Part II

A Note on Alberti's Rank-One Theorem

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1 Introduction

The aim of these notes is to illustrate a proof of the following remarkable Theorem of Alberti (first proved in [1]). Here, when μ is a Radon measure on $\Omega \subset \mathbb{R}^n$, we denote by μ^a its absolutely continuous part (with respect to the Lebesgue measure $\mathscr{L}^n$), by $\mu^s := \mu - \mu^a$ its singular part, and by $|\mu|$ its total variation measure. Clearly, $|\mu|^a = |\mu^a|$ and $|\mu|^s = |\mu^s|$. When $\mu = Du$ for some $u \in BV(\Omega, \mathbb{R}^k)$, we will write $D^s u$ and $D^a u$. If ν is a nonnegative measure, μ/ν will denote the Radon–Nykodim derivative of μ with respect to ν. Finally we recall the polar decomposition of Radon measures, namely the identity $\mu = \frac{\mu}{|\mu|}|\mu|$, which implies that the vetor Borel map $\mu/|\mu|$ has modulus 1 μ-a.e.

Theorem 1.1. *Let $u \in BV(\Omega, \mathbb{R}^k)$ for some open set $\Omega \subset \mathbb{R}^n$. Then* $\mathrm{rank}(Du/|Du|(x)) = 1$ *for $|D^s u|$-a.e. $x \in \Omega$.*

We start by discussing what can be inferred from the "standard theory" of *BV* functions without much effort. A first conclusion can be drawn from the *BV* Structure Theorem (see Sect. 3.6, Theorem 3.77, and Proposition 3.92 of [3]) for which we first need some terminology. Given an L^1 function u we say that u is approximately continuous at x if there exists $\tilde{u}(x) \in \mathbb{R}^k$ such that $\lim_r r^{-n} \int_{B_r(x)} |u(y) - \tilde{u}(x)|\, dy = 0$. We denote by S_u the set of points where u is not approximately continuous and we say that $x \in S_u$ is an approximate jump point if there exists $\nu(x) \in \mathbf{S}^{n-1}$

and $u^{\pm}(x) \in \mathbb{R}^k$ such that

$$\lim_{r\downarrow 0} \frac{1}{r^n} \left(\int_{B_r^+(x)} |u(y) - u^+(x)|\, dy + \int_{B_r^-(x)} |u(y) - u^-(x)|\, dy \right) = 0,$$

where $B_r^{\pm}(x) = \{y \in B_r(x) : \pm(y-x)\cdot \nu(x) > 0\}$. The triple $(\nu(x), u^+(x), u^-(x))$ is unique up to a change of sign of $\nu(x)$ and a permutation of $u^+(x)$ and $u^-(x)$. The set of approximate jump points is denoted by J_u.

Finally, we recall that an $(n-1)$-dimensional rectifiable set $R \subset \mathbb{R}^n$ is a Borel set which can be covered $\mathscr{H}^{n-1}$-almost all by a countable family of C^1 $(n-1)$-dimensional surfaces. Here, $\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure.

Theorem 1.2 (Structure Theorem for *BV* functions). *If $\Omega \subset \mathbb{R}^n$ is open and $u \in BV(\Omega, \mathbb{R}^k)$, then J_u is a rectifiable $(n-1)$-dimensional set, $\mathscr{H}^{n-1}(S_u \setminus J_u) = |Du|(S_u \setminus J_u) = 0$ and $D^s u$ can be decomposed as $D^c u + D^j u$, where*

- *$|D^c u|(E) = 0$ for every Borel set E with $\mathscr{H}^{n-1}(E) < \infty$;*
- *$D^j u = (u^+ - u^-) \otimes \nu \mathscr{H}^{n-1} \llcorner J_u$.*

Here and in what follows, given a measure μ and a Borel set E we denote by $\mu \llcorner E$ the measure given by $\mu \llcorner E(A) = \mu(E \cap A)$. Following [5], we call $D^c u$ and $D^j u$ respectively Cantor part and Jump part of the measure Du. Thus, Theorem 1.2 implies the statement of Theorem 1.1 when we replace $|D^s u|$ with $|D^j u|$.

A second fact that can be inferred from the "standard theory" of *BV* functions is the following dimensional reduction:

Proposition 1.3. *Theorem 1.1 holds if and only if it holds for $\Omega = B_1(0) \subset \mathbb{R}^2$ and $\mathbb{R}^k = \mathbb{R}^2$.*

This proposition will be proved in Sect. 2. Thus, the key point of Theorem 1.1 is to show that M has rank one $|D^c u|$-a.e. when u is a *BV* planar map. A first heuristic idea of why this property indeed holds is given in Sect. 3. The key remark of that section is the following lemma, which has a quite simple proof.

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^2$ be connected and $u \in BV(\Omega, \mathbb{R}^2)$ be such that $Du/|Du|$ is a constant matrix M of rank 2. Then, $Du = cM\mathscr{L}^2 \llcorner \Omega$ for some $c > 0$.*

Building on this lemma and on a "blow-up" argument, we prove in Sect. 3 a particular case of Theorem 1.1. However, this simple strategy cannot prove Theorem 1.1 in its full generality (see Sect. 3, in particular Proposition 3.3). Alberti's strategy relies on replacing Lemma 1.4 with Lemma 1.5 below. From now on a set $C \subset \mathbb{R}^2$ will be called a closed convex cone if there exist $e \in \mathbf{S}^1$ and $0 < a < 1$ such that $C = C(e,a) := \{x : x \cdot e \geq a|x|\}$.

Lemma 1.5. *Let C_1 and C_2 be two closed convex cones such that $C_1 \cap C_2 = (-C_1) \cap C_2 = \{0\}$. Let $\Omega \subset \mathbb{R}^2$ be open and $v_1, v_2 \in BV(\Omega)$ be two scalar functions such that $Dv_i/|Dv_i|(x) \in C_i$ for $|Dv_i|$-a.e. x. If $\mu \geq 0$ is a measure such that $\mu << |Dv_i|$ for $i = 1, 2$, then $\mu << \mathscr{L}^2 \llcorner \Omega$.*

This lemma will be proved in Sect. 4. We want to stress here the analogies with Lemma 1.4. Set $v = (v_1, v_2)$. By the polar factorization, the main assumption of Lemma 1.5 could be restated as $Dv/|Dv|$ *belongs* (*$|Dv|$-almost everywhere*) *to a suitably small neighborhood of a constant matrix M of rank 2*. Moreover the last sentence is equivalent to $|Dv| << \mathscr{L}^2 \llcorner \Omega$. Thus, we can consider Lemma 1.4 as a rigidity result and Lemma 1.5 as its quantitative counterpart.

Now consider $u \in BV(\Omega, \mathbb{R}^2)$ and the Borel set $E := \{x : \operatorname{rank}(Du/|Du|(x)) = 2\}$. Standard arguments show that E can be decomposed in countably many Borel pieces E_i where $Du/|Du|$ is very close to a single constant matrix M_i. Thus the relaxed assumption of Lemma 1.5 suggests that we could use a "decomposition" approach, in contrast with the "blow-up" argument which builds on the rigidity Lemma 1.4. More precisely, we will show in Sect. 5 that the decomposition in Borel pieces E_is can be chosen so that

- If we fix any i and set $\mu := |Du| \llcorner E_i$, then there are two BV scalar functions v_1 and v_2 such that v_1, v_2 and μ satisfy the hypotheses of Lemma 1.5.

Clearly, the decomposition stated above and Lemma 1.5 show that μ is absolutely continuous, i.e. they prove Theorem 1.1. The construction of the v_is is the second key idea of Alberti's proof. The argument combines a simple geometric consideration on the level sets of the u_is together with a clever use of the coarea formula for BV scalar functions.

Recently, Alberti, Csorniey and Preiss, (see [2]) have proposed a different proof of the Rank-One Theorem. This new proof uses as well the coarea formula, but it avoids Lemma 1.5, and relies instead on a general covering result for Lebesgue-null sets of the plane. Let us mention, in passing, that this last result has many other deep implications in real analysis and geometric measure theory; see [2].

2 Dimensional Reduction

Proof of Proposition 1.3. Assume that Theorem 1.1 holds for maps $u \in BV(B_1(0), \mathbb{R}^2)$ with $B_1(0) \subset \mathbb{R}^2$. Clearly, by translating and rescaling, we immediately conclude the theorem when $u \in BV(B, \mathbb{R}^2)$ for any two-dimensional ball B. The statement of Theorem 1.1 is trivially true if $\Omega \subset \mathbb{R}$ or if $k = 1$ Moreover, any open set $\Omega \subset \mathbb{R}^n$ can be written as countable union of balls. Hence it suffices to prove the theorem when Ω is a ball of $\mathbb{R}^n$, $n \geq 2$, and $k \geq 2$.

From $n = 2$ to n generic. Here we prove Theorem 1.1 for maps $u \in BV(B, \mathbb{R}^2)$ whenever B is an n-dimensional ball. We argue by contradiction and let $u \in BV(B, \mathbb{R}^2)$ be such that $\operatorname{rank}(Du/|Du|(x)) = 2$ on some set E with $|D^s u|(E) > 0$. Set $M = Du/|Du|$ and choose coordinates $x_1, \ldots, x_n$ on B and u_1, u_2 on $\mathbb{R}^2$. Clearly, M has $n(n-1)/2$ different minors, corresponding to the choice of coordinates x_i, x_j with $1 \leq i < j \leq n$: We denote them by M^{ij}. If we set $E_{ij} := \{x : \operatorname{rank}(M^{ij}(x)) = 2\}$, then $E = \bigcup_{ij} E_{ij}$, and hence $|D^s u|(E_{ij}) > 0$ for some i and j. Without loss of generality we assume $i = 1$ and $j = 2$. Consider the matrix valued measure $(\mu)_{l\alpha} = (\partial_{x_l} u_\alpha)_{l\alpha}$ with $l, \alpha = 1,2$. Then, $\mathrm{rank}\,(\mu/|\mu|(x)) = 2$ for $|\mu|$-a.e. $x \in E_{12}$ and $|\mu^s|(E_{12}) > 0$.

For any $y \in \mathbb{R}^{n-2}$ we define $B_y = \{(x_1,x_2) \in \mathbb{R}^2 : (x_1,x_2,y) \in B\}$. Clearly, B_y is either empty or it is an open two-dimensional ball. Moreover, we define

$$v_y : B_y \to \mathbb{R}^2 \qquad \text{by} \qquad v_y(x_1,x_2) = u(x_1,x_2,y).$$

By the slicing theory of BV functions (see Theorem 3.103, Theorem 3.107, and Theorem 3.108 of [3]) we have:

(a) $v_y \in BV(B_y,\mathbb{R}^2)$ for $\mathscr{L}^{n-2}$-a.e. $y \in \mathbb{R}^{n-2}$;
(b) $\mu = Dv_y \otimes \mathscr{L}^{n-2}$ and $|\mu| = |Dv_y| \otimes \mathscr{L}^{n-2}$.

(Here, when α is a measure on Y and $y \mapsto \beta_y$ a weakly measurable map from Y into the space $\mathscr{M}(X)$ of Radon measures on X, the symbol $\beta_y \otimes \alpha$ denotes the measure γ on $X \times Y$ which satisfies

$$\int_{X\times Y} \varphi(x,y)\, d\gamma(x,y) = \int_Y \int_X \varphi(x,y)\, d\beta_y(x)\, d\alpha(y)$$

for every $\varphi \in C_c(X \times Y)$.)

(b) implies two things. First of all,

$$\frac{Dv_y}{|Dv_y|}(x_1,x_2) = \frac{\mu}{|\mu|}(x_1,x_2,y) \quad \text{for } \mathscr{L}^{n-2}\text{-a.e. } y \text{ and } |Dv_y|\text{-a.e. } (x_1,x_2). \tag{1}$$

Second, if for every y we set $E_y := \{(x_1,x_2) : (x_1,x_2,y) \in E\}$, then

$$\int_{\mathbb{R}^{n-2}} |Dv_y^s|(E_y)\, d\mathscr{L}^{n-2}(y) = |\mu^s|(E) > 0. \tag{2}$$

Thus, from (a), (1) and (2), we conclude that there exists a y such that $v_y \in BV(B_y,\mathbb{R}^2)$, $|Dv_y^s|(E_y) > 0$, and $\mathrm{rank}\,(Dv_y/|Dv_y|(x)) = 2$ for $|Dv_y|$-a.e. $x \in E_y$. Such v_y contradicts our assumption that Theorem 1.1 holds for maps $u \in BV(B_y,\mathbb{R}^2)$.

From $k = 2$ to k generic. Fix any $u \in BV(B,\mathbb{R}^k)$, with $k \geq 2$ and B n-dimensional ball, and choose coordinates $u_1,\dots,u_k$ on $\mathbb{R}^k$. For any pair of integers $1 \leq i < j \leq k$, consider the map $u_{ij} := (u_i,u_j) \in BV(B,\mathbb{R}^2)$. If $M = Du/|Du|$ and M_{ij} is the corresponding $2\times n$ minor, then $Du_{ij} = M_{ij}|Du|$. Thus, by the previous step, $\mathrm{rank}\,(M_{ij}(x)) \leq 1$ for $|D^s u_{ij}|$-a.e. x, and hence for $|D^s u|$-a.e. x. Set

$$E_{ij} := \left\{x : \mathrm{rank}\,(M_{ij}(x)) \leq 1\right\} \qquad \text{and} \qquad E := \bigcap_{1\leq i<j\leq k} E_{ij}.$$

Then, $|D^s u|(\mathbb{R}^n \setminus E) = 0$ and $\mathrm{rank}\,(M(x)) \leq 1$ for every $x \in E$. This concludes the proof. $\square$

3 A Blow-Up Argument Leading to a Partial Result

We start this section by proving Lemma 1.4.

Proof of Lemma 1.4. We let M be the constant matrix $Du/|Du|$ and $\mu = |Du|$. By standard arguments, it suffices to prove the lemma when Ω is the unit ball $B_1(0)$. Denote by u_1 and u_2 the two components of u. Then $Du_i = v_i\mu$, where $v_1, v_2 \in \mathbb{R}^2$ are two linearly independent vectors. Let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a standard family of mollifiers supported in $B_\varepsilon(0)$ and consider the mollifications $u_i * \varphi_\varepsilon$ in $B_{1-\varepsilon}(0)$. Notice that $D(u_i * \varphi_\varepsilon) = v_i\mu * \varphi_\varepsilon$, and hence $u_i * \varphi_\varepsilon$ is constant on the direction orthogonal to v_i. Therefore the density of the absolutely continuous measure $\mu * \varphi_\varepsilon$ is a function f_ε which is constant along two linearly independent directions. Thus, f_ε is constant. Letting $\varepsilon \downarrow 0$ we complete the proof. $\square$

This simple remark leads to a partial answer to Theorem 1.1, given in Proposition 3.2.

Definition 3.1. Let μ be a measure on $\Omega \subset \mathbb{R}^2$ and for any x in the support of μ and any $r \in]0, \mathrm{dist}(x, \partial\Omega)[$ consider the measures $\mu_{x,r}$ on $B_1(0)$ given by

$$\mu_{x,r}(A) = \mu(x + rA)/|\mu|(B_r(x)) \qquad \text{for any Borel set } A \subset B_1(0).$$

We say that a measure μ_0 is tangent to μ at x if for some sequence $r_n \downarrow 0$ we have $\mu_{x,r_n} \rightharpoonup^* \mu_0$.

A nonnegative measure μ on $\Omega \subset \mathbb{R}^2$ is said to have only trivial blow-ups at x, if every tangent measure to μ at x is of the form $c\mathscr{L}^2 \llcorner B_1(0)$. For $u \in BV(\Omega, \mathbb{R}^2)$ we denote by T the set of points where $|D^s u|$ has only trivial blow-ups.

This definition of tangent measure is very similar to that introduced by Preiss in the fundamental paper [6]. We are now ready to state our

Proposition 3.2. *Let* $u \in BV(\Omega, \mathbb{R}^2)$. *Then* $\mathrm{rank}(Du/|Du|(x)) = 1$ *for* $|D^s u|$*-a.e.* $x \notin T$.

Proof. We argue by contradiction and assume that the proposition is false for some u. Denote by μ the measure $|D^s u|$. Then, by standard measure-theoretic arguments, it is possible to find a point $x \notin T$ and a sequence $r_n \downarrow 0$ such that the following properties hold:

(i) $\mu_{x,r_n} \rightharpoonup^* \mu_0$, and $\mu_0 \neq c\mathscr{L}^n \llcorner B_1(0)$;
(ii) $||Du| - \mu|(B_r(x)) = o(\mu(B_r(x)))$;
(iii) $M = Du/|Du|(x)$ is a matrix of rank > 1 and

$$\lim_{r\downarrow 0} \frac{1}{|Du|(B_r(x))} \int_{B_r(x)} \big|Du/|Du|(y) - M\big|\, d|Du|(y) = 0.$$

Let $\overline{u}_r$ be the average of u on $B_r(x)$ and define the function $u_r \in BV(B_1(0), \mathbb{R}^2)$ as

$$u_r(y) = \frac{r^{n-1}(u(x+ry) - \overline{u}_r)}{|Du|(B_r(x))}.$$

It follows that $Du_r = [Du]_{x,r}$, and hence $|Du_r|(B_1(0)) = 1$. Moreover, since the average of u_r is 0, the Poincaré inequality gives $\|u_r\|_{L^1} \leq C$. Thus, we can assume that a subsequence, not relabeled, of $\{u_{r_n}\}$ converges to some $u_0 \in BV(B_1(0), \mathbb{R}^2)$ strongly in L^1. Now, from (ii) we get $|Du|_{x,r} - \mu_{x,r} \rightharpoonup^* 0$ and from (iii) we conclude $[Du]_{x,r} - M|Du|_{x,r} \rightharpoonup^* 0$. Therefore, by (i), $Du_r = [Du]_{x,r} \rightharpoonup^* M\mu_0$. This implies $Du_0 = M\mu_0$, because u_{r_n} converges to u_0. Applying Lemma 1.4 we conclude $\mu_0 = c\mathscr{L}^2 \llcorner B_1(0)$, which contradicts (i).

Unfortunately, we cannot hope to prove Theorem 1.1 by showing that singular parts of BV functions have necessarily nontrivial blow-ups. More precisely we have

Proposition 3.3. *There exist BV maps u such that* $|D^s u|(T) > 0$.

Proof. The example 5.8(1) of [6] gives a nonnegative measure μ on a bounded interval I which is singular and such that $\mu_{x,r} \rightharpoonup^* \frac{1}{2}\mathscr{L}^1 \llcorner [-1,1]$ for μ-a.e. x. Clearly, any primitive of μ is a bounded BV function which satisfies the requirements of the proposition. □

4 The Fundamental Lemma

Before coming to the proof of the lemma, let us explain its basic ingredients. Assume for the moment that the v_is of the lemma are regular, and that $\mu = f\mathscr{L}^2 \leq C|\nabla v_i|$. Consider the map $v = (v_1, v_2)$. Since the gradients ∇v_i belong everywhere to the cones C_i and $C_1 \cap C_2 = (-C_1) \cap C_2 = \{0\}$, a simple algebraic consideration shows that $\det \nabla v$ controls, up to some constant depending on the C_is, the product $|\nabla v_1||\nabla v_2|$, and hence f^2. Thus we can bound the L^2 norm of f by the integral of $\det \nabla v$. A second key remark is that the geometric constraints on the C_is imply that v is almost injective (more precisely, v would be injective if $\nabla v_i \in C_i \setminus \{0\}$). Thus, $\int \det \nabla v$ can be computed using the area formula. This means that $\int f^2$ can be bound in terms, for instance, of the L^∞ norm of v, but independently of ∇v. In the proof below we will extend such a priori estimate to the general case, using truncations and a suitable regularization procedure.

Remark 4.1. In the rest of these notes we will use extensively the following elementary fact. Let $C = C(e,a) = \{x : x \cdot e \geq a|x|\}$ be a closed convex cone, $\Omega \subset \mathbb{R}^2$ an open set and $v \in BV(\Omega, \mathbb{R})$. Then, it follows easily from the polar decomposition of measures that $Dv/|Dv|(x) \in C$ for $|Dv|$-a.e. x if and only if $\partial_e v \geq a|Dv|$.

Proof of Lemma 1.5. We can assume without loss of generality that $v_1, v_2 \in L^\infty$. Indeed for every $k \in \mathbb{N}$ set $v_i^k = \min(\max(v_i, -k), k)$ and $E_k = \{|v_1| < k\} \cap \{|v_2| < k\}$. Then, by the locality of $|Dv|$ (see Remark 3.93 of [3]):

- v_1^k, v_2^k are bounded BV functions which satisfy the assumptions of the lemma;
- $\mu(\Omega \setminus \bigcup_k E_k) = 0$ and $\mu \llcorner E_k << |Dv_i| \llcorner E_k = |Dv_i^k| \llcorner E_k \le |Dv_i^k|$.

Therefore, if the lemma holds for bounded BV functions, then we conclude that $\mu \llcorner E_k << \mathscr{L}^2 \llcorner \Omega$, and hence that $\mu << \mathscr{L}^2 \llcorner \Omega$. In addition, since every open set Ω can be covered by a countable family of convex subsets, we will assume that Ω is convex. Finally, we can assume, without loss of generality, that $\mu \le N|Dv_i|$ for some constant N. Indeed, for any $N > 0$ let E_N be the set of points x where the Radon–Nykodim derivatives $\mu/|Dv_i|(x) \le N$. Then $\mu(\mathbb{R}^2 \setminus \bigcup_N E_N) = 0$ and $\mu \llcorner E_N \le N|Dv_i|$.

Let any such v_is and Ω satisfy all these assumptions, and let C_1 and C_2 be the cones of the lemma. Recall that $C_i = C(e_i, a_i)$ for some $1 > a_i > 0$ and $e_i \in \mathbf{S}^1$. Given two vectors $z_1, z_2 \in \mathbb{R}^2$ we measure the angle $\theta(z_1, z_2)$ between z_1 and z_2 in counterclockwise direction. By possibly exchanging the indices we can assume $\theta(e_1, e_2) < \pi$. Then, the assumptions $C_1 \cap C_2 = (-C_1) \cap C_2 = \{0\}$ translate into the existence of a constant $\delta_0 > 0$ such that $\delta_0 \le \theta(f_1, f_2) \le \pi - \delta_0$ for every pair $(f_1, f_2) \in C_1 \times C_2$. Therefore, for $\delta = \sin \delta_0 > 0$,

$$\det(f_1, f_2) = |f_1||f_2| \sin \theta(f_1, f_2) \ge \delta |f_1||f_2| \qquad \forall (f_1, f_2) \in C_1 \times C_2. \tag{3}$$

By Remark 4.1, $\partial_{e_i} v_i \ge a_i |Dv_i|$. Set $w_i(x) = v_i(x) + \arctan(x \cdot e_i)$ and $w = (w_1, w_2)$ and note that

(a) $\partial_{e_i} w_i \ge a_i |Dw_i|$;
(b) $[\partial_{e_i} w_i](B_r(x)) > 0$ for every ball $B_r(x) \subset \Omega$;
(c) $\mu \le N|Dv_i| \le N a_i^{-1} \partial_{e_i} v_i \le N a_i^{-1} \partial_{e_i} w_i$.

Let $\{\varphi_\varepsilon\}$ be a standard family of nonnegative mollifiers supported in $B_\varepsilon(0)$ and consider the mollifications $w * \varphi_\varepsilon$ in the open sets $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$. We claim that

(a') $\nabla(w_i * \varphi_\varepsilon)(x) \in C_i$ for any i and any $x \in \Omega_\varepsilon$;
(b') $w * \varphi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^2$ is injective;
(c') $\mu * \varphi_\varepsilon \le N a_i^{-1} \partial_{e_i}(w_i * \varphi_\varepsilon)$.

From (a) we get $\partial_{e_i}(w_i * \varphi_\varepsilon) \ge a_i |Dw_i| * \varphi_\varepsilon \ge a_i |D(w_i * \varphi_\varepsilon)|$, which, by Remark 4.1 and the smoothness of $w_i * \varphi_\varepsilon$, implies (a'). (c') follows from $\mu \le N a_i^{-1} \partial_{e_i} w_i$. We now come to (b'). Note that, by (b), $\partial_{e_i}(w_i * \varphi_\varepsilon) > 0$. So $\nabla w_i * \varphi_\varepsilon(x) \ne 0$ for every $x \in \Omega_\varepsilon$, and hence belongs to $C_i \setminus \{0\}$. Let $x \ne y \in \Omega_\varepsilon$, and set $f := (x - y)/|x - y|$. We claim that, for some i,

$$|f \cdot z| > 0 \text{ for all } z \in C_i \setminus \{0\}. \tag{4}$$

Otherwise, there are $z_1 \in C_1$ and $z_2 \in C_2$ with $|z_i| = 1$ and $z_i \perp f$. Therefore, either $z_1 = z_2$ or $z_1 = -z_2$, contradicting $C_1 \cap C_2 = (-C_1) \cap C_2 = \{0\}$. Next, write

$$w_i * \varphi_\varepsilon(y) - w_i * \varphi_\varepsilon(x) \;=\; \int_0^{|y-x|} \nabla w_i * \varphi_\varepsilon\,(x + \sigma f) \cdot f \, d\sigma. \tag{5}$$

Recall that $\nabla w_i * \varphi_\varepsilon\,(x+\sigma f) \in C_i \setminus \{0\}$. Moreover, since $C_i \setminus \{0\}$ is connected, (4) implies that the integrand in (5) is either strictly positive, or strictly negative. In any case, $w_i * \varphi_\varepsilon(y) \neq w_i * \varphi_\varepsilon(x)$, which gives (b').

We are now ready for the final step. (a'), (b'), (c') and the area formula give

$$\begin{aligned}
\|w_1\|_\infty \|w_2\|_\infty &\geq \|w_1 * \varphi_\varepsilon\|_\infty \|w_2 * \varphi_\varepsilon\|_\infty \;\geq\; \mathscr{L}^2(w * \varphi_\varepsilon(\Omega_\varepsilon))\\
&\overset{(b')}{=} \int_{\Omega_\varepsilon} \det(\nabla(w * \varphi_\varepsilon)(x))\,dx\\
&\overset{(a')+(3)}{\geq} \delta \int_{\Omega_\varepsilon} |\nabla(w_1 * \varphi_\varepsilon)(x)||\nabla(w_2 * \varphi_\varepsilon)(x)|\,dx\\
&\geq \delta \int_{\Omega_\varepsilon} [\partial_{e_1}(w_1 * \varphi_\varepsilon)](x)\,[\partial_{e_2}(w_2 * \varphi_\varepsilon)](x)\,dx\\
&\overset{(c')}{\geq} \delta N^{-2} a_1 a_2 \int_{\Omega_\varepsilon} (\mu * \varphi_\varepsilon(x))^2\,dx.
\end{aligned}$$

Hence, $\|\mu * \varphi_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq N^2 (a_1 a_2 \delta)^{-1} \|w_1\|_\infty \|w\|_\infty$, which, letting $\varepsilon \downarrow 0$, gives $\mu = f\mathscr{L}^2$ for some $f \in L^2(\Omega)$. □

5 Proof of Theorem 1.1 in the Planar Case

We will argue by contradiction, and hence in a different way with respect to what said in Sect. 1. However, this is only to make the presentation more transparent: The ideas presented in this section can be easily adapted to prove the general decomposition property claimed at the end of the introduction.

So, let $u = (u_1, u_2) \in BV(B, \mathbb{R}^2)$ where B is a two-dimensional disk. Define

$$E \;:=\; \big\{x : \mathrm{rank}\,(Du/|Du|(x)) = 2\big\}, \tag{6}$$

and assume that $|D^s u|(E) > 0$. Without loss of generality, we can assume $u \in L^\infty$. Indeed, for every k truncate u_1 and u_2 by setting $u_i^k = \min\{\max\{u_i, -k\}, k\}$, and define

$$u^k := (u_1^k, u_2^k) \qquad \text{and} \qquad E_k := \big\{x : \mathrm{rank}\,(Du^k/|Du^k|(x)) = 2\big\}.$$

Then, $|D^s u^k|(E_k) \to |D^s u|(E)$ as $k \uparrow \infty$.

Hence, from now on we assume that $u \in BV \cap L^\infty$. For each point $x \in E$, we set $w_i(x) := Du_i/|Du|(x)$, which must be nonzero vectors. Thus, we can define $e_i(x) := w_i(x)/|w_i(x)|$, which is paralell to $Du_i/|Du_i|(x)$ and pointing in the same direction. Next, let

- $\mathscr{F}_k$ be the set of pairs $(f_1, f_2) \in \mathbf{S}^1 \times \mathbf{S}^1$ which form an angle $\geq 1/k$ and $\leq \pi - 1/k$;
- $F_k := \{x \in E : (e_1(x), e_2(x)) \in \mathscr{F}_k\}$.

Since $E = \bigcup_k F_k$, obviously $|D^s u|(F_k) > 0$ for some k. Fix any such k and for any $(f_1, f_2) \in \mathscr{F}_k$ and any $\varepsilon > 0$ define

$$F(f_1, f_2, \varepsilon) := \Big\{ x \in F_k : e_1(x) \in C(f_1, 1-\varepsilon), e_2(x) \in C(f_2, 1-\varepsilon) \Big\}.$$

We claim that there exist $(f_1, f_2) \in \mathscr{F}_k$ such that $|D^s u|(F(f_1, f_2, \varepsilon)) > 0$ for every $\varepsilon > 0$. Otherwise, by compactness of $\mathscr{F}_k$, we can find N pairs (f_1^j, f_2^j) and N positive numbers $\varepsilon_j > 0$ such that

$$\mathscr{F}_k \subset \bigcup_{j=1}^{N} C(f_1^j, 1-\varepsilon_j) \times C(f_2^j, 1-\varepsilon_j)$$

and $|D^s u|(F(f_1^j, f_2^j, \varepsilon_j)) = 0$. This would give $|D^s u|(F_k) \leq \sum_j |D^s u|(F(f_1^j, f_2^j, \varepsilon_j)) = 0$.

Therefore, fix $(f_1, f_2) \in \mathscr{F}_k$ such that $|D^s u|(F(f_1, f_2, \varepsilon)) > 0$ for every positive ε. Note that, since f_1 and f_2 are linearly independent, for ε sufficiently small the closed convex cones $C_i = C(f_i, 1-\varepsilon)$ satisfy $C_1 \cap C_2 = (-C_1) \cap C_2 = \{0\}$. We choose such an ε and we define

$$F' := \left\{ x : \frac{Du_i}{|Du_i|}(x) \in C_i \qquad \text{for both } is \right\}. \tag{7}$$

Theorem 1.1 is then implied by the following

Proposition 5.1. *Let $C = C(e, a)$ be a closed convex cone, $v \in BV \cap L^\infty(B, \mathbb{R})$ and*

$$G := \left\{ x : \frac{Dv}{|Dv|}(x) \in C \right\}. \tag{8}$$

For any convex cone $C' = C(e, a')$ with $a' < a$ there exists $w \in BV \cap L^\infty(B, \mathbb{R})$ such that $|Dv| \llcorner G << |Dw|$, and

$$\frac{Dw}{|Dw|}(x) \in C' \qquad \text{for } |Dw|\text{-a.e. } x. \tag{9}$$

Proof of Theorem 1.1. We recall that we argue by contradiction. The discussion above gives a bounded BV map $u : B \to \mathbb{R}^2$ and two closed convex cones C_1 and C_2 such that

- $C_1 \cap C_2 = (-C_1) \cap C_2 = \{0\}$;
- If E and F' are defined as in (6) and (7), then $|D^s u|(E \cap F') > 0$.

Now, by definition of E, $|D^s u| \llcorner E << |Du_i|$ for both $i = 1, 2$. Thus, if we set $\mu := |D^s u| \llcorner (E \cap F')$, then μ is a singular measure such that $\mu << |Du_i| \llcorner F'$ for both $i = 1, 2$.

Next choose two larger closed convex cones C_1' and C_2' so that $C_1' \cap C_2' = (-C_1') \cap C_2' = \{0\}$. Apply Proposition 5.1 to find v_1 and v_2 such that $Dv_i/|Dv_i|(x) \in C_i'$ for

$|Dv_i|$-a.e. x, and $|Du_i|\llcorner F' << |Dv_i|$. Thus, we have $\mu << |Dv_i|$ for both $i = 1,2$. Applying Lemma 1.5 we conclude that μ is absolutely continuous, which is the desired contradiction. □

Therefore, we are left with the task of proving Proposition 5.1. A special case of this proposition is when v is the indicator function of a set (which therefore is a Caccioppoli set). This case turns out to be an elementary geometric remark, but it is the key to prove the proposition in its full generality, via the coarea formula.

Proof of Proposition 5.1 when v is the indicator function of a set A. Since v is a BV function, A is a Caccioppoli set. We denote by $\partial^* A$ its reduced boundary (see Sect. 3.5 of [3] for the definition) and by η the approximate exterior unit normal to $\partial^* A$. Since $Dv = \eta \mathscr{H}^1 \llcorner \partial^* A$, the set G is given by $\{x \in \partial^* A : \eta(x) \in C\}$. Since $\partial^* A$ is rectifiable (cp. with Theorem 3.59 of [3]), G can be decomposed as $G_0 \cup \bigcup_{i=1}^{\infty} G_i$, where:

- $\mathscr{H}^1(G_0) = 0$ and for $i \geq 1$ each G_i is the subset of a C^1 curve γ_i;
- $\eta|_{G_i}$ coincides with the normal to the curve γ_i.

Step 1. For each i we claim that there are Lipschitz open sets $\{S_{i,j}\}_{j\in\mathbb{N}}$ such that: the exterior normal to $\partial S_{i,j}$ belongs $\mathscr{H}^1$-a.e. to C' and $\{\partial S_{i,j}\}_j$ is a covering of G_i.

Recall that $C' = C(e,a')$, and choose coordinates x_1, x_2 in $\mathbb{R}^2$ in such a way that $e = (0,1)$. For any $x \in G_i$, the normal $\nu_i(x)$ belongs to $C(e,a)$, and thus it is transversal to $(1,0)$. Since γ_i is C^1, this implies that we can choose an open ball B_x centered at x such that $\gamma_i \cap B_x$ is the graph $\{(x_1, f(x_1))\}$ of a C^1 function $f : I \to \mathbb{R}$, where I is some bounded open interval of $\mathbb{R}$. Moreover, by continuity of the normal ν_i, we can choose B_x so that $\nu_i(y) \in C'$ for every $y \in \gamma_i \cap B_x$.

Fix any such y. Note that the angle θ between e and $\nu_i(y)$ is equal to the angle between $(0,1)$ and the tangent to γ_i at y. Since $\nu_i(y) \in C(e,a')$, we conclude that

$$\theta \;=\; \arccos(\nu_i(y)\cdot e) \leq \arccos(1/a').$$

Thus $|f'| \leq \tan(\arccos(1/a')) \leq \sqrt{a'^2 - 1}$, and hence f is a Lipschitz function with constant less than $\sqrt{a'^2 - 1}$. It is an elementary well-known fact that f can be extended to a function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with the same Lipschitz constant. If we define $S_x := \{(x_1, x_2) : x_2 < \tilde{f}(x_1)\}$, then S_x is a Lipschitz open set, the normal to S_x belongs everywhere to the cone C', and ∂S_x covers $B_x \cap \gamma_i$ (Fig. 1). Since we can cover γ_i with a countable family of these balls B_x, the corresponding S_x form the desired countable covering $\{S_{i,j}\}_j$

Step 2. Consider the sets $\{S_{i,j}\}_{i,j}$. Their boundaries have all finite lengths, which we denote by $\ell_{i,j}$ and they cover $\mathscr{H}^1$-a.e. G. Let $\lambda_{i,j}$ be a collection of positive numbers such that $\sum_{i,j} \lambda_{i,j} \leq 1$ and $\sum_{i,j} \lambda_{i,j}\ell_{i,j} \leq 1$. Let w be the function

$$\sum_{i,j} \lambda_{i,j} \mathbf{1}_{S_{i,j}}.$$

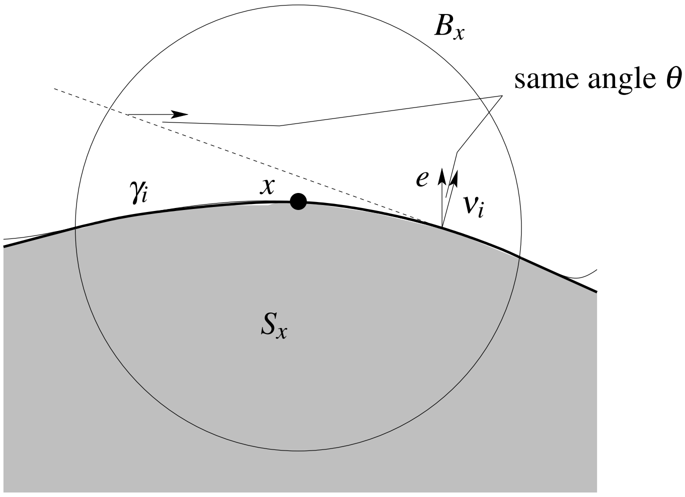

Fig. 1 The set S_x and the ball B_x

First of all, $\|w\|_\infty \le \sum_{i,j} \lambda_{i,j} \le 1$. Second, $w \in BV$ and $|Dw|$ is the nonnegative measure $\sum_{i,j} \lambda_{i,j} \mathscr{H}^1 \llcorner \partial S_{i,j}$. Thus, $|Dv| \llcorner G = \mathscr{H}^1 \llcorner G << |Dw|$. Finally,

$$\|w\|_{BV} = \|w\|_{L^1} + |Dw|(B) \le 2\pi + \sum_{i,j} \lambda_{i,j} \ell_{i,j} \le 2\pi + 1.$$

□

Proof of Proposition 5.1. Fix v, C, G and C' as in the statement, set $c := \|v\|_\infty$ and for every $t \in [-c, c]$ consider the function $v_t := \mathbf{1}_{\{v>t\}}$. Then, it follows from the coarea formula (see Theorem 3.40 of [3]) that:

(i) v_t is a BV function for $\mathscr{L}^1$-a.e. t, i.e. $\{v > t\}$ is a Caccioppoli set, and we denote by ν_t its exterior unit normal;
(ii) $\nu_t(x) = Dv/|Dv|(x)$ for $\mathscr{L}^1$-a.e. t and $\mathscr{H}^1$-a.e. $x \in \partial^*\{v > t\}$;
(iii) $|Dv| = \int_{-c}^{c} |Dv_t| \, d\mathscr{L}^1(t)$.

(Here, when α is a measure on Y and $y \mapsto \beta_y$ a weakly measurable map from Y into the space $\mathscr{M}(X)$ of Radon measures on X, the symbol $\int \beta_y \, d\alpha(y)$ denotes the measure γ on X which satisfies

$$\int \varphi(x) \, d\gamma(x) = \int_Y \int_X \varphi(x) \, d\beta_y(x) \, d\alpha(y)$$

for every $\varphi \in C_c(X)$.)

Therefore, for $\mathscr{L}^1$-a.e. t, v_t, C, G, and C' satisfy the hypotheses of the proposition. We denote by w_t the corresponding BV function given by the special case of this proposition, proved above. We will show below that w_t can be selected in such a way that the map $t \mapsto w_t \in L^\infty$ is weakly* measurable, i.e. that $t \mapsto \int \varphi w_t$ is measurable for every $\varphi \in L^1(B)$. Having a map with this property, we choose $\lambda \in L^1([-c,c])$ such that $\lambda > 0$ and

$$\int_{-c}^{c} \lambda(t)\big(\|w_t\|_\infty + \|w_t\|_{BV}\big)\,dt < \infty.$$

Assuming this fact, we set $w(x) := \int_{-c}^{c} \lambda(t) w_t(x)\,d\mathscr{L}^1(t)$. Then w is bounded, $|Dw|$ is a measure and $|Dw| \leq \int_{-c}^{c} \lambda(t)|Dw_t|d\mathscr{L}^1(t)$, which is a finite measure. Therefore $w \in BV \cap L^\infty$. Next, recall that $C' = C(e,a')$ for some real a' and some $e \in \mathbf{S}^1$. By Remark 4.1, $\partial_e w_t \geq a'|Dw_t|$. Thus $\partial_e w_t$ is a nonnegative measure for $\mathscr{L}^1$-a.e. t. From this we conclude

$$\partial_e w = \int_{-c}^{c} \lambda(t)\partial_e w_t d\mathscr{L}^1(t) \geq a' \int_{-c}^{c} \lambda(t)|Dw_t|d\mathscr{L}^1(t) \geq a'|Dw|,$$

which, by Remark 4.1, gives $Dw/|Dw|(x) \in C'$ for $|Dw|$-a.e. x. Finally, $|Dv_t|\llcorner G << |Dw_t|$, from which we get

$$\begin{aligned}|Dv|\llcorner G = \int_{-c}^{c} |Dv_t|\llcorner G d\mathscr{L}^1(t) &<< \int_{-c}^{c} \lambda(t)|Dw_t|d\mathscr{L}^1(t)\\ \leq a' \int_{-c}^{c} \lambda(t)\partial_e w_t d\mathscr{L}^1(t) &= a'\,\partial_e w \leq a'\,|Dw|.\end{aligned}$$

Thus w satisfies the requirements of the proposition.

Proof of the existence of a measurable selection $t \mapsto w_t$. In order to show the existence of such a selection, we will use a general Measurable Selection Theorem due to Aumann (see Theorem III.2 in [4]). More precisely, consider the set S of functions z such that $z \in BV \cap L^\infty$ and $Dz/|Dz|(x) \in C'$ for $|Dz|$-a.e. x. We endow S with the L^∞ weak* topology.

Next, set $F_t := \{z \in S : |Dv_t|\llcorner G << |Dz|\}$ if $v_t \in BV$, and $F_t = \emptyset$ otherwise. In order to apply Aumann's Theorem we need that:

- S and $[-c,c]$ are both locally compact and separable;
- The set $F := \{(t,u) : u \in F_t\} \subset [-c,c] \times S$ is a Borel set;
- $F_t \neq \emptyset$ for $\mathscr{L}^1$-a.e. t.

This last condition has been already shown. Moreover, $[-c,c]$ is compact and separable. Thus it remains to show that S is locally compact and separable and that F is a Borel set.

S is locally compact and separable. For every $N \in \mathbb{N}$ consider the set $S_N := S \cap \{\|z\|_\infty \leq N\}$. Since on bounded sets the L^∞ weak* topology is metrizable, clearly S_N is separable. Therefore, S is separable. We next show that S_N is compact, which implies that S is locally compact. Indeed consider any sequence $\{z_n\} \subset S_N$. By weak* compactness we can assume that $z_n \rightharpoonup^* z$ for some $z \in L^\infty$: Our task is to show that $z \in S$. Recall that $a'\partial_e z_n \geq |Dz_n|$. Thus $\{\partial_e z_n\}$ is a sequence of nonnegative measures which converge distributionally to $\partial_e z$. Therefore, these measures

are uniformly bounded, i.e. $\|z_n\|_{BV}$ is uniformly bounded. Thus $Dz_n \rightharpoonup^* Dz$. Up to extraction of a subsequence we can assume that $|Dz_n|$ converges in the sense of measures to some ν. Then,

$$|Dz| \leq \nu \;=\; w^* \lim_n |Dz_n| \;\leq\; a' w^* \lim_n \partial_e z_n \;=\; a' \partial_e z.$$

This implies that $z \in S$.

F is a Borel set. Denote by $\mathscr{M}^2$ the set of $\mathbb{R}^2$-valued Radon measures on B and by $\mathscr{M}^+$ the set of nonnegative Radon measures. Define $T : \mathscr{M}^+ \times \mathscr{M}^2 \to \mathbb{R}$ by

$$T(\nu,\mu) \;:=\; \int \frac{\nu}{|\mu|}(x)\, d|\mu|(x).$$

Note that $\nu << |\mu|$ if and only if $T(\nu,\mu) = \nu(B)$. Thus,

$$F \;=\; \big\{(t,z) \in [-c,c] \times S : T(|Dv_t| \llcorner G, Dz) = |Dv_t|(B \cap G)\big\}.$$

Since the map $t \mapsto |Dv_t|$ can be chosen Borel-measurable, in order to prove that F is a Borel set it suffices to show that T is a Borel function.

First of all, note that

$$T(\nu,\mu) \;=\; \sup_{n\in\mathbb{N}} \int \min\left\{n, \frac{\nu}{|\mu|}(x)\right\} d|\mu|(x) \;=\; \sup_{n\in\mathbb{N}} n \int \min\left\{1, \frac{\nu/n}{|\mu|}(x)\right\} d|\mu|(x).$$

Therefore, it suffices to show that the map $\tilde{T} : \mathscr{M}^+ \times \mathscr{M}^2 \to \mathbb{R}$ given by

$$\tilde{T}(\alpha,\mu) \;=\; \int \min\left\{1, \frac{\alpha}{|\mu|}(x)\right\} d|\mu|(x)$$

is Borel measurable. Note that $\tilde{T}(\alpha,\mu) = \inf\{\alpha(A) + |\mu|(B \setminus A) : A \subset B$ is measurable$\}$. Therefore,

$$\begin{aligned}\tilde{T}(\alpha,\mu) &= \inf_{f\in C_c(B),\, 0<f<1} \left[\int (1-f)\, d\alpha + \int f\, d|\mu|\right] \\ &= \inf_{f\in C_c(B),\, 0<f<1} \left[\int (1-f)\, d\alpha + \sup_{g\in C_c(B,\mathbb{R}^2),\, 0\leq |g|<f} \int g \cdot d\mu\right].\end{aligned}$$

Let $\mathscr{F}_1$ be a countable dense subset of $\{f \in C_c(B) : 0 < f < 1\}$ and $\mathscr{F}_2$ a countable dense subset of $C_c(B,\mathbb{R}^2)$. Then

$$\tilde{T}(\alpha,\mu) \;=\; \inf_{f\in\mathscr{F}_1} \sup_{g\in\mathscr{F}_2,\, 0\leq|g|<f} \left[\int (1-f)\, d\alpha + \int g \cdot d\mu\right]. \tag{10}$$

Since for each $(f,g) \in \mathscr{F}_1 \times \mathscr{F}_2$ the map

$$(\alpha, \mu) \mapsto \int (1-f)\, d\alpha + \int g \cdot d\mu$$

is weakly* continuous, (10) implies that $\tilde{T}$ is a Borel function. $\square$

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