

# G Porous Absorbers

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If no extra reference is given in the Sections of this chapter, see [G.1]

The aims of the Sections in this chapter are twofold: 1<sup>st</sup> derive the characteristic propagation constant  $\Gamma_a$  and the characteristic wave impedance  $Z_a$  of a plane wave in the porous material as functions of structure data, 2<sup>nd</sup> derive the flow resistivity  $\Xi$  of the material as function of structure data, because the flow resistivity is the most useful material parameter for the evaluation of  $\Gamma_a, Z_a$ . Different models of a porous material will be displayed; special attention will be given to fibrous materials. See also the Sections about scattering in random media in the chapter “Scattering of Sound” for propagation constant and wave impedance in fibrous and granular media.

## G.1 Structure Parameters of Porous Materials

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The definition of structure parameters depends on the model theory in which they are applied (see the relevant Sections for specific definitions). Especially, see the chapter about Biot’s theory for special parameters of that theory. This Section describes the structure parameters which come from the theory of the “quasi-homogeneous material” (see next Section), because they are most often used to describe qualitatively porous absorber materials; the theory of the quasi-homogeneous material is the most simple theory.

*Volume porosity*  $\sigma_V$ , *massivity*  $\mu$ :

The *volume porosity*  $\sigma_V$  is the ratio of air volume contained in the porous material to the total volume; it is given by  $\sigma_V = 1 - \rho_a/\rho_m$  with  $\rho_a$  = bulk density of the porous absorber material,  $\rho_m$  = density of the (dense) matrix material. For glass or mineral fibre materials a value  $\rho_m \approx 2250$  [kg/m<sup>3</sup>] may be used.

For some considerations it is advantageous to apply the *massivity*  $\mu = 1 - \sigma_V$ .

The following table gives ranges of  $\sigma_V$  for some materials.

Material	$\sigma_v$ from	$\sigma_v$ to
Mineral fibre materials	0,92	0,99
Foams	0,95	0,995
Felts	0,83	0,95
Wood-fibre board	0,65	0,80
Wood-wool board	0,50	0,65
Porous render	0,60	0,65
Pumice concrete	0,25	0,50
Pumice fill	0,65	0,85
Gravel and stone chip fill	0,25	0,45
Ceramic filters	0,33	0,42
Brick	0,25	0,30
Sinter metal	0,10	0,25
Fire-clay	0,15	0,35
Sand stone	0,02	0,06
Marble		ca. 0,005

### Structure factor $\chi$ :

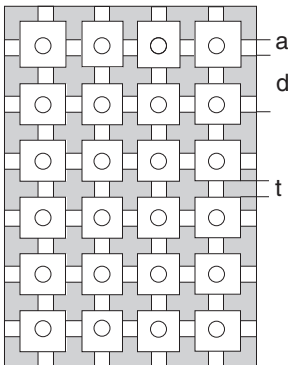
The *structure factor* is the most ambiguous quantity in porous material theories. It is defined in the theory of the quasi-homogeneous material as  $\chi = \sigma_v / \sigma_s$  with  $\sigma_s$  = surface porosity of a cut through the material. Its value depends on the type of the pore shapes. The pore volume  $V_p$  is given by the integral of the pore surface  $S_p$  over a distance  $x$  normal to the considered pore surface:

$$V_p = \int_x S_p(x) dx \quad ; \quad \chi = \frac{\langle \sigma_s \rangle_x}{\sigma_s} \quad (1)$$

A model of an open-cellular foam consisting of cubic cells with connecting pores has the values:

$$\sigma_v \approx 1 - 3 \frac{t}{d} \quad ; \quad \chi = \frac{(1 + \pi a/2)(d - 3t)}{\pi a^2} \quad (2)$$

Two end corrections  $\Delta \ell = \pi a/2$  have been added to the neck length  $t$ .



The following table gives the volume and surface porosities  $\sigma_v$ ,  $\sigma_s$  and the structure factor  $\chi$  of some regular model structures;  $a$  is the width of the pores,  $d$  is their distance.

Structure	$\sigma_v$	$\sigma_s$	$\chi$
Flat pores, longitudinal or inclined	$a/d$	$a/d$	1
Square pores, longitudinal or inclined	$(a/d)^2$	$(a/d)^2$	1
Round pores, longitudinal or inclined	$\pi(a/d)^2/4$	$\pi(a/d)^2/4$	1
Square fibres, longitudinal or inclined	$2a/d - (a/d)^2$	$2a/d - (a/d)^2$	1
Round fibres, longitudinal or inclined	$1 - \pi(a/d)^2/4$	$1 - \pi(a/d)^2/4$	1
Square fibres, transversal	$2a/d - (a/d)^2$	$a/d$	$2 - a/d$
Round fibres, transversal	$1 - \pi(a/d)^2/4$	$1 - a/d$	$1 + a/d + (1 - \pi/4)(a/d)^2$
Array of cubes	$3a/d - 3(a/d)^2 + (a/d)^3$	$2a/d - (a/d)^2$	$3/2 - 3a/(4d) + (a/d)^2/8$

Flow resistivity  $\Xi$ ; absorber variable  $E$ :

The *flow resistivity*  $\Xi$  of a porous material is its flow resistance per unit thickness for stationary flow with low velocity  $V$  (about  $V = 0.05$  [cm/s]). For a material test sample of thickness  $dx$ :

$$\Xi = -\frac{1}{dx} \frac{dP}{V} \quad (3)$$

with  $dP$  = static pressure difference across the sample in flow direction.

Theories mostly determine the interior velocity  $V_i$  in a pore for a given pressure difference. This “internal” flow resistivity  $\Xi_i$  is related with the flow resistivity  $\Xi$  of the sample by  $\Xi = \Xi_i/\sigma$ .

A suitable non-dimensional quantity  $R$  is the ratio of the flow resistance  $\Xi \cdot d$  of a layer of thickness  $d$  with the free field wave impedance  $Z_0$ :  $R = \Xi \cdot d / Z_0$ . According to:

$$R = \frac{\Xi d}{\omega \rho_0 \cdot \lambda_0 / 2\pi} \quad (4)$$

this is the ratio of the flow resistance to the mass reactance of a layer of air with thickness  $\lambda_0 / 2\pi$ . An important non-dimensional quantity for the evaluation of the characteristic data  $\Gamma_a, Z_a$  is the *absorber variable* ( $f$  = frequency):

$$E = \frac{\rho_0 f}{\Xi} \quad (5)$$

One distinguishes with fibrous materials consisting of parallel fibres the flow resistivity  $\Xi_{||}$  if the flow is parallel to the fibres, and  $\Xi_{\perp}$  if the flow is transversal to the fibres. Empirical data by SULLIVAN for parallel fibre materials with mono-valued fibre radii  $a$  follow the relations:

$$\Xi_{||} = 3.94 \cdot \frac{\eta}{a^2} \frac{\mu^{1.413}}{1 - \mu} [1 + 27 \mu^3] \quad (6)$$

and

$$\Xi_{\perp} = \begin{cases} 10.56 \frac{\eta}{a^2} \frac{\mu^{1.531}}{(1 - \mu)^3} & ; \quad a \approx 6 - 10 [\mu\text{m}] \\ 6.8 \frac{\eta}{a^2} \frac{\mu^{1.296}}{(1 - \mu)^3} & ; \quad a \approx 20 - 30 [\mu\text{m}] \end{cases} \quad (7)$$

$a$  = fibre radius;  
 $\eta$  = dynamic viscosity;  
 $\mu = 1 - \sigma$  = massivity

Semi-empirical data (analytical relation fitted to experimental values) for fibre materials with mono-valued fibre radii  $a$  and random fibre orientation give:

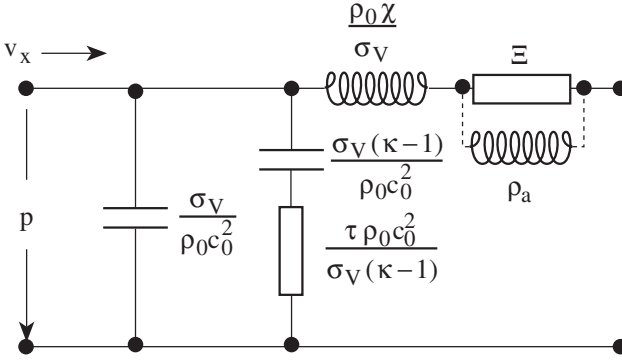
$$\Xi = 4 \cdot \frac{\eta}{a^2} \left[ 0.55 \frac{\mu^{4/3}}{(1 - \mu)} + \sqrt{2} \frac{\mu^2}{(1 - \mu)^3} \right] \quad (8)$$

Empirical data for fibrous materials with random fibre radius distribution and random fibre orientation can be approximated by:

$$\Xi = \frac{\eta}{\langle a^2 \rangle} \cdot \begin{cases} 3.2 \mu^{1.42} & ; \text{glass fiber material} \\ 4.4 \mu^{1.59} & ; \text{mineral fiber material} \end{cases} \quad (9)$$

## G.2 Theory of the Quasi-Homogeneous Material

An equivalent network is designed for a homogeneous material, taking into account a finite volume porosity  $\sigma_v$ , a structure factor  $\chi$  of randomly oriented pores, and a relaxation time constant  $\tau$  for heat exchange between air and matrix material. A possible vibration of the matrix, induced by the friction between air and matrix, can be included also.



**Fig. 1** Equivalent network for unit length of a porous material

The 1<sup>st</sup> transversal branch represents the compressibility of the air in the material; the 2<sup>nd</sup> transversal branch represents the relaxation due to heat exchange with the matrix; the 1<sup>st</sup> longitudinal element represents the inertia of the air in the pores (modified by the structure factor  $\chi$ ); the 2<sup>nd</sup> longitudinal element represents the friction. If matrix vibration shall be included, the bulk density  $\rho_a$  of the material is parallel to  $\Xi$ . This can be taken into account by an effective resistivity:

$$\Xi \rightarrow \Xi_{\text{eff}} = \frac{j\omega\rho_a \cdot \Xi}{j\omega\rho_a + \Xi} \quad (10)$$

Characteristic values:

$$\frac{\Gamma_a}{k_0} = j\sqrt{\chi \left(1 + \frac{\kappa - 1}{1 + j\omega\tau}\right) \cdot \left(1 - j\frac{\sigma_v \Xi}{\omega\rho_0\chi}\right)} \quad (11)$$

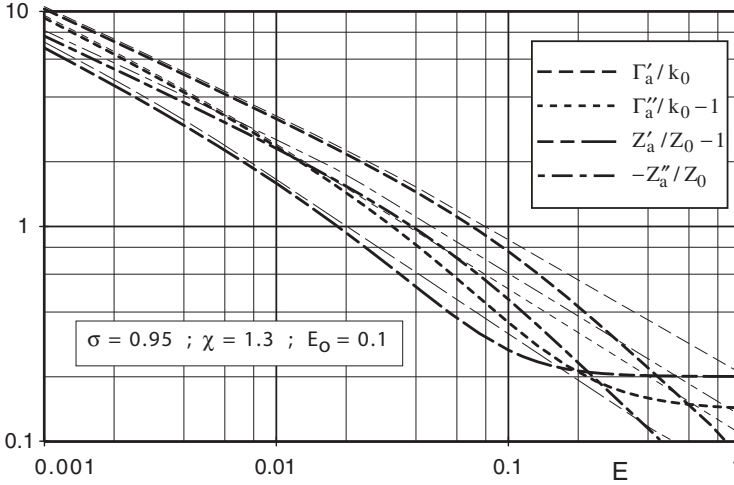
$$\frac{Z_a}{Z_0} = \frac{1}{\sigma_v} \sqrt{\chi \left(1 - j\frac{\sigma_v \Xi}{\omega\rho_0\chi}\right) / \left(1 + \frac{\kappa - 1}{1 + j\omega\tau}\right)} \quad (12)$$

$k_0$  = free field wave number;  
 $Z_0$  = free field wave impedance;  
 $\rho_0$  = density of air;  
 $\omega$  = angular frequency;  
 $\kappa$  = adiabatic exponent of air;  
 $\Xi$  = flow resistivity;  
 $\chi$  = structure factor;  
 $\sigma_v$  = volume porosity;  
 $\tau$  = heat relaxation time constant;

Introducing  $E = \rho_0 f / \Xi$  and  $E_0 = \rho_0 f_0 / \Xi$  with  $f_0$  = relaxation frequency, the characteristic values are:

$$\frac{\Gamma_a}{k_0} = j\sqrt{\frac{\kappa + jE/E_0}{1 + jE/E_0} \cdot \left(\chi - j\frac{\sigma_v}{2\pi E}\right)} \quad ; \quad \frac{Z_a}{Z_0} = \frac{1}{\sigma_v} \sqrt{\frac{1 + jE/E_0}{\kappa + jE/E_0} \cdot \left(\chi - j\frac{\sigma_v}{2\pi E}\right)} \quad (13)$$

$E_0 \rightarrow 0$  belongs to an isothermal sound wave in the material;  $E_0 \rightarrow \infty$  belongs to an adiabatic sound wave;  $E_0 = 0.1$  is a typical value for mineral fibre materials.



**Fig. 2** Thick lines: theory of quasi-homogeneous material; thin lines: measured values

### G.3 Rayleigh Model With Round Capillaries

The RAYLEIGH model has been for a long time the only existing model for porous materials.

This model consists of parallel circular capillaries with radius  $a$  and mutual distance  $d$  in a bloc of the matrix material. The sound propagation in the capillary is determined with viscous and thermal losses at the capillary wall taken into account. See the chapter “Duct Acoustics” for sound propagation in capillaries. The arrangement and mutual distance of the capillaries is supposed to be so, that a prescribed porosity  $\sigma$  is obtained (e.g.  $\sigma = \pi a^2/d^2$  for a square arrangement), even if the value of the porosity cannot be realised physically (in a square arrangement the realisable porosity is  $\sigma \leq \pi/4 = 0.785$ ), because only a single capillary is considered. The theory gives the propagation constant  $\Gamma_a$  of the density wave (which is considered to be the propagation constant of sound in the porous material), and the “interior” axial wave impedance  $Z_i$  of the density wave in a capillary. Its relation to the wanted wave impedance  $Z_a$  of the porous material is  $Z_a = Z_i/\sigma$ .

The normalised characteristic values of a cell are:

$$\frac{\Gamma_a}{k_0} = j \sqrt{\frac{\rho_{\text{eff}}}{\rho_0} \cdot \frac{C_{\text{eff}}}{C_0}} \quad ; \quad \frac{Z_i}{Z_0} = \sqrt{\frac{\rho_{\text{eff}}}{\rho_0} \bigg/ \frac{C_{\text{eff}}}{C_0}} \quad (1)$$

with the ratios of the effective air density and air compressibility (index 0 indicates free field values):

$$\frac{\rho_{\text{eff}}}{\rho_0} = \frac{1}{1 - J_{1,0}(k_{va})} \quad ; \quad \frac{C_{\text{eff}}}{C_0} = 1 + (\kappa - 1) \cdot J_{1,0}(k_{\alpha 0} a) \quad (2)$$

and the function:

$$J_{1,0}(z) = 2 \frac{J_1(z)}{z \cdot J_0(z)} = 2 \cdot \frac{1}{2-} \frac{z^2}{4-} \frac{z^2}{6-} \frac{z^2}{8-} \dots \frac{z^2}{2 \cdot n - \dots} ; n = 1, 2, 3 \dots \quad (3)$$

with Bessel functions  $J_n(z)$  and the continued-fraction expansion of their ratio in the last term (the expansion must go until  $|z|^2/(2n) \ll 1$ ). The asymptotic approximation

$$J_{1,0}(z) \approx \frac{2}{z} \frac{\tan(z - \frac{\pi}{4}) + \frac{3}{8z}}{1 + \frac{1}{8z} \tan(z - \frac{\pi}{4})} \quad (4)$$

may be applied for large arguments  $|z|$ . The squares of the used wave numbers are

$$k_v^2 = -j \omega / \nu ; \quad k_{\alpha 0}^2 = k_v^2 \cdot \sqrt{\kappa \text{Pr}} \quad (5)$$

$\kappa$  = adiabatic exponent;  
 $\nu$  = kinematic viscosity;  
 $\eta$  = dynamic viscosity;  
 $\text{Pr}$  = Prandtl number;  
 $\mu$  = cell massivity;  
 $C_0 = 1/(\rho_0 c_0^2)$ ;  
 $\Xi$  = flow resistivity

The argument  $k_v a$  should be replaced with an argument, which can be applied also for material with random pore radius distribution. This is the “absorber variable”  $E = \rho_0 f / \Xi$ . For its application, the flow resistivity  $\Xi_i$  in a pore must be determined.

The flow velocity profile in a circular capillary is, with the static pressure gradient  $dP/dx$ :

$$V(r) = -\frac{1}{4\eta} \frac{dP}{dx} (a^2 - r^2) \quad (6)$$

The flow resistivity  $\Xi_i$  is defined with the volume flow  $Q$  through the capillary cross-section area  $S$ :

$$\Xi_i = \frac{-dP/dx \cdot S}{Q} = \frac{8\eta}{a^2} ; \quad Q = -\frac{\pi a^4}{8\eta} \frac{dP}{dx} \quad (7)$$

Thus the wanted relations between the arguments are:

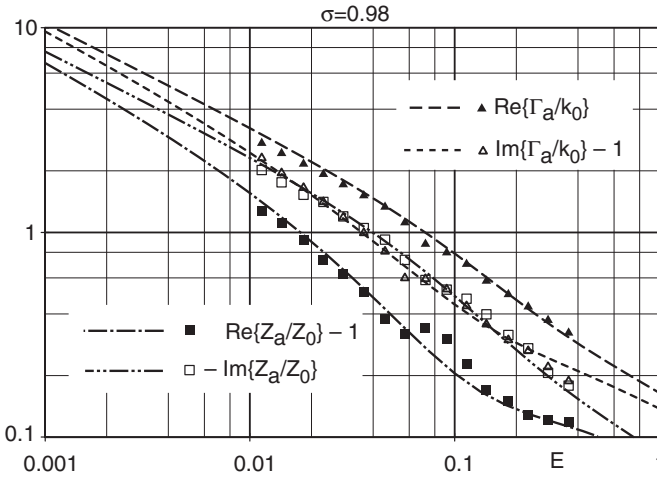
$$(k_v a)^2 = -16\pi j E ; \quad (k_{\alpha 0} a)^2 = -16\pi j \kappa \text{Pr} E \quad (8)$$

If in a material test the (exterior) flow resistivity  $\Xi$  is determined, its relation to the interior flow resistivity is  $\Xi = \Xi_i / \sigma$  with the porosity  $\sigma$  of the material.

The continued-fraction approximation (up to the 5<sup>th</sup> fraction) of the characteristic values is with the variable  $E$ :

$$\left( \frac{\Gamma_i}{k_0} \right)^2 = -\kappa \frac{75 + 30\pi j [8 + \text{Pr}(5 + 3\kappa)] E - \pi^2 [90 + \text{Pr}(480 + 288\kappa) + \text{Pr}^2 \kappa(80 + 10\kappa)] E^2 - \dots}{2\pi E [75j - 40\pi(1 + 6\kappa \text{Pr}) E - 2\pi^2 j \kappa \text{Pr}(64 + 45\kappa \text{Pr}) E^2 + \dots] - 4\pi^3 j \text{Pr}[45 + 27\kappa + \kappa \text{Pr}(84 + 8\kappa)] E^3 + 12\pi^4 \kappa \text{Pr}^2(8 + \kappa) E^4 + 48\pi^3 (\kappa \text{Pr})^2 E^3} \quad (9)$$

$$\left(\frac{Z_i}{Z_0}\right)^2 = \frac{1}{\kappa} \frac{225 + 720\pi j (1 + \kappa \text{Pr}) E - 18\pi^2 [15 + \kappa \text{Pr}(128 + 15\kappa \text{Pr})] E^2 - \dots}{2\pi E \{225j - 30\pi[4 + \text{Pr}(15 + 9\kappa)] E - 6\pi^2 j \text{Pr}[40 + 24\kappa + \kappa \text{Pr}(40 + 5\kappa)] E^2 + \dots} \\ - \frac{864\pi^3 j \kappa \text{Pr}(1 + \kappa \text{Pr}) E^3 + 324\pi^4 (\kappa \text{Pr})^2 E^4}{+ 16\pi^3 \kappa \text{Pr}^2 (8 + \kappa) E^3} \quad (10)$$



**Fig. 3** Comparison of the characteristic values from the Rayleigh model (curves) with measurements at a technical mineral fibre absorber (points)

## G.9 Effective Wave Multiple Scattering in Transversal Fibre Bundle

As in Section G.8 (with which this Section has a number of similarities) the porous material consists of parallel fibres (at rest) with fibre radius  $a$  and fibre number density  $N$ . A sound wave propagates inside the fibre bundle normal to the fibres. Whereas it was assumed in Section G.8 that the propagating wave is a density wave, here it is supposed to be a less specified “effective” wave with potential  $\Phi_E$  with propagation constant  $\Gamma = jk_e$ . The propagating wave (exciting wave) may be some hybrid wave of a density wave, a thermal wave and a viscous wave. The propagation constant and wave impedance  $Z_e$  of this wave shall be determined. See Section G.8 for the supposed co-ordinates. As before, the index  $\beta = \rho, \alpha$  indicates a density or thermal wave with scalar potential functions  $\Phi_\beta$ , and  $\Psi_z$  is the  $z$  component of the vector potential of a viscous wave. It is further assumed as before, that the nearest neighbours of a reference fibre in the position  $(r_i, \varphi_i)$  have a distance  $d$  of it.



Exciting wave formulation:

$$\begin{aligned}\Phi_E(r, \varphi) &= A_E \cdot e^{-\Gamma x} = A_E \cdot e^{-jk_e r \cos \varphi} = A_E \cdot \sum_{n=0}^{\infty} (-j)^n \delta_n J_n(k_e r) \cos(n\varphi) \\ &= A_E \cdot \sum_{n=0}^{\infty} \Phi_{En}(r) \cdot \cos(n\varphi) \quad ; \quad \delta_n = \begin{cases} 1 ; n = 0 \\ 2 ; n > 0 \end{cases}\end{aligned}\quad (1)$$

Scattered field formulation (marked with a prime):

$$\begin{aligned}\Phi'_\beta(r, \varphi) &= \sum_{n=0}^{\infty} A_{\beta n} H_n^{(2)}(k_\beta r) \cos(n\varphi) = \sum_{n=0}^{\infty} \Phi'_{\beta n}(r) \cdot \cos(n\varphi) \quad ; \quad \beta = \rho, \alpha \\ \Psi'_z(r, \varphi) &= \sum_{n=0}^{\infty} A_{vn} H_n^{(2)}(k_v r) \sin(n\varphi) = \sum_{n=0}^{\infty} \Psi'_{vn}(r) \cdot \sin(n\varphi) \quad ; \quad A_{v0} = 0\end{aligned}\quad (2)$$

Total scattered field at a reference fibre in  $(r_i, \varphi_i)$  (marked with a double prime):

$$\begin{aligned}\Phi''_\beta(r_i, \varphi_i) &= N \sum_{n=0}^{\infty} A_{\beta n} \int_0^{2\pi} d\varphi_{ij} \int_d^\infty e^{-jk_e r_{ij} \cos \varphi_{ij}} \cdot H_n^{(2)}(k_\beta r_j) \cos(n\varphi_j) \cdot r_{ij} \cdot dr_{ij} \\ \Psi''_z(r_i, \varphi_i) &= N \sum_{n=0}^{\infty} A_{vn} \int_0^{2\pi} d\varphi_{ij} \int_d^\infty e^{-jk_e r_{ij} \cos \varphi_{ij}} \cdot H_n^{(2)}(k_v r_j) \sin(n\varphi_j) \cdot r_{ij} \cdot dr_{ij}\end{aligned}\quad (3)$$

and after application of the addition theorem for Hankel functions and integration:

$$\begin{aligned}\Phi_\beta(r_i, \varphi_i) &= \\ &= \sum_{n=0}^{\infty} \cos(n\varphi_i) \cdot \{A_{\beta n} H_n^{(2)}(k_\beta r_i) - \frac{2\pi N}{k_\beta^2 - k_e^2} (-j)^n J_n(k_\beta r_i) \cdot \sum_{m=0}^{\infty} (j)^m B_{\beta mn} \cdot A_{\beta m}\} \\ &= \sum_{n=0}^{\infty} \Phi_{\beta rn}(r_i) \cdot \cos(n\varphi_i)\end{aligned}\quad (4)$$

$$\begin{aligned}\Psi_z(r_i, \varphi_i) &= \\ &= \sum_{n=0}^{\infty} \sin(n\varphi_i) \cdot \{A_{vn} H_n^{(2)}(k_v r_i) + \frac{2\pi N}{k_v^2 - k_e^2} (-j)^n J_n(k_v r_i) \cdot \sum_{m=0}^{\infty} (j)^m B_{vmn} \cdot A_{vm}\} \\ &= \sum_{n=0}^{\infty} \Psi_{zrn}(r_i) \cdot \sin(n\varphi_i)\end{aligned}\quad (5)$$

with the abbreviations for both  $\beta = \rho, \alpha$ :

$$\begin{aligned}
 B_{\beta mn} &= k_e d \cdot [H_{m+n}^{(2)}(k_\beta d) J_{m+n-1}(k_e d) + H_{m-n}^{(2)}(k_\beta d) J_{m-n-1}(k_e d)] - \\
 &\quad - k_\beta d \cdot [H_{m+n-1}^{(2)}(k_\beta d) J_{m+n}(k_e d) + H_{m-n-1}^{(2)}(k_\beta d) J_{m-n}(k_e d)] \\
 B_{v mn} &= k_e d \cdot [H_{m+n}^{(2)}(k_v d) J_{m+n-1}(k_e d) - H_{m-n}^{(2)}(k_v d) J_{m-n-1}(k_e d)] - \\
 &\quad - k_v d \cdot [H_{m+n-1}^{(2)}(k_v d) J_{m+n}(k_e d) - H_{m-n-1}^{(2)}(k_v d) J_{m-n}(k_e d)]
 \end{aligned} \tag{6}$$

These scattered waves plus the exciting wave have to satisfy the boundary conditions at the reference fibre (which is supposed to be isothermal):

$$v_\varphi(a, \varphi_i) = 0: \begin{cases} n \cdot [A_{\rho n} H_n^{(2)}(k_\rho a) - \frac{2\pi N}{k_\rho^2 - k_e^2} (-j)^n J_n(k_\rho a) \cdot \sum_{m=0} j^m B_{\rho mn} \cdot A_{\rho m}] + \\ + n \cdot [A_{\alpha n} H_n^{(2)}(k_\alpha a) - \frac{2\pi N}{k_\alpha^2 - k_e^2} (-j)^n J_n(k_\alpha a) \cdot \sum_{m=0} j^m B_{\alpha mn} \cdot A_{\alpha m}] - \\ - k_v a \cdot [A_{vn} H_n^{(2)}(k_v a) + \frac{2\pi N}{k_v^2 - k_e^2} (-j)^n J_n'(k_v a) \cdot \sum_{m=0} j^m B_{v mn} \cdot A_{vm}] = \\ = -n \delta_n (-j)^n J_n(k_e a) \cdot A_E \end{cases} \tag{7}$$

$$v_r(a, \varphi_i) = 0: \begin{cases} k_\rho a \cdot [A_{\rho n} H_n^{(2)}(k_\rho a) - \frac{2\pi N}{k_\rho^2 - k_e^2} (-j)^n J_n'(k_\rho a) \cdot \sum_{m=0} j^m B_{\rho mn} \cdot A_{\rho m}] + \\ + k_\alpha a \cdot [A_{\alpha n} H_n^{(2)}(k_\alpha a) - \frac{2\pi N}{k_\alpha^2 - k_e^2} (-j)^n J_n'(k_\alpha a) \cdot \sum_{m=0} j^m B_{\alpha mn} \cdot A_{\alpha m}] - \\ - n \cdot [A_{vn} H_n^{(2)}(k_v a) + \frac{2\pi N}{k_v^2 - k_e^2} (-j)^n J_n(k_v a) \cdot \sum_{m=0} j^m B_{v mn} \cdot A_{vm}] = \\ = -\delta_n (-j)^n k_e a J_n'(k_e a) \cdot A_E \end{cases} \tag{8}$$

$$\frac{T(a, \varphi_i)}{T_0} = 0: \begin{cases} [A_{\rho n} H_n^{(2)}(k_\rho a) - \frac{2\pi N}{k_\rho^2 - k_e^2} (-j)^n J_n(k_e a) \cdot \sum_{m=0} (j)^m B_{\rho mn} \cdot A_{\rho m}] + \\ + \frac{\Theta_\alpha}{\Theta_\rho} \cdot [A_{\alpha n} H_n^{(2)}(k_\alpha a) - \frac{2\pi N}{k_\alpha^2 - k_e^2} (-j)^n J_n(k_\alpha a) \cdot \sum_{m=0} (j)^m B_{\alpha mn} \cdot A_{\alpha m}] = \\ = -\delta_n (-j)^n J_n(k_e a) \cdot A_E \end{cases} \tag{9}$$

With a solution for the amplitudes  $A_{\beta n}$ ,  $A_{vn}$  evaluate:

$$\begin{aligned}
 \frac{\rho_{\text{eff}}}{\rho_0} &= \frac{[r \cdot (\Phi_{Er1}(r) + \Phi_{\rho r1}(r) + \frac{\Pi_\alpha}{\Pi_\rho} \Phi_{\alpha r1}(r))]_a^R}{[r \cdot (\Phi_{Er1}(r) + \Phi_{\rho r1}(r) + \Phi_{\alpha r1}(r) - \Psi_{zr1}(r))]_a^R} \\
 C_{\text{eff}} &= \frac{[\hat{\Phi}_{Er0}(r) + \hat{\Phi}_{\rho r0}(r) + \hat{\Phi}_{\alpha r0}(r)]_a^R}{C_0 \left[ \frac{\rho_{\text{eff}}}{\rho_0} \frac{k_\rho^2}{k_e^2} \hat{\Phi}_{Er0}(r) + \hat{\Phi}_{\rho r0}(r) + \frac{\Pi_\alpha k_\rho^2}{\Pi_\rho k_\alpha^2} \hat{\Phi}_{\alpha r0}(r) \right]_a^R}
 \end{aligned} \tag{10}$$

with  $[f(y)]_a^b = f(b) - f(a)$  and  $\hat{\Phi}_\beta(r) = \int_a^r k_\beta^2 r \cdot \Phi_\beta(r) dr$ .

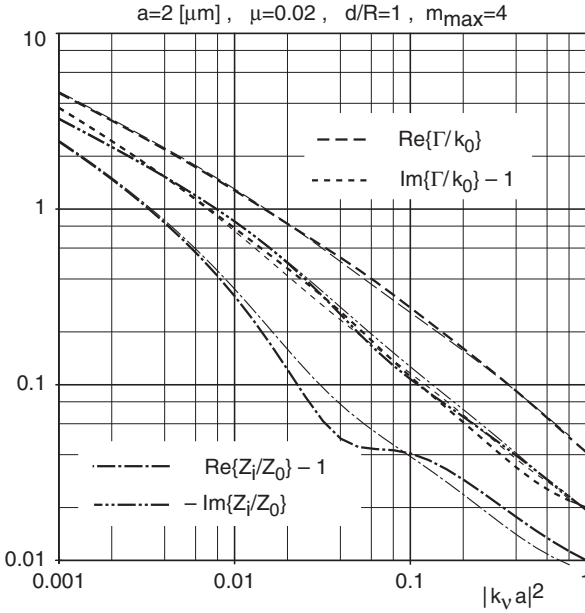
The evaluation must proceed iteratively, because the equations from the boundary conditions contain (besides the unknown amplitudes) the unknown wave number  $k_e$ . Begin the iteration with an approximation  $\Gamma = jk_e$  from Section G.8; solve for a 1<sup>st</sup> approximation of  $A_{\beta n}$ ,  $A_{vn}$ ; insert into the expressions for  $\rho_{\text{eff}}$ ,  $C_{\text{eff}}$ ; evaluate the next approximation for the propagating wave from:

$$\frac{\Gamma}{k_0} = j \sqrt{\frac{\rho_{\text{eff}}}{\rho_0} \cdot \frac{C_{\text{eff}}}{C_0}} \quad ; \quad \frac{Z_e}{Z_0} = \sqrt{\frac{\rho_{\text{eff}}}{\rho_0} / \frac{C_{\text{eff}}}{C_0}} \quad (11)$$

and resume the cycle of iteration. Apply in the evaluations, with the massivity  $\mu$ :

$$\frac{a}{R} = \sqrt{\mu} \quad ; \quad \frac{2\pi N}{k_\beta^2 - k_e^2} = \frac{2\mu}{(k_\beta a)^2 - (k_e a)^2} \quad (12)$$

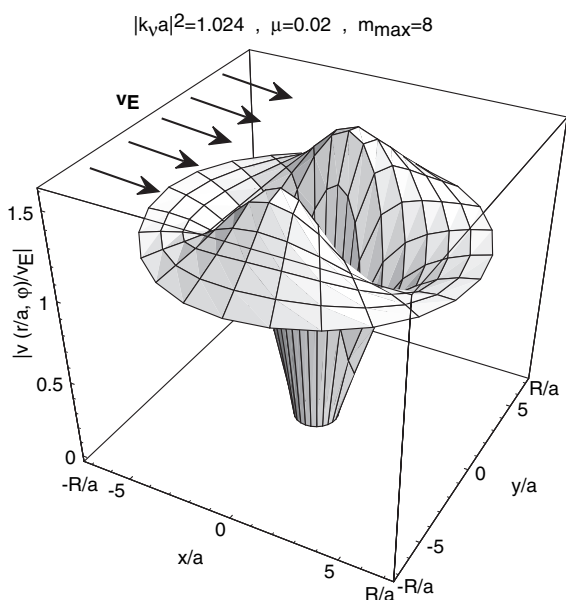
The following diagram compares the components of the characteristic values from the present iterative evaluation (with an “effective” propagating wave) with values from Section G.8 (with the density wave as the propagating wave). Up to  $m_{\text{max}} = 4$  orders of scattering were used.



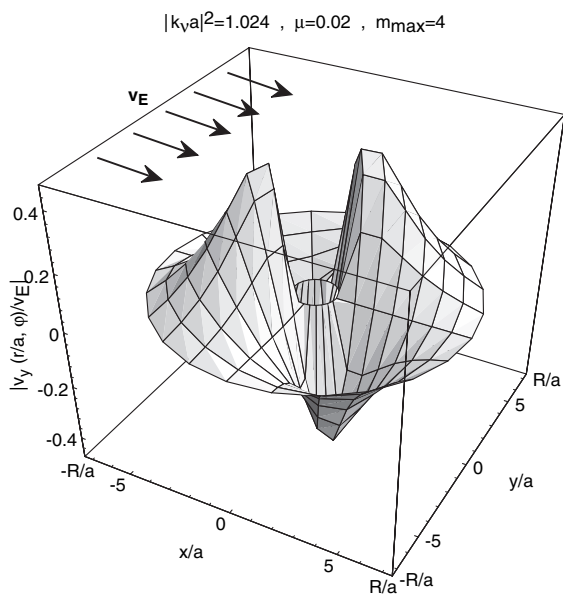
**Fig. 4** Comparison of the components of characteristic values in a transversal fibre bundle, evaluated iteratively for an effective propagating wave (thick lines, from this Section), and with a propagating density wave (thin lines, from Section G.8)

Notice that the theoretical curves of this Section reproduce the humps of the experimental points in the figure of Section G.4.

The present method, although much more complicated numerically than the method of Section G.8, is well suited to evaluate particle velocity profiles around a fibre in a transversal fibre bundle.



**Fig. 5** Profile of the magnitude of the total particle velocity around a fibre in a fibre bundle with transversal sound propagation



**Fig. 6** Profile of the real component of the transversal particle velocity around a fibre in a fibre bundle with transversal sound propagation

Formulas of Acoustics

Mechel, F.P. (Ed.)

2008, XXXVIII, 1275 p. In 2 volumes, not available  
separately., Hardcover

ISBN: 978-3-540-76832-6