6

Symplectic Grassmannian

In this chapter we give an exposition of the determinantal varieties in $Sym\, M_n$ (the space of symmetric $n \times n$ matrices) from the view-point of their relationship to Schubert varieties in the symplectic (or also Lagrangian) Grassmannian varieties. After reviewing some basic algebraic group-theoretic results on symplectic groups, we establish the identification of $Sym\, M_n$ with the "opposite cell" in a suitable symplectic Grassmannian, which then yields an identification of a determinantal variety in $Sym\, M_n$ with the "opposite cell" in a suitable Schubert variety in the symplectic Grassmannian (cf. [67]). Using the standard monomial basis for the homogeneous co-ordinate ring of a Schubert variety in the symplectic Grassmannian, we obtain a standard monomial basis for the ring of regular functions on a determinantal variety in $Sym\, M_n$. This basis in fact coincides with that of DeConcini-Procesi (cf. [22]). Thus Schubert-variety-theoretic approach gives a different proof of DeConcini-Procesi's basis for these varieties.

Let $V = K^{2n}$ together with a non-degenerate, skew-symmetric bilinear form $(\cdot,\cdot)$. Let $H = SL(V)$ and $G = Sp(V) = \{A \in SL(V) \mid A$ leaves the form $(\cdot,\cdot)$ invariant $\}$. Taking the matrix of the form (with respect to the standard basis $\{e_1, ..., e_{2n}\}$ of V) to be

$$E = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

where J is the anti-diagonal $(1, \dots, 1)$ of size $n \times n$, we may realize $Sp(V)$ as the fixed point set of a certain involution σ on $SL(V)$, namely $G = H^\sigma$, where $\sigma : H \longrightarrow H$ is given by $\sigma(A) = E({}^tA)^{-1}E^{-1}$. Thus

$$\begin{aligned} G = Sp(2n) &= \{A \in SL(2n) \mid {}^tAEA = E\} \\ &= \{A \in SL(2n) \mid E^{-1}({}^tA)^{-1}E = A\} \\ &= \{A \in SL(2n) \mid E({}^tA)^{-1}E^{-1} = A\} \\ &= H^\sigma. \end{aligned}$$

(Note that $E^{-1} = -E$). Denoting by T_H (resp. B_H) the maximal torus in H consisting of diagonal matrices (resp. the Borel subgroup in H consisting of upper triangular matrices) we see easily that T_H, B_H are stable under σ. We set $T_G = T_H^\sigma, B_G = B_H^\sigma$. Then it can be seen easily that T_G is a maximal torus in G and B_G is a Borel subgroup in G; in particular, T_G consists of diagonal matrices of the form diag $(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1})$. We note the following specific facts for this group. See [67] for details.

6.1 Some basic facts on $Sp(V)$

Let N_G (resp. N_H) denote the normalizer in G (resp. H) of T_G (resp. T_H). We have, $N_G \subset N_H$; further, N_H is stable under σ, and we have

$$N_G = N_H^\sigma$$
$$N_G/T_G \hookrightarrow N_H/T_H$$

Thus we obtain

$$W_G \hookrightarrow W_H$$

where W_G, W_H denote the Weyl groups of G, H respectively (with respect to T_G, T_H respectively). Further, σ induces an involution on W_H:

$$w = (a_1, \cdots, a_{2n}) \in W_H, \sigma(w) = (c_1, \cdots, c_{2n}),\ c_i = 2n + 1 - a_{2n+1-i}$$

and

$$W_G = W_H^\sigma$$

Thus we obtain

$$W_G = \{(a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n + 1 - a_{2n+1-i},\ 1 \le i \le 2n\}.$$

(here, S_{2n} is the symmetric group on $2n$ letters). Thus $w = (a_1 \cdots a_{2n}) \in W_G$ is known once $(a_1 \cdots a_n)$ is known. We shall denote an element $(a_1 \cdots a_{2n})$ in W_G by just $(a_1 \cdots a_n)$. For example, $(4231) \in S_4$ represents $(42) \in W_G, G = Sp(4)$.
Root system of type C_n: σ induces an involution on $X(T_H)$, the character group of T_H:

$$\chi \in X(T_H),\ \sigma(\chi)(D) = \chi(\sigma(D)),\ D \in T_H$$

Let ϵ_i, $1 \le i \le 2n$ be the character in $X(T_H)$, $\epsilon_i(D) = d_i$, the i-th entry in $D (\in T_H)$. We have

$$\sigma(\epsilon_i) = -\epsilon_{2n+1-i}$$

Now it is easily seen that under the canonical surjective map

$$\varphi : X(T_H) \to X(T_G)$$

we have

$$\varphi(\epsilon_i) = -\varphi(\epsilon_{2n+1-i}),\ 1 \le i \le 2n$$

Let $R_H := \{\epsilon_i - \epsilon_j, 1 \le i, j \le 2n\}$, the root system of G (relative to T_H), and $R_H^+ := \{\epsilon_i - \epsilon_j, 1 \le i < j \le 2n\}$, the set of positive roots (relative to B_H). We have the following:

1. σ leaves R_H (resp. R_H^+) stable.
2. For $\alpha, \beta \in R_H, \varphi(\alpha) = \varphi(\beta) \Leftrightarrow \alpha = \sigma(\beta)$.
3. φ is equivariant for the canonical action of W_G on $X(T_H), X(T_G)$.
4. $R_H^\sigma = \{\pm(\epsilon_i - \epsilon_{2n+1-i}), 1 \le i \le n\}$.

Let R_G (resp. R_G^+) the set of roots of G with respect to T_G (resp. the set of positive roots with respect to B_G). Using the above facts and the explicit nature of the adjoint representation of G on Lie G, we deduce that

$$R_G = \varphi(R_H),\ R_G^+ = \varphi(R_H^+)$$

In particular, R_G (resp. R_G^+) gets identified with the orbit space of R_H (resp. R_H^+) modulo the action of σ. Thus we obtain the following identification:

$$R_G = \{\pm(\epsilon_i \pm \epsilon_j),\ 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i,\ i = 1, ..., n\}$$

$$R_G^+ = \{(\epsilon_i \pm \epsilon_j),\ 1 \le i < j \le n\} \cup \{2\epsilon_i,\ i = 1, ..., n\}.$$

The set S_G of simple roots in R_G^+ is given by

$$S_G := \{\alpha_i = \epsilon_i - \epsilon_{i+1},\ 1 \le i \le n-1\} \cup \{\alpha_n = 2\epsilon_n\}.$$

Let us denote the simple reflections in W_G by $\{s_i,\ 1 \le i \le n\}$, namely, s_i= reflection with respect to $\epsilon_i - \epsilon_{i+1},\ 1 \le i \le n-1$, and s_n = reflection with respect to $2\epsilon_n$. Then we have (cf. [8])

$$s_i = \begin{cases} r_i r_{2n-i}, & \text{if } 1 \le i \le n-1 \\ r_n, & \text{if } i = n \end{cases}$$

where r_i denotes the transposition $(i, i+1)$ in $S_{2n},\ 1 \le i \le 2n-1$. Continuing the example above, we have, $(42) = s_1 s_2 s_1$.

Chevalley basis. For $1 \le i \le 2n$, set $i' = 2n + 1 - i$. The involution $\sigma : SL(2n) \to SL(2n), A \mapsto E({}^tA)^{-1}E^{-1}$, induces an involution $\sigma : sl(2n) \to sl(2n), A \mapsto -E({}^tA)E^{-1} (= E({}^tA)E$, since $E^{-1} = -E)$. In particular, we have, for $1 \le i, j \le 2n$

$$\sigma(E_{ij}) = \begin{cases} -E_{j'i'}, & \text{if } i, j \text{ are both } \le n \text{ or both } > n \\ E_{j'i'}, & \text{if one of } \{i, j\} \text{ is } \le n \text{ and the other } > n \end{cases}$$

where E_{ij} is the elementary matrix with 1 at the (i, j)-th place and 0 elsewhere. Further

$$\text{Lie } Sp(2n) = \{A \in \text{sl}(2n) \mid E({}^tA)E = A\}.$$

The Chevalley basis $\{H_{\alpha_i} : \alpha_i \in S_G\} \cup \{X_\alpha : \alpha \in R_G\}$ for Lie $Sp(2n)$ may be given as follows:

$$
\begin{aligned}
H_{\epsilon_i-\epsilon_{i+1}} &= E_{ii} - E_{i+1,i+1} + E_{(i+1)',(i+1)'} - E_{i'i'} \\
H_{2\epsilon_n} &= E_{nn} - E_{n'n'} \\
X_{\epsilon_j-\epsilon_k} &= E_{jk} - E_{k'j'} \\
X_{\epsilon_j+\epsilon_k} &= E_{jk'} + E_{kj'} \\
X_{2\epsilon_m} &= E_{mm'} \\
X_{-(\epsilon_j-\epsilon_k)} &= E_{kj} - E_{j'k'} \\
X_{-(\epsilon_j+\epsilon_k)} &= E_{k'j} + E_{j'k} \\
X_{-2\epsilon_m} &= E_{m'm}.
\end{aligned}
$$

Length Formula: For $w \in W_G$, let us denote $l(w, W_H)$ (resp. $l(w, W_G)$), the length of w as an element of W_H (resp. W_G). For $w = (a_1, \cdots, a_{2n}) \in W_H$, denote

$$m_w := \#\{i \le n \mid a_i > n\}$$

Proposition 6.1.0.1 *For $w = (a_1, \cdots, a_{2n}) \in W_G$, we have*

$$l(w, W_G) = \frac{1}{2}(l(w, W_H) + m_w)$$

Proof. Set

$$S_H(w) = \{\beta \in R_H^+ \mid w(\beta) < 0\},\ S_G(w) = \{\beta \in R_G^+ \mid w(\beta) < 0\}$$

We have

$$l(w, W_H) = \#S_H(w),\ l(w, W_G) = \#S_G(w)$$

The canonical map $\varphi : X(T_H) \to X(T_G)$ induces a surjective map $R_H \to R_G$. Further, we have,

$$\texttt{for}\ \ \alpha \in R_H, \alpha > 0 \Leftrightarrow \varphi(\alpha) > 0$$

Hence φ induces a surjective map

$$S_H(w) \to S_G(w)$$

Further, σ leaves $S_H(w)$ stable, and

$$S_H(w)^\sigma = \{\alpha \mid \alpha = \epsilon_i - \epsilon_{2n+1-i}, 1 \le i \le n,\ w(\alpha) < 0\ \ \texttt{in}\ \ R_H\}$$

Now if $\alpha = \epsilon_i - \epsilon_{2n+1-i}$, then $w(\alpha) = \epsilon_{a_i} - \epsilon_{a_{2n+1-i}}$. Hence

$$w(\alpha) < 0\ \ \texttt{in}\ \ R_H \Leftrightarrow a_{2n+1-i} < a_i,\ i \le n$$

But now $a_{2n+1-i} = 2n + 1 - a_i$ (since $w \in W_G$). Hence

$$w(\alpha) < 0\ \ \texttt{in}\ \ R_H \Leftrightarrow n < a_i, 1 \le i \le n$$

From this, we obtain

$$\#S_H(w)^\sigma = m(w)$$

Also, we have,

$$\#S_H(w) = 2\#S_G(w) - \#S_H(w)^\sigma$$

The required result now follows from this.

6.1.1 Schubert varieties in G/B_G

For $w \in W_H$, let $C_H(w)$ be the *Schubert cell* $B_H w B_H (mod\ B_H)$ in H/B_H; if $w \in W_G$, we shall denote by $C_G(w)$ the Schubert cell $B_G w B_G (mod\ B_G)$ in G/B_G. We observe that σ induces a natural involution on H/B_H.

Proposition 6.1.1.1 *Let $w \in W_G$. The Schubert cell $C_H(w)$ is stable under σ, and $C_H(w)^\sigma = C_G(w),\ w \in W_G$.*

Proof. The first part is clear. Let B_H^u denote the unipotent part of B_H. Clearly, B_H^u is σ-stable (since B_H is σ-stable). Let B_1 be the isotropy subgroup of B_H^u at the point $wH (\in C_H(w))$; then B_1 is stable under σ (since $\sigma(w) = w$). We have

$$(*) \qquad B_H^u = \prod_{\alpha \in R_H^+} U_\alpha$$

(here, U_α denotes the root subgroup of H associated to α). We see easily

$$B_1 = \prod_{\{\alpha \in R_H^+,\ w^{-1}(\alpha) > 0\}} U_\alpha$$

Let

$$B_2 = \prod_{\{\alpha \in R_H^+,\ w^{-1}(\alpha) < 0\}} U_\alpha$$

Then (*) together with the facts that B_H^u, B_1 are σ-stable implies that B_2 is σ-stable. Let $x \in C_H(w)$; then x has a presentation $x = bw$ for a unique $b \in B_2$. Hence $x \in C_H(w)^\sigma \Leftrightarrow \sigma(b) = b \Leftrightarrow b \in B_G$; and therefore $x \in C_H(w)^\sigma \Leftrightarrow x \in C_G(w)$.

Maximal parabolics. For $1 \le d \le n$, we let P_d be the maximal parabolic subgroup of G with $S_G \setminus \{\alpha_d\}$ as the associated set of simple roots. Then it can be seen easily that $W_G^{P_d}$, the set of minimal representatives of W_G/W_{P_d} can be identified with the set of all $(a_1 \cdots a_d)$ satisfying (1) and (2) below:

$$\begin{aligned} &(1)\ 1 \le a_1 < a_2 < \cdots < a_d \le 2n \\ &(2)\ \text{for}\ \ 1 \le i \le 2n,\ \ \text{if}\ \ i \in \{a_1, ..., a_d\} \\ &\quad\ \ \text{then}\ \ 2n + 1 - i \notin \{a_1, ..., a_d\} \end{aligned}$$

Bruhat–Chevalley order. For $w_1 = (a_1 \cdots a_{2n}) \in W_G$ and $w_2 = (b_1 \cdots b_{2n})$, w_1, w_2 we have $w_2 \ge w_1 \Leftrightarrow \{b_1, ..., b_d\} \uparrow \ge \{a_1, ..., a_d\} \uparrow$ for each $1 \le d \le n$ (cf. [102]). Here $\{a_1, \ldots, a_d\}\uparrow$, $\{b_1, \ldots, b_d\}\uparrow$ are the corresponding d-tuples arranged in ascending order. Hence for $w \in W_G$, denoting by $w^{(d)}$ the element in $W_G^{P_d}$ which represents the coset wW_{P_d}, we have for $w_1, w_2 \in W_G$ and $1 \le d \le n$,

$$w_2^{(d)} \ge w_1^{(d)},\ 1 \le d \le n \iff \{b_1, \ldots, b_d\} \uparrow \ge \{a_1, \ldots, a_d\} \uparrow .$$

Further, $w_2 \geq w_1 \iff w_2^{(d)} \geq w_1^{(d)},\ 1 \leq d \leq n$. But now, the latter condition is equivalent to $w_2 \geq w_1$ in W_H. Thus we obtain that the partial order on W_G is induced by the partial order on W_H (cf. [102]). In particular, for $w_1 = (a_1 \cdots a_d),\ w_2 = (b_1 \cdots b_d), w_1, w_2 \in W_G^{P_d}$, we have $w_2 \geq w_1 \Leftrightarrow \{b_1, \ldots, b_d\} \uparrow \geq \{a_1, \ldots, a_d\} \uparrow$.

In the sequel, we shall denote an element $(a_1 \cdots a_n)$ in $W_G^{P_d}$ by just $(a_1 \cdots a_d)$. The fact that the partial order on W_G is induced by the partial order on W_H together with Proposition 6.1.1.1 yields the following:

Proposition 6.1.1.2 *Let $w \in W_G$; let $X_G(w)$ (resp. $X_H(w)$) be the associated Schubert variety in G/B_G (resp. H/B_H). Under the canonical inclusion $G/B_G \hookrightarrow H/B_H$, we have $X_G(w) = X_H(w) \cap G/B_G$. Further, the intersection is scheme-theoretic.*

Proof. Denote $Z := X_H(w) \cap G/B_G$. We have $X_G(w) \subseteq X_H(w) \cap G/B_G$ (clearly), and is in fact an irreducible component of Z (in view of Proposition 6.1.1.1). Let Z' be another irreducible component of Z. Then $Z' = X_G(w')$, for some $w' \in W_G$ (Z being closed and B_G-stable in G/B_G, is a union of Schubert varieties in G/B_G). Now the inclusion $X_G(w') \subset Z$ implies in particular that $C_G(w')(= C_H(w')^\sigma)$ is contained in Z. Hence we obtain that $C_H(w')^\sigma \subseteq X_H(w) \cap C_H(w')$; this implies in particular that $C_H(w') \subseteq X_H(w)$ (since $X_H(w) \cap C_H(w')$ is non-empty, and X_H is B_H-stable), and hence we obtain that $w' \leq w$ in W_H, and hence also in W_G (in view of the fact that the partial order on W_G is induced by the partial order on W_H). Thus we get that $Z'(= X_G(w')) \subset X_G(w)$. The first part of the Lemma follows from this. The second part follows from the fact (cf. Chapter 13, Theorem 13.2.1.10) that for any $\tau \in W_G, \tau \leq w$, we have that the tangent space to $X_H(w)$ at the T-fixed point τB_H is σ-stable, and its subspace of σ-fixed points is the tangent space to $X_G(w)$ at the T-fixed point τB_G.

6.2 The variety G/P_n

Let Q_n denote the maximal parabolic subgroup of H associated to the simple root $\epsilon_n - \epsilon_{n+1}$. As above, let P_n denote the maximal parabolic subgroup of G associated to the simple root $2\epsilon_n$. It is easily seen that Q_n is σ-stable so that σ induces an involution on H/Q_n. Let Z be the subgroup of H consisting of matrices of the form

$$\begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix},\ Y \in M_n$$

(here, M_n denotes the $n \times n$ matrices with entries in K). The canonical morphism $H \to H/Q_n$ induces a morphism

$$\psi_H : Z \to H/Q_n$$

We have (cf. Chapter 5)

Fact: ψ_H is an open immersion, and $\psi_H(Z)$ gets identified with the opposite big cell O_H^- in H/Q_n.

Lemma 6.2.0.1 *O_H^- is σ- stable, and we have an identification $Sym\ M_n \cong (O_H^-)^\sigma$.*

Proof. For $z = \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} \in Z$, we have,

$$\sigma(z) = \begin{pmatrix} Id_n & 0 \\ J\,{}^tYJ & Id_n \end{pmatrix}$$

(where recall that J is the anti-diagonal $(1, \cdots, 1)$ of size $n \times n$). Hence, Z is σ-stable, and the first part of the Lemma follows (in view of Fact above).
We have,

$$Z^\sigma = \{z \in Z \mid J\,{}^tYJ = Y\}$$

Letting $Y = JX$, we have

$$J\,{}^tYJ = Y \Leftrightarrow {}^t X = X$$

The second part of Lemma follows from this.

Consider H/Q_n as the Grassmannian of n-dimensional subspaces of $V := K^{2n}$. It is well-known that the set of maximal totally isotropic subspaces (for the skew-symmetric form $(,)$ on V) is a closed subvariety of H/Q_n isomorphic to G/P_n, the *symplectic* (or also *the Lagrangian*) *Grassmannian.*

Lemma 6.2.0.2 *ψ_H induces an inclusion $\psi_H : Z^\sigma \hookrightarrow G/P_n$.*

Proof. Now $z = \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix}$ considered as a point of H/Q_n (= the Grassmannian of n dimensional subspaces of K^{2n}) corresponds to the n-dimensional subspace spanned by the columns of $\begin{pmatrix} Id_n \\ Y \end{pmatrix}$. Let us denote this n-dimensional subspace by U_z. We have,

$$J\,{}^tYJ = Y \Leftrightarrow \begin{pmatrix} Id_n & {}^tY \end{pmatrix} \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \begin{pmatrix} Id_n \\ Y \end{pmatrix} = 0$$

Hence we obtain that $z \in Z^\sigma \Leftrightarrow U_z$ is a maximal totally isotropic subspace. The required result follows from this.

Let us denote the restriction of the morphism $\psi_H : Z \to H/Q_n, Y \mapsto \begin{pmatrix} Id_n & 0 \\ JY & Id_n \end{pmatrix}$ to $Sym\ M_n$ by ψ_G. Let O_G^- the opposite big cell in G/P_n.

6.2.1 Identification of $Sym\ M_n$ with O_G^-

Let L_n denote the tautological line bundle on $H/Q_n (= G_{n,2n})$ for the Plücker embedding. Then $H^0(H/Q_n, L_n)$ is the dual Weyl H-module with highest weight ω_n.

Proposition 6.2.1.1 *Let $Id = (1, \cdots, n) (\in I_{n,2n})$; denote $f := p_{Id}$, the Plücker co-ordinate corresponding to Id. We have*

1. *f is a lowest weight vector in $H^0(H/Q_n, L_n)$.*
2. *O_H^- is the principal open set $(H/Q_n)_f$.*

Proof. (1). We have

$$H^0(H/Q_n, L_n) = (\Lambda^n(K^{2n}))^*$$

Further, for $\mathbf{i} = (i_1, \cdots, i_n) \in I_{n,2n}$,

$$p_{\mathbf{i}} = e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$$

(here $\{e_t^*, 1 \le t \le 2n\}$ denotes the basis of V^* dual to the standard basis $\{e_t, 1 \le t \le 2n\}$ of $V = K^{2n}$). Hence it follows that $p_{\mathbf{i}}$ is a T_H-weight vector of weight $-(\epsilon_{i_1} + \cdots + \epsilon_{i_n})$. Now (1) follows from this.
(2) follows from the fact that the opposite big cell is precisely the set of points in H/Q_n where a lowest weight vector (in $H^0(H/Q_n, L_n)$) does not vanish.

6.2.2 Canonical dual pair

Let $Y \in M_n$. Given two s-tuples $A := (a_1, \cdots, a_s)$, $B := (b_1, \cdots, b_s)$, in $I_{s,n}$, $s \le n$, let us denote by $p(A, B)(Y)$, the s-minor of Y with row and column indices given by A, B respectively. Consider the identification

$$(*) \qquad M_n \cong \left\{ \begin{pmatrix} Id_n \\ JY \end{pmatrix}, Y \in M_n \right\}$$

where J is as above. Let $\mathbf{i} := (i_1, \cdots, i_n) \in I_{n,2n}$, and let $p_{\mathbf{i}}$ be the associated Plücker co-ordinate on $G_{n,2n} (= H/Q_n)$. Let $f_{\mathbf{i}}$ denote the restriction of $p_{\mathbf{i}}$ to M_n (under the identification (*)). Then $f_{\mathbf{i}} = p(\mathbf{i}(A), \mathbf{i}(B))$, where $\mathbf{i}(A), \mathbf{i}(B)$ are given as follows:

Let r be such that $i_r \le n, i_{r+1} > n$; let $s = n - r$. Then $\mathbf{i}(A)$ is the s-tuple given by $(2n + 1 - i_n, \cdots, 2n + 1 - i_{r+1})$, while $\mathbf{i}(B)$ is the s-tuple given by the complement of $(i_1, \cdots, i_r)$ in $(1, \cdots, n)$. We refer to $(\mathbf{i}(A), \mathbf{i}(B))$ as the *canonical dual pair* associated to $\mathbf{i}$.

6.2.3 The bijection θ

Define a partial order $\ge$ on the set of minors of $Y \in M_n$ as follows: Given A, B, A', B' where $A, B \in I_{s,n}$, and $A', B' \in I_{s',n}$, $p(A, B) > p(A', B')$ if $s \le s'$, and $a_t \ge a'_t, b_t \ge b'_t, 1 \le t \le s$. Then the map $\mathbf{i} \mapsto (\mathbf{i}(A), \mathbf{i}(B))$ defines an order-reversing bijection θ between $I_{n,2n}$ and minors of Y. In the above bijection, the n-tuple $(1, \cdots, n)$ will correspond to the constant function 1 (corresponding to the minor with the set of row (resp. column) indices being the empty set).

6.2.4 The dual Weyl G-module with highest weight ω_n

Let L'_n be the restriction of L_n to G/P_n. For $\mathbf{i} \in I_{n,2n}$, let $p'_{\mathbf{i}}$ be the restriction of $p_{\mathbf{i}}$ to G/P_n; note that $p'_{\mathbf{i}} \in H^0(G/P_n, L'_n)$. Let $p(\mathbf{i}(A), \mathbf{i}(B))$ be the restriction of $p_{\mathbf{i}}$ to M_n, M_n being identified with the opposite cell O_H^- as above (cf. (*)); let $p'(\mathbf{i}(A), \mathbf{i}(B))$ be the restriction of $p(\mathbf{i}(A), \mathbf{i}(B))$ to $Sym\, M_n$. Let f be a lowest weight vector in $H^0(H/Q_n, L_n)$, and f' the restriction of f to G/P_n.

Proposition 6.2.4.1 *$H^0(G/P_n, L'_n)$ is the dual Weyl G-module with highest weight ω_n.*

Proof. We first observe that $p'_{\mathbf{i}}, \mathbf{i} \in I_{n,2n}$ is non-zero; this follows from the fact that $p'(\mathbf{i}(A), \mathbf{i}(B))$ is non-zero on $Sym\, M_n$ ($Sym\, M_n = Z^\sigma$ being identified with an open subset of G/P_n). Now f being a lowest weight vector in $H^0(H/Q_n, L_n)$ (cf. Proposition 6.2.1.1), the one-dimensional span Kf is B_H^--stable, where B_H^- is the Borel subgroup of H opposite to B_H. Also, one sees easily that B_H^- is σ-stable, and $(B_H^-)^\sigma (= G \cap B_H^-)$ is precisely B_G^-, the Borel subgroup of G opposite to B_G. This fact together with the fact that Kf is stabilized by B_G^- implies that f' is a lowest weight vector in $H^0(G/P_n, L'_n)$; further,

$$\texttt{weight of } f' (= \texttt{weight of } f) = -(\epsilon_1 + \cdots + \epsilon_n) = -\omega_n$$

Hence we obtain that a highest weight vector in $H^0(G/P_n, L'_n)$ has weight $w_0(G)(-\omega_n)$ (note that the Weyl involution $i = -w_0(G)$ is Id on $X(T_G)$—here, $w_0(G)$ is the element of largest length in W_G). Hence we obtain that L'_n is the ample generator of Pic G/P_n so that $H^0(G/P_n, L'_n)$ is the fundamental representation of G with highest weight ω_n. The result now follows from this.

As an immediate consequence we obtain

Theorem 6.2.4.2 *We have an identification $Z^\sigma \cong O_G^-$.*

Proof. Under the identification $\psi : Z \cong O_H^-$, we have that $Z \cong (H/Q_n)_f$ (cf. Proposition 6.2.1.1,(2)); thus Z is precisely the set of points in H/Q_n where the lowest weight vector $f (\in H^0(H/Q_n, L_n))$ does not vanish. Hence we obtain that

$$(G/P_n)_{f'} = Z \cap G/P_n$$

Let $z \in Z$, say $z = \begin{pmatrix} Id \\ JY \end{pmatrix}$. Then $z \in G/P_n$ if and only if the n-dimensional subspace spanned by the columns of $\begin{pmatrix} Id \\ JY \end{pmatrix}$ is totally isotropic, i.e, if and only if

$$\begin{pmatrix} Id_n & {}^tY \end{pmatrix} \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \begin{pmatrix} Id_n \\ Y \end{pmatrix} = 0$$

i.e, if and only if

$$J\,{}^tYJ = Y$$

Hence, letting $Y = JX$, we obtain that

$$O_G^-(= (G/P_n)_{f'}) = Z \cap G/P_n = Sym\, M_n = Z^\sigma$$

This implies the required result.

Corollary 6.2.4.3 $O_G^- = (O_H^-)^\sigma = Sym\, M_n$.
(Thus the opposite cell in H/Q_n induces the opposite cell in G/P_n.)

Proof. This follows from the above Theorem, the identification $Z \cong O_H^-$, and Lemma 6.2.0.1.

Corollary 6.2.4.4 $dim\, G/P_n = \frac{1}{2}n(n+1)$.

6.2.5 Identification of $D_t(Sym\, M_n)$ with $Y_{P_n}(\varphi)$

Fix an integer t where $1 \le t \le n$. Let $D_t(Sym\, M_n)$ denote the subscheme of $Sym\, M_n$ defined by the vanishing of t minors in $Sym\, M_n$. In this subsection, we shall describe an identification of $D_t(Sym\, M_n)$ with the opposite cell $Y_{P_n}(\varphi)$ in a suitable Schubert variety $X_{P_n}(\varphi)$.

As a consequence of Proposition 6.1.1.2, we have the following:

Proposition 6.2.5.1 *Let $w \in W^{P_n}$. Let $X_{P_n}(w)$ (resp. $X_{Q_n}(w)$) be the associated Schubert variety in G/P_n (resp. H/Q_n). Then under the canonical inclusion $G/P_n \subset H/Q_n$, we have, $X_{P_n}(w) = X_{Q_n}(w) \cap G/P_n$ (scheme-theoretically).*

Fix a t as above. Let $\varphi \in W^{P_n}$ be defined by

$$\varphi = (t, t+1, \cdots, n, 2n+2-t, 2n+3-t, \cdots, 2n)$$

(note that φ consists of two blocks $[t, n]$, $[2n+2-t, 2n]$ of consecutive integers—here, for $i < j$, $[i, j]$ denotes the set $\{i, i+1, \cdots, j\}$). Let $Y_{P_n}(\varphi) := X_{P_n}(\varphi) \cap O_G^-$, the *opposite cell in $X_G(\varphi)$*.

Theorem 6.2.5.2 *The isomorphism $Sym\, M_n \cong O_G^-$ induces an isomorphism*

$$D_t(Sym\, M_n) \cong Y_{P_n}(\varphi)$$

Proof. We have scheme-theoretic identifications

$$D_t(Sym\, M_n) \cong D_t(M_n) \cap Sym\, M_n$$
$$Sym\, M_n \cong O_G^-$$
$$Y_{P_n}(\varphi) \cong Y_{Q_n}(\varphi) \cap O_G^-$$

On the other hand, we have an isomorphism (cf. Chapter 5)

$$D_t(M_n) \cong Y_{Q_n}(\varphi)$$

The required result now follows.

Corollary 6.2.5.3 *$D_t(Sym\, M_n)$ is normal, Cohen-Macaulay of dimension $n(t-1) - \frac{1}{2}(t-1)(t-2)$.*

Proof. The normality and Cohen-Macaulayness follow since Schubert varieties in G/P_n are normal and Cohen-Macaulay (cf. §A.12 of Appendix). The assertion on the dimension follows in view of Proposition 6.1.0.1 (note that for $w \in W^P$, we have, $\dim X_P(w) = l(w)$).

6.2.6 Admissible pairs and canonical pairs

Let (τ, φ) be an admissible pair in W^{P_n} (see Appendix A.2.4 for the definition of admissible pairs). We have ([56]) that (τ, φ) is an admissible pair with $X_{P_n}(\varphi)$ being a divisor in $X_{P_n}(\tau)$ if and only if there is a pair $i, (i+1)'$ present in φ (as an element of $I_{n,2n}$ - also recall that for $1 \leq j \leq 2n$, $j' = 2n+1-j$), for some $i \leq n$, and τ is obtained from φ by replacing $i, (i+1)'$ by $(i+1), i'$. Let us refer to this as a *elementary double move*. Hence, (τ, φ) is an admissible pair if and only if either $\tau = \varphi$(in which case it is a trivial admissible pair) or τ is obtained from φ by a sequence of elementary double moves. From this we obtain the following

Lemma 6.2.6.1 *Let $\tau, \varphi \in W^{P_n}$. Then (τ, φ) is a non-trivial admissible pair if and only if the following hold:*
(i) $\tau > \varphi$ (as elements of $I_{n,2n}$)
(ii) $m_\tau > 0$
(iii) $m_\tau = m_\varphi$
where recall that for $\theta := (a_1, \cdots, a_{2n}) \in I_{n,2n}$, $m_\theta = \#\{i \leq n \mid a_i > n\}$.

6.2.7 Canonical pairs

We now define an injection of $I_{n,2n} (= W^{Q_n})$ into $W^{P_n} \times W^{P_n}$. We first associate to $\mathbf{i} \in I_{n,2n}$, a pair $(\mathbf{i}(\alpha), \mathbf{i}(\beta))$ to be called *the canonical pair* as follows: This is defined similar to the canonical dual pair $(\mathbf{i}(A), \mathbf{i}(B))$ (cf. §6.2.1). In §6.2.1, we used the identification

$$(*) \qquad M_n(K) \cong \left\{ \begin{pmatrix} Id_n \\ JY \end{pmatrix}, Y \in M_n(K) \right\}$$

where J is the anti-diagonal $(1, \cdots, 1)$ of size $n \times n$. If $\mathbf{i} \in I_{n,2n}$, then the restriction to M_n of the Plücker co-ordinate $p_\mathbf{i}$ (under the identification (*)) evaluated at Y corresponds to the minor of Y with row and column indices given by $\mathbf{i}(A), \mathbf{i}(B)$ respectively. To define the canonical pair for an $\mathbf{i} \in I_{n,2n}$ we use the identification:

$$(**) \qquad M_n(K) \cong \left\{ \begin{pmatrix} Y \\ J \end{pmatrix}, Y \in M_n(K) \right\}$$

For $\mathbf{i} \in I_{n,2n}$, the pair $(\mathbf{i}(\alpha), \mathbf{i}(\beta))$ is defined to be the set of row and column indices of the minor of Y which is the evaluation at Y of the restriction to $M_n(K)$ of the Plücker co-ordinate $p_{\mathbf{i}}$ (under the identification (**)). Let $\mathbf{i} = (i_1, \cdots, i_n)$. Then $\mathbf{i}(\alpha), \mathbf{i}(\beta)$ have the following description:

Let r be such that $i_r \leq n, i_{r+1} > n$. For $1 \leq j \leq 2n$, let $j' = 2n + 1 - j$. Then $\mathbf{i}(\alpha)$ is the r-tuple $(i_1, \cdots, i_r)$, while $\mathbf{i}(\beta)$ is the r-tuple which is the complement in $(1, \cdots, n)$ of $i'_{r+1}, i'_{r+2}, \cdots, i'_n$. Note that the canonical pair $(\mathbf{i}(\alpha), \mathbf{i}(\beta))$, and the canonical dual pair $(\mathbf{i}(A), \mathbf{i}(B))$ are related as follows:

$$\mathbf{i}(A) = \texttt{ complement of } \mathbf{i}(\beta) \texttt{ in } (1, \cdots, n),$$
$$\mathbf{i}(B) = \texttt{ complement of } \mathbf{i}(\alpha) \texttt{ in } (1, \cdots, n)$$

Further we have the map

$$\mathbf{i} \mapsto (\mathbf{i}(\alpha), \mathbf{i}(\beta))$$

is an order-preserving bijection between $I_{n,2n}$ and minors of Y; while the map

$$\mathbf{i} \mapsto (\mathbf{i}(A), \mathbf{i}(B))$$

is an order-reversing bijection between $I_{n,2n}$ and minors of Y (the partial order among the minors of Y being as in §6.2.3). In the above bijections, the n-tuple $(1, \cdots, n)$ will correspond to the constant function 1 (corresponding to the minor with the set of row (resp. column) indices being the empty set).

6.2.8 The inclusion $\eta : I_{n,2n} \hookrightarrow W^{P_n} \times W^{P_n}$

First observe that for $\tau := (a_1, \cdots, a_r, b_1, \cdots, b_s) \in W^{P_n}$, where $a_1 < \cdots < a_r \leq n < b_1 < \cdots < b_s$, we have, $\{b_1, \cdots, b_s\}$ is just the complement in $\{n + 1, n+2, \cdots, 2n\}$ of $\{a'_1, \cdots, a'_r\}$ (arranged in ascending order). Hence, τ is completely determined by $(a_1, \cdots, a_r)$. Denoting

$$\mathcal{I} := \bigcup_{1 \leq r \leq n} I_{r,n}$$

the map $\tau \mapsto (a_1, \cdots, a_r)$ defines a bijection

$$\upsilon : W^{P_n} \overset{\texttt{bij}}{\to} \mathcal{I}$$

Set

$$\mathcal{A} := \{(I, J) \in \bigcup_{r \leq n} I_{r,n} \times I_{r,n}, \mid I \geq J\}$$

Then the bijection υ induces a bijection:

$$\rho : \{\texttt{admissible pairs in } W^{P_n}\} \overset{\texttt{bij}}{\to} \mathcal{A}$$

(cf. Lemma 6.2.6.1; note that for $I = J$, we obtain W^{P_n} (being identified with trivial admissible pairs)). We shall refer to the elements of $\mathcal{A}$ as *admissible elements* in $\mathcal{I} \times \mathcal{I}$. Using the bijection υ, we define η to be the injection

$$\eta : I_{n,2n} \hookrightarrow W^{P_n} \times W^{P_n}, \mathbf{i} \mapsto (\mathbf{i}(\alpha), \mathbf{i}(\beta))$$

Identifying W^{P_n} with the diagonal set in $W^{P_n} \times W^{P_n}$, we have that $\eta(\mathbf{i}) \in W^{P_n}$ if and only if $\mathbf{i}(\alpha) = \mathbf{i}(\beta)$. Further, if $\mathbf{i}(\alpha) \neq \mathbf{i}(\beta)$, then $\eta(\mathbf{i})$ is an admissible pair if and only if the following hold:
(i) $\mathbf{i}(\alpha) \geq \mathbf{i}(\beta))$
(ii) $\#\mathbf{i}(\alpha)(= \#\mathbf{i}(\beta)) < n$.

We shall refer to an $\mathbf{i} \in I_{n,2n}$ such that $\eta(\mathbf{i}) \in \mathcal{A}$ as *an admissible element of* $I_{n,2n}$. Let us denote

$$\mathcal{A}_{n,2n} := \{\texttt{admissible elements in } I_{n,2n}\}$$

Then we have a bijection

$$\mathcal{A}_{n,2n} \overset{\texttt{bij}}{\to} \{\texttt{admissible pairs in } W^{P_n}\}$$

The comparison order $\geq$ among admissible pairs in W^{P_n} (namely, $(\tau, \varphi) \geq (\tau', \varphi') \Leftrightarrow \varphi \geq \tau'$) induces a comparison order on $\mathcal{A}_{n,2n}$:

$$\mathbf{i} \geq \mathbf{j}, \quad \texttt{if } \mathbf{i}(\beta) \geq \mathbf{j}(\alpha)$$

(here, an s-tuple $(a_1, \cdots, a_s) \geq$ an t-tuple $(b_1, \cdots, b_t)$, if $s \leq t$ and $a_r \geq b_r$, $1 \leq r \leq s$).

6.2.9 A standard monomial basis for $D_t(Sym\ M_n)$

Fix $1 \leq t \leq n$. For $A := (a_1, \cdots, a_s)$, $B := (b_1, \cdots, b_s)$, in $I_{s,n}$, $s \leq n$, let $p'(A, B)$ denote the restriction of $p(A, B)(\in K[M_n])$ to $Sym\ M_n$; if $\mathbf{i}$ is the element of $I_{n,2n}$ corresponding to (A, B) under the bijection θ (cf. §6.2.3), we shall denote $p'(A, B)$ also by $p'(\mathbf{i})$.

Definition 6.2.9.1 *A monomial* $p'(\mathbf{i}_1) \cdots p'(\mathbf{i}_r)$, $\mathbf{i}_t \in \mathcal{A}_{n,2n}$ *is standard if* $\mathbf{i}_1 \geq \cdots \geq \mathbf{i}_r$*, i.e., if* $\mathbf{i}_1(\alpha) \geq \mathbf{i}_1(\beta) \geq \mathbf{i}_2(\alpha) \geq \cdots \geq \mathbf{i}_r(\beta)$.
Note that for any $t \leq r$*, we have,* $\mathbf{i}_t(\alpha) \geq \mathbf{i}_t(\beta)$ *(since* $\mathbf{i}_t \in \mathcal{A}_{n,2n}$*).*

Using the identification

$$M_n \cong \left\{ \begin{pmatrix} Y \\ J \end{pmatrix}, Y \in M_n \right\}$$

and the isomorphism (cf. Theorem 6.2.5.2)

$$D_t(Sym\ M_n) \cong Y_G(\varphi)$$

we obtain

Theorem 6.2.9.2 $K[D_t(Sym\ M_n)]$ *has a basis consisting of standard monomials of the form* $p'(A_1, B_1) \cdots p'(A_r, B_r)$, $A_1 \geq B_1 \geq A_2 \geq \cdots \geq B_r$, $r \in \mathbb{N}$, $A_1 \leq (t, t+1, \cdots, n)$.

Proof. We have following facts:

• $K[X_{P_n}(\varphi)]$ has a basis consisting of standard monomials on $X_{P_n}(\varphi)$ of the form $p(\tau_1, \delta_1)\cdots p(\tau_r, \delta_r), r \in \mathbb{N}, \varphi \geq \tau_1 \geq \delta_1 \geq \tau_2 \geq \cdots \geq \delta_r, (\tau_i, \delta_i)$ being admissible pairs (cf. Chapter 8, Theorem 8.2.0.7).

• $\mathbf{i} \mapsto (\mathbf{i}(\alpha), \mathbf{i}(\beta))$ is an order-preserving bijection between $I_{n,2n}$ and minors of (a generic matrix) $Y \in M_n$. Further, under this bijection, the admissible elements of $I_{n,2n}$ are mapped bijectively onto admissible pairs in W^{P_n}.

• Under the bijection $\upsilon : W^{P_n} \overset{\mathtt{bij}}{\to} \mathcal{I}$, we have, $\upsilon(\varphi) = ([t, n], [2n + 2 - t, 2n])$ (cf. Theorem 6.2.5.2).

• The canonical pair corresponding to $\varphi(= ([t, n], [(t-1)', 1']))$ is $((t, t+1, \cdots, n), (t, t+1, \cdots, n))$.

These facts together with Theorem 6.2.5.2 identify $K[D_t(Sym\, M_n)]$ with $K[X_G(\varphi)]_{(p_{id})}$ (the homogeneous localization of $K[X_{P_n}(\varphi)]$ at p_{id}). Now, the fact that $p_{id} \leq p_{\tau,\delta}$ (for all admissible pairs (τ, δ)) implies that $p(\tau_1, \delta_1)\cdots p(\tau_r, \delta_r)$, $r \in \mathbb{N}$ is standard if and only if $\frac{p(\tau_1,\delta_1)}{p_{id}} \cdots \frac{p(\tau_r,\delta_r)}{p_{id}}$ is. Also, the fact that the standard monomials

$$\{p(\tau_1, \delta_1)\cdots p(\tau_r, \delta_r), \varphi \geq \tau_1, r \in \mathbb{N}\}$$

generate $K[X_{P_n}(\varphi)]$ implies that the standard monomials

$$\{\frac{p(\tau_1, \delta_1)}{p_{id}} \cdots \frac{p(\tau_r, \delta_r)}{p_{id}}, \varphi \geq \tau_1, r \in \mathbb{N}\}$$

generate $K[D_t(Sym\, M_n)](= K[X_G(\varphi)]_{(p_{id})})$; further, the linear independence of the standard monomials $\{p(\tau_1, \delta_1)\cdots p(\tau_r, \delta_r), \varphi \geq \tau_1, r \in \mathbb{N}\}$ clearly implies the linear independence of the standard monomials $\{\frac{p(\tau_1,\delta_1)}{p_{id}} \cdots \frac{p(\tau_r,\delta_r)}{p_{id}}, \varphi \geq \tau_1, r \in \mathbb{N}\}$. The result now follows from this.

6.2.10 De Concini-Procesi's basis for $D_t(Sym\, M_n)$

In arriving at Theorem 6.2.9.2, we have used the order-preserving bijection between $I_{n,2n}$ and minors of $Y \in M_n(K)$ given by the canonical pairs of elements of $I_{n,2n}$. If instead we use the order-reversing bijection between $I_{n,2n}$ and minors of $Y \in M_n(K)$ given by the canonical dual pairs of elements of $I_{n,2n}$, then we obtain De Concini-Procesi's basis for $D_t(Sym\, M_n)$ (cf. [22]) described below.
First, we note (or recall) the following:
(i) The canonical dual pair of $\mathbf{i} := ([t, n], [(t-1)', 1'])(= \varphi)$ is given by

$$\mathbf{i}(A) = (1, \cdots, t-1),\ \mathbf{i}(B) = (1, \cdots, t-1)$$

(ii) Under the identification

$$M_n \cong \left\{ \begin{pmatrix} Id_n \\ JY \end{pmatrix}, Y \in M_n \right\}$$

we have

$$p'_{\mathbf{i}}(Y) = p(\mathbf{i}(A), \mathbf{i}(B))(Y)$$

(iii) Defining η' to be the injection

$$\eta' : I_{n,2n} \hookrightarrow W^{P_n} \times W^{P_n}, \mathbf{i} \mapsto (\mathbf{i}(A), \mathbf{i}(B))$$

we have that $\eta'(\mathbf{i}) \in W^{P_n}$ if and only if $\mathbf{i}(A) = \mathbf{i}(B)$. Further, if $i(A) \neq i(B)$, then $\eta'(\mathbf{i})$ is an admissible pair if and only if the following hold:
(1) $\mathbf{i}(A) \leq \mathbf{i}(B))$
(2) $\# \mathbf{i}(\alpha)(= \# \mathbf{i}(\beta)) < n$.

We shall refer to such $\mathbf{i}$'s as *dual admissible elements of* $I_{n,2n}$, and denote

$$\mathcal{A}'_{n,2n} := \{\texttt{dual admissible elements in } I_{n,2n}\}$$

(iv) The comparison order $\geq$ among admissible pairs in W^{P_n} induces a comparison order on $\mathcal{A}'_{n,2n}$:

$$\mathbf{i} \leq \mathbf{j}, \quad \texttt{if } \mathbf{i}(B) \leq \mathbf{j}(A)$$

Definition 6.2.10.1 *A monomial* $p'(\mathbf{i}_1) \cdots p'(\mathbf{i}_r), \mathbf{i}_t \in \mathcal{A}'_{n,2n}$, *is standard if* $\mathbf{i}_1 \leq \cdots \leq \mathbf{i}_r$, *i.e., if* $\mathbf{i}_1(A) \leq \mathbf{i}_1(B) \leq \mathbf{i}_2(A) \leq \cdots \leq \mathbf{i}_r(B)$.
Note that for any $t \leq r$, *we have,* $\mathbf{i}_t(A) \leq \mathbf{i}_t(B)$ *(since* $\mathbf{i}_t \in \mathcal{A}'_{n,2n}$*).*

Theorem 6.2.10.2 $K[D_t(Sym\ M_n)]$ *has a basis consisting of standard monomials of the form* $p'(A_1, B_1) \cdots p'(A_r, B_r), A_1 \leq B_1 \leq A_2 \leq \cdots \leq B_r, r \in \mathbb{N}, A_1 \geq (1, \cdots, t-1)$.

(Note that the condition that $(1, \cdots, t-1) \leq A \leq B$ *amounts to the condition that* $\#A(= \#B) \leq t-1$, *i.e.,* $p'(A, B)$*'s are minors of size* $\leq t-1$.*)*

Proof. Proof is similar to that of Theorem 6.2.9.2. As in the proof of Theorem 6.2.9.2, we have the following facts:

- $K[X_{P_n}(\varphi)]$ has a basis consisting of standard monomials on $X_{P_n}(\varphi)$ of the form $p(\tau_1, \delta_1) \cdots p(\tau_r, \delta_r), r \in \mathbb{N}, \varphi \geq \tau_1 \geq \delta_1 \geq \tau_2 \geq \cdots \geq \delta_r, (\tau_i, \delta_i)$ being admissible pairs.
- $\mathbf{i} \mapsto (\mathbf{i}(A), \mathbf{i}(B))$ is an order-reversing bijection between $I_{n,2n}$ and minors of (a generic matrix) $Y \in M_n$. Further, under this bijection, the dual admissible elements of $I_{n,2n}$ are mapped bijectively onto admissible pairs in W^{P_n}.
- Under the bijection $\upsilon : W^{P_n} \overset{\texttt{bij}}{\to} \mathcal{I}$, we have, $\upsilon(\varphi) = [t, n](= (t, t+1, \cdots, n))$ (cf. Theorem 6.2.5.2).
- The canonical dual pair corresponding to $\varphi(= ([t, n], [(t-1)', 1']))$ is $((1, \cdots, t-1), (1, \cdots, t-1))$.

These facts together with Theorem 6.2.5.2 imply the required result.

Taking $t = n+1$, we have $D_t(Sym\ M_n)$ equals $Sym\ M_n$, and we obtain

Theorem 6.2.10.3 $K[Sym\ M_n]$ *has a basis consisting of standard monomials of the form* $p'(A_1, B_1) \cdots p'(A_r, B_r), A_1 \leq B_1 \leq A_2 \leq \cdots \leq B_r, r \in \mathbb{N}$.

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