^{2+\varepsilon}<\infty$, $t>0$ *for some* $\varepsilon>0$, $E\sup_{0\leqslant s\leqslant t}\left[\left(\frac{\partial F}{\partial x}(s,Y_s)\right)^2+\left(\frac{\partial^2 F}{\partial x^2}(s,Y_s)\right)^2\right]<\infty$, $t>0$. *Then*

$$F(t, Y_t) - F(0,0) = \int_0^t \frac{\partial F}{\partial t}(s, Y_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, Y_s) \diamondsuit dY_s$$

$$+ \sum_{i,k=1}^{m} \sigma_i \sigma_k \tilde{C}_{H_i,H_k}(H_i + H_k) \int_0^t \frac{\partial^2 F}{\partial x^2}(s, Y_s) s^{H_i+H_k-1} ds. \quad (2.7.4)$$

### 2.7.4 The Itô Formula for $H \in (0, 1/2)$

We use the integral representation of fBm via the underlying Wiener process $B$ on the finite interval $[0,t]$ :

$$B_t^H = \int_0^t m_H(t,s) dB_s$$

$$= C_H^{(6)} t^\alpha \int_0^t u^{-\alpha}(t-u)^\alpha dB_u - C_H^{(6)} \alpha \int_0^t s^{\alpha-1} \left( \int_0^s u^{-\alpha}(s - u^{-\alpha}) dB_u \right) ds.$$

Let the function $F \in C^3(\mathbb{R})$ and we want to expand $F(B_t^H)$. Note that $B_t^H = B_{t,t}^H$, where for $0 < z < t$ $B_{z,t}^H = C_H^{(6)} z^\alpha \int_0^z u^{-\alpha}(t-u)^\alpha dB_u$ $- C_H^{(6)} \alpha \int_0^z s^{\alpha-1} \left( \int_0^s u^{-\alpha}(s-u)^{-\alpha} dB_u \right) ds$. Therefore

$$F(B_t^H) = F(0) + \int_0^t F'(B_{z,t}^H) d_z B_{z,t}^H + \frac{1}{2}(C_H^{(6)})^2 \int_0^t F''(B_{z,t}^H)(t-z)^{2\alpha} dz$$

$$= F(0) + \alpha C_H^{(6)} \int_0^t F'(B_{z,t}^H) z^{\alpha-1} \int_0^z u^{-\alpha}(t-u)^\alpha dB_u dz$$

$$+ C_H^{(6)} \int_0^t F'(B_{z,t}^H)(t-z)^\alpha dB_z$$

$$- \alpha C_H^{(6)} \int_0^t F'(B_{z,t}^H) z^{\alpha-1} \Big( \int_0^z u^{-\alpha}(t - u^{-\alpha}) dB_u \Big) dz$$

$$+ \frac{1}{2}(C_H^{(6)})^2 \int_0^t F''(B_{z,t}^H)(t-z)^{2\alpha} dz. \quad (2.7.5)$$

Further,

$$B_{z,t}^H = B_z^H + \alpha C_H^{(6)} z^\alpha \int_0^z u^{-\alpha} \int_z^t (v-u)^{\alpha-1} dv \, dB_u$$

$$= B_z^H + \alpha C_H^{(6)} z^\alpha \int_z^t \int_0^z u^{-\alpha}(v-u)^{\alpha-1} dB_u dv, \quad (2.7.6)$$

whence

$$F'(B_{z,t}^H) = F'(B_z^H) + \int_z^t F'' \Big( B_z^H + \alpha C_H^{(6)} z^\alpha \int_z^r \int_0^z u^{-\alpha}(v-u)^{\alpha-1} dB_u dv \Big)$$

$$\times \alpha C_H^{(6)} z^\alpha \int_0^z u^{-\alpha}(r-u)^{\alpha-1} dB_u dr =: F'(B_z^H) + \phi(F'', z, t), \quad (2.7.7)$$

and similar relation holds for $F''(B^H_{z,t})$. But

$$\int_z^r \int_0^z u^{-\alpha}(v-u)^{\alpha-1} dB_u dv = \frac{1}{\alpha}\int_0^z u^{-\alpha}\left[(r-u)^\alpha - (z-u)^\alpha\right] dB_u. \quad (2.7.8)$$

Substituting (2.7.6)–(2.7.8) into (2.7.5), we obtain the following result.

**Theorem 2.7.6.** *Let $H \in (0,1/2)$, $B^H$ be an fBm with Hurst index $H$, represented as $B^H_t = \int_0^t m_H(t,s)dB_s$. Denote $Y_{r,z} := C_H^{(6)} \int_0^z u^{-\alpha}(r-u)^\alpha dB_u$, $0 \leqslant z \leqslant r$, $Y_z := Y_{z,z}$. Then*

$$\begin{aligned} F(B^H_t) = F(0) &+ \int_0^t F'(B^H_z)\alpha z^{\alpha-1} Y_{t,z} dz + C_H^{(6)} \int_0^t F'(B^H_z)(t-z)^\alpha dB_z \\ &- \alpha \int_0^t F'(B^H_z) z^{\alpha-1} Y_{t,z} dz + \frac{1}{2}(C_H^{(6)})^2 \int_0^t F''(B^H_z)(t-z)^{2\alpha} dz + R_t, \end{aligned}$$

*where*

$$\begin{aligned} R_t = \alpha &\int_0^t \phi(F'',z,t)\alpha z^{\alpha-1} Y_{t,z} dz + C_H^{(6)} \int_0^t \phi(F'',z,t)(t-z)^\alpha dB_z \\ &- \alpha \int_0^t \phi(F'',z,t) z^{\alpha-1} Y_{t,z} dz + \frac{1}{2}(C_H^{(6)})^2 \int_0^t \phi(F''',z,t)(t-z)^{2\alpha} dz. \end{aligned}$$

*Remark* 2.7.7. The different approaches to the Itô formula for fBm with $H \in (1/2,1)$ are contained in the papers (Lin95), (DH96), (DU99), (AN02), (DHP00), (BO04), (CCM03), (FdP01). An elegant version of the Itô formula for $F(B^H_t)$ for any $H \in (0,1)$ was obtained by C. Bender in (Ben03a) and (Ben03c), but in terms of distributions. If the distribution $F$ is of function type, continuous at 0 and of polynomial growth, the form of such an Itô formula coincides with (2.7.4) for $m = 1$. For the other forms of the Itô formula for fBm with $H \in (0,1/2)$ see also (Nua03), (GRV03), (ALN01), (AMN00), (CN05).

### 2.7.5 Itô Formula for Fractional Brownian Fields

First, we prove one auxiliary result for Hölder two-parameter functions. Let the function

$$\begin{aligned} F: \mathbb{R} \to \mathbb{R},\ F \in C^3(\mathbb{R}), F''' \text{ is the Lipschitz function, } f(t) := F(g(t)), \\ g \in C^{\mu_1\mu_2}(\mathbb{R}^2_+) \text{ with } \mu_i > 1/2,\ i = 1,2. \quad (2.7.9) \end{aligned}$$

Let the rectangle $\mathcal{P}_t = [0,t] \subset \mathbb{R}^2_+$ be fixed, $\pi^i_n := \left\{0 = t^{i,n}_0 < \cdots t^{i,n}_{2^n} = t_i\right\}$, where $t^{i,n}_k = \frac{kt_i}{2^n}$, $f_{ik} = f(\frac{it_1}{2^n}, \frac{kt_2}{2^n})$,

$$\Delta^1_{ik} f = f_{i+1k} - f_{ik}, \Delta^2_{ik} f = f_{ik+1} - f_{ik}, \Delta_{ik} f = \Delta^1_{ik+1} f - \Delta^1_{ik} f.$$

**Lemma 2.7.8.** *Under assumption* (2.7.9) $\lim_{n\to\infty} I_j^n = 0$, $1 \leqslant j \leqslant 7$, *where* $I_1^n = \sum_{i,k=0}^{2^n-1} \Delta_{ik}^1 f \Delta_{ik} g$, $I_2^n = \sum_{i,k=0}^{2^n-1} \Delta_{ik}^2 f \Delta_{ik} g$, $I_3^n = \sum_{i,k=0}^{2^n-1} f_{ik} \Delta_{ik} g \Delta_{ik}^{\mathbf{1}} g$, $I_4^n = \sum_{i,k=0}^{2^n-1} f_{ik} \Delta_{ik} g \Delta_{ik}^2 g$, $I_5^n = \sum_{i,k=0}^{2^n-1} \Delta_{ik}^1 f (\Delta_{ik}^2 g)^2$, $I_6^n = \sum_{i,k=0}^{2^n-1} (\Delta_{ik}^1 f)^2 \Delta_{ik}^2 g$, $I_7^n = \sum_{i,k=0}^{2^n-1} F'''(g_{i,k}) \Delta_{ik}^1 g (\Delta_{ik}^2 g)^2$.

*Proof.* Consider $I_1^n$ ($I_2^n$ is similar). We can rewrite $I_1^n = \int_{\mathcal{P}_t} \tilde{f}_n dg$, where $\tilde{f}_n = \Delta_{ik}^1 f$ for $s \in \Delta_{ik}^n := \left[\frac{it_1}{2^n}, \frac{(i+1)t_1}{2^n}\right) \times \left[\frac{kt_2}{2^n}, \frac{(k+1)t_2}{2^n}\right)$. Further,

$$\int_{\mathcal{P}_t} \tilde{f}_n dg = \int_{\mathcal{P}_t} (D_{0+}^{\alpha_1 \alpha_2} \tilde{f}_n)(s)(D_{1-}^{1-\alpha_1 1-\alpha_2} g_{1-})(s) ds,$$

where $1-\mu_1 < \alpha_i < \mu_i$, $i = 1, 2$. Since $\left|(D_{1-}^{1-\alpha_1 1-\alpha_2} g_{1-})(s)\right| \leqslant C$ for some $C > 0$, it is sufficient to prove that $\lim_{n\to\infty} \int_{\mathcal{P}_t} \left|(D_{0+}^{\alpha_1\alpha_2} \tilde{f}_n)(s)\right| ds = 0$, and in turn, for this purpose it is sufficient to prove that $\int_{\mathcal{P}_t} |\phi_{n,i}(s)|\, ds \to 0$, $1 \leqslant i \leqslant 4$, where $\phi_{n,1}(s) = s_1^{-\alpha} s_2^{-\alpha} \tilde{f}_n(s)$, $\phi_{n,2}(s) = s_2^{-\alpha_2} \int_0^{s_1} (\tilde{f}_n(s) - \tilde{f}_n(u, s_2))(s_1 - u)^{-1-\alpha_1} du$, $\phi_{n,3}(s) = s_1^{-\alpha_1} \int_0^{s_2} (\tilde{f}_n(s) - \tilde{f}_n(s_1, v))(s_2 - v)^{-1-\alpha_2} dv$, $\phi_{n,4}(s) = \int_{[0,s]} \Delta_{u,v} \tilde{f}_n(s)(s_1 - u)^{-1-\alpha_1}(s_2 - v)^{-1-\alpha_2} du\, dv$. The relation $\int_{\mathcal{P}_t} |\phi_{n,1}(s)|\, ds \to 0$ is evident. Further, if $\frac{it_1}{2^n} \leqslant s < \frac{(i+1)t_1}{2^n}$, then $|\phi_{n,2}(s)| \leqslant C s_2^{-\alpha_2} \int_0^{i2^{-n}} (s_1 - u_1)^{-1-\alpha_1} du \cdot 2^{-n\mu_1}$, whence $\int_{\mathcal{P}_t} |\phi_{n,2}(s)|\, ds \leqslant C \int_0^{t_2} s_2^{-\alpha_2} ds_2 \cdot 2^{n(\alpha_1 - \mu_1)} \to 0$, $n \to \infty$. Similarly, $\int_{\mathcal{P}_t} |\phi_{n,3}(s)|\, ds \to 0$, $n \to \infty$. Finally, $\int_{\mathcal{P}_t} |\phi_{n,4}(s)|\, ds \leqslant C 2^{-n\mu_1}$ $\times \sum_{i,k=0}^{2^n-1} \int_{\Delta_{ik}^n} \int_{[0,t_k^{i,n}]} (s_1 - u)^{-1-\alpha_1}(s_2 - v + 2^{-n})^{\mu_2 - \alpha_2 - 1} du\, dv\, ds_1\, ds_2$ $= C 2^{n(\alpha_1 + \alpha_2 - \mu_1 - \mu_2)} \to 0$, $n \to \infty$. Of course, similar estimates hold for $I_3^n$ and $I_4^n$. As to $I_5^n, I_6^n$ and $I_7^n$, their estimates resemble each other, so, we consider only $I_5^n$. Note that

$$\lim_{n\to\infty} S_n := \lim_{n\to\infty} \sum_{i=0}^{2^n-1} f(t_{i2^n}^n)(\Delta_{i2^n}^1 g_{i+12^n})^2 \leqslant \lim_{n\to\infty} C \cdot 2^n \cdot 2^{-2n\mu_1} = 0.$$

Now, present the sum $S_n$ as

$$S_n = \sum_{i,k=0}^{2^n-1} (f_{ik}(\Delta_{ik} g)^2 + 2 f_{ik} \Delta_{ik} g \Delta_{ik}^1 g + \Delta_{ik}^2 f (\Delta_{ik}^1 g)^2 + \Delta_{ik}^2 f (\Delta_{ik} g)^2 + 2 \Delta_{ik}^2 f \Delta_{ik}^1 g \Delta_{ik} g) =: \sum_{1 \leqslant i \leqslant 5} S_{n,i},$$

where $S_{n,1} \leqslant C \cdot 2^{-2n(\mu_1+\mu_2-1)} \to 0$, $n \to \infty$, similarly, $S_{n,4} \to 0$, $S_{n,5} \to 0$, $n \to \infty$. According to previous estimates $\lim_{n\to\infty} S_{n,2} = \lim_{n\to\infty} I_3^n = 0$. Therefore, $\lim_{n\to\infty} I_5^n = \lim_{n\to\infty} S_{n,3} = 0$. □

*Remark* 2.7.9. Let $F : \mathbb{R} \to \mathbb{R}$, $F \in C^3(\mathbb{R})$ and $F'''$ is the Lipschitz function, the field $g(t)$ is a linear combination of the fractional Brownian fields,

$$g(t) = \sum_{i=1}^{m} \sigma_i B_t^{H_1^i H_2^i} \text{ with } H_j^i > \frac{1}{2},\ j = 1, 2,\ 1 \leqslant i \leqslant m.$$

Clearly, the previous lemma holds for such $g(t)$ and $f(t) = F(g(t))$.

**Theorem 2.7.10.** *For any* $t \in \mathbb{R}_+^2$

$$F(g(t)) = F(0) + \int_{\mathcal{P}_t} F'(g)dg + \int_{\mathcal{P}_t} F''(g)d_1g\, d_2g.$$

*Proof.* According to the one-parameter Itô formula (Theorem 2.7.3)

$$\begin{aligned} F(g(t)) = F(0) + \int_0^{t_1} F'(g(s_1, t_2))d_1g(s_1, t_2) \\ = F(0) + \lim_{n\to\infty} \sum_{i=0}^{2^n} f(t_{i,2^n}^n)\Delta_{i,2^n}^1 g_{i+1,2^n} \text{ a.s.} \end{aligned}$$

The prelimit sum can be presented as

$$\begin{aligned} &\sum_{i,k=0}^{2^n-1} F'(g(t_{ik}^n))\Delta_{ik}g + \sum_{i,k=0}^{2^n-1} F''(g(t_{ik}^n))\Delta_{ik}^1 g\Delta_{ik}^2 g + \sum_{i,k=0}^{2^n-1} F''(g(s_n^{ik}))\Delta_{ik}g\Delta_{ik}^2 g \\ &\quad + \frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''(g(\theta_{ik}^n))(\Delta_{ik}^2 g)^2\Delta_{ik}^1 g + \frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''(g(\theta_{ik}^n))(\Delta_{ik}^2 g)^2\Delta_{ik} g, \end{aligned} \tag{2.7.10}$$

where $\theta_{ik}^n \in \Delta_{ik}^n$. According to Theorem 2.2.9, $\sum_{i,k=0}^{2^n-1} F'(g(t_{ik}^n))\Delta_{ik}g \to \int_{\mathcal{P}_t} F'(g)dg$ a.s. Furthermore, according to Theorem 2.2.17 and Lemma 2.7.8, $\sum_{i,k=0}^{2^n-1} F''(g(t_{ik}^n))\Delta_{ik}^1 g\Delta_{ik}^2 g \to \int_{\mathcal{P}_t} F''(g)d_1g\, d_2g$, $\sum_{i,k=0}^{2^n-1} F''(g(s_n^{ik}))\Delta_{ik}g\Delta_{ik}^2 g \to 0$, $\frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''(g(t_{ik}^n))(\Delta_{ik}^2 g)^2\Delta_{ik}^1 g \to 0$, $\frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''(g(t_{ik}^n))(\Delta_{ik}^2 g)^2\Delta_{ik} g \to 0$, and due to the Lipschitz properties of $F'''$, $\frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''(g(\theta_{ik}^n))(\Delta_{ik}^2 g)^2\Delta_{ik}^1 g \to 0$,

$\frac{1}{2}\sum\limits_{i,k=0}^{2^n-1} F'''(g(\theta_{ik}^n))(\Delta_{ik}^2 g)^2 \Delta_{ik} g \to 0$, $n \to \infty$, a.s., and the assertion of the theorem is proved. □

*Remark* 2.7.11. The theorem holds even for $F \in C^2(\mathbb{R})$, such that $F''$ is the Lipschitz function. To prove this, we must rewrite the sum of second and fourth term on the right-hand side of (2.7.10) as $\sum\limits_{i,k=0}^{2^n-1} F''(g(\theta_{ik}^n))\Delta_{ik}^1 g \Delta_{ik}^2 g$. Then we can prove that this sum has a limit $\int_{\mathcal{P}_t} F''(g) d_1 g\, d_2 g$, similarly to Theorem 2.2.17. Also, the sum of third and fifth terms can be rewritten as $\sum\limits_{i,k=0}^{2^n-1} F''(g(\theta_{ik}^n))\Delta_{ik} g \Delta_{ik}^2 g$, and we can prove that its limit is zero.

### 2.7.6 The Itô Formula for $H \in (0,1)$ in Terms of Isometric Integrals, and Its Applications

#### Definitions

If $f \in L^2(P \otimes \lambda)$, $f$ is predictable, $\pi$ is a partition, then $f_\pi$ is the step function $f_\pi = \sum\limits_i f(t_{i-1}) \mathbf{1}_{[t_{i-1}, t_i)}(t)$.

Define the class of functions $\Phi$ as follows: $\overrightarrow{f} \in \Phi$ if the following conditions are satisfied:

(i) $\overrightarrow{f} := (f^i : i \geq 1)$, where $f^i \in L^2(P \otimes \lambda)$, $f^i$ is predictable and $\sum\limits_i \left\|f^i\right\|_{L^2(P\otimes\lambda)} < \infty$.

(ii) $\overrightarrow{f}$ is uniformly tight: $P\{\sup_{t \leq T} \sup_i |f^i(t)| > C\} \to 0$ as $C \to \infty$.

(iii) The random variable $u$ defined by $u := \sum\limits_i (f_\pi^i, (B^H)^{(i)})$ (for the notations see Section 2.5.2) does not depend on the partition $\pi$, and the series converges absolutely with probability one, when $\overrightarrow{f} \in \Phi$.

Write $(\overrightarrow{f}, \overrightarrow{B^H})$ for the sum $\sum\limits_i (f_\pi^i, (B^H)^{(i)})$, and put $\mathcal{U} := \{u : u = (\overrightarrow{f}, \overrightarrow{B^H}), \overrightarrow{f} \in \Phi\}$. Let $\Phi_p$ be the projection of $\Phi$ to the first $p$ coordinates.

The following example shows that $\mathcal{U}$ is nonempty.

*Example* 2.7.12. Assume that $f \in C_b^\infty(\mathbb{R})$: then

$$f(B_T^H) - f(0) = \sum_{i=1}^{n} \Delta f(B_{t_i}^H)$$

and if $f^k := (1/k!) f^{(k)}, k \geq 1$, then

$$f(B_T^H) - f(0) = (\overrightarrow{f}, \overrightarrow{B^H}),$$

$f(B_T^H) - f(0) \in \mathcal{U}$ and $\overrightarrow{f} \in \Phi, (f^1, \ldots, f_p) \in \Phi_p$ for any $p \geq 1$.

**Lemma 2.7.13.** *If $u \in \mathcal{U}, u = (\overrightarrow{f}, \overrightarrow{B^H})$ with $u = 0$, then $f^i = 0, i \geq 1$.*

*Proof.* Since $u$ does not depend on the partition, take first the partition $\{0, T\}$. The random variable $u$ has a representation

$$u = \sum_i f_0^i (B_T^H)^i, \tag{2.7.11}$$

where $f_0^i$ are real numbers, since $\mathcal{F}_0$ is the trivial $\sigma$-algebra. But since $u = 0$ from (2.7.11) it follows that for almost all $y \in \mathbb{R}$ we have that $\sum\limits_i f_0^i y^i = 0$ and hence $f_0^i = 0$ for all $i \geq 1$.

Next, consider the partition $\{0, t, T\}$. We have that

$$u = \sum_i f_0^i (B_t^H)^i + \sum_i f_t^i (B_T^H - B_t^H)^i = 0.$$

From the above we get that $f_0^i = 0$ for all $i \geq 1$ and hence also $f_t^i = 0$ for all $i \geq 1$. □

### The Itô Formula for Isometric Integrals

The following is an analogue of the Itô formula in this context.

**Theorem 2.7.14.** *Assume that the Hurst index $H$ satisfies $H \in (0, 1/2)$. There exists one-to-one correspondence between $\mathcal{U}$ and the set*

$$\mathcal{V} := \Big\{ v : v := \sum_{i=1}^{[1/H]} (f^i, (B^H)^{(i)}) \Big\}.$$

*Proof.* We must show that there exists one-to-one correspondence between $\mathcal{U}$ and $\Phi_{[1/H]}$. Assume that $f \in \Phi_{[1/H]}$. Then there exists a vector $\overrightarrow{g} \in \Phi$ such that $f^i = g^i$ for $i \leq [1/H]$. Assume that $\overrightarrow{h}$ is another element from $\Phi$ such that $f^i = h^i$ for $i \leq [1/H]$. Put $u := (\overrightarrow{g}, \overrightarrow{B^H})$ and $v := (\overrightarrow{h}, \overrightarrow{B^H})$. Then

$$u - v = \sum_{i=\lfloor 1/H \rfloor + 1}^{\infty} (g^i - h^i, (B^H)^{(i)}).$$

On one hand, since $u$ and $v$ are independent of the partition $\pi$, we can take a partition $\pi$ such that $|\pi| < 1$. Then for any $\varepsilon > 0$ we have that

$$P\{|u - v| > \varepsilon\} \leq P(D) + P\{|u - v| > \varepsilon, \Omega \setminus D\} \tag{2.7.12}$$

and $D$ is the set $D := \{\sup_{t \leq T} \sup_i |f_t^i - g_t^i| \geq C\}$. But

$$P\{|u - v| > \varepsilon, \Omega \setminus D\} \leq \frac{C}{\varepsilon} \sum_{i > 1/H} E \sum_k |\Delta B_k^H|^i$$

and since

$$E\sum_k |\Delta B_k^H|^i \le CT(|\pi|)^{Hi-1}$$

we have that

$$P\{|u-v|>\varepsilon, \Omega\setminus D\}\to 0$$

as $|\pi|\to 0$. By property (iii) of $\Phi$ we can choose $C$ such that $P(D)<\delta$ for any $\delta>0$. Use these estimates in (2.7.12) to conclude that $u=v$. On the other hand, if $u=(\overrightarrow{f},\overrightarrow{B^H})=(\overrightarrow{h},\overrightarrow{B^H})$ we have from Lemma 2.7.13 that $\overrightarrow{f}=\overrightarrow{h}$. To finish, note that from Example 2.7.12 it follows that the random variable $f(B_T^H)-f(0)$ is a representative of $\sum\limits_{i=1}^{[} 1/H](1/i!)\int_0^T f^{(i)}(x_s)dB_s^{H(i)}$. □

*Example* 2.7.15 (Fractional Doleans exponent). Assume that $[1/H]=2p$, where $p\in\mathbb{N}$. Then the random variable $y_t=\exp(B_t^H-t/(2p)!)-1$ is a representative of

$$\sum_{i=1}^{2p-1}\frac{1}{i!}\int_0^t y_s d(B_s^H)^{(i)}.$$

We say that $y$ is the Doleans exponent of $B^H$.

## 2.8 The Girsanov Theorem for fBm and Its Applications

### 2.8.1 The Girsanov Theorem for fBm

Consider the kernel $l_H(t,s)=C_H^{(5)}s^{-\alpha}(t-s)^{-\alpha}$, $0<s<t$. Let $\mathcal{F}_t=\sigma\left\{B_s^H, 0\leqslant s\leqslant t\right\}=\sigma\left\{B_s, 0\leqslant s\leqslant t\right\}$, where $B$ is underlying Wiener process in the representation

$$M_t^H=\int_0^t l_H(t,s)dB_s^H,\ B_t=\widehat{\alpha}\int_0^t s^{\alpha}dM_s^H.$$

Assume that the random process $\{\phi_t, t\geqslant 0\}$ is adapted to filtration $\mathcal{F}_t$ and satisfies

$$\int_0^t l_H(t,s)\,|\phi_s|\,ds<\infty,\ t>0,\ P\text{-a.s.} \tag{2.8.1}$$

Assume also that we have the representation

$$\int_0^t l_H(t,s)\phi_s ds=\widetilde{\alpha}\int_0^t \delta_s ds,\ t>0, \tag{2.8.2}$$

with some $\mathcal{F}_t$-adapted process $\delta$ satisfying

$$\int_0^t |\delta_s|\,ds<\infty,\ P\text{-a.s.},\ t>0, \tag{2.8.3}$$

and

$$E\int_0^t s^{2\alpha}\delta_s^2 ds < \infty,\ t > 0. \tag{2.8.4}$$

Define a square-integrable martingale $L$ by $L_t := \int_0^t s^\alpha \delta_s dB_s$.

**Theorem 2.8.1.** *Assume that we have* (2.8.1)–(2.8.4) *and the martingale $L$ satisfies*

$$E\exp\{L_t - 1/2\langle L\rangle_t\} = 1,\ t > 0.$$

*Then the process $\widetilde{B}_t^H := B_t^H - \int_0^t \phi_s ds$ is an fBm with respect to measure $Q$, where the measure $Q$ is defined by*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\}.$$

*Proof.* Note first that the integral

$$\widetilde{M}_t^H := \int_0^t l_H(t,s)d\widetilde{B}_s^H = \int_0^t l_H(t,s)dB_s^H - \int_0^t l_H(t,s)\phi_s ds \tag{2.8.5}$$

exists, since both integrals exist as pathwise integrals (the first integral was studied in Section 1.8 and (2.8.2) ensures the existence of the second integral). Moreover, from (2.8.2) it follows that

$$\widetilde{M}_t^H = M_t^H - \widetilde{\alpha}\int_0^t \delta_s ds = \widetilde{\alpha}\Big(\int_0^t s^{-\alpha}dB_s - \int_0^t \delta_s ds\Big).$$

Evidently, $\left[\widetilde{M}^H\right]_t := P\text{-}\lim_{|\pi|\to 0}\sum_{t_i\in\pi}(\widetilde{M}_{t_i}^H - \widetilde{M}_{t_{i-1}}^H)^2$ exists and equals $\left[\widetilde{M}^H\right]_t = t^{1-2\alpha}$. Therefore, for any $\theta\in\mathbb{R}$ we have for $\widehat{M}_t^H := \widehat{\alpha}\widetilde{M}_t^H$ that

$$\begin{aligned}\theta\widehat{M}_t^H - \frac{\theta^2}{2}\left[\widehat{M}^H\right]_t + L_t - \frac{1}{2}\langle L\rangle_t &= \theta\int_0^t s^{-\alpha}dB_s - \theta\int_0^t \delta_s ds - \frac{\theta^2}{2}\frac{t^{1-2\alpha}}{1-2\alpha}\\ &+ \int_0^t s^\alpha\delta_s dB_s - \frac{1}{2}\int_0^t s^{2\alpha}\delta_s^2 ds = \int_0^t (\theta s^{-\alpha} + s^\alpha\delta_s)dB_s\\ &- \frac{1}{2}\int_0^t (\theta^2 s^{-2\alpha} - 2\delta_s\theta + \delta_s^2 s^{2\alpha})ds =: R_t - \frac{1}{2}\langle R\rangle_t, \end{aligned} \tag{2.8.6}$$

where $R$ is a square-integrable martingale given by $R_t := \int_0^t(\theta s^{-\alpha} + s^\alpha\delta_s)dB_s$. But (2.8.6) means that the process

$$K_t := \exp\left\{\theta\widehat{M}_t^H - \frac{\theta^2}{2}\left[\widehat{M}^H\right]_t + L_t - \frac{1}{2}\langle L\rangle_t\right\}$$

is a local $P$-martingale. This implies, in turn, that the process $\exp\left\{\theta\widehat{M}_t^H - \frac{\theta^2}{2}\left[\widehat{M}^H\right]_t\right\}$ is a local $Q$-martingale. From (Ell82, Theorem

13.22), we can conclude that $\widehat{M}^H$ is a local $Q$-martingale with the angle bracket $\langle \widehat{M}^H \rangle_t = \int_0^t s^{-2\alpha} ds$ and so $\widetilde{M}_t = \widetilde{\alpha} \int_0^t s^{-\alpha} d\widetilde{B}_s$, where $\widetilde{B}$ is a standard Brownian motion with respect to $Q$ (and is obtained from $B$ by subtracting a drift). This means that

$$\int_0^t l_H(t,s) d\widetilde{B}_s^H = \widetilde{\alpha} \int_0^t s^{-\alpha} d\widetilde{B}_s. \tag{2.8.7}$$

Now, using two representations for $\widetilde{B}^H$, (2.8.5) and (2.8.7), we can obtain (1.8.17) for $\widetilde{B}^H$ and then conclude from Remark 1.8.2 that it is the fBm with respect to the measure $Q$. □

### 2.8.2 When the Conditions of the Girsanov Theorem Are Fulfilled? Differentiability of the Fractional Integrals

If we analyze the conditions of the Girsanov theorem, we see that condition (2.8.2) is a principal concern. Now we shall establish that in one particular but important case this condition holds. Let the process $I(t) := \int_0^t l_H(t,s)\phi(s)ds$ with $\phi(t) = \int_0^t a(s,\omega)dB_s^H$, where the integrand $a = a(s,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ is measurable in its variables and for a.a. $\omega \in \Omega$ is Hölder in $s$ with some index $\beta \in (1/2, 1)$. According to Theorem 2.1.7, the integral $\phi(t)$ exists as a pathwise integral for $\omega \in \Omega'$, $P(\Omega') = 1$. Moreover, according to Lemma 2.6.6, there exists a repeated integral $J(t) := \int_0^t a(u,\omega) \int_u^t l_H(t,s) ds\, dB_u^H$ and the equality $I(t) = J(t)$ holds for $\omega \in \Omega'$.

**Lemma 2.8.2.** *Let $a \in C^\rho[0,t]$ for any $t > 0$ and for any $\omega \in \Omega'$, $P(\Omega') = 1$, $\rho \in (1/2, 1)$. Then for any $t > 0$ $I(t)$ admits the representation*

$$I(t) = C_H^{(5)} t^{1-2\alpha} \int_0^t \delta_s ds,$$

*where $\delta_s = s^{2\alpha-2} \int_0^s u^{1-\alpha}(s-u)^{-\alpha} a(u,\omega) dB_u^H$, and $\delta \in L_1[0,t]$ for any $t > 0$, $\omega \in \Omega'$.*

*Proof.* Further we suppose everywhere that $\omega \in \Omega'$ and argument $\omega$ will be omitted. We rewrite $J(t)$ as

$$\begin{aligned} J(t) &= t^{1-2\alpha} \int_0^t \int_{u/t}^1 a(u) l_H(1,s) ds\, dB_u^H \\ &= C_H^{(5)} t^{1-2\alpha} \int_0^t \int_u^t s^{2\alpha-2}(s-u)^{-\alpha} u^{1-\alpha} a(u) ds\, dB_u^H =: C_H^{(5)} t^{1-2\alpha} M(t). \end{aligned}$$

Consider now the function

$$N(t) := \int_0^t s^{2\alpha-2} \int_0^s (s-u)^{-\alpha} u^{1-\alpha} a(u) dB_u^H ds.$$

The following results ensure its existence:
(i) According to (NVV99, Lemma 2.1), for the function $g \in C^\beta[0,T]$ with $0<\gamma+\beta<1$, $f(0)=0$ the integral $\int_0^t (t-u)^\gamma dg(u)$ exists and equals

$$\int_0^t (t-u)^\gamma dg(u) = \lim_{\varepsilon\to 0}(\varepsilon^\gamma(g(t-\varepsilon)-g(t)) + t^\gamma g(t) + \gamma\int_0^{t-\varepsilon}(g(u)-g(t))(t-u)^{\gamma-1}du). \quad (2.8.8)$$

(ii) According to Lemma 2.6.3, for $f\in C^\gamma[a,b]$, $g\in C^\beta[a,b]$, $\gamma+\beta>1$, $0<\varepsilon'<\gamma+\beta-1$

$$\left|\int_a^b f(t)dg(t)\right| \leqslant C\,\|f\|_{C^\gamma[a,b]}\,\|g\|_{C^\beta[a,b]}\,((b-a)^{1+\varepsilon'}\vee(b-a)^\beta), \quad (2.8.9)$$

where $C$ does not depend of $f$ and $g$. Using (2.8.8)–(2.8.9), we obtain the following estimates for $0<s_1<s_2<t$:

$$\begin{aligned}\left|\int_{s1}^{s_2} a(z)(s_2-z)^{-\alpha}dB_z^H\right| &= \left|\lim_{\varepsilon\to 0}\left(-\varepsilon\int_{s_2-\varepsilon}^{s_2} a(v)dB_v^H\right.\right.\\ &\quad\left.\left. +\ (s_2-s_1)^{-\alpha}\int_{s_1}^{s_2}a(z)dB_z^H + \alpha\int_{s_1}^{s_2-\varepsilon}(s_2-z)^{-1-\alpha}\int_z^{s_2}a(v)dB_v^H dz\right)\right|\\ &\leqslant \lim_{\varepsilon\to 0}\left(C\,\|a\|_{C^\rho[0,t]}\left\|B^H\right\|_{C^{H'}[0,t]}\left((s_2-s_1)^{1-\alpha+\varepsilon'}\vee(s_2-s_1)^{-\alpha+H'}\right)\right.\\ &\quad\left. +\ \alpha\int_{s_1}^{s_2-\varepsilon}(s_2-z)^{-1-\alpha}\left((s_2-z)^{1+\varepsilon'}\vee(s_2-z)^{H'}\right)dz\right), \quad (2.8.10)\end{aligned}$$

where $H'$ is any constant not exceeding $H$ and $0<\varepsilon<\rho+H-1$. Evidently, the right-hand side of (2.8.10) can be estimated by $CK_1(t)(s_2-s_1)^{-\alpha+H'}$, where $K_1(t)\leqslant\|a\|_{C^\rho[0,t]}\left\|B^H\right\|_{C^{H'}[0,t]}(t\vee 1)^{1+\varepsilon-H'}$, $C$ does not depend on $\rho, B^H, t$. Further,

$$\begin{aligned}\int_{s_1}^{s_2}(s_2-u)^{-\alpha}u^{1-\alpha}a(u)dB_u^H &= \int_{s_1}^{s_2}u^{1-\alpha}d\left(\int_{s_1}^u(s_2-z)^{-\alpha}a(z)dB_z^H\right)\\ = s_2^{1-\alpha}\int_{s_1}^{s_2}(s_2-z)^{-\alpha}a(z)dB_z^H &- (1-\alpha)\int_{s_1}^{s_2}u^{-\alpha}\int_{s_1}^u(s_2-z)^{-\alpha}a(z)dB_z^H du\\ &=: L(s_1,s_2).\end{aligned}$$

The estimate

$$\begin{aligned}|L(s_1,s_2)| &\leqslant Cs_2^{1-\alpha}K_1(t)(s_2-s_1)^{-\alpha+H'}\\ &\quad + C(1-\alpha)K_1(t)\int_{s_1}^{s_2}u^{-\alpha}(u-s_1)^{-\alpha+H'}du\\ &\leqslant CK_1(t)\left(s_2^{1-\alpha}(s_2-s_1)^{-\alpha+H'}+(s_2-s_1)^{1-2\alpha+H'}\right) \quad (2.8.11)\end{aligned}$$

means that $|L(0,s)| \leqslant CK_1(t)s^{1-2\alpha+H'}$.

Now it is clear that

$$|N_t| \leqslant CK_1(t) \int_0^t s^{2\alpha-2} s^{1-2\alpha+H'} ds \leqslant CK_1(t) t^{H'} < \infty.$$

Consider the function

$$N_\varepsilon(t) := \int_0^t s^{2\alpha-2} \mathbf{1}_{\{s\in[\varepsilon,t]\}} \int_0^{s-\varepsilon} u^{1-\alpha}(s-u)^{-\alpha} a(u) dB_u^H ds.$$

Evidently, for any $\varepsilon > 0$ the function

$$\phi_\varepsilon(s,u) := \mathbf{1}_{\{s\in[\varepsilon,t],0\leqslant u\leqslant s-\varepsilon\}} s^{2\alpha-2} u^{1-\alpha}(s-u)^{-\alpha} a(u)$$

is piecewise-Hölder in $u$ with index $\rho \wedge (1-\alpha) > 1/2$ ($u = s-\varepsilon$ is the point of Hölder discontinuity), and the function

$$\psi_\varepsilon(s) := \int_0^t \phi_\varepsilon(s,u) dB_u^H = s^{2\alpha-2} \mathbf{1}_{\{s\in[\varepsilon,t]\}} \int_0^{s-\varepsilon} (s-u)^{-\alpha} u^{1-\alpha} a(u) dB_u^H$$

is Riemann integrable on $[0,t]$. Therefore, $\phi_\varepsilon(s,u)$ satisfies the conditions of the stochastic Fubini Theorem 2.6.5, whence $N_\varepsilon(t)$ exists and equals

$$M_\varepsilon(t) := \int_0^{t-\varepsilon} u^{1-\alpha} a(u) \int_{u+\varepsilon}^t s^{2\alpha-2}(s-u)^{-\alpha} ds\, dB_u^H.$$

Further,

$$\begin{aligned}
|N(t) - N_\varepsilon(t)| &\leqslant \left| \int_\varepsilon^t s^{2\alpha-2} \int_{s-\varepsilon}^s u^{1-\alpha}(s-u)^{-\alpha} a(u) dB_u^H ds \right| \\
&\quad + \left| \int_0^\varepsilon s^{2\alpha-2} \int_0^s u^{1-\alpha}(s-u)^{-\alpha} a(u) dB_u^H ds \right| \\
&\leqslant \int_\varepsilon^t s^{2\alpha-2} CK_1(t)(s^{1-\alpha}\varepsilon^{-\alpha+H'} + \varepsilon^{1-2\alpha+H'}) ds \\
&\quad + \int_0^\varepsilon s^{2\alpha-2} CK_1(t) s^{1-2\alpha+H'} ds \\
&\le CK_1(t)(\varepsilon^{-\alpha+H'} + \varepsilon^{H'}) \to 0,\ \varepsilon \to 0.
\end{aligned}$$

For $M(t) - M^\varepsilon(t)$ we use one of the integral transformations from (NVV99, Lemma 2.2): for $\mu \in \mathbb{R}$, $\nu > -1$, $c > 1$ the integral $\int_1^c t^\mu (t-1)^\nu dt = \int_0^{1-1/c} s^\nu (1-s)^{-\mu-\nu-2} ds$, and as a result obtain the bound

$$
\begin{aligned}
|M(t)-M_\varepsilon(t)| &\leqslant C\left|\int_0^{t-\varepsilon} a(u)u^{1-\alpha}\int_u^{u+\varepsilon} s^{2\alpha-2}(s-u)^{-\alpha}ds\,dB_u^H\right| \\
&\quad + C\left|\int_{t-\varepsilon}^{t} a(u)u^{1-\alpha}\int_u^{t} s^{2\alpha-2}(s-u)^{-\alpha}ds\,dB_u^H\right| \\
&= C\left|\int_0^{t-\varepsilon} a(u)\int_0^{\frac{\varepsilon}{u+\varepsilon}} s^{-\alpha}(1-s)^{-\alpha}ds\,dB_u^H\right| \\
&\quad + C\left|\int_{t-\varepsilon}^{t} a(u)\int_0^{1-\frac{u}{t}} s^{-\alpha}(1-s)^{-\alpha}ds\,dB_u^H\right| =: A_1(\varepsilon)+A_2(\varepsilon).
\end{aligned}
$$

According to the stochastic Fubini theorem 2.6.5,

$$
\begin{aligned}
A_1(\varepsilon) = C\int_0^{\varepsilon/t} s^{-\alpha}(1-s)^{-\alpha}\int_0^{t-\varepsilon} a(u)dB_u^H ds \\
+ C\int_{\varepsilon/t}^{t} s^{-\alpha}(1-s)^{-\alpha}\int_0^{\frac{\varepsilon(1-s)}{s}} a(u)dB_u^H ds
\end{aligned}
$$

and

$$
A_2(\varepsilon) = C\int_0^{\varepsilon/t} s^{-\alpha}(1-s)^{-\alpha}\int_{t-\varepsilon}^{t(1-s)} a(u)dB_u^H ds.
$$

Therefore,

$$
\begin{aligned}
|A_1(\varepsilon)| \leqslant C\left|\int_0^{t-\varepsilon} a(u)sB_u^H\right|\left(1-\frac{\varepsilon}{t}\right)^{-\alpha}\left(\frac{\varepsilon}{t}\right)^{1-\alpha} \\
+ CK_1(t)\int_{\varepsilon/t}^{1} s^{-\alpha}(1-s)^{\alpha}\left(\frac{\varepsilon(1-s)}{s}\right)^{H'} ds \to 0,\ \varepsilon\to 0,
\end{aligned}
$$

and

$$
|A_2(\varepsilon)| \leqslant CK_1(t)\int_0^{\varepsilon/t} s^{-\alpha}(1-s)^{-\alpha}(\varepsilon - ts)^{H'}ds \to 0,\ \varepsilon\to 0.
$$

Therefore, $N(t)=M(t)$, and our lemma is proved. $\square$

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