
Contents

1	Introduction	1
1.1	The subject matter in a nutshell	1
1.1.1	What is CIT?	1
1.1.2	What is SMT?	2
1.1.3	The SMT approach to CIT	2
1.2	The subject matter in detail	2
1.2.1	Proof by the SMT approach	3
1.2.2	$SL_n(K)$, $SO_n(K)$ actions	5
1.3	Why this book?	6
1.4	A brief history of SMT	7
1.5	Some features of the SMT approach	7
1.6	The organization of the book	9
2	Generalities on algebraic varieties	11
2.1	Some basic definitions	11
2.2	Algebraic varieties	12
2.2.1	Affine varieties	12
3	Generalities on algebraic groups	17
3.1	Abstract root systems	17
3.2	Root systems of algebraic groups	19
3.2.1	Linear algebraic groups	19
3.2.2	Parabolic subgroups	21
3.3	Schubert varieties	22
3.3.1	Weyl and Demazure modules	24
3.3.2	Line bundles on G/Q	25
3.3.3	Equations defining a Schubert variety	27
4	Grassmannian	29
4.1	The Plücker embedding	29
4.1.1	The partially ordered set $I_{d,n}$	30

4.1.2	Plücker embedding and Plücker coordinates	30
4.1.3	Plücker quadratic relations	31
4.1.4	More general quadratic relations	32
4.1.5	The cone $\widehat{G_{d,n}}$ over $G_{d,n}$	33
4.1.6	Identification of G/P_d with $G_{d,n}$	33
4.2	Schubert varieties of $G_{d,n}$	34
4.2.1	Bruhat decomposition	34
4.2.2	Dimension of X_i	35
4.2.3	Further results on Schubert varieties	35
4.3	Standard monomial theory for Schubert varieties in $G_{d,n}$	36
4.3.1	Standard monomials	36
4.3.2	Linear independence of standard monomials	36
4.3.3	Generation by standard monomials	37
4.3.4	Equations defining Schubert varieties	38
4.4	Standard monomial theory for a union of Schubert varieties	39
4.4.1	Linear independence of standard monomials	39
4.4.2	Standard monomial basis	39
4.4.3	Consequences	40
4.5	Vanishing theorems	41
4.6	Arithmetic Cohen-Macaulayness, normality and factoriality	44
4.6.1	Factorial Schubert varieties	46
5	Determinantal varieties	47
5.1	Recollection of facts	47
5.1.1	Equations defining Schubert varieties in the Grassmannian	48
5.1.2	Evaluation of Plücker coordinates on the opposite big cell in $G_{d,n}$	48
5.1.3	Ideal of the opposite cell in $X(w)$	49
5.2	Determinantal varieties	49
5.2.1	The variety D_t	49
5.2.2	Identification of D_t with Y_ϕ	49
5.2.3	The bijection θ	51
5.2.4	The partial order \geq	51
5.2.5	Cogeneration of an ideal	52
5.2.6	The monomial order \prec and Gröbner bases	53
6	Symplectic Grassmannian	55
6.1	Some basic facts on $Sp(V)$	56
6.1.1	Schubert varieties in G/B_G	59
6.2	The variety G/P_n	60
6.2.1	Identification of $Sym M_n$ with O_G^-	61
6.2.2	Canonical dual pair	62
6.2.3	The bijection θ	62
6.2.4	The dual Weyl G -module with highest weight ω_n	63
6.2.5	Identification of $D_t(Sym M_n)$ with $Y_{P_n}(\varphi)$	64

6.2.6	Admissible pairs and canonical pairs	65
6.2.7	Canonical pairs	65
6.2.8	The inclusion $\eta : I_{n,2n} \hookrightarrow W^{P_n} \times W^{P_n}$	66
6.2.9	A standard monomial basis for $D_t(\text{Sym } M_n)$	67
6.2.10	De Concini-Procesi's basis for $D_t(\text{Sym } M_n)$	68
7	Orthogonal Grassmannian	71
7.1	The even orthogonal group $SO(2n)$	71
7.1.1	Schubert varieties in G/B_G	74
7.2	The variety G/P_n	77
7.2.1	Identification of $Sk M_n$ with O_G^-	78
7.2.2	Canonical dual pair	78
7.2.3	The bijection θ	79
7.2.4	The dual Weyl G -module with highest weight ω_n	79
7.2.5	Identification of $D_t(Sk M_n)$ with $Y_G(\varphi)$	80
7.2.6	A standard monomial basis for $D_t(Sk M_n)$	82
8	The standard monomial theoretic basis	85
8.1	SMT for the even orthogonal Grassmannian	86
8.2	SMT for the symplectic Grassmannian	89
9	Review of GIT	95
9.1	G -spaces	95
9.1.1	Reductive groups	95
9.2	Affine quotients	98
9.2.1	Affine actions	99
9.3	Categorical quotients	101
9.3.1	Examples	102
9.4	Good quotients	103
9.4.1	Some results on good quotients	105
9.5	Stable and semi-stable points	108
9.5.1	Stable, semistable, and polystable points	108
9.5.2	Other characterizations of stability, semistability	110
9.6	Projective quotients	114
9.7	L -linear actions	117
9.8	Hilbert-Mumford criterion	117
10	Invariant theory	121
10.1	Preliminary lemmas	121
10.2	$SL_d(K)$ -action	124
10.2.1	The functions f_τ	124
10.2.2	The first and second fundamental theorems	126
10.3	$GL_n(K)$ -action:	128
10.3.1	The first and second fundamental theorems	129
10.4	$O_n(K)$ -action	132

10.5	$Sp_{2\ell}(K)$ -action	136
11	$SL_n(K)$-action	137
11.1	Quadratic relations	138
11.1.1	The partially ordered set $H_{r,d}$	139
11.2	The K -algebra S	140
11.2.1	The $SL_n(K)$ -action	141
11.3	Standard monomials in the K -algebra S	142
11.3.1	Quadratic relations	143
11.3.2	Linear independence of standard monomials	145
11.3.3	The algebra $S(D)$	146
11.3.4	A standard monomial basis for $R(D)$	148
11.3.5	Standard monomial bases for $M(D), S(D)$	149
11.4	Normality and Cohen-Macaulayness of the K -algebra S	150
11.4.1	The algebra associated to a distributive lattice	150
11.4.2	Flat degenerations of certain K -algebras	151
11.4.3	The distributive lattice D	152
11.4.4	Flat degeneration of $\text{Spec } R(D)$ to the toric variety $\text{Spec } A(D)$	154
11.5	The ring of invariants $K[X]^{SL_n(K)}$	155
12	$SO_n(K)$-action	159
12.1	Preliminaries	160
12.1.1	The Lagrangian Grassmannian variety	161
12.1.2	Schubert varieties in L_m	161
12.1.3	The opposite big cell in L_m	162
12.1.4	The functions $f_{\tau,\varphi}$ on O_G^-	163
12.1.5	The opposite cell in $X(w)$	164
12.1.6	Symmetric determinantal varieties	164
12.1.7	The set H_m	165
12.2	The algebra S	167
12.2.1	Standard monomials and their linear independence	168
12.2.2	Linear independence of standard monomials	169
12.3	The algebra $S(D)$	169
12.3.1	Quadratic relations	170
12.3.2	A standard monomial basis for $R(D)$	171
12.3.3	Standard monomial bases for $S(D)$	172
12.4	Cohen-Macaulayness of S	173
12.4.1	A doset algebra structure for $R(D)$	175
12.5	The equality $R^{SO_n(K)} = S$	176
12.6	Application to moduli problem	180
12.7	Results for the adjoint action of $SL_2(K)$	181

13 Applications of standard monomial theory	187
13.1 Tangent space and smoothness	187
13.1.1 The Zariski tangent space	187
13.1.2 Smooth and non-smooth points	188
13.1.3 The space $T(w, \tau)$	188
13.1.4 A canonical affine neighborhood of a T -fixed point	188
13.1.5 The affine variety $Y(w, \tau)$	189
13.1.6 Equations defining $Y(w, \tau)$ in O_τ^-	189
13.1.7 Jacobian criterion for smoothness	189
13.1.8 T -stable curves	190
13.2 Singularities of Schubert varieties in the flag variety	190
13.2.1 Ideal of $Y(w, \tau)$	191
13.2.2 A criterion for smoothness	193
13.2.3 Components of the singular locus	193
13.3 Singular loci of Schubert varieties in the Grassmannian	195
13.3.1 Multiplicity at a singular point	196
13.4 Results for Schubert varieties in a minuscule G/P	200
13.4.1 Homogeneity of $\mathcal{I}_P(w)$	201
13.4.2 A basis for $(M_{\tau,w})^r / (M_{\tau,w})^{r+1}$	202
13.5 Applications to other varieties	202
13.5.1 Ladder determinantal varieties	202
13.5.2 The varieties V_i , $1 \leq i \leq l$	204
13.5.3 Quiver varieties	205
13.6 Variety of complexes	209
13.6.1 A partial order on $\{(k_1, k_2, \dots, k_h)\}$	210
13.7 Degenerations of Schubert varieties to toric varieties	210
13.7.1 Generalities on distributive lattices	210
13.7.2 An important example	211
13.7.3 Generalities on toric varieties	212
13.7.4 An example	213
13.7.5 The algebra associated to a distributive lattice	213
13.7.6 Varieties defined by binomials	214
13.7.7 Degenerations of Schubert varieties in the Grassmannian to toric varieties	215
Appendix: Proof of the main theorem of SMT	219
A.1 Notation	219
A.2 Admissible pairs and the first basis theorem	220
A.2.1 More notation	220
A.2.2 Chevalley multiplicity	220
A.2.3 Minuscule and classical type parabolics	220
A.2.4 Admissible pairs	221
A.3 The three examples	221
A.3.1 Example A	222
A.3.2 Example B	222

A.3.3 Example C	223
A.4 Tableaux and the statement of the main theorem	224
A.5 Preparation	225
A.6 The tableau character formula	226
A.7 The structure of admissible pairs	226
A.8 The procedure	227
A.9 The basis	229
A.10 The first basis theorem	230
A.11 Linear independence	232
A.12 Arithmetic Cohen-Macaulayness and arithmetic normality	235
A.12.1 Arithmetic Cohen-Macaulayness	236
References	243
Index	249
Index of notation	260
Author index	263

Standard Monomial Theory
Invariant Theoretic Approach
Lakshmibai, V.; Raghavan, K.N.
2008, XIV, 266 p., Hardcover
ISBN: 978-3-540-76756-5