# Preface

This book collects the lecture notes of two courses and one mini-course held in a winter school in Bologna in January 2005. The aim of this school was to popularize techniques of geometric measure theory among researchers and PhD students in hyperbolic differential equations. Though initially developed in the context of the calculus of variations, many of these techniques have proved to be quite powerful for the treatment of some hyperbolic problems. Obviously, this point of view can be reversed: We hope that the topics of these notes will also capture the interest of some members of the elliptic community, willing to explore the links to the hyperbolic world.

The courses were attended by about 70 participants (including post-doctoral and senior scientists) from institutions in Italy, Europe, and North-America. This initiative was part of a series of schools (organized by some of the people involved in the school held in Bologna) that took place in Bressanone (Bolzano) in January 2004, and in SISSA (Trieste) in June 2006. Their scope was to present problems and techniques of some of the most promising and fascinating areas of research related to nonlinear hyperbolic problems that have received new and fundamental contributions in the recent years. In particular, the school held in Bressanone offered two courses that provided an introduction to the theory of control problems for hyperbolic-like PDEs (delivered by Roberto Triggiani), and to the study of transport equations with irregular coefficients (delivered by Francois Bouchut), while the conference hosted in Trieste was organized in two courses (delivered by Laure Saint-Raymond and Cedric Villani) and in a series of invited lectures devoted to the main recent advancements in the study of Boltzmann equation. Some of the material covered by the course of Triggiani can be found in [17, 18, 20], while the main contributions of the conference on Boltzmann will be collected in a forthcoming special issue of the journal DCDS, of title “Boltzmann equations and applications”.

The three contributions of the present volume gravitate all around the theory of BV functions, which play a fundamental role in the subject of hyperbolic conservation laws. However, so far in the hyperbolic community little attention has been paid to some typical problems which constitute an old topic in geometric measure theory: the structure and fine properties of BV functions in more than one space dimension.

The lecture notes of Luigi Ambrosio and Gianluca Crippa stem from the remarkable achievement of the first author, who recently succeeded in extending the so-called DiPerna–Lions theory for transport equations to the BV setting. More precisely, consider the Cauchy problem for a transport equation with variable coefficients

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x) \;=\; 0\,, \\ \\ u(0,x) \;=\; u_0(x)\,. \end{cases} \tag{1}$$

When $b$ is Lipschitz, (1) can be explicitly solved via the method of characteristics: a solution $u$ is indeed constant along the trajectories of the ODE

$$\begin{cases} \frac{d\Phi_x}{dt} \;=\; b(t,\Phi_x(t)) \\ \\ \Phi(0,x) \;=\; x\,. \end{cases} \tag{2}$$

Transport equations appear in a wealth of problems in mathematical physics, where usually the coefficient is coupled to the unknowns through some nonlinearities. This already motivates from a purely mathematical point of view the desire to develop a theory for (1) and (2) which allows for coefficients $b$ in suitable function spaces. However, in many cases, the appearance of singularities is a well-established central fact: the development of such a theory is highly motivated from the applications themselves.

In the 1980s, DiPerna and Lions developed a theory for (1) and (2) when $b \in W^{1,p}$ (see [16]). The task of extending this theory to BV coefficients was a long-standing open question, until Luigi Ambrosio solved it in [2] with his Renormalization Theorem. Sobolev functions in $W^{1,p}$ cannot jump along a hypersurface: this type of singularity is instead typical for a BV function. Therefore, not surprisingly, Ambrosio's theorem has found immediate application to some problems in the theory of hyperbolic systems of conservation laws (see [3, 5]).

Ambrosio's result, together with some questions recently raised by Alberto Bressan, has opened the way to a series of studies on transport equations and their links with systems of conservation laws (see [4,6–13]). The notes of Ambrosio and Crippa contain an efficient introduction to the DiPerna–Lions theory, a complete proof of Ambrosio's theorem and an overview of the further developments and open problems in the subject.

The first proof of Ambrosio's Renormalization Theorem relies on a deep result of Alberti, perhaps the deepest in the theory of BV functions (see [1]).

Consider a regular open set $\Omega \subset \mathbb{R}^2$ and a map $u : \mathbb{R}^2 \to \mathbb{R}^2$ which is regular in $\mathbb{R}^2 \setminus \partial\Omega$ but jumps along the interface $\partial\Omega$. The distributional derivative of $u$ is then the sum of the classical derivative (which exists in $\mathbb{R}^2 \setminus \partial\Omega$) and a singular matrix-valued radon measure $\nu$, supported on $\partial\Omega$. Let $\mu$ be the nonnegative measure on $\mathbb{R}^2$ defined by the property that $\mu(A)$ is the length of $\partial\Omega \cap A$. Moreover, denote by $n$ the exterior unit normal to $\partial\Omega$ and by $u^-$ and $u^+$, respectively, the interior and exterior traces of $u$ on $\partial\Omega$. As a straightforward application of Gauss' theorem, we then conclude that the measure $\nu$ is given by $[(u^+ - u^-)\otimes n]\,\mu$.

Consider now the singular portion of the derivative of *any* BV vector-valued map. By elementary results in measure theory, we can always factorize it into a matrix-valued function $M$ times a nonnegative measure $\mu$. Alberti's Rank-One Theorem states that the values of $M$ are always rank-one matrices. The depth of this theorem can be appreciated if one takes into account how complicated the singular measure $\mu$ can be.

Though the most recent proof of Ambrosio's Renormalization Theorem avoids Alberti's result, the Rank-One Theorem is a powerful tool to gain insight in subtle further questions (see for instance [6]). The notes of Camillo De Lellis is a short and self-contained introduction to Alberti's result, where the reader can find a complete proof.

As already mentioned above, the space of BV functions plays a central role in the theory of hyperbolic conservation laws. Consider for instance the Cauchy problem for a scalar conservation law

$$\begin{cases} \partial_t u + \mathrm{div}_x[f(u)] \;=\; 0\,, \\ \\ u(0,\cdot) \;=\; u_0\,. \end{cases} \tag{3}$$

It is a classical result of Kruzhkov that for bounded initial data $u_0$ there exists a unique entropy solution to (3). Furthermore, if $u_0$ is a function of bounded variation, this property is retained by the entropy solution.

Scalar conservation laws typically develop discontinuities. In particular jumps along hypersurfaces, the so-called *shock waves*, appear in finite time, even when starting with smooth initial data. These discontinuities travel at a speed which can be computed through the so-called Rankine–Hugoniot condition. Moreover, the admissibility conditions for distributional solutions (often called *entropy conditions*) are in essence devised to rule out certain "non-physical" shocks. When the entropy solution has BV regularity, the structure theory for BV functions allows us to identify a jump set, where all these assertions find a suitable (measure-theoretic) interpretation.

What happens if instead the initial data are merely bounded? Clearly, if $f$ is a linear function, i.e. $f''$ vanishes, (3) is a transport equation with constant coefficients: extremely irregular initial data are then simply preserved. When we are far from this situation, loosely speaking when the range of $f''$ is "generic", $f$ is called genuinely nonlinear. In one space dimension an extensively studied case of genuine nonlinearity is that of convex fluxes $f$. It is then an old result of Oleinik that, under this assumption, entropy solutions are BV functions for any bounded initial data. The assumption of genuine nonlinearity implies a regularization effect for the equation.

In more than one space dimension (or under milder assumptions on $f$) the BV regularization no longer holds true. However, Lions, Perthame, and Tadmore gave in [19] a kinetic formulation for scalar conservation laws and applied velocity averaging methods to show regularization in fractional Sobolev spaces. The notes of Gianluca Crippa, Felix Otto, and Michael Westdickenberg start with an introduction to entropy solutions, genuine nonlinearity, and kinetic formulations. They then discuss the regularization effects in terms of linear function spaces for a "generalized Burgers" flux, giving optimal results.

From a structural point of view, however, these estimates (even the optimal ones) are always too weak to recover the nice picture available for the BV framework, i.e. a solution which essentially has jump discontinuities behaving like shock waves. Guided by the analogy with the regularity theory developed in [14] for certain variational problems, De Lellis, Otto, and Westdickenberg in [15] showed that this picture is an outcome of an appropriate "regularity theory" for conservation laws. More precisely, the property of being an entropy solution to a scalar conservation law (with a genuinely nonlinear flux $f$) allows a fairly detailed analysis of the possible singularities. The information gained by this analysis is analogous to the fine properties of a generic BV function, even when the BV estimates fail. The notes of Crippa, Otto, and Westdickenberg give an overview of the ideas and techniques used to prove this result.

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*Fabio Ancona*
*Stefano Bianchini*
*Rinaldo M. Colombo*
*Camillo De Lellis*
*Andrea Marson*
*Annamaria Montanari*

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