Chapter 3
Variational Formulations

The purpose of this chapter is to develop and analyse the variational formulations for a one-dimensional case, on the one hand and to apply the $\mathbf{P_1}$ elements of Lagrange, on the other hand.

In fact, in accordance with the degree of complexity of the problems tackled, it will be suggested to apply the following program of analysis, which is common to all partial differential equations, whose approximation is sought by the finite elements method:

- **Step A**: This first step aims at obtaining a variational formulation (**VP**) (or weak formulation) for a given continuous problem (**CP**) (or strong formulation).
- **Step B**: The existence and uniqueness of weak solutions of the problem (**VP**) are studied, (essentially the application of the Lax-Milgram theorem).
- **Step C**: This step is devoted to the analysis of the regularity of the weak solutions of the variational problem (**VP**).
- **Step D**: This part deals with the equivalence of strong and weak formulations. It is particularly shown that a weak solution of the variational problem (**VP**), which moreover shows the regularity property as obtained in Step **C**, is a strong solution of the continuous problem (**CP**).
- **Step E**: This step is devoted to the writing of the nodal equation system that provides an approximation of the weak solution of the variational problem (**VP**).

Furthermore, nodal equations obtained will be compared to those obtained with the finite differences scheme that are associated with the continuous problem (**CP**).

Hence, two distinct parts may be studied separately. A first theoretical part will be devoted to the analysis of the existence, uniqueness and regularity of the solutions to variational formulations as well as to the notion of equivalence between continuous and variational problems (Steps **A**, **B**, **C** and **D**).

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The second part, which is totally different, will be devoted to obtaining an approximation of the different nodal equations, mainly using the Lagrange finite element $\mathbf{P_1}$ and to the analysis of some schemes having finite differences.

Therefore, the reader may, at leisure, deal with the whole aspect of the problem (both theoretical and numerical) or study either of these parts.

However, in the case where only the numerical part needs to be studied, it would be suitable to refer to the theoretical part, or at least the first question, so as to elaborate and determine the proper variational formulation for the numerical application of finite elements.

3.1 Dirichlet's Problem

3.1.1 Statement

The aim of this problem is to propose a mathematical and numerical study of the solution to a linear differential equation subjected to Dirichlet boundary conditions.

Find $u \in H^2(0,1)$ being the solution to:

$$(\mathbf{CP}) \begin{cases} -u''(x) + u(x) = f(x),\ 0 \leq x \leq 1\ , \\ u(0) = u(1) = 0\ , \end{cases} \tag{3.1}$$

in which f is a given function belonging to $L^2(0,1)$.

Besides, it is pointed out that Sobolev's space $H^2(0,1)$ is defined as:

$$H^2(0,1) = \left\{ v :]0,1[\to \mathbf{R},\ \frac{\mathrm{d}^k v}{\mathrm{d}x^k} \in L^2(0,1),\ \forall k = 0,1,2 \right\}. \tag{3.2}$$

► **Variational Formulation – Theoretical Part**

1) Let v be a test function, defined from $[0,1]$ to $\mathbf{R}$, belonging to a functional space V whose characteristics will be determined *a posteriori*.

Show that the continuous problem (**CP**) may be expressed in a variational formulation (**VP**) in the form:

$$a(u,v) = L(v), \quad \forall v \in V\ .$$

The bilinear form $a(.,.)$, the linear form $L(.)$ as well as the functional space V need to be specified.

2) Establish the existence and uniqueness of the weak solution of the variational problem (**VP**) in $H_0^1(0,1)$, in which $H_0^1(0,1)$ is defined as:

$$H_0^1(0,1) = \left\{v :]0,1[\to \mathbf{R}\,,\ v \text{ and } v' \in L^2(0,1),\ v(0) = v(1) = 0\right\} \,. \tag{3.3}$$

3) Show that any weak solution of the variational problem (**VP**) also belongs to $H^2(0,1)$.

4) To infer the equivalence between the strong formulation (**CP**) set in $H^2(0,1)$ and the weak formulation (**VP**) considered in:

$$H_0^1(0,1) \cap H^2(0,1) \,.$$

► Numerical Part – Lagrange Finite $\mathbf{P_1}$ Elements

5) The approximation of the variational problem (**VP**) is worked out using the Lagrange finite elements $\mathbf{P_1}$.

This is performed by introducing a regular mesh of $[0,1]$ interval, of constant step h, such as:

$$\begin{cases} x_0 = 0, \quad x_{N+1} = 1\,, \\ x_{i+1} = x_i + h\,,\ i = 0 \text{ to } N\,. \end{cases} \tag{3.4}$$

The approximation space $\tilde{V}$ can now be defined as:

$$\tilde{V} = \left\{\tilde{v}/\tilde{v} \in C^0([0,1]),\ \tilde{v}|_{[x_i,x_{i+1}]} \in P_1,\ \tilde{v}(0) = \tilde{v}(1) = 0\right\} \,, \tag{3.5}$$

in which $P_1 \equiv P_1([x_i,x_{i+1}])$ refers to the polynomial space which is defined over $[x_i,x_{i+1}]$, having a degree less or equal to one.

– What is the dimension of $\tilde{V}$?

6) Let $\varphi_i, (i = 1 \text{ to } \dim\tilde{V})$, be the canonical basis of $\tilde{V}$ establishing $\varphi_i(x_j) = \delta_{ij}$, in which δ_{ij} refers to the Krönecker symbol.

After having written the approximate variational formulation ($\widetilde{\mathbf{VP}}$), of solution $\tilde{u}$, which is associated to the variational problem (**VP**), show that by choosing:

$$\tilde{v}(x) = \varphi_i(x),\ (i = 1 \text{ to } \dim\tilde{V}) \quad \text{and} \quad \tilde{u}(x) = \sum_{j=1,\dim\tilde{V}} \tilde{u}_j \varphi_j\,, \tag{3.6}$$

the following system ($\widetilde{\mathbf{VP}}$) is obtained:

$$(\widetilde{\mathbf{PV}}) \quad \sum_{j=1,\dim\tilde{V}} A_{ij}\tilde{u}_j = b_i, \quad \forall i \in \{1,\ldots,\dim\tilde{V}\}\,, \tag{3.7}$$

where it has been observed that:

$$A_{ij} = \int_0^1 (\varphi_i' \varphi_j' + \varphi_i \varphi_j) \mathrm{d}x, \quad b_i = \int_0^1 f \varphi_i \mathrm{d}x \,. \tag{3.8}$$

► Function φ_i Characteristic of a Node Strictly Interior at [0,1]

7) Given the regularity of the mesh, the generic nodal equation of the $(\widetilde{\mathbf{VP}})$ system associated to any basis function φ_i, which is characteristic of a node strictly interior at $[0,1]$, is expressed as:

$$(\widetilde{\mathbf{VP}}_{\mathrm{Int}}) \, A_{i,i-1}\tilde{u}_{i-1} + A_{i,i}\tilde{u}_i + A_{i,i+1}\tilde{u}_{i+1} = b_i, \quad (\forall i = 1 \text{ to } K_h) \,, \tag{3.9}$$

where it has been assumed that:

$$K_h = \dim \tilde{V} \,.$$

– Using the trapezium formula, calculate the 4 coefficients of $(A_{ij},\ b_i)$.

8) Group the results together by writing down the corresponding nodal equation.

9) Show that the centrered finite differences scheme associated with the differential equation of the continuous problem (**CP**) is obtained again. What is its degree of precision?

It is pointed out that the trapezium quadrature formula is written as:

$$\int_a^b \xi(s) \mathrm{d}s \simeq \frac{(b-a)}{2} \{\xi(a) + \xi(b)\} \,.$$

3.1.2 Solution

► Variational Formulation – Theoretical Part

A.1) Let v be a test function, defined on $[0,1]$ having real values and "sufficiently regular". Each time a variational formulation is needed, the regularity of the functions v will be specified *a posteriori*, so that the formulation is significant enough to be understood.

The differential equation of the continuous problem (**CP**) is then multiplied by v and is integrated along the interval $[0,1]$.

$$-\int_0^1 u'' v \,\mathrm{d}x + \int_0^1 uv \,\mathrm{d}x = \int_0^1 fv \,\mathrm{d}x, \quad \forall v \in V \,. \tag{3.10}$$

An integration by parts moreover leads to:

$$\int_0^1 u'v' \,\mathrm{d}x + u'(0)v(0) - u'(1)v(1) + \int_0^1 uv \,\mathrm{d}x = \int_0^1 fv \,\mathrm{d}x, \quad \forall v \in V \,. \tag{3.11}$$

It is now observed that the homogeneous boundary conditions for u, $(u(0) = u(1) = 0)$ do not appear in the integral formulation (3.11).

In order to retain the whole information of the continuous problem (**CP**) in the future variational formulation (**VP**), it would therefore be suitable to impose that test functions v fulfil the boundary conditions:

$$v(0) = v(1) = 0 \,. \tag{3.12}$$

Such a method indeed ensures that the solution u, as one of the functions v of the searched variational space V, will have all the properties required at the boundary conditions on $[0,1]$.

The following formal variational formulation is thus obtained:

Find u belonging to V being the solution of:

$$\int_0^1 (u'v' + uv)\mathrm{d}x = \int_0^1 fv \,\mathrm{d}x, \quad \forall v \text{ such that:} \quad v(0) = v(1) = 0 \,. \tag{3.13}$$

In fact, this variational formulation is indeed formal, since it is necessary to specify the regularity of the test functions, which enables the equ. (3.13) to acquire significance, especially the convergence of the integrals of the equation.

This consequently leads to the specification of the functional space V within which the solution u of the integral formulation (3.13) would be found out.

This is performed by making use of the Cauchy-Schwartz Inequality that produces the following inequality of control:

$$\left| \int_0^1 u'v' \mathrm{d}x \right| \le \int_0^1 |u'v'| \mathrm{d}x \le \left[\int_0^1 |u'|^2 \mathrm{d}x \right]^{1/2} \cdot \left[\int_0^1 |v'|^2 \mathrm{d}x \right]^{1/2} , \tag{3.14}$$

$$\left|\int_0^1 f v \mathrm{d}x\right| \leq \int_0^1 |fv| \mathrm{d}x \leq \left[\int_0^1 |f|^2 \mathrm{d}x\right]^{1/2} \cdot \left[\int_0^1 |v|^2 \mathrm{d}x\right]^{1/2} . \tag{3.15}$$

Given that the inequality (3.14) can be rewritten by substituting u' by u and v' by v, so as to process the integral bearing on the product uv by the same method.

Therefore, if the variational space V is determined as being the set of the functions v belonging to $L^2(0,1)$ and whose first derivative also belongs to $L^2(0,1)$, the variational equation (3.13) is correctly defined.

To conclude, and by adding the homogenous Dirichlet boundary conditions (3.12), the variational space V, in which the solution u of the variational formulation (**VP**) will be sought is nothing else but the Sobolev space $H_0^1(0,1)$, which is defined by (3.3). Finally, the variational formulation (**VP**) is written as:

$$(\mathbf{VP}) \begin{cases} \text{Find } u \text{ belonging to } V \text{ being the solution of: } a(u,v) = L(v),\ \forall v \in V, \text{ where:} \\ a(u,v) \equiv \displaystyle\int_0^1 \left[u'(x)v'(x) + u(x)v(x)\right] \mathrm{d}x\,, \\ L(v) \equiv \displaystyle\int_0^1 f(x)v(x)\mathrm{d}x\,, \\ V \equiv H_0^1(0,1)\,. \end{cases} \tag{3.16}$$

A.2) In order to prove the existence and uniqueness of the solution pertaining to the variational problem (**VP**) (3.16), the application of Lax-Milgram theorem 4 requires the choice of a norm to be defined in the functional space $H_0^1(0,1)$.

Yet, as $H_0^1(0,1) \subset H^1(0,1)$, it is natural to evaluate the size of the functions of $H_0^1(0,1)$ using the natural norm of $H^1(0,1)$.

In other words, the following formula is proposed:

$$\forall v \in H_0^1(0,1) : \|v\|_{H^1}^2 \equiv \int_0^1 v^2 \mathrm{d}x + \int_0^1 v'^2 \mathrm{d}x \equiv \|v\|_{L^2}^2 + \|v'\|_{L^2}^2 . \tag{3.17}$$

It is then seen that the bilinear form $a(.,.)$ is none other than the inner product from which the H^1 norm (3.17) is obtained:

$$\forall v \in H^1(0,1) : a(v,v) \equiv (v,v)_{H^1} = \|v\|_{H^1}^2 \,,$$

in which $(.,.)_{H^1}$ has been written as the inner product in H^1.

Under these conditions, the Sobolev spaces $H^1(0,1)$ and $H_0^1(0,1)$, (as a subspace of $H^1(0,1)$, also is a Hilbert space for the norm (3.17)), (the works of P. A. Raviart [7] or H. Brézis [1] may be referred to for clarifications).

Moreover, the continuity constant for the bilinear form $a(.,.)$ can be then easily be obtained since it only necessary to use the Cauchy-Schwartz Inequality to find the inner product $(.,.)_{H^1}$:

$$|a(u,v)| \equiv |(u,v)_{H^1}| \leq (u,u)_{H^1}^{1/2} \cdot (v,v)_{H^1}^{1/2} = \|u\|_{H^1} \cdot \|v\|_{H^1} . \tag{3.18}$$

In other words, the continuity constant for the form $a(.,.)$ is equal to one.

The continuity for the linear form L is furthermore obtained by interpreting the inequality check (3.15) with the use of the H^1 norm:

$$| L(v) |^2 \leq \left[\int_0^1 |f(x)v(x)| \mathrm{d}x \right]^2 \leq \|f\|_{L^2}^2 \|v\|_{L^2}^2 \leq \|f\|_{L^2}^2 \|v\|_{H^1}^2 \ . \tag{3.19}$$

Hence, the continuity constant that arises for the linear form L is equal to the L^2-norm of the second member f.

Finally, the property of the V-ellipticity for the bilinear form $a(.,.)$ is immediate if it is observed that:

$$a(v,v) = \|v'\|_{L^2}^2 + \|v\|_{L^2}^2 = \|v\|_{H^1}^2 \geq \|v\|_{H^1}^2 \ , \tag{3.20}$$

which means that the ellipticity constant is also equal to one.

The Lax-Milgram theorem then implies that there is strictly only one function belonging to $H_0^1(0,1)$, which is the solution of the variational problem (**VP**).

► *Remark*

The application of the Lax-Milgram could have been perfectly performed by choosing a different norm. Specifically, for this variational problem (**VP**) established in $H_0^1(0,1)$, a more precise norm exists to describe the elements of this Sobolev space.

Indeed, if the norm is changed by establishing:

$$\forall v \in H_0^1(0,1): \ \|v\|_{H_0^1} \equiv \left[\int_0^1 v'(x)^2 \mathrm{d}x \right]^{1/2} \ . \tag{3.21}$$

It is then easily shown that (3.21) is a norm for $H_0^1(0,1)$.

In particular, the first property of the norms is fulfilled, given that any function v belonging to $H_0^1(0,1)$ is zero on the border of its definition interval.

Hence, if v is a function of $H_0^1(0,1)$ such that $\|v\|_{H_0^1} = 0$, v' is therefore zero on $[0,1]$. Consequently, v is a constant on the entire interval $[0,1]$ that can be measured, especially when $x=0$. This then implies that v is identically zero on $[0,1]$.

It will be noted that the other properties of the norm $H_0^1(0,1)$ may be immediately established.

A.3) To achieve results of complementary regularity for weak solutions to a variational problem (**VP**) is often difficult and therefore requires the availability of a fairly sophisticated mathematical tool.

For the present work, the study is limited to simple one-dimensional case for which the mathematical tools that will be used are referred to in Chap. 1, paragraph 1.1, sect. 1.1.4.

Thus, let u be an element of $H_0^1(0,1)$, solution to the variational problem (**VP**) and the following is obtained:

$$\int_0^1 u'v'\mathrm{d}x = \int_0^1 (f-u)\,v\,\mathrm{d}x, \quad \forall v \in H_0^1(0,1)\,. \tag{3.22}$$

The function space $C_0^1(]0,1[)$ is then introduced and defined by:

$$C_0^1(]0,1[) \equiv \left\{v\colon\ [0,1] \to \mathbf{R},\ \ v \in C^1(]0,1[),\ \ \mathit{Supp}\,v \subset]0,1[\right\}\,, \tag{3.23}$$

where *Supp* v denotes the support of function v.

Therefore function v from $C_0^1(]0,1[)$ can be chosen in the equality (3.22), since this equality is in fact, a series of variational equations valid for all functions v belonging to $H_0^1(0,1)$ and containing $C_0^1(]0,1[)$.

Moreover, if function g is introduced and defined by $g = f - u$, the result is:

$$g \in L^2(0,1) \subset L^1(0,1) \subset L^1_{\mathrm{loc}}(0,1)\,, \tag{3.24}$$

where space $L^1_{\mathrm{loc}}(0,1)$ is defined by:

Let any K be a closed subset strictly included in $[0,1]$, then:

$$\text{Given} \quad v \in L^1_{\mathrm{loc}}(0,1) \quad \text{then} \quad v \in L^1(K)\,. \tag{3.25}$$

Thus, the variational equations family (3.22) can be expressed in $C_0^1(]0,1[)$, (according to lemma 1), in the form of:

$$\int_0^1 u'v'\mathrm{d}x = \int_0^1 gv\mathrm{d}x = -\int_0^1 Gv'\mathrm{d}x, \quad \forall v \in C_0^1(]0,1[)\,, \tag{3.26}$$

where G is a primitive of g.

It is then expressed (3.26) in the form:

$$\int_0^1 (u'+G)v'\mathrm{d}x = 0, \quad \forall v \in C_0^1(]0,1[)\,. \tag{3.27}$$

Having finally obtained $u'+G \in L^2(0,1)$, since u belongs to $H^1(0,1)$, on one part, and G belongs to $C^0(]0,1[) \cap H^1(0,1)$ on the other, the result obtained, according to the same lemma 2, is $u'+G$ which belongs to $L^1_{\mathrm{loc}}(0,1)$.

It is then possible to apply the lemma 1 to the variational equation (3.27), which leads to:

$$u' + G = C^{\mathrm{te}}\,. \tag{3.28}$$

Thus, it is seen that u' is a function of $H^1(0,1)$, as the difference of a constant and of the G function.

It is finally inferred that u belongs to $H^2(0,1)$.

A.4) It is now important to establish the equivalence between the solution of the continuous problem (**CP**) and that of the variational problem (**VP**).

Clearly, if u is a solution to the continuous problem (**CP**) looked for in $H^2(0,1)$ then u is a weak solution to the variational problem (**VP**).

For this to happen, it is necessary that the construction process of the variational formulation (**VP**) be re-examined and be stated to be licit, (specially using the integration formula by parts, cf. Chapter 1, theorem 4), being given that u belongs to $H^2(0,1)$ and v to $H_0^1(0,1)$.

The reciprocal is then established. Let u be the solution belonging to $H_0^1(0,1) \cap H^2(0,1)$ of the variational problem (**VP**).

The integration formula is used by parts in the reverse order to the one used to obtain the variational formulation.

This results in:

$$\int_0^1 (-u'' + u - f) v \, \mathrm{d}x = 0, \quad \forall v \in H_0^1(0,1)\,. \tag{3.29}$$

The particular functions v belonging to $\mathscr{D}(0,1)$, which are the functions v belonging to $C^\infty(]0,1[)$ whose support is strictly included in interval $]0,1[$, are then chosen from the variational equations (3.29). This choice is legitimate since $\mathscr{D}(0,1) \subset H_0^1(0,1)$.

The density theorem 2 is then used as follows, rewriting equ. (3.29) in space $\mathscr{D}(0,1)$:

$$\int_0^1 (-u'' + u - f) v \, \mathrm{d}x = 0, \quad \forall v \in \mathscr{D}(0,1)\,. \tag{3.30}$$

It would be better to have the last equality family (3.30) for any v function belonging to $L^2(0,1)$ in order to choose $v = -u'' + u - f$, as special function of $L^2(0,1)$ to conclude.

Therefore let φ be any function belonging to $L^2(0,1)$ and according to the density theorem 2, $\mathscr{D}(0,1)$ is dense in $L^2(0,1)$. Then, there exists a sequence of functions φ_n belonging to $\mathscr{D}(0,1)$ and that converge towards φ, according to the L^2 norm:

$$\lim_{n \to \infty} \left[\int_0^1 |\varphi_n - \varphi|^2 \, \mathrm{d}x \right] = 0\,. \tag{3.31}$$

However, for any function of the sequence φ_n belonging to $\mathscr{D}(0,1)$, the equality (3.30) occurs by choosing $v = \varphi_n$:

$$\int_0^1 (-u'' + u - f) \varphi_n \, \mathrm{d}x = 0, \quad \forall n \in \mathbf{N}\,. \tag{3.32}$$

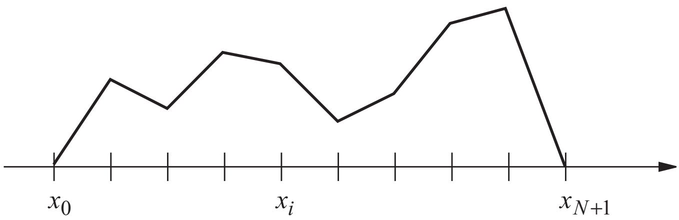

Fig. 3.1 Profile of a Piecewise Affine Function

It is then possible to obtain the same property for the functions φ of $L^2(0,1)$ by the following method:

$$\left|\int_0^1 \psi \cdot \varphi\right| = \left|\int_0^1 \psi \cdot (\varphi - \varphi_n)\right| \leq \left[\int_\Omega |\psi|^2\right]^{1/2} \cdot \left[\int_0^1 |\varphi_n - \varphi|^2\right]^{1/2}, \tag{3.33}$$

where it was stated: $\psi = -u'' + u - f$.

n is made to tend towards $+\infty$ in inequality (3.33), which demonstrates that:

$$\int_0^1 \psi \cdot \varphi \, \mathrm{d}x = 0, \quad \forall \varphi \in L^2(0,1)\,. \tag{3.34}$$

The demonstration now ends by choosing function as one of all functions of $L^2(0,1)$ which is equal to: $\varphi^* = -u'' + u - f$.

It is then deduced that:

$$-u'' + u + f = 0 \quad \text{in} \quad L^2(0,1)\,. \tag{3.35}$$

Moreover, if the second member f belongs to $L^2(0,1) \cap C^0(]0,1[)$, then the differential equation is satisfied for any x belonging to $]0,1[$ and the solution u is the classical solution to the continuous problem (**CP**) belonging to $C^2(]0,1[)$.

► Numerical part – Lagrange Finite Elements $\mathbf{P_1}$

A.5) The dimension of the approximation space $\tilde{V}$ can be determined in various ways. The simplest and smartest way is to state that the functions $\tilde{v}$ of $\tilde{V}$ are basically pecked lines affined by full mesh $[x_i, x_{i+1}]$ and cancelling each one when $x = 0$ and when $x = 1$, (see Fig. 3.1).

Hence, having $(N+2)$ points of discretisation for the entire mesh of interval $[0,1]$, two $\tilde{V}$ functions stand out because of their difference in values that may be seen at N interior points $(x_1, \ldots, x_N)$.

Any function $\tilde{v}$ of $\tilde{V}$ also needs to satisfy, $\tilde{v}_0 = \tilde{v}_{N+1} = 0$.

In other words, a function $\tilde{v}$ belonging to $\tilde{V}$ is entirely determined by the N-tuple $(\tilde{v}_1, \ldots, \tilde{v}_N)$.

This implies that the space is isomorphic to $\mathbf{R}^N$. In conclusion, it can be deduced that the dimension of $\tilde{V}$ is equal to N.

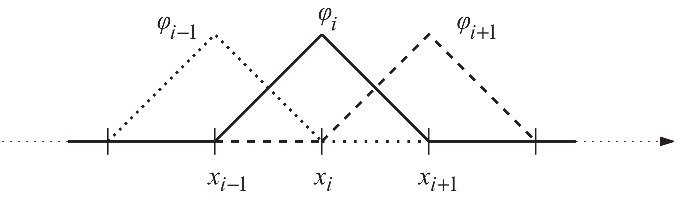

Fig. 3.2 Basis Functions φ_{i-1}, φ_i and φ_{i+1}

A.6) The approximated variational formulation is obtained by substituting the approximation functions $(\tilde{u},\tilde{v})$ to the (u,v) functions in the variational formulation (**VP**).

Moreover, the approximation expressions given by (3.6) are used and the following is obtained:

Find the numerical sequence $(\tilde{u}_j),(j=1 \text{ to } N)$, solution to:

$$\sum_{j=1,N}\left[\int_0^1 \left(\varphi_i'\varphi_j'+\varphi_i\varphi_j\right)\mathrm{d}x\right]\tilde{u}_j=\int_0^1 f\varphi_i\,\mathrm{d}x,\quad \forall i=1 \text{ to } N\,. \tag{3.36}$$

The expressions of A_{ij}, and b_j corresponding to the formulas (3.8) are then obtained by identification.

► Function φ_i Characteristic of a Node Strictly Interior at [0,1]

A.7) The basis functions φ_i, characteristic of nodes strictly interior to integration interval $[0,1]$, are now considered.

The generic equation of system (3.36) has, *a priori*, non zero terms, except those corresponding to φ_j functions whose support intercepts those of the φ_i function considered (see Fig. 3.2).

Thus, the basis functions concerned are: φ_{i-1}, φ_i and φ_{i+1}.

This explains why the equation ($\widetilde{\mathbf{VP}}_{\mathbf{int}}$), only has terms $A_{i,i-1}$, $A_{i,i}$ and $A_{i,i+1}$ and is expressed according to (3.9).

► Approximate Calculation of Coefficients $A_{ij}, j=i-1, i, i+1$.

a) *Approximation of coefficient A_{ii}.*

$$\begin{aligned}
A_{ii} &= \int_0^1 \left(\varphi_i'^2+\varphi_i^2\right)\mathrm{d}x &&= \int_{Supp\,\varphi_i}\left(\varphi_i'^2+\varphi_i^2\right)\mathrm{d}x\,,\\
&= \int_{x_{i-1}}^{x_i}\left(\varphi_i'^2+\varphi_i^2\right)\mathrm{d}x &&+ \int_{x_i}^{x_{i+1}}\left(\varphi_i'^2+\varphi_i^2\right)\mathrm{d}x\,,\\
&\simeq \left(\frac{1}{h^2}\times h\right)+\frac{h}{2}(0+1) + && \left(\frac{1}{h^2}\times h\right)+\frac{h}{2}(1+0)\,,\\
A_{ii} &\simeq \frac{2}{h}+h\,.
\end{aligned} \tag{3.37}$$

This was achieved by considering the fact that the basis functions φ_i of are piecewise affines. Thereafter, the derivatives φ_i' are constant on each mesh having the form $[x_i, x_{i+1}]$.

The integrals bearing on those derivatives can then be calculated either exactly or by using the trapezium quadrature formula being exact for constants functions.

b) *Approximation of coefficient* $A_{i,i-1}$.

$$\begin{aligned}
A_{i,i-1} &= \int_0^1 \left(\varphi_i'\varphi_{i-1}' + \varphi_i\varphi_{i-1}\right) \mathrm{d}x \,, \\
&= \int_{Supp\ \varphi_{i-1} \cap Supp\ \varphi_i} \left(\varphi_i'\varphi_{i-1}' + \ \varphi_i\varphi_{i-1}\right) \mathrm{d}x \,, \\
&\simeq \left(-\frac{1}{h^2} \times h\right) + \frac{h}{2}\left[(0 \times 1) + (1 \times 0)\right] \,, \\
A_{i,i-1} &\simeq -\frac{1}{h} \,.
\end{aligned} \tag{3.38}$$

c) *Approximation of coefficient* $A_{i,i+1}$.

Calculation of the coefficient $A_{i,i+1}$ is easily obtained as long as the following symmetrical properties are observed:

- Matrix A of coefficient A_{ij} is symmetrical: $A_{i,j} = A_{j,i}$.
- The mesh over interval $[0, L]$ is translation invariant as a consequence of its uniform step of constant discretisation h.

It then becomes:

$$A_{i,i-1} \overset{\substack{\textbf{Symmetry}\\ \downarrow}}{=} A_{i-1,i} \overset{\substack{\textbf{Invariant}\\ \downarrow}}{=} A_{i,i+1} \simeq -\frac{1}{h} \,.$$

▶ Estimation of the Second Member b_i

The second member b_i is calculated by considering that every basis function φ_i, characteristic of a strictly interior node has a support consisting of the union of the $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ intervals, (see Fig. 3.2).

It then becomes:

$$\begin{aligned}
b_i &= \int_0^1 f\varphi_i \,\mathrm{d}x = \int_{x_{i-1}}^{x_i} f\varphi_i \,\mathrm{d}x + \int_{x_i}^{x_{i+1}} f\varphi_i \,\mathrm{d}x \,, \\
&\simeq \frac{h}{2}\left[0 + f_i\right] + \frac{h}{2}\left[f_i + 0\right] \,, \\
b_i &\simeq hf_i \,.
\end{aligned} \tag{3.40}$$

A.8) The previous results (3.37)–(3.40) are then grouped to obtain the corresponding nodal equation:

$$-\frac{\tilde{u}_{i-1}-2\tilde{u}_i+\tilde{u}_{i+1}}{h^2}+\tilde{u}_i=f_i, \quad (i=1 \text{ to } N) \ . \tag{3.41}$$

A.9) Discretization by finite differences of second order differential equation of the continuous problem (**CP**) is classical.

The method uses the Taylor's formula after choosing to express the differential equation at the discretisation point x_i:

$$-u''(x_i)+u(x_i)=f(x_i),(i=1 \text{ to } N) \ . \tag{3.42}$$

Taylor's formula enables the substitution of the second derivative u'' at point x_i by algebraic combination, using different values of the unknown u in different proximal points of the mesh.

To obtain an order which is consistent with the finite elements method, the progressive form and the regressive form of the Taylor's formula are used:

$$u(x_{i+1})=u(x_i)+hu'(x_i)+\frac{h^2}{2}u''(x_i)+\frac{h^3}{6}u'''(x_i)+O(h^4) \ , \tag{3.43}$$

$$u(x_{i-1})=u(x_i)-hu'(x_i)+\frac{h^2}{2}u''(x_i)-\frac{h^3}{6}u'''(x_i)+O(h^4) \ . \tag{3.44}$$

The addition of equs. (3.43) and (3.44) is then performed.

It then becomes:

$$u''(x_i)=\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2}+O(h^2) \ . \tag{3.45}$$

The expression of the second derivative of u (3.45) is substituted at point x_i in the differential equation (3.20) to obtain:

$$-\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2}+u(x_i)=f(x_i)+O\left(h^2\right), \quad (i=1 \text{ to } N) \ . \tag{3.46}$$

The traces u_i of u, $(u_i \equiv u(x_i))$ are replaced at nodes x_i, by the approximations $\tilde{u}_i,(\tilde{u}_i \approx u_i)$, in order to preserve the equality between both members of (3.46) when suppressing the infinitely small $O(h^2)$.

This substitution leads to an exact correspondence between the scheme with finite differences and the nodal equation (3.41).

It is obvious that the scheme with finite differences (3.41) is of the second order, considering the approximation process that was explained earlier.

As a matter of fact, if the $u(x_i)$ values were substituted by the $\tilde{u}_i$ values, the result would be the differential equation (3.42) as closely as possible by $O(h^2)$.

This is the reason why the scheme having finite differences (3.41) is the second order.

3.2 The Neumann Problem

3.2.1 Statement

The aim of this problem is to propose a mathematical and numerical study of the solution to a linear differential problem, subjected to Neumann boundary conditions.

Thus, let u be a function of the real variable defined on $[0,1]$ and has values in $\mathbf{R}$. The interest is on the solution to the continuous problem (**CP**) defined by:

Find $u \in H^2(0,1)$ as solution to:

$$(\mathbf{CP})\begin{cases} -u''(x)+u(x)=f(x), \quad 0 \leq x \leq 1\ , \\ u'(0)=u'(1)=0\ , \end{cases} \tag{3.47}$$

where f is a given function belonging to $L^2(0,1)$.

▶ Variational Formulation – Theoretical Part

1) Let v be a test function defined on $[0,1]$ having real values and belonging to a variational space V.

Show that the continuous problem (**CP**) can be written in a variational formulation (**VP**) like the following:

$$a(u,v)=L(v), \quad \forall v \in V\ .$$

The bilinear form $a(.,.)$, the linear form $L(.)$ and the functional space V will be specified.

2) Establish the existence and uniqueness of the weak solution of the variational problem (**VP**) in $H^1(0,1)$.

3) Show that any weak solution to the variational problem (**VP**) also belongs to $H^2(0,1)$.

4) Deduce from it the equivalence between the strong formulation of the problem (**CP**) set in $H^2(0,1)$ and the weak formulation of the variational problem (**VP**) considered in $H^1(0,1) \cap H^2(0,1)$.

▶ Numerical Part – Lagrange Finite Elements $\mathbf{P_1}$

5) The approximation of the variational problem (**VP**) is performed by using Lagrange finite elements $\mathbf{P_1}$.

To make that happen, we introduce a regular mesh of the interval $[0,1]$ with a constant step h, so that:

$$\begin{cases} x_0 = 0,\ x_{N+1} = 1\ , \\ x_{i+1} = x_i + h\ ,\ i = 0 \text{ to } N\ . \end{cases} \tag{3.48}$$

The approximation space $\tilde{V}$ is now defined using:

$$\widetilde{V} = \left\{ \tilde{v} \colon [0,1] \to \mathbf{R},\ \tilde{v} \in C^0([0,1]),\ \tilde{v}|_{[x_i, x_{i+1}]} \in P_1 \right\}\ , \tag{3.49}$$

where $P_1 \equiv P_1([x_i, x_{i+1}])$ refers to the space of polynomials defined over $[x_i, x_{i+1}]$ having a degree less than or equal to one.

– What is the dimension of $\tilde{V}$?

6) Let $\varphi_i, (i = 1 \text{ to } \dim \tilde{V})$, be the canonical basis of $\tilde{V}$ verifying $\varphi_i(x_j) = \delta_{ij}$, where δ_{ij} refers to the Kronecker symbol.

After having written the approximate variational formulation ($\widetilde{\mathbf{VP}}$), having a solution and associated with the variational problem (**VP**), show that by selecting:

$$\tilde{v}(x) = \varphi_i(x), \quad (i = 1 \text{ to } \dim \tilde{V}) \quad \text{and} \quad \tilde{u}(x) = \sum_{j=1,\dim \tilde{V}} \tilde{u}_j \varphi_j\ , \tag{3.50}$$

the following ($\widetilde{\mathbf{VP}}$) system is obtained:

$$(\widetilde{\mathbf{VP}}) \sum_{j=1,\dim \tilde{V}} A_{ij} \tilde{u}_j = b_i, \quad \forall i \in \{1, \ldots, \dim \tilde{V}\}\ , \tag{3.51}$$

where the following was noted:

$$A_{ij} = \int_0^1 (\varphi_i' \varphi_j' + \varphi_i \varphi_j) \mathrm{d}x, \quad b_i = \int_0^1 f \varphi_i \mathrm{d}x\ . \tag{3.52}$$

► Function φ_i Characteristic of Node Strictly Interior at [0,1]

7) Given the mesh regularity, the generic nodal equation of the ($\widetilde{\mathbf{VP}}$) system associated with any basis function $\varphi_i, (i = 1 \text{ to } \dim \tilde{V} - 2)$, characteristic of a node interior at $[0,1]$ is written as:

$$(\widetilde{\mathbf{VP}}_{\mathbf{Int}})\ A_{i,i-1} \tilde{u}_{i-1} + A_{i,i} \tilde{u}_i + A_{i,i+1} \tilde{u}_{i+1} = b_i, \quad (\forall i = 1 \text{ to } \dim \widetilde{V} - 2)\ . \tag{3.53}$$

– Using the trapezium formula, calculate the 4 coefficients $(A_{ij},\ b_i)$.

8) Group the results together by writing the corresponding nodal equation.

9) Show that the centred finite differences scheme associated with the differential equation of the continuous problem (**CP**) is obtained again. What is its order of precision?

Remember that the trapezium quadrature is written as:

$$\int_a^b \xi(s)\mathrm{d}s \simeq \frac{(b-a)}{2}\{\xi(a)+\xi(b)\} .$$

► Function φ_0 Characteristic of the Node $x_0 = 0$

10) When considering the basis function φ_0 characteristic of the node $x_0 = 0$, show that the corresponding nodal equation of the ($\widetilde{\mathbf{VP}}$) system is written as:

$$(\widetilde{\mathbf{VP}}_\mathbf{0})\; A_{0,0}\tilde{u}_0 + A_{0,1}\tilde{u}_1 = b_0 . \tag{3.54}$$

– Using the trapezium formula, calculate the 3 coefficients $A_{0,0}$, $A_{0,1}$ and b_0.

11) Write the corresponding nodal equation.

12) Show that the finite differences scheme associated with the Neumann condition in $x = 0$ is obtained again. What is its order of precision?

► Function φ_{N+1} Characteristic of the Node $x_{N+1} = 1$

13) Now, consider the basis function φ_{N+1} characteristic of the node $x_{N+1} = 1$.

Show that the nodal equation associated with the ($\widetilde{\mathbf{VP}}$) system is written as:

$$(\widetilde{\mathbf{VP}}_\mathbf{N+1})\; A_{N+1,N}\tilde{u}_N + A_{N+1,N+1}\tilde{u}_{N+1} = b_{N+1} . \tag{3.55}$$

– Using the trapezium formula, calculate the 3 coefficients $A_{N+1,N+1}$, $A_{N+1,N}$ and b_{N+1}.

14) Write the corresponding nodal equation.

15) Show that the finite differences scheme associated with the Neumann condition in $x = 1$ is obtained again. What is its order of precision?

3.2.2 Solution

► Variational Formulation – Theoretical Part

A.1) Let v be a test function defined on $[0,1]$ having real values and "sufficiently regular".

As already mentioned in the presentation of the Dirichlet problem (see paragraph [3.1]), the regularity of functions v will be specified *a posteriori* in order to give sense to the variational formulation, when the latter is established.

The differential equation of the continuous problem (**CP**) is multiplied by v then integrated over the interval $[0,1]$.

$$-\int_0^1 u''v\mathrm{d}x + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x, \quad \forall v \in V\ . \tag{3.56}$$

An integration by parts then gives the following:

$$\int_0^1 u'v'\mathrm{d}x + u'(0)v(0) - u'(1)v(1) + \int_0^1 uv\mathrm{d}x = \int_0^1 fv\mathrm{d}x, \quad \forall v \in V\ . \tag{3.57}$$

Here, the homogenous Neumann boundary conditions defined in the continuous problem (**CP**), $(u'(0) = u'(1) = 0)$, appear in the integral formulation (3.57).

As a result and by considering the above two boundary conditions, the following formulation is obtained:

Find u belonging to V being the solution to:

$$\int_0^1 (u'v' + uv)\mathrm{d}x = \int_0^1 fv\mathrm{d}x, \quad \forall v \in V\ . \tag{3.58}$$

At this stage, the formulation (3.58) is only formal since the various integrals appearing in it have no reason to be convergent.

It is then observed that this variational formulation is strictly analogous to the one obtained within the framework of the Dirichlet problem – see paragraph [3.1], (3.13) – except the boundary conditions that should no longer be imposed on the test functions v within the framework of the Neumann problem treated here.

That is the reason why, if the functional analysis presented in paragraph [3.1] is used, a sufficient condition guaranteeing the convergence of the integrals in the variational formulation (3.58) consists in defining the variational space V as follows:

$$V \equiv H^1(0,1) \equiv \{v\colon [0,1] \to \mathbf{R},\ v \in L^2(0,1),\ v' \in L^2(0,1)\}\ . \tag{3.59}$$

Finally, the variational problem (**VP**) is written as:

$$
(\mathbf{VP})\begin{cases}
\text{Find } u \text{ belonging to } V \text{ solution of: } a(u,v)=L(v), \quad \forall v\in V, \text{ where:}\\
a(u,v)\equiv \displaystyle\int_0^1 \left[u'(x)v'(x)+u(x)v(x)\right]\mathrm{d}x\,,\\
L(v)\equiv \displaystyle\int_0^1 f(x)v(x)\mathrm{d}x\,,\\
V\equiv H^1(0,1)\,.
\end{cases}
\tag{3.60}
$$

A.2) The existence and uniqueness of the variational problem (**VP**) (3.60) are demonstrated by applying the Lax-Milgram theorem (lemma 4).

To make that happen, choosing one norm to be defined on the functional space $H^1(0,1)$ represents one of the key points in the application of the Lax-Milgram Theorem.

Then the choice is to measure the dimension of functions v belonging to $H^1(0,1)$ by the natural norm defined by:

$$
\forall v\in H^1(0,1):\ \|v\|_{H^1}^2 \equiv \int_0^1 v(x)^2\mathrm{d}x+\int_0^1 v'(x)^2\mathrm{d}x \equiv \|v\|_{L^2}^2+\|v'\|_{L^2}^2\,. \tag{3.61}
$$

The norm being selected, the process that consists in verifying the various hypotheses of the Lax-Milgram theorem is strictly similar to the one presented for the Dirichlet problem, (see paragraph [3.1]).

The following is a point-by-point summary of this verification:

- The space $H^1(0,1)$ is a Hilbert space for the norm (3.61); the inner product resulting from this norm coincides exactly with the bilinear norm $a(.,.)$ defined by (3.60).
- The bilinear form $a(.,.)$ is continuous on $H^1(0,1)\times H^1(0,1)$; the continuity constant being equal to one.
- The linear form $L(.)$ defined by (3.60) is continuous on $H^1(0,1)$; the continuity constant being equal to the L^2-norm of the second member f.
- The form $a(.,.)$ is H^1-elliptic and the ellipticity constant is equal to one.

The application of the Lax-Milgram theorem thus implies the existence of one and only one function u belonging to $H^1(0,1)$, the solution to the variational problem (**VP**) defined by (3.60).

A.3) Once again, the Dirichlet problem [3.1] will be used to treat this question.

In fact, as mentioned above, achieving complementary regularity results for weak solutions to the variational problem (**VP**) may require sufficiently sophisticated mathematical tools.

That is the reason why the present study, being limited to a one-dimensional case, refers to mathematical results mentioned in Chap. 1, paragraph 1.1, sect. 1.1.4.

So, let u be an element of $H^1(0,1)$, solution to the variational problem (**VP**), and the following is obtained:

$$\int_0^1 u'v'\mathrm{d}x = \int_0^1 (f-u)v\,\mathrm{d}x, \quad \forall v \in H^1(0,1)\,. \tag{3.62}$$

Among the functions v belonging to $H^1(0,1)$, only those belonging to $C_0^1(]0,1[)$, $(C_0^1(]0,1[) \subset H^1(0,1))$, are then selected.

Moreover, by introducing the function g defined by $g = f - u$, the following is obtained:

$$g \in L^2(0,1) \subset L^1(0,1) \subset L^1_{\mathrm{loc}}(0,1)\,, \tag{3.63}$$

where the space $L^1_{\mathrm{loc}}(0,1)$ is defined by:

Let any K be a closed subset strictly included in $[0,1]$, then:

$$\text{Given } v \in\ L^1_{\mathrm{loc}}(0,1) \text{ then } v \in L^1(K)\ . \tag{3.64}$$

Thus the family of variational equations (3.62) can be written within $C_0^1(]0,1[)$ in the following form:

$$\int_0^1 u'v'\mathrm{d}x = \int_0^1 gv\,\mathrm{d}x = -\int_0^1 Gv'\mathrm{d}x, \quad \forall v \in C_0^1(]0,1[)\,, \tag{3.65}$$

where G is a primitive of g. (To make that happen, the lemma 2 would be used).

(3.65) is then written in the following form:

$$\int_0^1 (u'+G)v'\mathrm{d}x = 0, \quad \forall v \in C_0^1(]0,1[)\,. \tag{3.66}$$

Having finally obtained $u'+G \in L^2(0,1)$, (since, on one hand u belongs to $H^1(0,1)$ and on the other hand G belongs to $C^0(]0,1[) \cap H^1(0,1)$), still according to lemma 2, $u'+G$ belonging to $L^1_{\mathrm{loc}}(0,1)$ is obtained in the same manner.

Then, lemma 1 may be applied to the variational equation (3.66), leading to:

$$u' + G = C^{\mathrm{te}}\,. \tag{3.67}$$

Thus, it happens that u' is a function belonging to $H^1(0,1)$ as the difference of a constant and of the function G.

Finally, it is inferred that u belongs to $H^2(0,1)$.

A.4) This last question of the theoretical part is dedicated to the equivalence between the solution to the continuous problem (**CP**) and the solution to the variational problem (**VP**).

The direct way is simple since if u is a solution to the continuous problem (**CP**) searched for in $H^2(0,1)$, then u is a weak solution to the variational problem (**VP**).

To do so, it is only necessary to revert back to the process that enabled the establishment of the variational formulation (**VP**) and to note that the latter makes sense, (in particular by using the integration-by-parts formula, see Chap. 1, theorem 4), given that u belongs to $H^2(0,1)$ and v to $H^1(0,1)$.

The reciprocal is now calculated. In the variational problem (**VP**), let u be a solution belonging to $H^2(0,1)$ – any solution u to the variational problem henceforth belongs to $H^2(0,1)$ according to the previous question.

The integration-by-parts formula is then used in the inverse direction to the one that enabled the variational formulation to be obtained and this gives:

$$\int_0^1 (-u'' + u - f)v\,\mathrm{d}x = 0, \quad \forall v \in H^1(0,1)\,. \tag{3.68}$$

It is then noticed that the formulation (3.68) is identical to the one considered in the Dirichlet problem [3.1] except the functional framework (H_0^1 in the Dirichlet problem and H^1which is here considered in the Neumann problem).

It is then only necessary to observe the functional inclusion $H_0^1 \subset H^1$ even if it is trivial, in order to strictly apply the whole methodology that has been presented. (see Dirichlet problem [3.1], question No. **4**).

The essential points treated in the rest of the demonstration is thus pointed out:

- In the equality (3.68), functions v belonging to $\mathscr{D}(0.1)$ are selected since $\mathscr{D}(0,1) \subset H^1(0,1)$.
- The density theorem 2 is used: $\mathscr{D}(0,1)$ is dense in $L^2(0,1)$.
- Then, it is shown that the equality (3.68) no longer takes place in $H^1(0,1)$ but in a bigger space i. e. in $L^2(0,1)$.
- It is then possible to choose from all the v functions belonging to $L^2(0,1)$ and involved in the equation (3.68), the one that exactly equals: $v^* = -u'' + u - f$.

If in addition, the second member f belongs to $L^2(0,1) \cap C^0(]0,1[)$ then the differential equation is satisfied for any $x \in]0,1[$ and the solution u is the classical solution to the continuous problem (**CP**) belonging to $C^2(]0,1[)$.

▶ Numerical Part – Lagrange Finite Elements $\mathbf{P_1}$

A.5) To calculate the dimension of space $\tilde{V}$, the following remark is necessary:

The definition (3.49) of the approximation space is almost similar to the one considered in the Dirichlet problem (see problem of [3.1], question No.**5**, (3.5)).

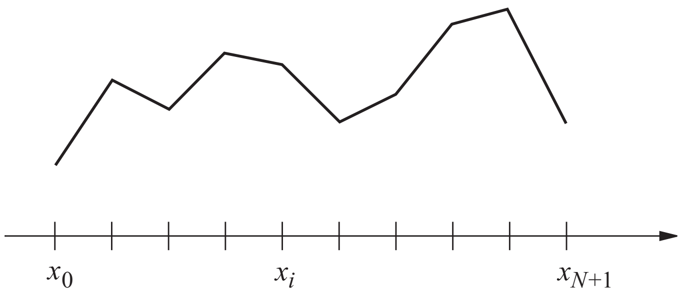

Fig. 3.3 Profile of a Piecewise Affine Function

Thus, using the demonstration performed within the framework of the Dirichlet problem, it is only necessary to note that in space $\tilde{V}$ defined by (3.49), two liberty degrees, due to the two values of any function $\tilde{v}$ of $\tilde{V}$ in $x = 0$ and in $x = 1$, add two units to the dimension found in the Dirichlet problem.

In other words, finding any function $\tilde{v}$ of $\tilde{V}$ means finding its trace $(\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_N, \tilde{v}_{N+1})$ in $(N+2)$ discretisation points of the mesh in the interval $[0,1]$, i. e. $(x_0, x_1, \ldots, x_N, x_{N+1})$ which on their own fix the definition of $\tilde{v}$ (see Fig. 3.3).

As a result, space $\tilde{V}$ is isomorphic to $\mathbf{R}^{N+2}$ and the dimension of $\tilde{V}$ is equal to $N+2$.

A.6) As usual, the approximate variational formulation $(\widetilde{\mathbf{VP}})$ is obtained by substituting the approximate functions $(\tilde{u}, \tilde{v})$ for functions (u, v) in the variational formulation (**VP**).

Moreover, the expressions supplied by the formula (3.50) are used.

Thus, the approximate variational formulation $(\widetilde{\mathbf{VP}})$ is written as:

$$(\widetilde{\mathbf{VP}}) \left[\begin{array}{l} \text{Find the numerical sequence } (\tilde{u}_j)\,,\ (j = 0 \text{ to } N+1)\,, \text{ solution to:} \\ \displaystyle\sum_{j=0}^{N+1} \left[\int_0^1 \left(\varphi_i' \varphi_j' + \varphi_i \varphi_j \right) \right] \tilde{u}_j = \int_0^1 f \varphi_i(x), \quad \forall i = 0 \text{ to } N+1\,. \end{array} \right. \tag{3.69}$$

The expressions of A_{ij}, and b_j corresponding to the formulas (3.52) are then obtained by identification.

► Function φ_i characteristic of a node strictly interior at [0,1]

A.7) When observing variational formulation $(\widetilde{\mathbf{VP}})$ defined by (3.69), it appears that, in the linear system consecutive equations $(N+2)$, the "interior" N equations corresponding to the values of j ranging from 1 to N, are totally identical to those found in the Dirichlet problem, (see paragraph 3.1, question **7**).

As mentioned previously in the theoretical part, only the functional frame differs between the two formulation in order to consider the change in boundary conditions.

It is then expected to find the same approximation described by the nodal equations associated to the basis functions φ_i, characteristic of nodes strictly interior to $[0,1]$ mesh interval.

In other words, the nodal equation (3.53) coefficients have the following value:

$$A_{ii} \equiv \frac{2}{h} + h,\ A_{i,i-1} = A_{i,i+1} \equiv -\frac{1}{h},\ b_i \equiv hf_i\ . \tag{3.70}$$

This is the result of the variational formulation identical formalism between the Dirichlet and the Neumann problem, for basis functions φ_i characteristic of nodes strictly interior at interval $[0,1]$.

A.8) The nodal equation of the approximate variational problem ($\widetilde{\mathbf{VP}}$) corresponding to the basis function φ_i, which is characteristic of a strictly interior node x_i and written as:

$$-\frac{\tilde{u}_{i-1} - 2\tilde{u}_i + \tilde{u}_{i+1}}{h^2} + \tilde{u}_i = f_i,\ (i = 1 \text{ to } N)\ . \tag{3.71}$$

A.9) For the same reasons previously mentioned in the last questions, the analogy made with the Dirichlet problem ensures that the results, concerning the finite differences scheme obtained similarly in the present case, are at one's disposal.

Therefore, the finite differences scheme, with application of a second order discretisation to the differential equation of the continuous problem (**CP**), precisely corresponds to the nodal equation (3.71).

► Function φ_0 Characteristic of the Node $x_0 = 0$

A 10) The generic equation (3.69) of the approximate variational problem ($\widetilde{\mathbf{VP}}$), corresponding to the basis function φ_0, characteristic of the node x_0, is written as:

$$(\widetilde{\mathbf{VP}_0})\ A_{00}\tilde{u}_0 + A_{01}\tilde{u}_1 = b_0\ . \tag{3.72}$$

This results from the fact that when considering the approximate variational problem ($\widetilde{\mathbf{VP}}$) (3.69) in the case of the basis function φ_0, the summation upon the other basis functions φ_j only leads to zero contribution.

This is again the consequence of the position relative to the support of each of the basis functions φ_j as regards to the one of the basis function φ_0, (see Fig. 3.4).

► Approximate Calculation of the Coefficients $\mathbf{A_{00}}$ and $\mathbf{A_{01}}$

a) *Approximation of the coefficient* A_{00}.

The calculation of the coefficient is performed in a way analogous to that presented for the calculation of the coefficient A_{ii} in the answer to the question **7**.

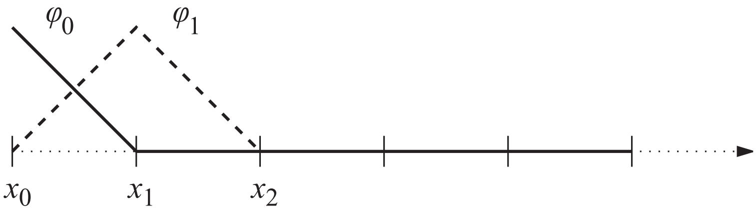

Fig. 3.4 Basis Functions φ_0 and φ_1

However, there is a difference since the basis function φ_0 comprises a support, which is solely constituted of the mesh $[x_0, x_1]$ while the basis functions φ_i, $(i = 1, N)$, have a support which is made up of two meshes, $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$.

The working out of the coefficient A_{00} is therefore performed as follows:

$$
\begin{aligned}
A_{00} &= \int_0^1 \left(\varphi_0'^2 + \varphi_0^2\right) \mathrm{d}x = \int_{Supp\ \varphi_0} \left(\varphi_0'^2 + \varphi_0^2\right) \mathrm{d}x \\
&= \int_{x_0}^{x_1} \left(\varphi_0'^2 + \varphi_0^2\right) \mathrm{d}x\,, \\
&\simeq \left(\frac{1}{h^2} \times h\right) + \frac{h}{2}(0+1)\,.
\end{aligned}
\tag{3.73}
$$

Thus finally:

$$
A_{00} \simeq \frac{1}{h} + \frac{h}{2}\,. \tag{3.74}
$$

b) *Approximation of the coefficient A_{01}.*

$$
\begin{aligned}
A_{01} &= \int_0^1 \left(\varphi_0'\varphi_1' + \varphi_0\varphi_1\right) \mathrm{d}x = \int_{Supp\ \varphi_0 \cap\ Supp \varphi_1} \left(\varphi_0'\varphi_1' + \varphi_0\varphi_1\right) \mathrm{d}x\,, \\
&\simeq -\frac{1}{h^2} \times h + \frac{h}{2}\left[(0 \times 1) + (1 \times 0)\right].
\end{aligned}
\tag{3.75}
$$

Results consequently in:

$$
A_{01} \simeq -\frac{1}{h}\,. \tag{3.76}
$$

It will be noted that the approximation of the coefficient A_{01} represents a particular case of the generic calculation presented above (see paragraph 3.1, (3.38), in so far as the discretisation step h is constant.

Therefore, the integration of the basis function φ_0 against the basis function φ_1 along the interval $[x_0, x_1]$ is completely equivalent to the integration of φ_i against φ_{i+1} along the interval $[x_i, x_{i+1}]$.

To prove this, it should be possible to carry out the substitution of the adequate variable by associating the interval $[x_i, x_{i+1}]$ with the interval $[x_0, x_1]$ and to observe the equality between the coefficients $A_{i,i+1}$ and A_{01}.

► **Evaluation of the Second Member b_0**

The evaluation of the second member b_0 is performed according to the scheme similar to the one presented for the second member b_i (see paragraph 3.1, (3.40)).

Therefore, the following is obtained:

$$b_0 = \int_0^1 f\varphi_0 \, \mathrm{d}x = \int_{x_0}^{x_1} f\varphi_0 \, \mathrm{d}x \simeq \frac{h}{2}[0 + f_0] \simeq \frac{h}{2} f_0 \, . \tag{3.77}$$

A.11) The results (3.73)–(3.77) are consequently gathered to obtain the corresponding nodal equation associated with the function of φ_0, characteristic of the node x_0.

$$\left(\frac{1}{h} + \frac{h}{2}\right)\tilde{u}_0 - \frac{1}{h}\tilde{u}_1 = \frac{h}{2} f_0 \, . \tag{3.78}$$

A.12) To determine the finite differences scheme which discretises the Neumann condition when $x_0 = 0, (u'(x_0) = 0)$, it can be observed that, if it were necessary to maintain the second order of discretisation obtained for the finite differences scheme (3.72), associated with differential equation of the continuous problem (**CP**), inside the interval $[0,1]$, the Taylor's expansion that will be studied must be written till the third order.

Therefore, the following Progressive Taylor's Expansion will be written as follows:

$$u(x_1) = u(x_0) + hu'(x_0) + \frac{h^2}{2}u''(x_0) + O(h^3) \, . \tag{3.79}$$

It is then possible to replace the value of the first derivative which is zero when $x_0 = 0$ (since it concerns the Neumann condition), still, as usual, in the case of the application of such a method, there appears the second derivative of u at the point x_0.

Therefore, the differential equation of the continuous problem (**CP**) can be assumedly written till the border of the interval, namely, here, when $x_0 = 0$:

$$-u''(x_0) + u(x_0) = f(x_0) \, . \tag{3.80}$$

The equ. (3.80) can express the second derivative u'' at the point x_0 and inserted in the Taylor's expansion (3.79).

There consequently results:

$$u(x_1) = u(x_0) + hu'(x_0) + \frac{h^2}{2}[u(x_0) - f(x_0)] + O(h^3) \, . \tag{3.81}$$

The approximations can then be worked out, while omitting the rest $O(h^3)$ in the equ. (3.81).

Following the process of discretisation when $x_0 = 0$, the equation is then written as:

$$\tilde{u}_1 = \tilde{u}_0 + \frac{h^2}{2}\left[\tilde{u}_0 - f_0\right] . \qquad (3.82)$$

The nodal equation (3.78), corresponding to the basis function φ_0, characteristic of the node x_0 of the discretisation is found.

► **Function φ_{N+1} characteristic of the node $x_{N+1} = 1$**

A.13) Considerations analogous to those presented above, for the working out of the nodal equation when $x_0 = 0$, are also relevant for the nodal equation when $x_{N+1} = 1$.

As such, it is only necessary to see to it that the situation of the basis functions φ_N and φ_{N+1} is symmetrical to the situation of the basis functions φ_0 and φ_1, (see Fig. 3.5).

That is why the equation of the approximate variational formulation ($\widetilde{\mathbf{VP}}$) corresponding to the basis function φ_{N+1} is written as:

$$(\widetilde{\mathbf{VP}}_{\mathbf{N+1}}) \;\; A_{N+1,N}\tilde{u}_N + A_{N+1,N+1}\tilde{u}_{N+1} = b_{N+1} . \qquad (3.83)$$

Likewise, provided that a great care is taken to replace the integration interval $[x_0, x_1]$ by the one which corresponds to the support of the function φ_{N+1} i. e. $[x_N, x_{N+1}]$, the following results for the evaluation of coefficients $A_{N+1,N}$, $A_{N+1,N+1}$ and b_{N+1} are obtained.

► **Approximate Calculation of Coefficients $A_{N+1,N}$ and $A_{N+1,N+1}$**

a) *Approximation of coefficient* $A_{N+1,N+1}$.

$$\begin{aligned} A_{N+1,N+1} &= \int_0^1 \left(\varphi_{N+1}'^2 + \varphi_{N+1}^2\right) \mathrm{d}x = \int_{Supp\ \varphi_{N+1}} \left(\varphi_{N+1}'^2 + \varphi_{N+1}^2\right) \mathrm{d}x , \\ &= \int_{x_N}^{x_{N+1}} \left(\varphi_{N+1}'^2 + \varphi_{N+1}^2\right) \mathrm{d}x , \\ &\simeq \left(\frac{1}{h^2} \times h\right) + \frac{h}{2}(0+1) , \\ &\simeq \frac{1}{h} + \frac{h}{2} . \end{aligned} \qquad (3.84)$$

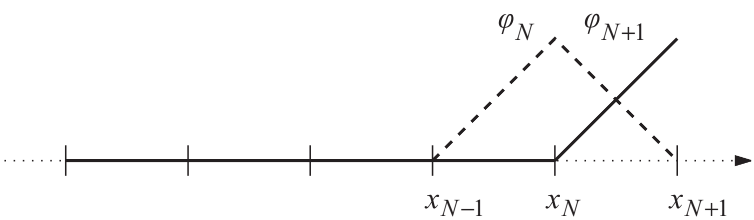

Fig. 3.5 Basis Functions φ_N and φ_{N+1}

b) *Approximation of coefficient* $A_{N+1,N}$.

$$\begin{aligned} A_{N+1,N} &= \int_0^1 \left(\varphi_N' \varphi_{N+1}' + \varphi_N \varphi_{N+1}\right) \mathrm{d}x \,, \\ &= \int_{Supp\ \varphi_N \cap Supp\ \varphi_{N+1}} \left(\varphi_N' \varphi_{N+1}' + \varphi_N \varphi_{N+1}\right) \mathrm{d}x \,, \\ &\simeq \left(-\frac{1}{h^2} \times h\right) + \frac{h}{2}\left[(0 \times 1) + (1 \times 0)\right] \simeq -\frac{1}{h} \,. \end{aligned} \tag{3.85}$$

► **Estimation of the Second Member b_{N+1}**

The approximation of the second member b_{N+1} is obtained by using the trapezium quadrature formula:

$$\begin{aligned} b_{N+1} &= \int_0^1 f \varphi_{N+1} \mathrm{d}x \,, \\ &= \int_{x_N}^{x_{N+1}} f \varphi_{N+1} \mathrm{d}x \simeq \frac{h}{2}\left[0 + f_{N+1}\right] \simeq \frac{h}{2} f_{N+1} \,. \end{aligned} \tag{3.86}$$

A.14) The nodal equation associated with the basis function φ_{N+1} is written by grouping the results of (3.84)–(3.86).

This equation is perfectly symmetrical compared to the one associated with the basis function φ_0:

$$-\frac{1}{h}\tilde{u}_N + \left(\frac{1}{h} + \frac{h}{2}\right)\tilde{u}_{N+1} = \frac{h}{2} f_{N+1} \,. \tag{3.87}$$

A.15) Given the symmetry mentioned in the previous question, finding the nodal equation (3.87) by using finite differences is conceivable.

In fact, at abscissa $x_{N+1} = 1$, if a progressive expansion was considered when $x_0 = 0$, this time a regressive Taylor's expansion must be considered in the following way:

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2} u''(x_{N+1}) + O(h^3) \,. \tag{3.88}$$

Then, the second derivative when x_{N+1} is replaced by writing the differential equation of the continuous problem (**CP**) at the point x_{N+1}:

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2}\left[u(x_{N+1}) - f(x_{N+1})\right] + O(h^3) \,. \tag{3.89}$$

Now, by exploiting the information concerning the homogenous Neumann condition when x_{N+1}, the pendant of the discrete equation (3.82) is obtained provided that the values $u(x_i)$ are replaced by the respective approximations $\tilde{u}_i$:

$$\tilde{u}_N = \tilde{u}_{N+1} + \frac{h^2}{2}\left[\tilde{u}_{N+1} - f_{N+1}\right] \,. \tag{3.90}$$

3.3 The Fourier-Dirichlet Problem

3.3.1 Statement

The aim of this problem is to propose a mathematical and numerical study of the solution to a second order linear differential problem subjected to Fourier-Dirichlet mixed boundary conditions.

Let u be a function of a real variable, defined from values $[0,1]$ in $\mathbf{R}$.

The considered continuous problem (**CP**) is defined by:

To find $u \in H^2(0,1)$ which is the solution to:

$$(\mathbf{CP}) \begin{cases} -u''(x) + u(x) = f(x), & 0 \leq x \leq 1 \, , \\ u(0) = 0, \quad u'(1) + ku(1) = 1 \, , \end{cases} \tag{3.91}$$

where f is a given function belonging to $L^2(0,1)$ and k a given positive or zero real parameter.

► Variational Formulation – Theoretical Part

1) Let v be a test function, defined by $[0,1]$, and having real values, belonging to the variational space V. Show that the continuous problem (**CP**) can be expressed as a variational formulation (**VP**) in the form:

$$a(u,v) = L(v), \quad \forall v \in V \, .$$

The bilinear form $a(.,.)$, the linear form $L(.)$ and the functional space V need to be specified.

2) Establish the existence and uniqueness of a weak solution of the variational problem (**VP**) in $H^1_*(0,1)$ defined by:

$$H^1_*(0,1) = \left\{ v\colon \,]0,1[\to \mathbf{R}, v \text{ and } v' \in L^2(0,1), \; v(0) = 0 \right\} \, . \tag{3.92}$$

3) Show that any weak solution to $H^1_*(0,1)$ the variational problem (**VP**) also belongs to $H^2(0,1)$.

4) Infer from therein, the equivalence between the strong formulation presented in $H^2(0,1)$ and the weak formulation considered in $H^1_*(0,1) \cap H^2(0,1)$.

► Lagrange Finite Element $\mathbf{P_1}$ – Numerical Part

5) Approximation of the variational problem (**VP**) is performed using Lagrange finite elements $\mathbf{P}_1$.

To achieve this, a regular mesh of interval $[0,1]$ of constant step h is introduced, such as:

$$\begin{cases} x_0 = 0,\ x_{N+1} = 1\ , \\ x_{i+1} = x_i + h,\ i = 0 \text{ to } N\ . \end{cases} \tag{3.93}$$

The approximation space $\tilde{V}$ is now defined using:

$$\tilde{V} = \left\{ \tilde{v}\colon [0,1] \to \mathbf{R},\ \tilde{v} \in C^0([0,1]),\ \tilde{v}|_{[x_i,x_{i+1}]} \in P_1([x_i,x_{i+1}]),\ \tilde{v}(0) = 0 \right\}\ , \tag{3.94}$$

where $P_1([x_i,x_{i+1}])$ denotes the polynomial space defined over $[x_i,x_{i+1}]$, of degree less than or equal to one.

– What is the dimension of $\tilde{V}$?

6) Let $\varphi_i, (i = 1 \text{ to } \dim \tilde{V})$, be the canonical basis $\tilde{V}$ of establishing $\varphi_i(x_j) = \delta_{ij}$.

After having written the approximate variational formulation $(\widetilde{\mathbf{VP}})$, of solution $(\widetilde{\mathbf{VP}})$, which is associated to the variational problem $(\mathbf{VP})$, show that by choosing:

$$\tilde{v}(x) = \varphi_i(x),\ (i = 1 \ \text{ to} \dim \tilde{V}) \ \text{ and } \ \tilde{u}(x) = \sum_{j=1,\, \dim \tilde{V}} \tilde{u}_j \varphi_j\ , \tag{3.95}$$

the following $(\widetilde{\mathbf{VP}})$ system is obtained:

$$(\widetilde{\mathbf{VP}}) \quad \sum_{j=1,\, \dim \tilde{V}} A_{ij} \tilde{u}_j = b_i, \quad \forall i \in \{1, \ldots, \dim \tilde{V}\}\ , \tag{3.96}$$

where the following was noted:

$$A_{ij} = \int_0^1 (\varphi_i' \varphi_j' + \varphi_i \varphi_j) \mathrm{d}x + k\varphi_i(1)\varphi_j(1), \quad b_i = \int_0^1 f \varphi_i \mathrm{d}x + \varphi_i(1)\ . \tag{3.97}$$

► **Function φ_i Characteristic of a Node Strictly Interior at [0,1]**

7) Given the regularity of the mesh, the generic nodal equation of the $(\widetilde{\mathbf{VP}})$ system associated to any function with basis φ_i, which is characteristic of a node strictly interior at $[0,1]$, is expressed as:

$$(\widetilde{\mathbf{VP}}_{\mathrm{Int}}) \ A_{i,i-1}\tilde{u}_{i-1} + A_{i,i}\tilde{u}_i + A_{i,i+1}\tilde{u}_{i+1} = b_i, \quad (\forall i = 1,\ \dim \tilde{V} - 1)\ . \tag{3.98}$$

– Using the trapezium rule, calculate the 4 coefficients (A_{ij}, b_i).

8) Group the results together by writing down the corresponding nodal equation.

9) Show that the centred finite differences scheme associated with the differential equation of the continuous problem (**CP**) is obtained again. What is its order of precision? It is pointed out that the trapezium quadrature formula is written as:

$$\int_a^b \xi(s)\mathrm{d}s \simeq \frac{(b-a)}{2}\{\xi(a)+\xi(b)\} .$$

► Function φ_{N+1} Characteristic of the Node $x_{N+1} = 1$

10) Now, consider the basis function φ_{N+1} characteristic of the node $x_{N+1} = 1$.

Show that the nodal equation associated with the $(\widetilde{\mathbf{VP}})$ system is written as:

$$(\widetilde{\mathbf{VP}}_{\mathbf{N+1}}) \quad A_{N+1,N}\tilde{u}_N + A_{N+1,N+1}\tilde{u}_{N+1} = b_{N+1} . \tag{3.99}$$

– Using the trapezium formula, calculate the 3 coefficients $A_{N+1,N}$, $A_{N+1,N+1}$ and b_{N+1}.

11) Write the corresponding nodal equation.

12) Show that the finite differences scheme associated to the Fourier boundary conditions when $x = 1$ is obtained again. What is its order of precision?

3.3.2 Solution

► Variational Formulation – Theoretical Part

A.1) Let v be a test function defined by $[0,1]$ having real values and "sufficiently regular".

As already mentioned in the presentation of the Dirichlet problem (see paragraph [3.1]), the regularity of functions v will be specified *a posteriori* in order to give sense to the variational formulation, when the latter is established.

The differential equation of the continuous problem (**CP**) is multiplied by v then integrated over the interval $[0,1]$.

$$-\int_0^1 u''(x)v(x)\mathrm{d}x+\int_0^1 u(x)v(x)\mathrm{d}x=\int_0^1 f(x)v(x)\mathrm{d}x,\quad \forall v\in V\ . \tag{3.100}$$

The integration by parts then results in:

$$\begin{aligned}\int_0^1 u'(x)v'(x)\mathrm{d}x + u'(0)v(0) - u'(1)v(1) + \int_0^1 u(x)v(x)\mathrm{d}x,\\ = \int_0^1 f(x)v(x)\mathrm{d}x,\quad \forall v\in V\ .\end{aligned} \tag{3.101}$$

Here the Fourier boundary conditions, defined in the continuous problem (**CP**), $(u'(1)+ku(1)=1)$, appears in the integral formulation (3.101). In fact, this may be confirmed by re-writing the Fourier condition in the form:

$$u'(1)=1-ku(1)\ , \tag{3.102}$$

in order to replace the first derivative of solution u, at the abscissa $x=1$ in equation (3.101).

Moreover, as previously observed for this type of second order differential equation, the homogenous Dirichlet condition when $x=0$ cannot be directly taken into account in formulation (3.101).

This explains why this homogenous Dirichlet condition is imposed on test functions v whose solution u constitutes a specific case.

This method guarantees that the full memory of the information contained in the continuous problem (**CP**) is maintained in the future variational formulation.

These two boundary conditions, (the first one bearing on u via its relationship with (3.102) and the second one concerning the zero v test functions when $x=0$) lead to the following variational formulation (**VP**):

Find u belonging to V, solution to:

$$\int_0^1 (u'v'+uv)\mathrm{d}x+ku(1)v(1)=\int_0^1 fv\mathrm{d}x+v(1),\quad \forall v\in V. \tag{3.103}$$

At this stage of the study, the Vspace is composed of v functions subjugated to the homogenous Dirichlet condition when $x = 0 : v(0) = 0$.

Meanwhile, formulation (3.103) is only formal since the different integrals do not need to be convergent.

Actually, this variational formulation is structurally analogous to the one obtained within the framework of the Dirichlet problem – see (3.13), paragraph [3.1] – except for the boundary conditions that need to be modified to suit the test functions v within the framework of the Fourier-Dirichlet problem being examined here.

Thus, referring to the functional analysis previously shown in paragraph [3.1], a sufficient condition securing the convergence of integrals of the variational formulation (3.103) is to consider the following functional framework:

$$V \equiv H^1(0,1) \equiv \{v\colon [0,1] \to \mathbf{R}, v \in L^2(0,1), v' \in L^2(0,1)\}. \tag{3.104}$$

The homogenous Dirichlet condition at abscissa $x = 0$ is added to the above functional space to finally lead to the following variational formulation:

$$(\mathbf{VP}) \begin{cases} \text{Find } u \text{ belonging to } V \text{ solution of: } a(u,v) = L(v), \quad \forall v \in V, \text{ where:} \\ a(u,v) \equiv \displaystyle\int_0^1 (u'v' + uv)\mathrm{d}x + ku(1)v(1)\,, \\ L(v) \equiv \displaystyle\int_0^1 fv\mathrm{d}x + v(1)\,, \\ V \equiv H^1_*(0,1)\,, \end{cases} \tag{3.105}$$

where space $H^1_*(0,1)$ is defined by (3.92).

A.2) The existence and uniqueness of the solution to the variational problem (**VP**) defined by (3.105), is obtained by applying the Lax-Milgram theorem 4.

This is achieved by again choosing the $H^1(0,1)$ norm (see paragraph 3.1, (3.61) and (3.106) defined hereafter), by trading off measurement of the "size" of as any function of $H^1(0,1), (H^1_*(0,1) \subset H^1(0,1))$.

The $H^1(0,1)$ natural norm, previously defined, is as follows:

$$\forall v \in H^1(0,1)\colon \|v\|^2_{H^1} \equiv \int_0^1 v(x)^2\mathrm{d}x + \int_0^1 v'(x)^2\mathrm{d}x \equiv \|v\|^2_{L^2} + \|v'\|^2_{L^2}\,. \tag{3.106}$$

Once the norm is chosen, the different points described below need to be validated in order to apply the Lax-Milgram theorem:

a) $H^1_*(0,1)$ space is a Hilbert space for norm (3.106).

This is achieved by showing, for example, that $H^1_*(0,1)$ is a closed vector subspace of $H^1(0,1)$ for norm (3.106), (see H. Brézis, [1]).

b) The bilinear form $a(.,.)$ is continuous on $H^1_*(0,1) \times H^1_*(0,1)$ for norm (3.106).

In fact, let be $(u,v) \in H^1_*(0,1) \times H^1_*(0,1)$:

$$\begin{aligned} |a(u,v)| \leq & |(u,v)_{H^1}| + k|u(1)v(1)| \\ \leq & (u,u)^{1/2}_{H^1} \cdot (v,v)^{1/2}_{H^1} + k|u(1)v(1)| \\ \leq & \|u\|_{H^1} . \|v\|_{H^1} + k|u(1)v(1)| \,, \end{aligned} \tag{3.107}$$

where $(.,.)_{H^1}$ denotes the inner product of H^1 from which norm (3.106) arises.

Moreover, for any function v belonging to $H^1(0,1)$, the result is:

$$\forall x \in [0,1]:\ v(x) = v(0) + \int_0^x v'(t)\mathrm{d}t \,. \tag{3.108}$$

Thus, for the specific case of v functions belonging to $H^1_*(0,1)$, $v(0) = 0$.

It then becomes:

$$\forall x \in [0,1]:\ v(x) = \int_0^x v'(t)\mathrm{d}t \,. \tag{3.109}$$

Equation (3.109) is then expressed as $x = 1$ and the Cauchy-Schwartz inequality is used to obtain the following control inequality:

$$|v(1)| \leq \left[\int_0^1 v'^2(t)\mathrm{d}t \right]^{1/2} \equiv \|v'\|_{L^2} \,. \tag{3.110}$$

Inequality (3.107) is then used to obtain:

$$|a(u,v)| \leq \|u\|_{H^1} \cdot \|v\|_{H^1} + k\|u'\|_{L^2}\|v'\|_{L^2} \leq (1+k)\|u\|_{H^1} \cdot \|v\|_{H^1} \,. \tag{3.111}$$

The bilinear form $a(.,.)$ is thus continuous for the H^1-norm defined by (3.106). The continuity constant is actually equal to $(1+k)$.

c) The linear form $L(.)$ defined by (3.105) is continuous on $H^1_*(0,1)$ for norm (3.106).

The same scheme of analysis previously presented for the bilinear form $a(.,.)$ is applied.

$$| L(v) | \leq \int_0^1 |fv|\mathrm{d}x + |v(1)| \leq \|f\|_{L^2}\|v\|_{L^2} + |v(1)| \,. \tag{3.112}$$

The inequality control (3.110), which is valid for any v function belonging to $H^1_*(0,1)$, is used again to obtain:

$$| L(v) | \leq \|f\|_{L^2}\|v\|_{L^2} + \|v'\|_{L^2} \leq (1 + \|f\|_{L^2})\|v\|_{H^1} \,. \tag{3.113}$$

The linear form $L(.)$ is therefore continuous on space $H^1_*(0,1)$ with the norm (3.106) and the continuity constant is equal to $(1+\|f\|_{L^2})$.

d) The $a(.,.)$ form is H^1-elliptic and the ellipticity constant is equal to one.

In fact, the ellipticity inequality is immediate if it is reckoned that:

$$a(v,v) = \|v'\|^2_{L^2} + \|v\|^2_{L^2} + kv(1)^2 = \|v\|^2_{H^1} + kv(1)^2 \geq \|v\|^2_{H^1} \ . \tag{3.114}$$

The fact that parameter k is a real positive number has been used.

The application of the Lax-Milgram theorem (see theorem 10) thus implies that there exists one and only one function belonging to $H^1_*(0,1)$ being the solution to the variational problem (**VP**) defined by (3.105).

A.3) The method presented for the Dirichlet problem [3.1] is used to prove that any weak solution belonging to $H^1_*(0,1)$ is also a function of $H^2(0,1)$.

Thus, let u be an element of $H^1_*(0,1)$, solution to the variational problem (**VP**) and the following is obtained:

$$\int_0^1 u'v'\mathrm{d}x = \int_0^1 (f-u)v\mathrm{d}x + [1-ku(1)]v(1), \quad \forall v \in H^1_*(0,1) \ . \tag{3.115}$$

Among the functions v belonging to $H^1_*(0,1)$, those belonging to $C^1_0(]0,1[)$, $(C^1_0(]0,1[) \subset H^1_*(0,1))$, are then selected.

This choice implies that the retained functions v are null when $x=0$ and when $x=1$.

Moreover, the function g defined by $g = f-u$ is introduced and the following is obtained:

$$g \in L^2(0,1) \subset L^1(0,1) \subset L^1_{\mathrm{loc}}(0,1) \ , \tag{3.116}$$

where space $L^1_{\mathrm{loc}}(0,1)$ is defined by:

Let any K be a closed subset strictly included in $]0,1[$, then:

$$\text{Given} \quad v \in L^1_{\mathrm{loc}}(0,1) \quad \text{then} \quad v \in L^1(K) \ . \tag{3.117}$$

Thus, the family of variational equations (3.115) can be written within $C^1_0(]0,1[)$ in the form:

$$\int_0^1 u'(x)v'(x)\mathrm{d}x = \int_0^1 g(x)v(x)\mathrm{d}x = -\int_0^1 G(x)v'(x)\mathrm{d}x, \quad \forall v \in C^1_0(]0,1[) \ , \tag{3.118}$$

where G is a primitive of g (Lemma 2 is used to achieve this).

(3.118) is then written in the form:

$$\int_0^1 \left[u'(x) + G(x)\right] v'(x)\mathrm{d}x = 0, \quad \forall v \in C_0^1(0,1)\,. \tag{3.119}$$

In this form, the family of variational equations (3.119) strictly corresponds to the family of equations demonstrated in the Dirichlet problem (see Dirichlet problem [3.1], (3.27)).

The rest of the analysis is inferred from it and the same arguments are used to state that any solution u to the variational problem belonging to $H_*^1(0,1)$ also belongs to the Sobolev space $H^2(0,1)$.

A.4) This last question of the theoretical part is dedicated to the equivalence between the solution to the continuous problem (**CP**) and the solution to the variational problem (**VP**).

The direct sense is simple as it is the result of the construction of a solution to the variational formulation (**VP**) using a given solution to the continuous problem (**CP**).

Then, it will be observed that this construction process is licit provided that the solution u to the continuous problem (**CP**) belongs to $H^2(0,1)$ and the test functions v that intervene in the variational formulation (**VP**) belong to $H_*^1(0,1)$.

It will notably be noticed that if u is a solution to the continuous problem **(CP)** satisfying the homogenous Dirichlet boundary conditions when $x = 0$, then this time u when considered as a solution to the variational problem (**VP**) belongs *de facto* to $H_*^1(0,1)$.

The reciprocal is now considered. Let u belong to $H^2(0,1) \cap H_*^1(0,1)$ a solution to the variational problem (**VP**). According to the previous question, it is known that any solution u of the variational problem (**VP**) belonging to $H_*^1(0,1)$ also belongs to $H^2(0,1)$.

The integration-by-parts formula used in the reverse order to the one that yielded the variational formulation (**VP**) leads to the following:

$$\int_0^1 (-u'' + u - f)v\,\mathrm{d}x + \left[u'(1) + ku(1) - 1\right] v(1) = 0, \quad \forall v \in H_*^1(0,1)\,. \tag{3.120}$$

Then, consider the particular case of functions v in the equation (3.120) belonging to $H_0^1(0,1)$. This means that the functions v that are null when $x = 0$ and when $x = 1$.

In that case, the equation (3.120) is written as:

$$\int_0^1 (-u'' + u - f)\,v\,\mathrm{d}x = 0, \quad \forall v \in H_0^1(0,1)\,. \tag{3.121}$$

Here it is observed that, the formulation (3.121) is similar to the one considered in the Dirichlet problem [3.1].

That is why the rest of the reasoning is also similar. The essential points for the continuation of the demonstration are:

- In the equality (3.121), functions v belonging to $\mathscr{D}(0,1)$ are selected since $\mathscr{D}(0,1) \subset H^1(0,1)$.
- The density theorem 2 is used: $\mathscr{D}(0,1)$ is dense in $L^2(0,1)$.
- Then, it is shown that the equality (3.121) no longer takes place in $H^1(0,1)$ but in a bigger space i. e. in $L^2(0,1)$.
- It is then possible to choose from all the v functions belonging to $L^2(0,1)$ the one that exactly equals: $v^* = -u'' + u - f$.

Moreover, if the second member f belongs to $L^2(0,1) \cap C^0(]0,1[)$, the differential equation is satisfied for any x belonging to $]0,1[$ and the solution u is the classical solution of the continuous problem (**CP**) belonging to $C^2(]0,1[)$.

Once it is proved that the differential equation of the problem (**CP**) is satisfied by the solution to the variational problem (**VP**), the family of equations (3.120) is reduced and is written as:

$$\left[u'(1) + ku(1) - 1\right] v(1) = 0, \quad \forall v \in H^1_*(0,1)\,. \tag{3.122}$$

A last selection in equation (3.122) consists in considering the particular case of function v^* defined by: $v^{**}(x) = x, \forall x \in [0,1]$.

Of course, it will be easily verified that v^{**} belongs to $H^1_*(0,1)$. In that case, equation (3.122) implies that u satisfies the Fourier boundary conditions:

$$u'(1) + ku(1) = 1\,. \tag{3.123}$$

This ends the demonstration of the reciprocal and any solution of the variational problem (**VP**) belonging to $H^2(0,1) \cap H^1_*(0,1)$ is a solution to the continuous problem (**CP**).

► Lagrange Finite Elements $\mathbf{P_1}$ – Numerical Part

A.5) To calculate the dimension of space $\tilde{V}$, the following observation is necessary:

Once again, definition (3.94) of the approximation space $\tilde{V}$ is very close to the one considered in the Dirichlet problem (see problem of [3.1], question **5**, (3.5)).

Thus, by considering the demonstration presented above, it is only necessary to observe that space $\tilde{V}$ defined by (3.94) produces an additional degree of freedom attributable to the value of function $\tilde{v}$ of $\tilde{V}$ when $x = 1$.

As a result, the dimension of $\tilde{V}$ defined by (3.5) found in the Dirichlet problem should be increased by one unit.

In other words, for any function $\tilde{v}$ of $\tilde{V}$, knowledge of its trace $(\tilde{v}_1, \ldots, \tilde{v}_{N+1})$ at $(N+1)$ discretisation points of the mesh at the $[0,1]$ interval, namely in $(x_1, \ldots, x_{N+1})$, that fixes the definition of $\tilde{v}$ in a unique manner.

That is why space $\tilde{V}$ is isomorphic to $\mathbf{R}^{N+1}$ and the dimension of $\tilde{V}$ is equal to $N+1$.

A.6) In order to obtain the approximate variational formulation ($\widetilde{\mathbf{VP}}$), the approximation functions $(\tilde{u}, \tilde{v})$ are substituted in the (u, v) functions of the variational formulation (**VP**). Moreover, the expressions given by (3.95) are used.

The approximate variational formulation ($\widetilde{\mathbf{VP}}$) is thus written as:

Find the numerical sequence $(\tilde{u}_j), (j = 1 \text{ to } N+1)$, solution to:

$$\sum_{j=1}^{N+1} \left[\int_0^1 \left(\varphi_i' \varphi_j' + \varphi_i \varphi_j \right) \mathrm{d}x + k \varphi_i(1) \varphi_j(1) \right] \tilde{u}_j = \int_0^1 f \varphi_i(x) \mathrm{d}x + \varphi_i(1) \,,$$

$$(\forall i = 1 \text{ to } N+1) \,. \tag{3.124}$$

The expressions of A_{ij}, and b_j corresponding to the formulas (3.97) are then obtained by identification.

► Characteristic Function φ_j of a Node Strictly Interior at [0,1]

A.7) The characteristic basis functions φ_i of the nodes of the mesh that are strictly interior at the $[0,1]$ integration interval is now considered.

The generic equation of system (3.124) is strictly similar to the one obtained in the Dirichlet problem, (see Problem [3.1], (3.36)).

In fact, to be sure of that, it is only necessary to note that for any characteristic basis function φ_i of a node strictly interior at the $[0,1]$ interval, the following is obtained: $\varphi_i(1) = 0$, (see Fig. 3.6).

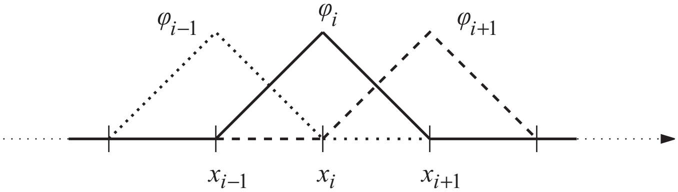

Fig. 3.6 Basis Functions φ_{i-1}, φ_i and φ_{i+1}

In that case, the nodal equ. (3.124) is then written as:

$$\sum_{j=1}^{N+1}\left[\int_0^1\left(\varphi_j'\varphi_i'+\varphi_j\varphi_i\right)\mathrm{d}x\right]\tilde{u}_j=\int_0^1 f\varphi_i(x)\mathrm{d}x,$$

$$(\forall i=1,N+1)\,, \tag{3.125}$$

which corresponds exactly to the nodal equation (3.36) of the Dirichlet problem.

That is why the results demonstrated in the Dirichlet problem are directly reused in order to exploit them directly in the Fourier-Dirichlet problem.

▶ Approximate Calculation of Coefficients $A_{ij}, j=i-1,i,i+1$

a) *Approximation of coefficient A_{ii}.*

$$A_{ii}\simeq\frac{2}{h}+h\,. \tag{3.126}$$

b) *Approximation of coefficient $A_{i,i-1}$.*

$$A_{i,i-1}=A_{i,i+1}\simeq-\frac{1}{h}\,. \tag{3.127}$$

▶ Estimation of the Second Member b_i

$$b_i\simeq hf_i\,. \tag{3.128}$$

A.8) The nodal equation associated with any characteristic function φ_i of a strictly interior node is obtained by grouping results (3.126)–(3.128):

$$-\frac{\tilde{u}_{i-1}-2\tilde{u}_i+\tilde{u}_{i+1}}{h^2}+\tilde{u}_i=f_i,\ i=1\text{ to }N\,. \tag{3.129}$$

A.9) Discretisation of the second order differential equation of the continuous problem (**CP**) by finite differences is strictly similar to the one presented in the Dirichlet problem.

This discretisation is written as:

$$-\frac{u\left(x_{i-1}\right)-2u\left(x_i\right)+u\left(x_{i+1}\right)}{h^2}+u\left(x_i\right)=f\left(x_i\right)+O\left(h^2\right),(i=1\text{ to }N)\,. \tag{3.130}$$

Then, the traces $u_i\equiv u(x_i)$ of function u, having nodes x_i are replaced by the respective approximations $(\tilde{u}_i\approx u_i)$ in order to keep equality between the two members of (3.130) during the elimination of the infinitesimal $O(h^2)$.

The result of this substitution is that the finite differences scheme obtained corresponds exactly to the nodal equation (3.129) associated with any characteristic function φ_i of a node x_i strictly interior at the $[0,1]$ interval.

Moreover, the finite differences scheme (3.129) is of the second order, given that the approximation consists in neglecting the term in $O(h^2)$ in the equation (3.130).

▶ Characteristic Basis Function φ_{N+1} of the Node $x_{N+1} = 1$

A.10) The nodal equation associated with the characteristic basis function φ_{N+1} of the node $x_{N+1} = 1$ is written as:

$$(\widetilde{\mathbf{VP}}_{\mathbf{N+1}}) \;\; A_{N+1,N}\tilde{u}_N + A_{N+1,N+1}\tilde{u}_{N+1} = b_{N+1} \,. \tag{3.131}$$

The coefficients $A_{N+1,N}$, $A_{N+1,N+1}$ and the second member b_{N+1} whose estimations are obtained below intervene in this equation (3.131).

▶ Approximate Calculation of Coefficients $A_{N+1,N}$ and $A_{N+1,N+1}$

a) *Approximation of coefficient* $A_{N+1,N+1}$.

$$\begin{aligned} A_{N+1,N+1} &= \int_0^1 \left(\varphi'^2_{N+1} + \varphi^2_{N+1}\right) \mathrm{d}x + k\varphi^2_{N+1}(1) \,, \\ &= \int_{x_N}^{x_{N+1}} \left(\varphi'^2_{N+1} + \varphi^2_{N+1}\right) \mathrm{d}x + k \,, \\ &\simeq \left(\frac{1}{h^2} \times h\right) + \frac{h}{2}(0+1) + k \,, \\ &\simeq \frac{1}{h} + \frac{h}{2} + k \,. \end{aligned} \tag{3.132}$$

It will be noticed that the characteristic property of the basis function φ_{N+1} has been used at abscissa x_{N+1}: $\varphi_{N+1}(x_{N+1}) = 1$.

b) *Approximation of coefficient* $A_{N+1,N}$.

$$\begin{aligned} A_{N+1,N} &= \int_0^1 \left(\varphi'_N\varphi'_{N+1} + \varphi_N\varphi_{N+1}\right) \mathrm{d}x + k\varphi_N(1)\varphi_{N+1}(1) \,, \\ &= \int_{Supp\ \varphi_N \,\cap\, Supp\ \varphi_{N+1}} \left(\varphi'_N\varphi'_{N+1} + \varphi_N\varphi_{N+1}\right) \mathrm{d}x \,, \\ &\simeq -\frac{1}{h^2} \times h + \frac{h}{2}\left[(0\times 1) + (1\times 0)\right] \simeq -\frac{1}{h} \,. \end{aligned} \tag{3.133}$$

Likewise, it will be noticed that the property of the basis function φ_N has been used: $\varphi_N(x_{N+1}) = 0$.

▶ Estimation of the Second Member b_{N+1}

Starting from:

$$\begin{aligned} b_{N+1} &= \int_0^1 f\varphi_{N+1}\mathrm{d}x + \varphi_{N+1}(1) = \int_{x_N}^{x_{N+1}} f\varphi_{N+1}\mathrm{d}x + 1 \,, \\ &\simeq \frac{h}{2}\left[0 + f_{N+1}\right] + 1 \simeq \frac{h}{2}f_{N+1} + 1 \,. \end{aligned} \tag{3.134}$$

A.11) The nodal equation associated with the basis function φ_{N+1} is written by grouping the results of (3.132)–(3.134).

This equation is written as:

$$-\frac{1}{h}\tilde{u}_N + \left[\frac{1}{h} + \frac{h}{2} + k\right] \tilde{u}_{N+1} = \frac{h}{2} f_{N+1} + 1 \,. \tag{3.135}$$

A.12) The Fourier boundary conditions at abscissa $x_{N+1} = 1$ are now discretised using finite differences.

To achieve this, a regressive Taylor's expansion is considered at abscissa x_{N+1} expressing the solution u of the continuous problem (**CP**) at abscissa x_N according to the values of u and of its derivatives at abscissa x_{N+1}.

This expansion is written as:

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2}u''(x_{N+1}) + O(h^3) \,. \tag{3.136}$$

Then, supposing that the differential equation of the continuous problem (**CP**) can be written at the border of the integration domain $]0,1[$ i. e. here at abscissa x_{N+1}, the second derivative of the solution u at abscissa x_{N+1} appearing in the equ. (3.136) is replaced as below:

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2}\left[u(x_{N+1}) - f(x_{N+1})\right] + O(h^3) \,. \tag{3.137}$$

Moreover, the first derivative of the solution u at abscissa x_{N+1}, is expressed by using the Fourier boundary conditions:

$$u'(x_{N+1}) = 1 - ku(x_{N+1}) \,. \tag{3.138}$$

Then equ. (3.137) takes the following form:

$$u(x_N) = u(x_{N+1}) - h\left[1 - ku(x_{N+1})\right] + \frac{h^2}{2}\left[u(x_{N+1}) - f(x_{N+1})\right] + O(h^3) \,. \tag{3.139}$$

Then, some algebraic manipulations are operated to write equation (3.139) in the following form:

$$-\frac{1}{h}u(x_N) + \left[\frac{1}{h} + \frac{h}{2} + k\right] u(x_{N+1}) = \frac{h}{2} f_{N+1} + 1 + O(h^2) \,. \tag{3.140}$$

The nodal equ. (3.135) associated with the basis function φ_{N+1} is then obtained, provided that the values $u(x_i)$ of the solution u to the continuous problem (**CP**) are replaced by the respective approximations $\tilde{u}_i$ in equ. (3.140):

$$-\frac{1}{h}\tilde{u}_N + \left[\frac{1}{h} + \frac{h}{2} + k\right] \tilde{u}_{N+1} = \frac{h}{2} f_{N+1} + 1 \,. \tag{3.141}$$

3.4 Periodic Problem

3.4.1 Statement

This objective of the problem is to initiate the finite element method in a second order differential problem showing periodic boundary conditions.

Actually, interest is axed on the solutions to the following continuous problem:

Find $u \in H^2(0,1)$ which is the solution to:

$$(\mathbf{CP}) \begin{cases} -u''(x) + u(x) = f(x), 0 \leq x \leq 1 \, , \\ u(0) = u(1), u'(0) = u'(1) \, , \end{cases} \tag{3.142}$$

where f is a given function belonging to $L^2(0,1)$.

► **Variational Formulation – Theoretical Part**

1) Let v be a test function, defined from $[0,1]$ to $\mathbf{R}$, belonging to variational space V.

Show that the continuous problem (**CP**) can be expressed as a variational formulation (**VP**) under the form of:

$$a(u,v) = L(v), \quad \forall v \in V \, .$$

The bilinear form $a(.,.)$, the linear form $L(.)$ and the functional space V need to be determined.

2) Establish the existence and uniqueness of a weak solution of the variational problem (**VP**) in $H^1_{\text{per}}(0,1)$ defined by:

$$H^1_{\text{per}}(0,1) = \left\{ v\colon\,]0,1[\to \mathbf{R},\ v \in L^2(0,1),\ v' \in L^2(0,1),\ v(0) = v(1) \right\} \, .$$

3) Show that any weak solution to the variational problem (**VP**) belongs also to $H^2(0,1)$.

4) Deduce the equivalence between the strong formulation presented in $H^2(0,1)$ and the weak formulation (**VP**) considered in $H^1_{\text{per}}(0,1) \cap H^2(0,1)$.

► **Lagrange finite element $\mathbf{P_1}$ – Numerical Part**

5) The approximation to the variational problem (**VP**) is done by Lagrange finite elements P_1. To do so, a regular mesh is introduced at interval $[0,1]$

of constant step h, such as:

$$\begin{cases} x_0 = 0, \;\; x_{N+1} = 1 \;, \\ x_{i+1} = x_i + h, \;\; i = 0 \text{ to } N \;. \end{cases} \tag{3.143}$$

The approximation space $\tilde{V}$ is now defined by:

$$\tilde{V} = \left\{ \tilde{v}\colon [0,1] \to \mathbf{R}, \; \tilde{v} \in C^0([0,1]), \; \tilde{v}|_{[x_i,x_{i+1}]} \in P_1([x_i,x_{i+1}]), \; \tilde{v}(0) = \tilde{v}(1) \right\} \;, \tag{3.144}$$

where $P_1([x_i,x_{i+1}])$ denotes the polynomial space defined on $[x_i,x_{i+1}]$, of degree less than or equal to one.

– What is the dimension of $\tilde{V}$?

6) In order to numerically solve the variational problem (**VP**) by finite elements, the periodic boundary conditions bearing upon the values of the v and u functions, where $x = 0$ and $x = 1$, are temporarily “set aside”.

To achieve this, the $\tilde{W}$ approximation space is introduced and defined by:

$$\tilde{W} = \left\{ \tilde{w}\colon [0,1] \longrightarrow \mathbf{R}, \tilde{w} \in C^0([0,1]), \tilde{w} \in P_1([x_i,x_{i+1}]) \right\} \;. \tag{3.145}$$

– What is the dimension of $\tilde{W}$?

7) Let $\varphi_i (i = 0 \text{ to } \dim \tilde{W} - 1)$ be the basis of $\tilde{W}$ testing $\varphi_i(x_j) = \delta_{ij}$. After expressing the approximated variational formulation of solution $\tilde{u}$ (temporarily looked for in $\tilde{W}$) associated to the variational problem (**VP**), show that when choosing:

$$\tilde{w}(x) = \varphi_i(x), (i = 0, \dim \tilde{W} - 1) \quad \text{and} \quad \tilde{u}(x) = \sum_{j=0, \dim \tilde{W}-1} \tilde{u}_j \varphi_j \tag{3.146}$$

the following $(\widetilde{\mathbf{VP}})$ system is obtained :

$$(\widetilde{\mathbf{VP}}) \quad \sum_{j=0, \dim \tilde{W}-1} A_{ij} \tilde{u}_j = b_i, \quad \forall i \in \{0, \ldots, \dim \tilde{W} - 1\} \;, \tag{3.147}$$

where it was stated:

$$A_{ij} = \int_0^1 (\varphi_i' \varphi_j' + \varphi_i \varphi_j) \mathrm{d}x, \quad b_i = \int_0^1 f \varphi_i \mathrm{d}x \;. \tag{3.148}$$

► Characteristic Function φ_i of a Node Strictly Interior at [0,1]

8) Considering the mesh regularity, the generic nodal equation of the (**VP**) system associated to any basis function φ_i, $(i = 1 \dim \tilde{W} - 2)$, characteristic of a node interior at $[0,1]$, is expressed as:

$$(\widetilde{\mathbf{VP}}_{\mathbf{Int}}) \;\; A_{i,i-1}\tilde{u}_{i-1} + A_{i,i}\tilde{u}_i + A_{i,i+1}\tilde{u}_{i+1} = b_i, \\ (\forall i = 1 \text{ to } \dim \tilde{W} - 2) \;. \tag{3.149}$$

– Using the trapezium rule, calculate the 4 coefficients (A_{ij}, b_i).

9) Group the results by expressing them in a corresponding nodal equation.

10) Show that the centred finite differences scheme associated to the differential equation of the continuous problem (**CP**) is obtained again. What is its precision order?

For reminder, the trapezium quadrature formula is expressed as:

$$\int_a^b \xi(s)\mathrm{d}s \simeq \frac{(b-a)}{2}\{\xi(a)+\xi(b)\} \ .$$

▶ Characteristic Function φ_0 of the Abscissa Node $x_0 = 0$

11) The same process is used for the basis function φ_0, characteristic of the initial node x_0. The corresponding equation of the $(\widetilde{\mathbf{VP}})$ system is then expressed as:

$$(\widetilde{\mathbf{VP}}_\mathbf{0}) \ A_{00}\tilde{u}_0 + A_{01}\tilde{u}_1 = b_0 \ . \tag{3.150}$$

– Using the trapezium rule, calculate the A_{00}, A_{01} and b_0 coefficients.

12) Group the results by expressing them in a corresponding nodal equation.

▶ Characteristic Function φ_{N+1} of the Abscissa Node x_{N+1}

13) The same process is used for the basis function φ_{N+1}, characteristic of the final node x_{N+1}. The corresponding equation to the $(\widetilde{\mathbf{VP}})$ system is then expressed as:

$$(\widetilde{\mathbf{VP}}_\mathbf{N+1}) \ A_{N+1,N}\tilde{u}_N + A_{N+1,N+1}\tilde{u}_{N+1} = b_{N+1} \ . \tag{3.151}$$

– Using the trapezium rule, calculate the $A_{N+1,N}$, $A_{N+1,N+1}$ and b_{N+1} coefficients.

14) Group the results by expressing them in a corresponding nodal equation.

15) Considering the periodicity properties of the nodal equations characteristic of nodes x_0 and x_{N+1}, state an algebraic equation, noted (**R**), between the unknowns $(\tilde{u}_0, \tilde{u}_1, \tilde{u}_N)$ and the data (f_0, f_{N+1}).

16) Process a second-order discretization on the periodic boundary conditions of the continuous problem (**CP**) using the finite differences method, and show that the exact previous algebraic equation (**R**) is obtained.

3.4.2 Solution

▶ Variational Formation – Theoretical Part

A.1) Let v be a test function, defined from $[0,1]$ to $\mathbf{R}$, "sufficiently regular" belonging to a functional space V. The differential equation of the continuous problem (**CP**) is multiplied by v and integrated upon $[0,1]$ interval:

$$-\int_0^1 u''v\,\mathrm{d}x + \int_0^1 uv\,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x, \quad \forall v \in V\,. \tag{3.152}$$

The integration by parts then results in:

$$\int_0^1 u'v'\mathrm{d}x + u'(0)v(0) - u'(1)v(1) + \int_0^1 uv\,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x, \quad \forall v \in V\,. \tag{3.153}$$

It is now demonstrated that the periodic boundary conditions bearing on the u derivative $(u'(0) = u'(1))$, can be directly injected in the integral formulation (3.153).

Thus, the result obtained is:

$$\int_0^1 (u'v' + uv)\mathrm{d}x + u'(0)(v(0) - v(1)) = \int_0^1 fv\,\mathrm{d}x, \quad \forall v \in V\,. \tag{3.154}$$

Concerning the periodic boundary conditions bearing upon u, the test function v is bound to satisfy the same boundary conditions, being:

$$v(0) = v(1)\,. \tag{3.155}$$

Thus, the variational problem (**VP**) keeps all the information contained in the continuous problem (**CP**).

The result obtained is that u is solution to the following formal variational formulation:

$$\int_0^1 (u'v' + uv)\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x, \quad \forall v \text{ such that: } v(0) = v(1)\,. \tag{3.156}$$

Finally, it is shown that the Cauchy-Schwartz inequality secures, as usual, the convergence of the different integrals composing the variational formulation (3.156), if v and v' are functions belonging to $L^2(0,1)$. In other words, from then on, take the test functions v – and from there, solution u – as belonging to $H^1(0,1)$.

Considering the periodic boundary conditions (3.155) imposed in addition, the result is a variational space V defined by:

$$V \equiv H^1_{\text{per}}(0,1) = \left\{v\colon\,]0,1[\to \mathbf{R}, v \text{ and } v \in L^2(0,1),\ v' \in L^2(0,1),\ v(0) = v(1)\right\}\,. \tag{3.157}$$

Finally, the variational formulation (**VP**) is expressed as:

$$
(\mathbf{VP}) \begin{cases} \text{Find } u \text{ belonging to } V \text{ solution to: } a(u,v) = L(v),\ \forall v \in V,\ \text{where:} \\ a(u,v) \equiv \displaystyle\int_0^1 \left[u'(x)v'(x) + u(x)v(x)\right] \mathrm{d}x\,, \\ L(v) \equiv \displaystyle\int_0^1 f(x)v(x)\mathrm{d}x\,, \\ V \equiv H^1_{\mathrm{per}}(0,1)\,. \end{cases} \tag{3.158}
$$

A.2) The existence and uniqueness of the variational formulation (**VP**) solution is demonstrated by applying the Lax-Milgram theorem (theorem 10), using an analogous method to the one detailed in the Dirichlet problem (see problem [3.1]).

To achieve this, the space $H^1_{\mathrm{per}}(0,1)$ is fitted with the H^1-norm (3.17) and the aim is to prove that $H^1_{\mathrm{per}}(0,1)$ is a closed of $H^1(0,1)$, thus conferring it with the Hilbert structure for the H^1-norm.

The sequence v_n is then considered as belonging $H^1_{\mathrm{per}}(0,1)$ to converging for norm H^1 towards a v function of $H^1(0,1)$.

The closing property of $H^1_{\mathrm{per}}(0,1)$ in $H^1(0,1)$ consists to establish that the limit v is also an element of $H^1(0,1)$.

It is established that (Cf. H. Brézis, [1]) if v is a function of $H^1(0,1)$, v is also a continuous function (to be exact, a continuous representative in the function class equal to v exists almost everywhere).

Moreover, since v_n converges towards v in $H^1(0,1)$, it is inferred that (Cf. H. Brézis, [1]) a sub-sequence v_{n_k} exists, composed of v_n in such a way that v_{n_k} simply converges towards v, for nearly any x belonging to $[0,1]$ (actually, it can be established that the convergence of v_n towards v is uniform).

Since v_n and v are "continuous" functions, it is inferred that the simple convergence occurs at any x point of the interval $[0,1]$. Thus, it is possible to express the simple convergence when $x = 0$ and when $x = 1$:

$$
\lim_{n\to+\infty} v_{n_k}(0) = v(0) \quad \text{and} \quad \lim_{n\to+\infty} v_{n_k}(1) = v(1)\,. \tag{3.159}
$$

The only step left is to evaluate the $[v(0) - v(1)]$ difference where the aim is to establish that it is equal to zero, in order to certify that limit v and sequence v_n do belong to $H^1_{\mathrm{per}}(0,1)$:

$$
|v(0) - v(1)| = |v(0) - v_{n_k}(0) + v_{n_k}(0) - v(1)|\,, \tag{3.160}
$$

$$
\leq |v(0) - v_{n_k}(0)| + |v_{n_k}(1) - v(1)|\,. \tag{3.161}
$$

The periodicty property of the sequence v_n, $(v_n(0) = v_n(1))$, has been used and applied to the sub-series v_{n_k}.

To conclude, it suffices to perform a run at the boundary of inequality (3.61) to finally obtain:

$$v(0) = v(1)\,, \tag{3.162}$$

which ends the demonstration and confers a Hilbert structure to $H^1_{\text{per}}(0,1)$ together with the H^1-norm defined in (3.17), as a $H^1(0,1)$ closed vector sub-space.

The display of Lax-Milgram theorem is then performed for space $H^1_{\text{per}}(0,1)$ together with the H^1-norm, with no formal difference with the presentation of the Dirichlet problem.

Therefore, a unique solution exists to the variational formulation (**VP**) that belongs to $H^1_{\text{per}}(0,1)$.

A.3) The regularity result for the variational formulation (**VP**) solution will be obtained through the same procedures as the ones previously established for the Dirichlet problem (see Problem [3.1]).

It only needs to be fitted to the $H^1_{\text{per}}(0,1)$ functional frame.

Effectively, let u be a solution to the problem (**VP**) and the result is:

$$\int_0^1 u'(x)v'(x)\mathrm{d}x = \int_0^1 [f(x) - u(x)]\, v(x)\mathrm{d}x, \quad \forall v \in H^1_{\text{per}}(0,1)\,. \tag{3.163}$$

It is then possible to choose v in $C^1_0(]0,1[)$ which is well included in $H^1_{\text{per}}(0,1)$ and a situation of variational equations family (3.26) is obtained.

The following steps of the demonstration remain unchanged and it is inferred that the variational problem's (**VP**) solution belongs to $H^1_{\text{per}}(0,1) \cap H^2(0,1)$.

A.4) Using the same way as for the previous question, the equivalence between a weak and a strong solution is processed according to the demonstration proposed for the Dirichlet problem, while adapting it to the actual functional frame, being $H^1_{\text{per}}(0,1)$.

Effectively, let v be a variational problem's (**VP**) solution, where integration by parts leads to:

$$\int_0^1 (-u'' + u - f)v(x)\mathrm{d}x + u'(1)v(1) - u'(0)v(0) = 0, \quad \forall v \in H^1_{\text{per}}(0,1)\,, \tag{3.164}$$

or else, since $v(0) = v(1)$,

$$\int_0^1 (-u'' + u - f)v(x)\mathrm{d}x + [u'(1) - u'(0)]v(0) = 0, \quad \forall v \in H^1_{\text{per}}(0,1)\,. \tag{3.165}$$

Again, it is possible to choose v in $\mathscr{D}(0,1)$ (being legitimate since $\mathscr{D}(0,1) \subset H^1_{\text{per}}(0,1)$). The rest of the demonstration remains unchanged.

It is next proceeded by density and it is inferred that solution u of variational formulation (**VP**) verifies the differential equation of the continuous problem (**CP**) as a functional equation in $L^2(0,1)$.

Moreover, if the second member f belongs to $L^2(0,1) \cap C^0(]0,1[)$ then the differential equation is satisfied for any x belonging to $]0,1[$ and the solution u is the classical solution of the continuous problem (**CP**) belonging to $C^2(]0,1[)$.

▶ Lagrange Finite Element $\mathbf{P_1}$ – Numerical Part

A.5) A function $\tilde{v}$ belonging to $\tilde{V}$ is a continuous function over the interval $[0,1]$ and is piecewise affine. Therefore, in the absence of periodic boundary conditions, $\tilde{V}$ would be isomorphic to $\mathbf{R}^{N+2}$.

Such an explanation will prove convincing if it is observed that a function $\tilde{v}$ of $\tilde{V}$ is completely determined provided that its values at $(N+2)$ points x_i of the mesh are fixed.

Indeed, the difference between two functions of $\tilde{V}$ inevitably corresponds to a change in one of the values of theses functions in relation to one of the nodes of the mesh $x_i, (i = 0 \text{ to } N+1)$.

The periodicity constraint of the functions $\tilde{v}$ of $\tilde{V}$ consequently leads to the loss of a degree of freedom.

In other words, the following expression is finally obtained:

$$\dim \tilde{V} = N+1\,. \tag{3.166}$$

A.6) The previous question provides an immediate answer by showing that, considering the periodicity constraint, the dimension of $\tilde{W}$ is equal to $N+2$.

A.7) The approximate variational formulation ($\widetilde{\mathbf{VP}}$) is obtained by substituting the functions u and v in the variational formulation (**VP**) for the respective approximate functions $\tilde{u}$ and $\tilde{v}$.

The following is then obtained:

$$\int_0^1 \tilde{u}'\tilde{w}'\,\mathrm{d}x + \int_0^1 \tilde{u}\tilde{w}\,\mathrm{d}x = \int_0^1 f\tilde{w}\,\mathrm{d}x, \quad \forall \tilde{w} \in \tilde{W}. \tag{3.167}$$

Or, again, by using particular expressions defined by (3.146):

$$\sum_{j=0,N+1} \left[\int_0^1 (\varphi_i'\varphi_i' + \varphi_i\varphi_i)\,\mathrm{d}x\right] \tilde{u}_j = \int_0^1 f\varphi_i\,\mathrm{d}x, \quad \forall i = 0 \text{ to } N+1\,. \tag{3.168}$$

This expression precisely corresponds to what needs to be expounded, provided that the quantities A_{ij} and b_i, as defined by (3.148), are introduced.

► Function φ_i Characteristic of a Node Strictly Interior at [0,1]

A.8) The regularity of the mesh enables the constitution of a generic analysis of the nodal equation, associated with any function φ_i, which is characteristic of the interior node x_i.

Indeed, given the support properties of basis functions φ_i, $(i = 1 \text{ to } N)$, and for a fixed value i, only the values of $j = i-1, j = i, j = i+1$ in the sum of equation (3.168) can provide non-zero contributions, (see Fig. 3.7).

This is why the approximate variational equation $(\widetilde{\mathbf{VP}})$ is written in the form $(\widetilde{\mathbf{VP}}_{\text{Int}})$, for all values of i varying from 1 to N.

For the remainder, the same formalism, as considered for of the Dirichlet, Neumann and Fourier-Dirichlet problems, is observed.

Hence, direct use is made of the results obtained while solving these problems for the calculation of the coefficients of matrix $A_{ij,}$ as well as from the second member b_i.

In other words, the following formula is obtained:

$$A_{i,i} \simeq \frac{2}{h} + h,\ A_{i,i-1} = A_{i,i+1} \simeq -\frac{1}{h},\ b_i \simeq hf_i\ . \tag{3.169}$$

A.9) Likewise, the nodal equation corresponding to the above mentioned coefficients is once more used directly:

$$-\frac{\tilde{u}_{i-1} - 2\tilde{u}_i + \tilde{u}_{i+1}}{h^2} + \tilde{u}_i = f_i, \quad (i = 1 \text{ to } N)\ . \tag{3.170}$$

A.10) Discretisation by finite differences is also applied as for the Dirichlet problem to give the following:

$$-\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} + u(x_i) = f(x_i) + O\left(h^2\right), \quad (i = 1 \text{ to } N)\ . \tag{3.171}$$

The traces u_i of u, at the nodes x_i, are then replaced by the approximations $\tilde{u}_i$, $(\tilde{u}_i \approx u_i)$, so as to maintain equality between the two members of (3.171) when suppressing the infinitely small $O(h^2)$.

This substitution immediately leads to the nodal equation (3.170).

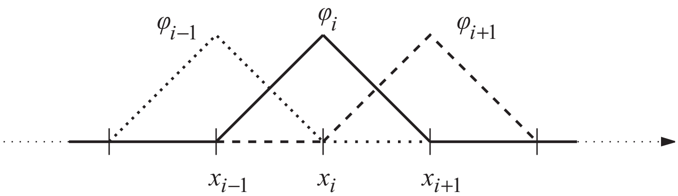

Fig. 3.7 Basis Functions φ_{i-1}, φ_i and φ_{i+1}

▶ Basis Function φ_0 Characteristic of the Node $x_0 = 0$

A.11) The first equation of the linear system (3.147) is now considered, that is when it corresponds to $i = 0$.

Given the support properties of basis functions φ_i, only the functions φ_0 and φ_1 can produce non-zero contributions in their integration against the function φ_0.

This is why the generic equation of system (3.147) is, in this particular case, written according to formula (3.150), namely:

$$A_{00}\tilde{u}_0 + A_{01}\tilde{u}_1 = b_0 \,. \tag{3.172}$$

Again, calculations of the coefficients A_{00}, A_{01} and b_0 have been presented in the Neumann problem (see Problem [3.2]).

The following is then obtained:

$$A_{0,0} \simeq \frac{1}{h} + \frac{h}{2}, \quad A_{0,1} \simeq -\frac{1}{h}, \quad b_0 \simeq \frac{h}{2} f_0 \,. \tag{3.173}$$

A.12) The resulting nodal equation is then written as:

$$\left[\frac{1}{h} + \frac{h}{2}\right] \tilde{u}_0 - \frac{1}{h}\tilde{u}_1 = \frac{h}{2} f_0 \,. \tag{3.174}$$

▶ Basis Function φ_{N+1} Characteristic of the Node $x_{N+1} = 1$

A.13) For analogous reasons to the ones previously described for the nodal equation associated with the basis function φ_0, the results obtained in the Neumann problem (see Problem [3.2]) are used directly.

Thus, the coefficients $A_{N,N+1}$, $A_{N+1,N+1}$ and b_{N+1} are given by:

$$A_{N+1,N+1} \simeq \frac{1}{h} + \frac{h}{2}, \quad A_{N+1,N} \simeq -\frac{1}{h}, \quad b_{N+1} \simeq \frac{h}{2} f_{N+1} \,. \tag{3.175}$$

A.14) The resulting nodal equation is then written as:

$$-\frac{1}{h}\tilde{u}_N + \left[\frac{1}{h} + \frac{h}{2}\right] \tilde{u}_{N+1} = \frac{h}{2} f_{N+1} \,. \tag{3.176}$$

A.15) The periodicity properties of the solution u to the continuous problem (**CP**) are now imposed into the approximation $\tilde{u}$, namely: $\tilde{u}_0 = \tilde{u}_{N+1}$.

By adding the two nodal equs. (3.174) and (3.176), the algebraic relationship (**R**) is obtained:

$$\left[\frac{2}{h} + h\right] \tilde{u}_0 - \frac{1}{h}\left[\tilde{u}_1 + \tilde{u}_N\right] = \frac{h}{2}\left[f_0 + f_{N+1}\right] \,. \tag{3.177}$$

A.16) The finite differences method, is applied to the periodic boundary conditions of the continuous problem (**CP**) by simultaneously expanding at the third order, the progressive Taylor formula at point x_0 and the regressive Taylor formula at point x_{N+1} :

$$u(x_1) = u(x_0) + hu'(x_0) + \frac{h^2}{2}u''(x_0) + O(h^3) \,, \tag{3.178}$$

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2}u''(x_{N+1}) + O(h^3) \,. \tag{3.179}$$

It is then assumed that the solution u of the continuous problem (**CP**) is sufficiently regular so as to correctly write the differential equ. (3.142) at points x_0 and x_{N+1}:

$$u''(x_0) = u(x_0) - f(x_0) \,, \tag{3.180}$$

$$u''(x_{N+1}) = u(x_{N+1}) - f(x_{N+1}) \,. \tag{3.181}$$

The respective expressions of the second derivative (3.180) and (3.181) at points x_0 and x_{N+1} are then replaced in equs. (3.178) and (3.179):

$$u(x_1) = u(x_0) + hu'(x_0) + \frac{h^2}{2}[u(x_0) - f(x_0)] + O(h^3), \tag{3.182}$$

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2}[u(x_{N+1}) - f(x_{N+1})] + O(h^3) \,. \tag{3.183}$$

The approximations $\tilde{u}_i$ are then considered and this enables one to discard the infinitely small ones while maintaining equality.

Finally, the periodicity conditions written for the sequence $\tilde{u}_i$ are applied, in order to obtain the algebraic relationship (**R**) by identifying (3.182) and (3.183):

$$\frac{h}{2}\left[f_0 + f_{N+1}\right] = \left[\frac{2}{h} + h\right]\tilde{u}_0 - \frac{1}{h}\left[\tilde{u}_1 + \tilde{u}_N\right] \,. \tag{3.184}$$

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