Ramifications of the Geometric Langlands Program

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Introduction

The Langlands Program, conceived as a bridge between Number Theory and Automorphic Representations [L], has recently expanded into such areas as Geometry and Quantum Field Theory and exposed a myriad of unexpected connections and dualities between seemingly unrelated disciplines. There is something deeply mysterious in the ways the Langlands dualities manifest themselves and this is what makes their study so captivating.

In this review we will focus on the geometric Langlands correspondence for complex algebraic curves, which is a particular brand of the general theory.

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Its origins and the connections with the classical Langlands correspondence are discussed in detail elsewhere (see, in particular, the reviews [F2, F6]), and we will not try to repeat this here. The general framework is the following: let X be a smooth projective curve over $\mathbb{C}$ and G be a simple Lie group over $\mathbb{C}$. Denote by ${}^L G$ the Langlands dual group of G (we recall this notion in Section 2.3). Suppose that we are given a principal ${}^L G$-bundle $\mathcal{F}$ on X equipped with a flat connection. This is equivalent to $\mathcal{F}$ being a holomorphic principal ${}^L G$-bundle equipped with a holomorphic connection ∇ (which is automatically flat as the complex dimension of X is equal to one). The pair $(\mathcal{F}, \nabla)$ may also be thought of as a ${}^L G$-local system on X, or as a homomorphism $\pi_1(X) \to {}^L G$ (corresponding to a base point in X and a trivialization of the fiber of $\mathcal{F}$ at this point).

The global Langlands correspondence is supposed to assign to $E = (\mathcal{F}, \nabla)$ an object Aut_E, called **Hecke eigensheaf with eigenvalue** E, on the moduli stack Bun_G of holomorphic G-bundles on X:

$$\boxed{\begin{array}{c}\text{holomorphic } {}^L G\text{-bundles} \\ \text{with connection on } X\end{array}} \quad \longrightarrow \quad \boxed{\text{Hecke eigensheaves on } \mathrm{Bun}_G}$$

$$E \mapsto \mathrm{Aut}_E$$

(see, e.g., [F6], Sect. 6.1, for the definition of Hecke eigensheaves). It is expected that there is a unique irreducible Hecke eigensheaf Aut_E (up to isomorphism) if E is sufficiently generic.

The Hecke eigensheaves Aut_E have been constructed, and the Langlands correspondence proved, in [FGV, Ga2] for $G = GL_n$ and an arbitrary irreducible GL_n-local system, and in [BD1] for an arbitrary simple Lie group G and those ${}^L G$-local systems which admit the structure of a ${}^L G$-oper (which is recalled below).

Recently, A. Kapustin and E. Witten [KW] have related the geometric Langlands correspondence to the S-duality of supersymmetric four-dimensional Yang-Mills theories, bringing into the realm of the Langlands correspondence new ideas and insights from quantum physics.

So far, we have considered the **unramified** ${}^L G$-local systems. In other words, the corresponding flat connection has no poles. But what should happen if we allow the connection to be singular at finitely many points of X?

This **ramified geometric Langlands correspondence** is the subject of this paper. Here are the most important adjustments that one needs to make in order to formulate this correspondence:

- The moduli stack Bun_G of G-bundles has to be replaced by the moduli stack of G-bundles together with the level structures at the ramification points. We call them the *enhanced* moduli stacks. Recall that a level structure of order N is a trivialization of the bundle on the Nth infinitesimal neighborhood of the point. The order of the level structure should be at least the order of the pole of the connection at this point.

- At the points at which the connection has regular singularity (pole of order 1) one can take instead of the level structure, a parabolic structure, i.e., a reduction of the fiber of the bundle to a Borel subgroup of G.
- The Langlands correspondence will assign to a flat ${}^L G$-bundle $E = (\mathcal{F}, \nabla)$ with ramification at the points $y_1, \ldots, y_n$ a **category** $\mathcal{A}ut_E$ of Hecke eigensheaves on the corresponding enhanced moduli stack with eigenvalue $E|_{X \backslash \{y_1, \ldots, y_n\}}$, which is a subcategory of the category of (twisted) $\mathcal{D}$-modules on this moduli stack.

If E is unramified, then we may consider the category $\mathcal{A}ut_E$ on the moduli stack Bun_G itself. We then expect that for generic E this category is equivalent to the category of vector spaces: its unique (up to isomorphism) irreducible object is Aut_E discussed above, and all other objects are direct sums of copies of Aut_E. Because this category is expected to have such a simple structure, it makes sense to say that the unramified geometric Langlands correspondence assigns to an unramified ${}^L G$-local system on X a single Hecke eigensheaf, rather than a category. This is not possible for general ramified local systems.

The questions that we are facing now are

(1) How to construct the categories of Hecke eigensheaves for ramified local systems?
(2) How to describe them in terms of the Langlands dual group ${}^L G$?

In this article I will review an approach to these questions which has been developed by D. Gaitsgory and myself in [FG2].

The idea goes back to the construction of A. Beilinson and V. Drinfeld [BD1] of the unramified geometric Langlands correspondence, which may be interpreted in terms of a localization functor. Functors of this type were introduced by A. Beilinson and J. Bernstein [BB] in representation theory of simple Lie algebras. In our situation this functor sends representations of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ to twisted $\mathcal{D}$-modules on Bun_G, or its enhanced versions. As explained in [F6], these $\mathcal{D}$-modules may be viewed as sheaves of conformal blocks (or coinvariants) naturally arising in the framework of Conformal Field Theory.

The affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ is the universal one-dimensional central extension of the loop algebra $\mathfrak{g}((t))$. The representation categories of $\widehat{\mathfrak{g}}$ have a parameter κ, called the level, which determines the scalar by which a generator of the one-dimensional center of $\widehat{\mathfrak{g}}$ acts on representations. We consider a particular value κ_c of this parameter, called the **critical level**. The completed enveloping algebra of an affine Kac-Moody algebra acquires an unusually large center at the critical level and this makes the structure of the corresponding category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod very rich and interesting. B. Feigin and I have shown [FF3, F3] that this center is canonically isomorphic to the algebra of functions on the space of ${}^L G$**-opers** on $D^\times$. Opers are bundles on $D^\times$ with flat connection and an additional datum (as defined by Drinfeld-Sokolov [DS] and Beilinson-Drinfeld [BD1]; we recall the definition below). Remarkably, their structure group turns out to be not G, but the Langlands dual group ${}^L G$, in agreement with the general Langlands philosophy.

This result means that the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod of (smooth) $\widehat{\mathfrak{g}}$-modules of critical level "lives" over the space $\mathrm{Op}_{{}^L G}(D^\times)$ of ${}^L G$-opers on the punctured disc $D^\times$. For each $\chi \in \mathrm{Op}_{{}^L G}(D^\times)$ we have a "fiber" category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ whose objects are $\widehat{\mathfrak{g}}$-modules on which the center acts via the central character corresponding to χ. Applying the localization functors to these categories, and their K-equivariant subcategories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$ for various subgroups $K \subset G[[t]]$, we obtain categories of Hecke eigensheaves on the moduli spaces of G-bundles on X with level (or parabolic) structures.

Thus, the localization functor gives us a powerful tool for converting **local** categories of representations of $\widehat{\mathfrak{g}}$ into **global** categories of Hecke eigensheaves. This is a new phenomenon which does not have any obvious analogues in the classical Langlands correspondence.

The simplest special case of this construction gives us the Beilinson-Drinfeld Hecke eigensheaves Aut_E on Bun_G corresponding to unramified ${}^L G$-local systems admitting the oper structure. Motivated by this, we wish to apply the localization functors to more general categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$ of $\widehat{\mathfrak{g}}$-modules of critical level, corresponding to opers on X with singularities, or ramifications.

These categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ are assigned to ${}^L G$-opers χ on the punctured disc $D^\times$. It is important to realize that the formal loop group $G(\!(t)\!)$ naturally acts on each of these categories via its adjoint action on $\widehat{\mathfrak{g}}_{\kappa_c}$ (because the center is invariant under the adjoint action of $G(\!(t)\!)$). Thus, we assign to each oper χ a categorical representation of $G(\!(t)\!)$ on $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$.

This is analogous to the classical **local Langlands correspondence**. Let F be a local non-archimedian field, such as the field $\mathbb{F}_q(\!(t)\!)$ or the field of p-adic numbers. Let W'_F be the Weil-Deligne group of F, which is a version of the Galois group of F (we recall the definition in Section 2.1). The local Langlands correspondence relates the equivalence classes of irreducible (smooth) representations of the group $G(F)$ (or "L-packets" of such representations) and the equivalence classes of (admissible) homomorphisms $W'_F \to {}^L G$. In the geometric setting we replace these homomorphisms by flat ${}^L G$-bundles on $D^\times$ (or by ${}^L G$-opers), the group $G(F)$ by the loop group $G(\!(t)\!)$ and representations of $G(F)$ by categorical representations of $G(\!(t)\!)$.

This analogy is very suggestive, as it turns out that the structure of the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ (and their K-equivariant subcategories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$) is similar to the structure of irreducible representations of $G(F)$ (and their subspaces of K-invariants). We will see examples of this parallelism in Sects. 7 and 8 below. This means that what we are really doing is developing a **local Langlands correspondence for loop groups**.

To summarize, our strategy [FG2] for constructing the global geometric Langlands correspondence has two parts:

(1) the local part: describing the structure of the categories of $\widehat{\mathfrak{g}}$-modules of critical level, and
(2) the global part: applying the localization functor to these categories to obtain the categories of Hecke eigensheaves on enhanced moduli spaces of G-bundles.

We expect that these localization functors are equivalences of categories (at least, in the generic situation), and therefore we can infer a lot of information about the global categories by studying the local categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ of $\widehat{\mathfrak{g}}$-modules. Thus, the local categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ take the center stage.

In this paper I review the results and conjectures of [FG1]–[FG6] with the emphasis on unramified and tamely ramified local systems. (I also discuss the case of irregular singularities at the end.) In particular, our study of the categories of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules leads us to the following conjecture. (For related results, see [AB, ABG, Bez1, Bez2].)

Suppose that $E = (\mathcal{F}, \nabla)$, where $\mathcal{F}$ is a ${}^L G$-bundle and ∇ is a connection on $\mathcal{F}$ with regular singularity at a single point $y \in X$ and unipotent monodromy (this is easy to generalize to multiple points). Let $M = \exp(2\pi i u)$, where $u \in {}^L G$ be a representative of the conjugacy class of the monodromy of ∇ around y. Denote by Sp_u the **Springer fiber** of u, the variety of Borel subalgebras of ${}^L\mathfrak{g}$ containing u. The category $\mathcal{A}ut_E$ of Hecke eigensheaves with eigenvalue E may then be realized as a subcategory of the category of $\mathcal{D}$-modules on the moduli stack of G-bundles on X with parabolic structure at the point y. We have the following conjectural description of the derived category of $\mathcal{A}ut_E$:

$$D^b(\mathcal{A}ut_E) \simeq D^b(\mathrm{QCoh}(\mathrm{Sp}_u^{\mathrm{DG}})),$$

where $\mathrm{QCoh}(\mathrm{Sp}_u^{\mathrm{DG}})$ is the category of quasicoherent sheaves on a suitable "DG enhancement" of $\mathrm{Sp}_u^{\mathrm{DG}}$. This is a category of differential graded (DG) modules over a sheaf of DG algebras whose zeroth cohomology is the structure sheaf of $\mathrm{Sp}_u^{\mathrm{DG}}$ (we discuss this in detail in Section 9).

Thus, we expect that the geometric Langlands correspondence attaches to a ${}^L G$-local system on a Riemann surface with regular singularity at a puncture, a category which is closely related to the variety of Borel subgroups containing the monodromy around the puncture. We hope that further study of the categories of $\widehat{\mathfrak{g}}$-modules will help us to find a similar description of the Langlands correspondence for connections with irregular singularities.

The paper is organized as follows. In Sect. 1 we review the Beilinson-Drinfeld construction in the unramified case, in the framework of localization functors from representation categories of affine Kac-Moody algebras to $\mathcal{D}$-modules on on Bun_G. This will serve as a prototype for our construction of more general categories of Hecke eigensheaves, and it motivates us to study categories of $\widehat{\mathfrak{g}}$-modules of critical level. We wish to interpret these categories in the framework of the local geometric Langlands correspondence for loop groups. In order to do that, we first recall in Sect. 2 the setup of the classical Langlands correspondence. Then in Sect. 3 we explain the passage to the geometric context. In Sect. 4 we describe the structure of the center at the critical level and the isomorphism with functions on opers. In Sect. 5 we discuss the connection between the local Langlands parameters (${}^L G$-local systems on the punctured disc) and opers. We introduce the categorical representations of loop groups corresponding to opers and the corresponding categories of Harish-Chandra modules in Sect. 6. We discuss these categories in detail in the unramified case in Sect. 7, paying particular attention to the analogies between the classical and the geometric settings. In Sect. 8 we do the same in the tamely ramified case. We then apply localization functor to these categories in Sect. 9 to obtain various results and conjectures on the global Langlands correspondence, both for regular and irregular singularities.

Much of the material of this paper is borrowed from my new book [F7], where I refer the reader for more details, in particular, for background on representation theory of affine Kac-Moody algebras of critical level.

Finally, I note that in a forthcoming paper [GW] the geometric Langlands correspondence with tame ramification is studied from the point of view of dimensional reduction of four-dimensional supersymmetric Yang-Mills theory.

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1 The Unramified Global Langlands Correspondence

Our goal in this section is to construct Hecke eigensheaves Aut_E corresponding to unramified ${}^L G$-local systems $E = (\mathcal{F}, \nabla)$ on X. By definition, Aut_E is a $\mathcal{D}$-module on Bun_G. We would like to construct Aut_E by applying a localization functor to representations of affine Kac-Moody algebra $\widehat{\mathfrak{g}}$.

Throughout this paper, unless specified otherwise, we let $\mathfrak{g}$ be a simple Lie algebra and G the corresponding connected and simply-connected algebraic group.

The key observation used in constructing the localization functor is that for a simple Lie group G the moduli stack Bun_G of G-bundles on X has a realization as a double quotient. Namely, let x be a point of X. Denote by $\mathcal{K}_x$ the completion of the field of rational functions on X at x, and by $\mathcal{O}_x$ its ring of integers. If we choose a coordinate t at x, then we may identify $\mathcal{K}_x \simeq \mathbb{C}((t)), \mathcal{O}_x \simeq \mathbb{C}[[t]]$. But in general there is no preferred coordinate, and so it is better not to use these identifications. Now let $G(\mathcal{K}_x) \simeq G((t))$ be the formal loop group corresponding to the punctured disc $D_x^\times$ around x.

It has two subgroups: one is $G(\mathcal{O}_x) \simeq G[[t]]$ and the other is G_{out}, the group of algebraic maps $X\backslash x \to G$. Then, according to [BeLa, DrSi], the algebraic stack Bun_G is isomorphic to the double quotient

$$\text{Bun}_G \simeq G_{\text{out}}\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x). \tag{1.1}$$

Intuitively, any G-bundle may be trivialized on the formal disc D_x and on $X\backslash x$. The transition function is then an element of $G(\mathcal{K}_x)$, which characterizes the bundle uniquely up to the right action of $G(\mathcal{O}_x)$ and the left action of G_{out} corresponding to changes of trivializations on D_x and $X\backslash x$, respectively.

The localization functor that we need is a special case of the following general construction. Let $\mathfrak{g}$ be a Lie algebra and K a Lie group $(\mathfrak{g}, K)$ whose Lie algebra is contained in $\mathfrak{g}$. The pair $(\mathfrak{g}, K)$ is called a Harish-Chandra pair. We will assume that K is connected. A $\mathfrak{g}$-module M is called K-equivariant if the action of the Lie subalgebra $\text{Lie}\, K \subset \mathfrak{g}$ on M may be exponentiated to an action of the Lie group K. Let $\mathfrak{g}\text{-mod}^K$ be the category of K-equivariant $\mathfrak{g}$-modules.

Now suppose that H is another subgroup of G. Let $\mathcal{D}_{H\backslash G/K}\text{-mod}$ be the category of $\mathcal{D}$-modules on $H\backslash G/K$. Then there is a localization functor [BB, BD1] (see also [F6, FB])

$$\Delta : \mathfrak{g}\text{-mod}^K \to \mathcal{D}_{H\backslash G/K}\text{-mod}\,.$$

Now let $\widehat{\mathfrak{g}}$ be a one-dimensional central extension of $\mathfrak{g}$ which becomes trivial when restricted to the Lie subalgebras $\text{Lie}\, K$ and $\text{Lie}\, H$. Suppose that this central extension can be exponentiated to a central extension $\widehat{G}$ of the corresponding Lie group G. Then we obtain a $\mathbb{C}^\times$-bundle $H\backslash\widehat{G}/K$ over $H\backslash G/K$. Let $\mathcal{L}$ be the corresponding line bundle and $\mathcal{D}_{\mathcal{L}}$ the sheaf of differential operators acting on $\mathcal{L}$. Then we have a functor

$$\Delta_{\mathcal{L}} : \widehat{\mathfrak{g}}\text{-mod}^K \to \mathcal{D}_{\mathcal{L}}\text{-mod}\,.$$

In our case we take the formal loop group $G(\mathcal{K}_x)$, and the subgroups $K = G(\mathcal{O}_x)$ and $H = G_{\text{out}}$ of $G(\mathcal{K}_x)$. We also consider the so-called critical central extension of $G(\mathcal{K}_x)$. Let us first discuss the corresponding central extension of the Lie algebra $\mathfrak{g} \otimes \mathcal{K}_x$. Choose a coordinate t at x and identify $\mathcal{K}_x \simeq \mathbb{C}((t))$. Then $\mathfrak{g} \otimes \mathcal{K}_x$ is identified with $\mathfrak{g}((t))$. Let κ be an invariant bilinear form on $\mathfrak{g}$. The **affine Kac-Moody algebra** $\widehat{\mathfrak{g}}_\kappa$ is defined as the central extension

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0.$$

As a vector space, it is equal to the direct sum $\mathfrak{g}((t)) \oplus \mathbb{C}\mathbf{1}$, and the commutation relations read

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\kappa(A, B)\, \text{Res}\, f dg)\mathbf{1}, \tag{1.2}$$

and $\mathbf{1}$ is a central element, which commutes with everything else. For a simple Lie algebra $\mathfrak{g}$ all invariant inner products are proportional to each other.

Therefore the Lie algebras $\widehat{\mathfrak{g}}_\kappa$ are isomorphic to each other for non-zero inner products κ.

Note that the restriction of the second term in (1.2) to the Lie subalgebra $\mathfrak{g} \otimes t^N \mathbb{C}[[t]]$, where $N \in \mathbb{Z}_+$, is equal to zero, and so it remains a Lie subalgebra of $\widehat{\mathfrak{g}}_\kappa$. A $\widehat{\mathfrak{g}}_\kappa$-module is called **smooth** if every vector in it is annihilated by this Lie subalgebra for sufficiently large N. We define the category $\widehat{\mathfrak{g}}_\kappa$-mod whose objects are smooth $\widehat{\mathfrak{g}}_\kappa$-modules on which the central element $\mathbf{1}$ acts as the identity. The morphisms are homomorphisms of representations of $\widehat{\mathfrak{g}}_\kappa$. Throughout this paper, unless specified otherwise, by a "$\widehat{\mathfrak{g}}_\kappa$-module" will always mean a module on which the central element $\mathbf{1}$ acts as the identity.[1] We will refer to κ as the **level**.

Now observe that formula (1.2) is independent of the choice of coordinate t at $x \in X$ and therefore defines a central extension of $\mathfrak{g} \otimes \mathcal{K}_x$, which we denote by $\widehat{\mathfrak{g}}_{\kappa,x}$. One can show that this central extension may be exponentiated to a central extension of the group $G(\mathcal{K}_x)$ if κ satisfies a certain integrality condition, namely, $\kappa = k\kappa_0$, where $k \in \mathbb{Z}$ and κ_0 is the inner product normalized by the condition that the square of the length of the maximal root is equal to 2. A particular example of the inner product which satisfies this condition is the **critical level** κ_c defined by the formula

$$\kappa_c(A, B) = -\frac{1}{2} \operatorname{Tr}_{\mathfrak{g}} \operatorname{ad} A \operatorname{ad} B. \tag{1.3}$$

Thus, κ_c is equal to minus one half of the Killing form on $\mathfrak{g}$.[2] When $\kappa = \kappa_c$ representation theory of $\widehat{\mathfrak{g}}_\kappa$ changes dramatically because the completed enveloping algebra of $\widehat{\mathfrak{g}}_\kappa$ acquires a large center (see below).

Let $\widehat{G}_x$ be the corresponding critical central extension of $G(\mathcal{K}_x)$. It is known (see [BD1]) that in this case the corresponding line bundle $\mathcal{L}$ is the square root $K^{1/2}$ of the canonical line bundle on Bun_G.[3] Now we are ready to apply the localization functor in the situation where our group is $G(\mathcal{K}_x)$, with the two subgroups $K = G(\mathcal{O}_x)$ and $H = G_{\mathrm{out}}$, so that the double quotient $H \backslash G / K$ is Bun_G.[4] We choose $\mathcal{L} = K^{1/2}$. Then we have a localization functor

$$\Delta_{\kappa_c,x} : \widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}^{G(\mathcal{O}_x)} \to \mathcal{D}_{\kappa_c}\text{-mod}.$$

We will apply this functor to a particular $\widehat{\mathfrak{g}}_{\kappa_c,x}$-module.

To construct this module, let us first define the vacuum module over $\widehat{\mathfrak{g}}_{\kappa_c,x}$ as the induced module

$$\mathbb{V}_{0,x} = \operatorname{Ind}_{\mathfrak{g} \otimes \mathcal{O}_x \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa,x}} \mathbb{C},$$

[1] Note that we could have $\mathbf{1}$ act instead as λ times the identity for $\lambda \in \mathbb{C}^\times$; but the corresponding category would just be equivalent to the category $\widehat{\mathfrak{g}}_{\lambda\kappa}$-mod.

[2] It is also equal to $-h^\vee \kappa_0$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

[3] Recall that by our assumption G is simply-connected. In this case there is a unique square root.

[4] Since Bun_G is an algebraic stack, one needs to be careful in applying the localization functor. The appropriate formalism has been developed in [BD1].

where $\mathfrak{g} \otimes \mathcal{O}_x$ acts by 0 on $\mathbb{C}$ and $\mathbf{1}$ acts as the identity. According to the results of [FF3, F3], we have

$$\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_{0,x} \simeq \mathrm{Fun}\, \mathrm{Op}_{^L G}(D_x),$$

where $\mathrm{Op}_{^L G}(D_x)$ is the space of $^L G$-opers on the formal disc $D_x = \mathrm{Spec}\, \mathcal{O}_x$ around x. We discuss this in detail in Section 4.

Now, given $\chi_x \in \mathrm{Op}_{^L G}(D_x)$, we obtain a maximal ideal $I(\chi_x)$ in the algebra $\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_{0,x}$. Let $\mathbb{V}_0(\chi_x)$ be the $\widehat{\mathfrak{g}}_{\kappa_c,x}$-module which is the quotient of $\mathbb{V}_{0,x}$ by the image of $I(\chi_x)$ (it is non-zero, as explained in Section 7.3). The module $\mathbb{V}_{0,x}$ is clearly $G(\mathcal{O}_x)$-equivariant, and hence so is $\mathbb{V}_0(\chi_x)$. Therefore $\mathbb{V}_0(\chi_x)$ is an object of the category $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}^{G(\mathcal{O}_x)}$.

We now apply the localization functor $\Delta_{\kappa_c,x}$ to $\mathbb{V}_0(\chi_x)$. The following theorem is due to Beilinson and Drinfeld [BD1].

Theorem 1. (1) *The $\mathcal{D}_{\kappa_c}$-module $\Delta_{\kappa_c,x}(\mathbb{V}_0(\chi_x))$ is non-zero if and only if there exists a global $^L\mathfrak{g}$-oper on X, $\chi \in \mathrm{Op}_{^L G}(X)$ such that $\chi_x \in \mathrm{Op}_{^L G}(D_x)$ is the restriction of χ to D_x.*

(2) *If this holds, then $\Delta_{\kappa_c,x}(\mathbb{V}_0(\chi_x))$ depends only on χ and is independent of the choice of x in the sense that for any other point $y \in X$, if $\chi_y = \chi|_{D_y}$, then $\Delta_{\kappa_c,x}(\mathbb{V}_0(\chi_x)) \simeq \Delta_{\kappa_c,y}(\mathbb{V}_0(\chi_y))$.*

(3) *For any $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{^L B}) \in \mathrm{Op}_{^L G}(X)$ the $\mathcal{D}_{\kappa_c}$-module $\Delta_{\kappa_c,x}(\mathbb{V}_0(\chi_x))$ is a non-zero Hecke eigensheaf with the eigenvalue $E_\chi = (F, \nabla)$.*

Thus, for any $\chi \in \mathrm{Op}_{^L G}(X)$, the $\mathcal{D}_{\kappa_c}$-module $\Delta_{\kappa_c,x}(\mathbb{V}_0(\chi_x))$ is the sought-after Hecke eigensheaf Aut_{E_χ} corresponding to the $^L G$-local system E_χ under the global geometric Langlands correspondence.[5] For an outline of the proof of this theorem from [BD1], see [F6], Sect. 9.4.

A drawback of this construction is that not all $^L G$-local systems on X admit the structure of an oper. In fact, under our assumption that G is simply-connected (and so $^L G$ is of adjoint type), the local systems, or flat bundles $(\mathcal{F}, \nabla)$, on a smooth projective curve X that admit an oper structure correspond to a unique $^L G$-bundle on X described as follows (see [BD1]). Let $\Omega_X^{1/2}$ be a square root of the canonical line bundle Ω_X. There is a unique (up to an isomorphism) non-trivial extension

$$0 \to \Omega_X^{1/2} \to \mathcal{F}_0 \to \Omega_X^{-1/2} \to 0.$$

Let $\mathcal{F}_{PGL_2}$ be the PGL_2-bundle corresponding to the rank two vector bundle $\mathcal{F}_0$. Note that it does not depend on the choice of $\Omega_X^{1/2}$. This is the oper bundle for PGL_2. We define the oper bundle $\mathcal{F}_{^L G}$ for a general simple Lie group $^L G$ of adjoint type as the push-forward of $\mathcal{F}_{PGL_2}$ with respect to a principal embedding $PGL_2 \hookrightarrow G$ (see Section 4.3).

[5] More precisely, Aut_{E_χ} is the $\mathcal{D}$-module $\Delta_{\kappa_c,x}(\mathbb{V}_0(\chi_x)) \otimes K^{-1/2}$, but here and below we will ignore the twist by $K^{1/2}$.

For each flat connection ∇ on the oper bundle $\mathcal{F}_{^LG}$ there exists a unique ^LB-reduction $\mathcal{F}_{^LB}$ satisfying the oper condition. Therefore $\mathrm{Op}_G(D)$ is a subset of $\mathrm{Loc}_{^LG}(X)$, which is the fiber of the forgetful map $\mathrm{Loc}_{^LG}(X) \to \mathrm{Bun}_{^LG}$ over $\mathcal{F}_{^LG}$.

Theorem 1 gives us a construction of Hecke eigensheaves for ^LG-local system that belong to the locus of opers. For a general ^LG-local system outside this locus, the above construction may be generalized as discussed at the end of Section 9.2 below.

Thus, Theorem 1, and its generalization to other unramified ^LG-local systems, give us an effective tool for constructing Hecke eigensheaves on Bun_G. It is natural to ask whether it can be generalized to the ramified case if we consider more general representations of $\widehat{\mathfrak{g}}_{\kappa_c,x}$. The goal of this paper is to explain how to do that.

We will see below that the completed universal enveloping algebra of $\widehat{\mathfrak{g}}_{\kappa_c,x}$ contains a large center. It is isomorphic to the algebra $\mathrm{Fun}\,\mathrm{Op}_{^LG}(D_x^\times)$ of functions on the space $\mathrm{Op}_{^LG}(D_x^\times)$ of ^LG-opers on the punctured disc $D_x^\times$. For $\chi_x \in \mathrm{Op}_{^LG}(D_x^\times)$, let $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}$ be the full subcategory of $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}$ whose objects are $\widehat{\mathfrak{g}}_{\kappa_c,x}$-modules on which the center acts according to the character corresponding to χ_x.

The construction of Hecke eigensheaves now breaks into two steps:

(1) we study the Harish-Chandra categories $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^K$ for various subgroups $K \subset G(\mathcal{O}_x)$;
(2) we apply the localization functors to these categories.

The simplest case of this construction is precisely the Beilinson-Drinfeld construction explained above. In this case we take χ_x to be a point in the subspace $\mathrm{Op}_{^LG}(D_x) \subset \mathrm{Op}_{^LG}(D_x^\times)$. Then the category $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^{G(\mathcal{O}_x)}$ is equivalent to the category of vector spaces: its unique up to an isomorphism irreducible object is the above $\mathbb{V}_0(\chi_x)$, and all other objects are direct sums of copies of $\mathbb{V}_0(\chi_x)$ (see [FG1] and Theorem 3 below). Therefore the localization functor $\Delta_{\kappa_c,x}$ is determined by $\Delta_{\kappa_c,x}(\mathbb{V}_0(\chi_x))$, which is described in Theorem 1. It turns out to be the desired Hecke eigensheaf Aut_{E_χ}. Moreover, we expect that the functor $\Delta_{\kappa_c,x}$ sets up an equivalence between $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^{G(\mathcal{O}_x)}$ and the category of Hecke eigensheaves on Bun_G with eigenvalue E_χ.

For general opers χ_x, with ramification, the (local) categories $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^K$ are more complicated, as we will see below, and so are the corresponding (global) categories of Hecke eigensheaves. In order to understand the structure of the global categories, we need to study first of local categories of $\widehat{\mathfrak{g}}_{\kappa_c,x}$-modules. Using the localization functor, we can then understand the structure of the global categories. We will consider examples of the local categories in the following sections.

It is natural to view our study of the local categories $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}$ and $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^K$ as a geometric analogue of the local Langlands correspondence. We will explain this point of view in the next section.

2 Classical Local Langlands Correspondence

The local Langlands correspondence relates smooth representations of reductive algebraic groups over local fields and representations of the Galois group of this field. In this section we define these objects and explain the main features of this correspondence. As the material of this section serves motivational purposes, we will only mention those aspects of this story that are most relevant for us. For a more detailed treatment, we refer the reader to the informative surveys [Vog, Ku] and references therein.

The local Langlands correspondence may be formulated for any local non-archimedian field. There are two possibilities: either F is the field $\mathbb{Q}_p$ of p-adic numbers or a finite extension of $\mathbb{Q}_p$, or F is the field $\mathbb{F}_q(\!(t)\!)$ of formal Laurent power series with coefficients in $\mathbb{F}_q$, the finite field with q elements (where q is a power of a prime number). For the sake of definiteness, in what follows we will restrict ourselves to the second case.

2.1 Langlands Parameters

Consider the group $GL_n(F)$, where $F = \mathbb{F}_q(\!(t)\!)$. A representation of $GL_n(F)$ on a complex vector space V is a homomorphism $\pi : GL_n(F) \to \operatorname{End} V$ such that $\pi(gh) = \pi(g)\pi(h)$ and $\pi(1) = \mathrm{Id}$. Define a topology on $GL_n(F)$ by stipulating that the base of open neighborhoods of $1 \in GL_n(F)$ is formed by the congruence subgroups

$$K_N = \{g \in GL_n(\mathbb{F}_q[[t]]) \,|\, g \equiv 1 \mod t^N\}, \qquad N \in \mathbb{Z}_+.$$

For each $v \in V$ we obtain a map $\pi(\cdot)v : GL_n(F) \to V, g \mapsto \pi(g)v$. A representation (V, π) is called **smooth** if the map $\pi(\cdot)v$ is continuous for each v, where we give V the discrete topology. In other words, V is smooth if for any vector $v \in V$ there exists $N \in \mathbb{Z}_+$ such that

$$\pi(g)v = v, \qquad \forall g \in K_N.$$

We are interested in describing the equivalence classes of irreducible smooth representations of $GL_n(F)$. Surprisingly, those turn out to be related to objects of a different kind: n-dimensional representations of the Galois group of F.

Recall that the algebraic closure of F is a field obtained by adjoining to F the roots of all polynomials with coefficients in F. However, in the case when $F = \mathbb{F}_q(\!(t)\!)$ some of the extensions of F may be non-separable. We wish to avoid the non-separable extensions, because they do not contribute to the Galois group. Let $\overline{F}$ be the maximal separable extension inside a given algebraic closure of F. It is uniquely defined up to an isomorphism.

Let $\mathrm{Gal}(\overline{F}/F)$ be the **absolute Galois group** of F. Its elements are the automorphisms σ of the field $\overline{F}$ such that $\sigma(y) = y$ for all $y \in F$.

Now set $F = \mathbb{F}_q((t))$. Observe that we have a natural map $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ obtained by applying an automorphism of F to $\overline{\mathbb{F}}_q \subset F$. The group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is isomorphic to the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ (see, e.g., [F6], Sect. 1.3). Its subgroup $\mathbb{Z} \subset \widehat{\mathbb{Z}}$ is generated by the **geometric Frobenius element** which is inverse to the automorphism $x \mapsto x^q$ of $\overline{\mathbb{F}}_q$. Let W_F be the preimage of the subgroup $\mathbb{Z} \subset \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. This is the **Weil group** of F. Denote by ν be the corresponding homomorphism $W_F \to \mathbb{Z}$.

Now let $W'_F = W_F \ltimes \mathbb{C}$ be the semi-direct product of W_F and the one-dimensional complex additive group $\mathbb{C}$, where W_F acts on $\mathbb{C}$ by the formula

$$\sigma x \sigma^{-1} = q^{\nu(\sigma)} x, \qquad \sigma \in W_F, x \in \mathbb{C}. \tag{2.1}$$

This is the **Weil-Deligne group** of F.

An n-dimensional complex representation of W'_F is by definition a homomorphism $\rho' : W'_F \to GL_n(\mathbb{C})$ which may be described as a pair (ρ, N), where ρ is an n-dimensional representation of W_F, $N \in GL_n(\mathbb{C})$, and we have $\rho(\sigma) N \rho(\sigma)^{-1} = q^{\nu(\sigma)} \rho(N)$ for all $\sigma \in W_F$. The group W_F is topological, with respect to the Krull topology (in which the open neighborhoods of the identity are the normal subgroups of finite index). The representation (ρ, N) is called **admissible** if ρ is continuous (equivalently, factors through a finite quotient of W_F) and semisimple, and N is a unipotent element of $GL_n(\mathbb{C})$.

The group W'_F was introduced by P. Deligne [De2]. The idea is that by adjoining the unipotent element N to W_F one obtains a group whose complex admissible representations are the same as continuous ℓ-adic representations of W_F (where $\ell \neq p$ is a prime).

2.2 The Local Langlands Correspondence for GL_n

Now we are ready to state the local Langlands correspondence for the group GL_n over a local non-archimedian field F. It is a bijection between two different sorts of data. One is the set of the equivalence classes of irreducible smooth representations of $GL_n(F)$. The other is the set of equivalence classes of n-dimensional admissible representations of W'_F. We represent it schematically as follows:

$$\boxed{\begin{array}{c} n\text{-dimensional admissible} \\ \text{representations of } W'_F \end{array}} \iff \boxed{\begin{array}{c} \text{irreducible smooth} \\ \text{representations of } GL_n(F) \end{array}}$$

This correspondence is supposed to satisfy an overdetermined system of constraints which we will not recall here (see, e.g., [Ku]).

The local Langlands correspondence for GL_n is a theorem. In the case when $F = \mathbb{F}_q((t))$ it has been proved in [LRS], and when $F = \mathbb{Q}_p$ or its finite extension in [HT] and also in [He].

2.3 Generalization to Other Reductive Groups

Let us replace the group GL_n by an arbitrary connected reductive group G over a local non-archimedian field F. The group $G(F)$ is also a topological group, and there is a notion of smooth representation of $G(F)$ on a complex vector space. It is natural to ask whether one can relate irreducible smooth representations of $G(F)$ to representations of the Weil-Deligne group W'_F. This question is addressed in the general local Langlands conjectures. It would take us too far afield to try to give here a precise formulation of these conjectures. So we will only indicate some of the objects involved referring the reader to the articles [Vog, Ku] where these conjectures are described in great detail.

Recall that in the case when $G = GL_n$ the irreducible smooth representations are parametrized by admissible homomorphisms $W'_F \to GL_n(\mathbb{C})$. In the case of a general reductive group G, the representations are conjecturally parametrized by admissible homomorphisms from W'_F to the so-called **Langlands dual group** ${}^LG$, which is defined over $\mathbb{C}$.

In order to explain the notion of the Langlands dual group, consider first the group G over the closure $\overline{F}$ of the field F. All maximal tori T of this group are conjugate to each other and are necessarily split, i.e., we have an isomorphism $T(\overline{F}) \simeq (\overline{F}^\times)$. For example, in the case of GL_n, all maximal tori are conjugate to the subgroup of diagonal matrices. We associate to $T(\overline{F})$ two lattices: the weight lattice $X^*(T)$ of homomorphisms $T(\overline{F}) \to \overline{F}^\times$ and the coweight lattice $X_*(T)$ of homomorphisms $\overline{F}^\times \to T(\overline{F})$. They contain the sets of roots $\Delta \subset X^*(T)$ and coroots $\Delta^\vee \subset X_*(T)$, respectively. The quadruple $(X^*(T), X_*(T), \Delta, \Delta^\vee)$ is called the root datum for G over $\overline{F}$. The root datum determines G up to an isomorphism defined over $\overline{F}$. The choice of a Borel subgroup $B(\overline{F})$ containing $T(\overline{F})$ is equivalent to a choice of a basis in Δ, namely, the set of simple roots Δ_s, and the corresponding basis $\Delta_s^\vee$ in $\Delta^\vee$.

Now, given $\gamma \in \mathrm{Gal}(\overline{F}/F)$, there is $g \in G(\overline{F})$ such that $g(\gamma(T(\overline{F}))g^{-1} = T(\overline{F})$ and $g(\gamma(B(\overline{F}))g^{-1} = B(\overline{F})$. Then g gives rise to an automorphism of the based root data $(X^*(T), X_*(T), \Delta_s, \Delta_s^\vee)$. Thus, we obtain an action of $\mathrm{Gal}(\overline{F}/F)$ on the based root data.

Let us now exchange the lattices of weights and coweights and the sets of simple roots and coroots. Then we obtain the based root data

$$(X_*(T), X^*(T), \Delta_s^\vee, \Delta_s)$$

of a reductive algebraic group over $\mathbb{C}$ which is denoted by ${}^LG^\circ$. For instance, the group GL_n is self-dual, the dual of SO_{2n+1} is Sp_{2n}, the dual of Sp_{2n} is SO_{2n+1}, and SO_{2n} is self-dual.

The action of $\mathrm{Gal}(\overline{F}/F)$ on the based root data gives rise to its action on ${}^LG^\circ$. The semi-direct product ${}^LG = \mathrm{Gal}(\overline{F}/F) \ltimes {}^LG^\circ$ is called the **Langlands dual group** of G.

According to the local Langlands conjecture, the equivalence classes of irreducible smooth representations of $G(F)$ are, roughly speaking, parameterized by the equivalence classes of admissible homomorphisms $W'_F \to {}^L G$. In fact, the conjecture is more subtle: one needs to consider simultaneously representations of all inner forms of G, and a homomorphism $W'_F \to {}^L G$ corresponds in general not to a single irreducible representation of $G(F)$, but to a finite set of representations called an ***L*-packet**. To distinguish between them, one needs additional data (see [Vog] and Section 8.1 below for more details). But in the first approximation one can say that the essence of the local Langlands correspondence is that

Irreducible smooth representations of $G(F)$ are parameterized in terms of admissible homomorphisms $W'_F \to {}^L G$.

3 Geometric Local Langlands Correspondence over $\mathbb{C}$

We now wish to find a generalization of the local Langlands conjectures in which we replace the field $F = \mathbb{F}_q((t))$ by the field $\mathbb{C}((t))$. We would like to see how the ideas and patterns of the Langlands correspondence play out in this new context, with the hope of better understanding the deep underlying structures behind this correspondence.

So let G be a connected simply-connected algebraic group over $\mathbb{C}$, and $G(F)$ the loop group $G((t)) = G(\mathbb{C}((t)))$. Thus, we wish to study smooth representations of the loop group $G((t))$ and try to relate them to some "Langlands parameters", which we expect, by analogy with the case of local non-archimedian fields described above, to be related to the Galois group of $\mathbb{C}((t))$ and the Langlands dual group ${}^L G$.

3.1 Geometric Langlands Parameters

Unfortunately, the Galois group of $\mathbb{C}((t))$ is too small: it is isomorphic to the pro-finite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$. This is not surprising from the point of view of the analogy between the Galois groups and the fundamental groups (see, e.g., [F6], Sect. 3.1). The topological fundamental group of the punctured disc is $\mathbb{Z}$, and the algebraic fundamental group is its pro-finite completion.

However, we may introduce additional Langlands parameters by using a more geometric perspective on homomorphisms from the fundamental group to ${}^L G$. Those may be viewed as ${}^L G$-local systems. In general, ${}^L G$-local systems on a compact variety Z are the same as flat ${}^L G$-bundles $(\mathcal{F}, \nabla)$ on Z. If the variety is not compact (as in the case of $D^\times$), then we should impose the additional condition that the connection has **regular singularities** (pole of order at most 1) at infinity. In our case we obtain ${}^L G$-bundles on $D^\times$ with a connection that has regular singularity at the origin. Then the monodromy of the connection gives rise to a homomorphism from $\pi_1(D^\times)$ to ${}^L G$. Now we generalize this by allowing connections with **arbitrary**, that is regular and **irregular**, singularities at the origin. Thus, we want to use as the general Langlands parameters, the equivalence classes of pairs $(\mathcal{F}, \nabla)$, where $\mathcal{F}$ is a ${}^L G$-bundle on $D^\times$ and ∇ is an arbitrary connection on $\mathcal{F}$.

Any bundle $\mathcal{F}$ on $D^\times$ may be trivialized. Then ∇ may be represented by the first-order differential operator

$$\nabla = \partial_t + A(t), \qquad A(t) \in {}^L\mathfrak{g}(\!(t)\!). \tag{3.1}$$

where ${}^L\mathfrak{g}$ is the Lie algebra of the Langlands dual group ${}^L G$. Changing the trivialization of $\mathcal{F}$ amounts to a gauge transformation

$$\nabla \mapsto \nabla' = \partial_t + gAg^{-1} - (\partial_t g)g^{-1}$$

with $g \in {}^L G(\!(t)\!)$. Therefore the set of equivalence classes of ${}^L G$-bundles with a connection on $D^\times$ is in bijection with the set of gauge equivalence classes of operators (3.1). We denote this set by $\mathrm{Loc}_{{}^L G}(D^\times)$. Thus, we have

$$\mathrm{Loc}_{{}^L G}(D^\times) = \{\partial_t + A(t),\ A(t) \in {}^L\mathfrak{g}(\!(t)\!)\}/{}^L G(\!(t)\!). \tag{3.2}$$

We declare that the local Langlands parameters in the complex setting should be the points of $\mathrm{Loc}_{{}^L G}(D^\times)$: the equivalence classes of flat ${}^L G$-bundles on $D^\times$ or, more concretely, the gauge equivalence classes (3.2) of first-order differential operators.

Having settled the issue of the Langlands parameters, we have to decide what it is that we will be parameterizing. Recall that in the classical setting the homomorphism $W'_F \to {}^L G$ parameterized irreducible smooth representations of the group $G(F)$, $F = \mathbb{F}_q(\!(t)\!)$. We start by translating this notion to the representation theory of loop groups.

3.2 Representations of the Loop Group

The loop group $G(\!(t)\!)$ contains the congruence subgroups

$$K_N = \{g \in G[[t]] \,|\, g \equiv 1 \mod t^N\}, \qquad N \in \mathbb{Z}_+. \tag{3.3}$$

It is natural to call a representation of $G(\!(t)\!)$ on a complex vector space V **smooth** if for any vector $v \in V$ there exists $N \in \mathbb{Z}_+$ such that $K_N \cdot v = v$. This condition may be interpreted as the continuity condition, if we define a topology on $G(\!(t)\!)$ by taking as the base of open neighborhoods of the identity the subgroups $K_N, N \in \mathbb{Z}_+$, as before.

But our group G is now a complex Lie group (not a finite group), and so $G(\!(t)\!)$ is an infinite-dimensional Lie group. More precisely, we view $G(\!(t)\!)$ as an ind-group, i.e., as a group object in the category of ind-schemes. At first glance, it is natural to consider the algebraic representations of $G(\!(t)\!)$. We observe that $G(\!(t)\!)$ is generated by the "parahoric" algebraic groups P_i

corresponding to the affine simple roots. For these subgroups the notion of algebraic representation makes perfect sense. A representation of $G((t))$ is then said to be algebraic if its restriction to each of the P_i's is algebraic.

However, this naive approach leads us to the following discouraging fact: an irreducible smooth representation of $G((t))$, which is algebraic, is necessarily trivial (see [BD1], 3.7.11(ii)). Thus, we find that the class of algebraic representations of loop groups turns out to be too restrictive. We could relax this condition and consider differentiable representations, i.e., the representations of $G((t))$ considered as a Lie group. But it is easy to see that the result would be the same. Replacing $G((t))$ by its central extension $\widehat{G}$ would not help us much either: irreducible integrable representations of $\widehat{G}$ are parameterized by dominant integral weights, and there are no extensions between them [K2]. These representations are again too sparse to be parameterized by the geometric data considered above. Therefore we should look for other types of representations.

Going back to the original setup of the local Langlands correspondence, we recall that there we considered representations of $G(\mathbb{F}_q((t)))$ on $\mathbb{C}$-vector spaces, so we could not possibly use the algebraic structure of $G(\mathbb{F}_q((t)))$ as an ind-group over $\mathbb{F}_q$. Therefore we cannot expect the class of algebraic (or differentiable) representations of the complex loop group $G((t))$ to be meaningful from the point of view of the Langlands correspondence. We should view the loop group $G((t))$ as an abstract topological group, with the topology defined by means of the congruence subgroups, in other words, consider its smooth representations as an *abstract* group.

So we need to search for some geometric objects that encapsulate representations of our groups and make sense both over a finite field and over the complex field.

3.3 From Functions to Sheaves

We start by revisiting smooth representations of the group $G(F)$, where $F = \mathbb{F}_q((t))$. We realize such representations more concretely by considering their matrix coefficients. Let (V, π) be an irreducible smooth representation of $G(F)$. We define the **contragredient** representation $V^\vee$ as the linear span of all smooth vectors in the dual representation V^*. This span is stable under the action of $G(F)$ and so it admits a smooth representation $(V^\vee, \pi^\vee)$ of $G(F)$. Now let ϕ be a K_N-invariant vector in $V^\vee$. Then we define a linear map

$$V \to C(G(F)/K_N), \qquad v \mapsto f_v,$$

where

$$f_v(g) = \langle \pi^\vee(g)\phi, v \rangle.$$

Here $C(G(F)/K_N)$ denotes the vector space of $\mathbb{C}$-valued locally constant functions on $G(F)/K_N$. The group $G(F)$ naturally acts on this space by the formula $(g \cdot f)(h) = f(g^{-1}h)$, and the above map is a morphism of representations, which is non-zero, and hence injective, if (V, π) is irreducible.

Thus, we realize our representation in the space of functions on the quotient $G(F)/K_N$. More generally, we may realize representations in spaces of functions on the quotient $G(\!(t)\!)/K$ with values in a finite-dimensional vector space, by considering a finite-dimensional subrepresentation of K inside V rather than the trivial one.

An important observation here is that $G(F)/K$, where $F = \mathbb{F}_q(\!(t)\!)$ and K is a compact subgroup of $G(F)$, is not only a set, but it is a set of points of an algebraic variety (more precisely, an ind-scheme) defined over the field $\mathbb{F}_q$. For example, for $K_0 = G(\mathbb{F}_q[[t]])$, which is the maximal compact subgroup, the quotient $G(F)/K_0$ is the set of $\mathbb{F}_q$-points of the ind-scheme called the **affine Grassmannian**.

Next, we recall an important idea going back to Grothendieck that functions on the set of $\mathbb{F}_q$-points on an algebraic variety X defined over $\mathbb{F}_q$ can often be viewed as the "shadows" of the so-called ℓ-adic sheaves on X. We will not give the definition of these sheaves, referring the reader to [Mi, FK]. The Grothendieck **fonctions-faisceaux** dictionary (see, e.g., [La]) is formulated as follows. Let $\mathcal{F}$ be an ℓ-adic sheaf and x be an $\mathbb{F}_{q_1}$-point of X, where $q_1 = q^m$. Then one has the Frobenius conjugacy class Fr_x acting on the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. Hence we can define a function $\mathtt{f}_{q_1}(\mathcal{F})$ on the set of $\mathbb{F}_{q_1}$-points of V, whose value at x is $\mathrm{Tr}(\mathrm{Fr}_x, \mathcal{F}_x)$. This function takes values in the algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. But there is not much of a difference between $\overline{\mathbb{Q}}_\ell$-valued functions and $\mathbb{C}$-valued functions: since they have the same cardinality, $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ may be identified as abstract fields. Besides, in most interesting cases, the values actually belong to $\overline{\mathbb{Q}}$, which is inside both $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$.

More generally, if $\mathcal{K}$ is a complex of ℓ-adic sheaves, one defines a function $\mathtt{f}_{q_1}(\mathcal{K})$ on $V(\mathbb{F}_{q_1})$ by taking the alternating sums of the traces of Fr_x on the stalk cohomologies of $\mathcal{K}$ at x. The map $\mathcal{K} \to \mathtt{f}_{q_1}(\mathcal{K})$ intertwines the natural operations on sheaves with natural operations on functions (see [La], Sect. 1.2).

Let $K_0(\mathcal{S}h_X)$ be the complexified Grothendieck group of the category of ℓ-adic sheaves on X. Then the above construction gives us a map

$$K_0(\mathcal{S}h_X) \to \prod_{m \geq 1} X(\mathbb{F}_{q^m}),$$

and it is known that this map is injective (see [La]).

Therefore we may hope that the functions on the quotients $G(F)/K_N$ which realize our representations come by this constructions from ℓ-adic sheaves, or more generally, from complexes of ℓ-adic sheaves, on X.

Now, the notion of constructible sheaf (unlike the notion of a function) has a transparent and meaningful analogue for a complex algebraic variety X, namely, those sheaves of $\mathbb{C}$-vector spaces whose restrictions to the strata of a stratification of the variety X are locally constant. The affine Grassmannian and more general ind-schemes underlying the quotients $G(F)/K_N$ may be defined both over $\mathbb{F}_q$ and $\mathbb{C}$. Thus, it is natural to consider the categories of such sheaves (or, more precisely, their derived categories) on these ind-schemes over $\mathbb{C}$ as the replacements for the vector spaces of functions on their points realizing smooth representations of the group $G(F)$.

We therefore naturally come to the idea, advanced in [FG2], that the representations of the loop group $G(\!(t)\!)$ that we need to consider are not realized on vector spaces, but on **categories**, such as the derived category of coherent sheaves on the affine Grassmannian. Of course, such a category has a Grothendieck group, and the group $G(\!(t)\!)$ will act on the Grothendieck group as well, giving us a representation of $G(\!(t)\!)$ on a vector space. But we obtain much more structure by looking at the categorical representation. The objects of the category, as well as the action, will have a geometric meaning, and thus we will be using the geometry as much as possible.

Let us summarize: to each local Langlands parameter $\chi \in \mathrm{Loc}_{{}^L G}(D^\times)$ we wish to attach a category $\mathcal{C}_\chi$ equipped with an action of the loop group $G(\!(t)\!)$. But what kind of categories should these $\mathcal{C}_\chi$ be and what properties do we expect them to satisfy?

To get closer to answering these questions, we wish to discuss two more steps that we can make in the above discussion to get to the types of categories with an action of the loop group that we will consider in this paper.

3.4 A Toy Model

At this point it is instructive to detour slightly and consider a toy model of our construction. Let G be a split reductive group over $\mathbb{Z}$, and B its Borel subgroup. A natural representation of $G(\mathbb{F}_q)$ is realized in the space of complex (or $\overline{\mathbb{Q}}_\ell$-) valued functions on the quotient $G(\mathbb{F}_q)/B(\mathbb{F}_q)$. It is natural to ask what is the "correct" analogue of this representation if we replace the field $\mathbb{F}_q$ by the complex field and the group $G(\mathbb{F}_q)$ by $G(\mathbb{C})$. This may be viewed as a simplified version of our quandary, since instead of considering $G(\mathbb{F}_q(\!(t)\!))$ we now look at $G(\mathbb{F}_q)$.

The quotient $G(\mathbb{F}_q)/B(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-points of the algebraic variety defined over $\mathbb{Z}$ called the flag variety of G and defined by Fl. Our discussion in the previous section suggests that we first need to replace the notion of a function on $\mathrm{Fl}(\mathbb{F}_q)$ by the notion of an ℓ-adic sheaf on the variety $\mathrm{Fl}_{\mathbb{F}_q} = \mathrm{Fl} \underset{\mathbb{Z}}{\otimes} \mathbb{F}_q$.

Next, we replace the notion of an ℓ-adic sheaf on Fl considered as an algebraic variety over $\mathbb{F}_q$, by the notion of a constructible sheaf on $\mathrm{Fl}_{\mathbb{C}} = \mathrm{Fl} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ which is an algebraic variety over $\mathbb{C}$. The complex algebraic group $G_{\mathbb{C}}$ naturally acts on $\mathrm{Fl}_{\mathbb{C}}$ and hence on this category. Now we make two more reformulations of this category.

First of all, for a smooth complex algebraic variety X we have a **Riemann-Hilbert correspondence** which is an equivalence between the derived category of constructible sheaves on X and the derived category of $\mathcal{D}$-modules on X that are holonomic and have regular singularities.

Here we consider the sheaf of algebraic differential operators on X and sheaves of modules over it, which we simply refer to as $\mathcal{D}$-modules. The simplest example of a $\mathcal{D}$-module is the sheaf of sections of a vector bundle on V equipped with a flat connection. The flat connection enables us to multiply any section by a function and we can use the flat connection to act on sections by vector fields. The two actions generate an action of the sheaf of differential operators on the sections of our bundle. The sheaf of horizontal sections of this bundle is then a locally constant sheaf of X. We have seen above that there is a bijection between the set of isomorphism classes of rank n bundles on X with connection having regular singularities and the set of isomorphism classes of locally constant sheaves on X of rank n, or equivalently, n-dimensional representations of $\pi_1(X)$. This bijection may be elevated to an equivalence of the corresponding categories, and the general Riemann-Hilbert correspondence is a generalization of this equivalence of categories that encompasses more general $\mathcal{D}$-modules.

The Riemann-Hilbert correspondence allows us to associate to any holonomic $\mathcal{D}$-module on X a complex of constructible sheaves on X, and this gives us a functor between the corresponding derived categories which turns out to be an equivalence if we restrict ourselves to the holonomic $\mathcal{D}$-modules with regular singularities (see [B2, GM] for more details).

Thus, over $\mathbb{C}$ we may pass from constructible sheaves to $\mathcal{D}$-modules. In our case, we consider the category of (regular holonomic) $\mathcal{D}$-modules on the flag variety $\mathrm{Fl}_{\mathbb{C}}$. This category carries a natural action of $G_{\mathbb{C}}$.

Finally, let us observe that the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$ acts on the flag variety infinitesimally by vector fields. Therefore, given a $\mathcal{D}$-module $\mathcal{F}$ on $\mathrm{Fl}_{\mathbb{C}}$, the space of its global sections $\Gamma(\mathrm{Fl}_{\mathbb{C}}, \mathcal{F})$ has the structure of $\mathfrak{g}$-module. We obtain a functor Γ from the category of $\mathcal{D}$-modules on $\mathrm{Fl}_{\mathbb{C}}$ to the category of $\mathfrak{g}$-modules. A. Beilinson and J. Bernstein have proved that this functor is an equivalence between the category of all $\mathcal{D}$-modules on $\mathrm{Fl}_{\mathbb{C}}$ (not necessarily regular holonomic) and the category $\mathcal{C}_0$ of $\mathfrak{g}$-modules on which the center of the universal enveloping algebra $U(\mathfrak{g})$ acts through the augmentation character.

Thus, we can now answer our question as to what is a meaningful geometric analogue of the representation of the finite group $G(\mathbb{F}_q)$ on the space of functions on the quotient $G(\mathbb{F}_q)/B(\mathbb{F}_q)$. The answer is the following: it is an **abelian category** equipped with an action of the algebraic group $G_{\mathbb{C}}$. This category has two incarnations: one is the category of $\mathcal{D}$-modules on the flag variety $\mathrm{Fl}_{\mathbb{C}}$, and the other is the category $\mathcal{C}_0$ of modules over the Lie algebra $\mathfrak{g}$ with the trivial central character. Both categories are equipped with natural actions of the group $G_{\mathbb{C}}$.

Let us pause for a moment and spell out what exactly we mean when we say that the group $G_{\mathbb{C}}$ acts on the category $\mathcal{C}_0$. For simplicity, we will describe the action of the corresponding group $G(\mathbb{C})$ of $\mathbb{C}$-points of $G_{\mathbb{C}}$.[6] This means the following: each element $g \in G$ gives rise to a functor F_g on $\mathcal{C}_0$ such that F_1 is the identity functor, and the functor $\mathcal{F}_{g^{-1}}$ is quasi-inverse to F_g. Moreover, for any pair $g, h \in G$ we have a fixed isomorphism of functors $i_{g,h} : F_{gh} \to F_g \circ F_h$ so that for any triple $g, h, k \in G$ we have the equality $i_{h,k} i_{g,hk} = i_{g,h} i_{gh,k}$ of isomorphisms $F_{ghk} \to F_g \circ F_h \circ F_k$.

The functors F_g are defined as follows. Given a representation (V, π) of $\mathfrak{g}$ and an element $g \in G(\mathbb{C})$, we define a new representation $F_g((V,\pi)) = (V, \pi_g)$, where by definition $\pi_g(x) = \pi(\mathrm{Ad}_g(x))$. Suppose that (V,π) is irreducible. Then it is easy to see that $(V,\pi_g) \simeq (V,\pi)$ if and only if (V,π) is integrable, i.e., is obtained from an algebraic representation of G.[7] This is equivalent to this representation being finite-dimensional. But a general representation (V,π) is infinite-dimensional, and so it will not be isomorphic to (V,π_g), at least for some $g \in G$.

Now we consider morphisms in $\mathcal{C}_0$, which are just $\mathfrak{g}$-homomorphisms. Given a $\mathfrak{g}$-homomorphism between representations (V,π) and (V',π'), i.e., a linear map $T : V \to V'$ such that $T\pi(x) = \pi'(x)T$ for all $x \in \mathfrak{g}$, we set $F_g(T) = T$. The isomorphisms $i_{g,h}$ are all identical in this case.

3.5 Back to Loop Groups

In our quest for a complex analogue of the local Langlands correspondence we need to decide what will replace the notion of a smooth representation of the group $G(F)$, where $F = \mathbb{F}_q((t))$. As the previous discussion demonstrates, we should consider representations of the complex loop group $G((t))$ on various categories of $\mathcal{D}$-modules on the ind-schemes $G((t))/K$, where K is a "compact" subgroup of $G((t))$, such as $G[[t]]$ or the Iwahori subgroup (the preimage of a Borel subgroup $B \subset G$ under the homomorphism $G[[t]] \to G$), or the categories of representations of the Lie algebra $\mathfrak{g}((t))$. Both scenarios are viable, and they lead to interesting results and conjectures which we will discuss in detail in Section 9, following [FG2]. In this paper we will concentrate on the second scenario and consider categories of modules over the loop algebra $\mathfrak{g}((t))$.

The group $G((t))$ acts on the category of representations of $\mathfrak{g}((t))$ in the way that we described in the previous section. An analogue of a smooth representation of $G(F)$ is a category of smooth representations of $\mathfrak{g}((t))$. Let us observe however that we could choose instead the category of smooth representations of the central extension of $\mathfrak{g}((t))$, namely, $\widehat{\mathfrak{g}}_\kappa$.

[6] More generally, for any $\mathbb{C}$-algebra R, we have an action of $G(R)$ on the corresponding base-changed category over R. Thus, we are naturally led to the notion of an algebraic group (or, more generally, a group scheme) acting on an abelian category, which is spelled out in [FG2], Sect. 20.

[7] In general, we could obtain a representation of a central extension of G, but if G is reductive, it does not have non-trivial central extensions.

The group $G((t))$ acts on the Lie algebra $\widehat{\mathfrak{g}}_\kappa$ for any κ, because the adjoint action of the central extension of $G((t))$ factors through the action of $G((t))$. We use the action of $G((t))$ on $\widehat{\mathfrak{g}}_\kappa$ to construct an action of $G((t))$ on the category $\widehat{\mathfrak{g}}_\kappa$ -mod, in the same way as in Section 3.4.

Now recall the space $\mathrm{Loc}_{{}^L G}(D^\times)$ of the Langlands parameters that we defined in Section 3.1. Elements of $\mathrm{Loc}_{{}^L G}(D^\times)$ have a concrete description as gauge equivalence classes of first order operators $\partial_t + A(t), A(t) \in {}^L\mathfrak{g}((t))$, modulo the action of ${}^L G((t))$ (see formula (3.2)).

We can now formulate the local Langlands correspondence over $\mathbb{C}$ as the following problem:

To each local Langlands parameter $\chi \in \mathrm{Loc}_{{}^L G}(D^\times)$ *associate a subcategory* $\widehat{\mathfrak{g}}_\kappa$ -mod$_\chi$ *of* $\widehat{\mathfrak{g}}_\kappa$ -mod *which is stable under the action of the loop group* $G((t))$.

We wish to think of the category $\widehat{\mathfrak{g}}_\kappa$ -mod as "fibering" over the space of local Langlands parameters $\mathrm{Loc}_{{}^L G}(D^\times)$, with the categories $\widehat{\mathfrak{g}}_\kappa$ -mod$_\chi$ being the "fibers" and the group $G((t))$ acting along these fibers. From this point of view the categories $\widehat{\mathfrak{g}}_\kappa$ -mod$_\chi$ should give us a "spectral decomposition" of the category $\widehat{\mathfrak{g}}_\kappa$ -mod over $\mathrm{Loc}_{{}^L G}(D^\times)$.

In the next sections we will present a concrete proposal made in [FG2] describing these categories in the special case when $\kappa = \kappa_c$, the critical level.

4 Center and Opers

In Section 1 we have introduced the category $\widehat{\mathfrak{g}}_\kappa$ -mod whose objects are smooth $\widehat{\mathfrak{g}}_\kappa$-modules on which the central element $\mathbf{1}$ acts as the identity. As explained at the end of the previous section, we wish to show that this category "fibers" over the space of the Langlands parameters, which are gauge equivalence classes of ${}^L G$-connections on the punctured disc $D^\times$ (or perhaps, something similar). Moreover, the loop group $G((t))$ should act on this category "along the fibers".

Any abelian category may be thought of as "fibering" over the spectrum of its center. Hence the first idea that comes to mind is to describe the center of the category $\widehat{\mathfrak{g}}_\kappa$ -mod in the hope that its spectrum is related to the Langlands parameters. As we will see, this is indeed the case for a particular value of κ.

4.1 Center of an Abelian Category

Let us first recall what is the center of an abelian category. Let $\mathcal{C}$ be an abelian category over $\mathbb{C}$. The center $Z(\mathcal{C})$ is by definition the set of endomorphisms of the identity functor on $\mathcal{C}$. Let us recall such such an endomorphism is a system of endomorphisms $e_M \in \mathrm{Hom}_{\mathcal{C}}(M, M)$, for each object M of $\mathcal{C}$, which is compatible with the morphisms in $\mathcal{C}$: for any morphism $f : M \to N$ in $\mathcal{C}$ we have $f \circ e_M = e_N \circ f$. It is clear that $Z(\mathcal{C})$ has a natural structure of a commutative algebra over $\mathbb{C}$.

Let $S = \operatorname{Spec} Z(\mathcal{C})$. This is an affine algebraic variety such that $Z(\mathcal{C})$ is the algebra of functions on S. Each point $s \in S$ defines an algebra homomorphism (equivalently, a character) $\rho_s : Z(\mathcal{C}) \to \mathbb{C}$ (evaluation of a function at the point s). We define the full subcategory $\mathcal{C}_s$ of $\mathcal{C}$ whose objects are the objects of $\mathcal{C}$ on which $Z(\mathcal{C})$ acts according to the character ρ_s. It is instructive to think of the category $\mathcal{C}$ as "fibering" over S, with the fibers being the categories $\mathcal{C}_s$.

Now suppose that $\mathcal{C} = A$-mod is the category of left modules over an associative $\mathbb{C}$-algebra A. Then A itself, considered as a left A-module, is an object of $\mathcal{C}$, and so we obtain a homomorphism

$$Z(\mathcal{C}) \to Z(\operatorname{End}_A A) = Z(A^{\mathrm{opp}}) = Z(A),$$

where $Z(A)$ is the center of A. On the other hand, each element of $Z(A)$ defines an endomorphism of each object of A-mod, and so we obtain a homomorphism $Z(A) \to Z(\mathcal{C})$. It is easy to see that these maps set mutually inverse isomorphisms between $Z(\mathcal{C})$ and $Z(A)$.

If $\mathfrak{g}$ is a Lie algebra, then the category $\mathfrak{g}$-mod of $\mathfrak{g}$-modules coincides with the category $U(\mathfrak{g})$-mod of $U(\mathfrak{g})$-modules, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Therefore the center of the category $\mathfrak{g}$-mod is equal to the center of $U(\mathfrak{g})$, which by abuse of notation we denote by $Z(\mathfrak{g})$.

Now consider the category $\widehat{\mathfrak{g}}_\kappa$-mod. Let us recall from Section 1 that objects of $\widehat{\mathfrak{g}}_\kappa$-mod are $\widehat{\mathfrak{g}}_\kappa$-modules M on which the central element $\mathbf{1}$ acts as the identity and which are **smooth**, that is for any vector $v \in M$ we have

$$(\mathfrak{g} \otimes t^N \mathbb{C}[[t]]) \cdot v = 0 \tag{4.1}$$

for sufficiently large N.

Thus, we see that there are two properties that its objects satisfy. Therefore it does not coincide with the category of all modules over the universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa)$ (which is the category of all $\widehat{\mathfrak{g}}_\kappa$-modules). We need to modify this algebra.

First of all, since $\mathbf{1}$ acts as the identity, the action of $U(\widehat{\mathfrak{g}}_\kappa)$ factors through the quotient

$$U_\kappa(\widehat{\mathfrak{g}}) \stackrel{\mathrm{def}}{=} U_\kappa(\widehat{\mathfrak{g}})/(\mathbf{1}-1).$$

Second, the smoothness condition (4.1) implies that the action of $U_\kappa(\widehat{\mathfrak{g}})$ extends to an action of its completion defined as follows.

Define a linear topology on $U_\kappa(\widehat{\mathfrak{g}})$ by using as the basis of neighborhoods for 0 the following left ideals:

$$I_N = U_\kappa(\widehat{\mathfrak{g}})(\mathfrak{g} \otimes t^N \mathbb{C}[[t]]), \qquad N \geqslant 0.$$

Let $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ be the completion of $U_\kappa(\widehat{\mathfrak{g}})$ with respect to this topology. Note that, equivalently, we can write

$$\widetilde{U}_\kappa(\widehat{\mathfrak{g}}) = \varprojlim U_\kappa(\widehat{\mathfrak{g}})/I_N.$$

Even though the I_N's are only left ideals (and not two-sided ideals), one checks that the associative product structure on $U_\kappa(\widehat{\mathfrak{g}})$ extends by continuity to an associative product structure on $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ (this follows from the fact that the Lie bracket on $U_\kappa(\widehat{\mathfrak{g}})$ is continuous in the above topology). Thus, $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ is a complete topological algebra. It follows from the definition that the category $\widehat{\mathfrak{g}}_\kappa$-mod coincides with the category of discrete modules over $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ on which the action of $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ is pointwise continuous (this is precisely equivalent to the condition (4.1)).

It is now easy to see that the center of our category $\widehat{\mathfrak{g}}_\kappa$-mod is equal to the center of the algebra $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$, which we will denote by $Z_\kappa(\widehat{\mathfrak{g}})$. The argument is similar to the one we used above: though $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ itself is not an object of $\widehat{\mathfrak{g}}_\kappa$-mod, we have a collection of objects $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})/I_N$. Using this collection, we obtain an isomorphism between the center of category $\widehat{\mathfrak{g}}_\kappa$-mod and the inverse limit of the algebras $Z(\operatorname{End}_{\widehat{\mathfrak{g}}_\kappa} \widetilde{U}_\kappa(\widehat{\mathfrak{g}})/I_N)$, which, by definition, coincides with $Z_\kappa(\widehat{\mathfrak{g}})$.

Now we can formulate our first question:

describe the center $Z_\kappa(\widehat{\mathfrak{g}})$ for all levels κ.

In order to answer this question we need to introduce the concept of G-opers.

4.2 Opers

Let G be a simple algebraic group of adjoint type, B its Borel subgroup and $N = [B, B]$ its unipotent radical, with the corresponding Lie algebras $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$.

Thus, $\mathfrak{g}$ is a simple Lie algebra, and as such it has the Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

We will choose generators $e_1, \dots, e_\ell$ (resp., $f_1, \dots, f_\ell$) of $\mathfrak{n}_+$ (resp., $\mathfrak{n}_-$). We have $\mathfrak{n}_{\alpha_i} = \mathbb{C}e_i, \mathfrak{n}_{-\alpha_i} = \mathbb{C}f_i$. We take $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ as the Lie algebra of B. Then $\mathfrak{n}$ is the Lie algebra of N. In what follows we will use the notation $\mathfrak{n}$ for $\mathfrak{n}_+$.

Let $[\mathfrak{n}, \mathfrak{n}]^\perp \subset \mathfrak{g}$ be the orthogonal complement of $[\mathfrak{n}, \mathfrak{n}]$ with respect to a non-degenerate invariant bilinear form κ_0. We have

$$[\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b} \simeq \bigoplus_{i=1}^{\ell} \mathfrak{n}_{-\alpha_i}.$$

Clearly, the group B acts on $\mathfrak{n}^\perp/\mathfrak{b}$. Our first observation is that there is an open B-orbit $\mathbf{O} \subset \mathfrak{n}^\perp/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{b}$, consisting of vectors whose projection on each subspace $\mathfrak{n}_{-\alpha_i}$ is non-zero. This orbit may also be described as the B-orbit of the sum of the projections of the generators $f_i, i = 1, \ldots, \ell$, of any possible subalgebra $\mathfrak{n}_-$, onto $\mathfrak{g}/\mathfrak{b}$. The action of B on $\mathbf{O}$ factors through an action of $H = B/N$. The latter is simply transitive and makes $\mathbf{O}$ into an H-torsor.

Let X be a smooth curve and x a point of X. As before, we denote by $\mathcal{O}_x$ the completed local ring and by $\mathcal{K}_x$ its field of fractions. The ring $\mathcal{O}_x$ is isomorphic, but not canonically, to $\mathbb{C}[[t]]$. Then $D_x = \operatorname{Spec} \mathcal{O}_x$ is the disc without a coordinate and $D_x^\times = \operatorname{Spec} \mathcal{K}_x$ is the corresponding punctured disc.

Suppose now that we are given a principal G-bundle $\mathcal{F}$ on a smooth curve X, or D_x, or $D_x^\times$, together with a connection ∇ (automatically flat) and a reduction $\mathcal{F}_B$ to the Borel subgroup B of G. Then we define the relative position of ∇ and $\mathcal{F}_B$ (i.e., the failure of ∇ to preserve $\mathcal{F}_B$) as follows. Locally, choose any flat connection ∇' on $\mathcal{F}$ preserving $\mathcal{F}_B$, and take the difference $\nabla - \nabla'$, which is a section of $\mathfrak{g}_{\mathcal{F}_B} \otimes \Omega_X$. We project it onto $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega_X$. It is clear that the resulting local section of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega_X$ are independent of the choice ∇'. These sections patch together to define a global $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$-valued one-form on X, denoted by $\nabla/\mathcal{F}_B$.

Let X be a smooth curve, or D_x, or $D_x^\times$. Suppose we are given a principal G-bundle $\mathcal{F}$ on X, a connection ∇ on $\mathcal{F}$ and a B-reduction $\mathcal{F}_B$. We will say that $\mathcal{F}_B$ is **transversal** to ∇ if the one-form $\nabla/\mathcal{F}_B$ takes values in $\mathbf{O}_{\mathcal{F}_B} \subset (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$. Note that $\mathbf{O}$ is $\mathbb{C}^\times$-invariant, so that $\mathbf{O} \otimes \Omega_X$ is a well-defined subset of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega_X$.

Now, a G-**oper** on X is by definition a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal G-bundle $\mathcal{F}$ on X, ∇ is a connection on $\mathcal{F}$ and $\mathcal{F}_B$ is a B-reduction of $\mathcal{F}$, such that $\mathcal{F}_B$ is transversal to ∇.

This definition is due to A. Beilinson and V. Drinfeld [BD1] (in the case when X is the punctured disc opers were introduced earlier by V. Drinfeld and V. Sokolov in [DS]).

Equivalently, the transversality condition may be reformulated as saying that if we choose a local trivialization of $\mathcal{F}_B$ and a local coordinate t then the connection will be of the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t), \tag{4.2}$$

where each $\psi_i(t)$ is a nowhere vanishing function, and $\mathbf{v}(t)$ is a $\mathfrak{b}$-valued function.

If we change the trivialization of $\mathcal{F}_B$, then this operator will get transformed by the corresponding B-valued gauge transformation. This observation allows us to describe opers on the disc $D_x = \operatorname{Spec} \mathcal{O}_x$ and the punctured disc $D_x^\times = \operatorname{Spec} \mathcal{K}_x$ in a more explicit way. The same reasoning will work on any sufficiently small analytic subset U of any curve, equipped with a local coordinate t, or on a Zariski open subset equipped with an étale coordinate. For the sake of definiteness, we will consider now the case of the base $D_x^\times$.

Let us choose a coordinate t on D_x, i.e., an isomorphism $\mathcal{O}_x \simeq \mathbb{C}[[t]]$. Then we identify D_x with $D = \operatorname{Spec} \mathbb{C}[[t]]$ and $D_x^\times$ with $D^\times = \operatorname{Spec} \mathbb{C}((t))$.

The space $\mathrm{Op}_G(D^\times)$ of G-opers on $D^\times$ is then the quotient of the space of all operators of the form (4.2), where $\psi_i(t) \in \mathbb{C}((t)), \psi_i(0) \neq 0, i = 1, \ldots, \ell$, and $\mathbf{v}(t) \in \mathfrak{b}((t))$, by the action of the group $B((t))$ of gauge transformations:

$$g \cdot (\partial_t + A(t)) = \partial_t + gA(t)g^{-1} - g^{-1}\partial_t g.$$

Let us choose a splitting $\imath : H \to B$ of the homomorphism $B \to H$. Then B becomes the semi-direct product $B = H \ltimes N$. The B-orbit $\mathbf{O}$ is an H-torsor, and so we can use H-valued gauge transformations to make all functions $\psi_i(t)$ equal to 1. In other words, there is a unique element of $H((t))$, namely, the element $\prod_{i=1}^{\ell} \check{\omega}_i(\psi_i(t))$, where $\check{\omega}_i : \mathbb{C}^\times \to H$ is the ith fundamental coweight of G, such that the corresponding gauge transformation brings our connection operator to the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} f_i + \mathbf{v}(t), \qquad \mathbf{v}(t) \in \mathfrak{b}((t)). \tag{4.3}$$

What remains is the group of N-valued gauge transformations. Thus, we obtain that $\mathrm{Op}_G(D^\times)$ is equal to the quotient of the space $\widetilde{\mathrm{Op}}_G(D^\times)$ of operators of the form (4.3) by the action of the group $N((t))$ by gauge transformations:

$$\mathrm{Op}_G(D^\times) = \widetilde{\mathrm{Op}}_G(D^\times)/N((t)).$$

Lemma 1 ([DS]). *The action of $N((t))$ on $\widetilde{\mathrm{Op}}_G(D^\times)$ is free.*

4.3 Canonical Representatives

Now we construct canonical representatives in the $N((t))$-gauge classes of connections of the form (4.3), following [BD1]. Observe that the operator $\mathrm{ad}\,\check{\rho}$ defines a gradation on $\mathfrak{g}$, called the **principal gradation**, with respect to which we have a direct sum decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$. In particular, we have $\mathfrak{b} = \bigoplus_{i \geq 0} \mathfrak{b}_i$, where $\mathfrak{b}_0 = \mathfrak{h}$.

Let now

$$p_{-1} = \sum_{i=1}^{\ell} f_i.$$

The operator $\mathrm{ad}\, p_{-1}$ acts from $\mathfrak{b}_{i+1}$ to $\mathfrak{b}_i$ injectively for all $i \geq 0$. Hence we can find for each $i \geq 0$ a subspace $V_i \subset \mathfrak{b}_i$, such that $\mathfrak{b}_i = [p_{-1}, \mathfrak{b}_{i+1}] \oplus V_i$. It is well-known that $V_i \neq 0$ if and only if i is an **exponent** of $\mathfrak{g}$, and in that case $\dim V_i$ is equal to the multiplicity of the exponent i. In particular, $V_0 = 0$.

Let $V = \bigoplus_{i \in E} V_i \subset \mathfrak{n}$, where $E = \{d_1, \ldots, d_\ell\}$ is the set of exponents of $\mathfrak{g}$ counted with multiplicity. They are equal to the orders of the generators of the center of $U(\mathfrak{g})$ minus one. We note that the multiplicity of each exponent is equal to one in all cases except the case $\mathfrak{g} = D_{2n}, d_n = 2n$, when it is equal to two.

There is a special choice of the transversal subspace $V = \bigoplus_{i \in E} V_i$. Namely, there exists a unique element p_1 in $\mathfrak{n}$, such that $\{p_{-1}, 2\check{\rho}, p_1\}$ is an $\mathfrak{sl}_2$-triple. This means that they have the same relations as the generators $\{e, h, f\}$ of $\mathfrak{sl}_2$. We have $p_1 = \sum_{i=1}^{\ell} m_i e_i$, where e_i's are generators of $\mathfrak{n}_+$ and m_i are certain coefficients uniquely determined by the condition that $\{p_{-1}, 2\check{\rho}, p_1\}$ is an $\mathfrak{sl}_2$-triple.

Let $V^{\mathrm{can}} = \bigoplus_{i \in E} V_i^{\mathrm{can}}$ be the space of $\operatorname{ad} p_1$-invariants in $\mathfrak{n}$. Then p_1 spans V_1^{can}. Let p_j be a linear generator of $V_{d_j}^{\mathrm{can}}$. If the multiplicity of d_j is greater than one, then we choose linearly independent vectors in $V_{d_j}^{\mathrm{can}}$.

Each $N(\!(t)\!)$-equivalence class contains a unique operator of the form $\nabla = \partial_t + p_{-1} + \mathbf{v}(t)$, where $\mathbf{v}(t) \in V^{\mathrm{can}}[[t]]$, so that we can write

$$\mathbf{v}(t) = \sum_{j=1}^{\ell} v_j(t) \cdot p_j, \qquad v_j(t) \in \mathbb{C}[[t]].$$

It is easy to find (see, e.g., [F6], Sect. 8.3) that under changes of coordinate t, v_1 transforms as a projective connection, and $v_j, j > 1$, transforms as a $(d_j + 1)$-differential on D_x. Thus, we obtain an isomorphism

$$\operatorname{Op}_G(D^\times) \simeq \mathcal{P}roj(D^\times) \times \bigoplus_{j=2}^{\ell} \Omega_{\mathcal{K}}^{\otimes(d_j+1)}, \tag{4.4}$$

where $\Omega_{\mathcal{K}}^{\otimes n}$ is the space of n-differentials on $D^\times$ and $\mathcal{P}roj(D^\times)$ is the $\Omega_{\mathcal{K}}^{\otimes 2}$-torsor of projective connections on $D^\times$.

We have an analogous isomorphism with $D^\times$ replaced by formal disc D or any smooth algebraic curve X.

4.4 Description of the Center

Now we are ready to describe the center of the completed universal enveloping algebra $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$. The following assertion is proved in [F7], using results of [K1]:

Proposition 1. *The center of* $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ *consists of the scalars for* $\kappa \neq \kappa_c$.

Let us denote the center of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ by $Z(\widehat{\mathfrak{g}})$. The following theorem was proved in [FF3, F3] (it was conjectured by V. Drinfeld).

Theorem 2. *The center* $Z(\widehat{\mathfrak{g}})$ *is isomorphic to the algebra* $\operatorname{Fun} \operatorname{Op}_{{}^L G}(D^\times)$ *in a way compatible with the action of the group of coordinate changes.*

This implies the following result. Let x be a point of a smooth curve X. Then we have the affine algebra $\widehat{\mathfrak{g}}_{\kappa_c,x}$ as defined in Section 1 and the corresponding completed universal enveloping algebra of critical level. We denote its center by $Z(\widehat{\mathfrak{g}}_x)$.

Corollary 1. *The center* $Z(\widehat{\mathfrak{g}}_x)$ *is isomorphic to the algebra* $\operatorname{Fun} \operatorname{Op}_{{}^L G}(D_x^\times)$ *of functions on the space of* ${}^L G$*-opers on* $D_x^\times$.

5 Opers vs. Local Systems

We now go back to the question posed at the end of Section 3: let

$$\mathrm{Loc}_{^LG}(D^\times) = \left\{ \partial_t + A(t),\ A(t) \in {}^L\mathfrak{g}(\!(t)\!) \right\} / {}^LG(\!(t)\!) \tag{5.1}$$

be the set of gauge equivalence classes of ${}^LG$-connections on the punctured disc $D^\times = \operatorname{Spec} \mathbb{C}(\!(t)\!)$. We had argued in Section 3 that $\mathrm{Loc}_{^LG}(D^\times)$ should be taken as the space of Langlands parameters for the loop group $G(\!(t)\!)$. Recall that the loop group $G(\!(t)\!)$ acts on the category $\widehat{\mathfrak{g}}_\kappa$-mod of (smooth) $\widehat{\mathfrak{g}}$-modules of level κ (see Section 1 for the definition of this category). We asked the following question:

Associate to each local Langlands parameter $\sigma \in \mathrm{Loc}_{^LG}(D^\times)$ *a subcategory* $\widehat{\mathfrak{g}}_\kappa\text{-mod}_\sigma$ *of* $\widehat{\mathfrak{g}}_\kappa$-mod *which is stable under the action of the loop group* $G(\!(t)\!)$.

Even more ambitiously, we wish to represent the category $\widehat{\mathfrak{g}}_\kappa$-mod as "fibering" over the space of local Langlands parameters $\mathrm{Loc}_{^LG}(D^\times)$, with the categories $\widehat{\mathfrak{g}}_\kappa\text{-mod}_\sigma$ being the "fibers" and the group $G(\!(t)\!)$ acting along these fibers. If we could do that, then we would think of this fibration as a "spectral decomposition" of the category $\widehat{\mathfrak{g}}_\kappa$-mod over $\mathrm{Loc}_{^LG}(D^\times)$.

At the beginning of Section 4 we proposed a possible scenario for solving this problem. Namely, we observed that any abelian category may be thought of as "fibering" over the spectrum of its center. Hence our idea was to describe the center of the category $\widehat{\mathfrak{g}}_\kappa$-mod (for each value of κ) and see if its spectrum is related to the space $\mathrm{Loc}_{^LG}(D^\times)$ of Langlands parameters.

We have identified the center of the category $\widehat{\mathfrak{g}}_\kappa$-mod with the center $Z_\kappa(\widehat{\mathfrak{g}})$ of the associative algebra $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$, the completed enveloping algebra of $\widehat{\mathfrak{g}}$ of level κ, defined in Section 4. Next, we described the algebra $Z_\kappa(\widehat{\mathfrak{g}})$. According to Proposition 1, if $\kappa \neq \kappa_c$, the critical level, then $Z_\kappa(\widehat{\mathfrak{g}}) = \mathbb{C}$. Therefore our approach cannot work for $\kappa \neq \kappa_c$. However, we found that the center $Z_{\kappa_c}(\widehat{\mathfrak{g}})$ at the critical level is highly non-trivial and indeed related to ${}^LG$-connections on the punctured disc.

Now, following the works [FG1]–[FG6] of D. Gaitsgory and myself, I will use these results to formulate more precise conjectures on the local Langlands correspondence for loop groups and to provide some evidence for these conjectures. I will then discuss the implications of these conjectures for the global geometric Langlands correspondence.[8]

According to Theorem 2, $Z_{\kappa_c}(\widehat{\mathfrak{g}})$ is isomorphic to $\operatorname{Fun} \mathrm{Op}_{^LG}(D^\times)$, the algebra of functions on the space of ${}^LG$-opers on the punctured disc $D^\times$. This isomorphism is compatible with various symmetries and structures on both

[8] Note that A. Beilinson has another proposal [Bei] for local geometric Langlands correspondence, using representations of affine Kac-Moody algebras of levels *less than critical*. It would be interesting to understand the connection between his proposal and ours.

algebras, such as the action of the group of coordinate changes. There is a one-to-one correspondence between points $\chi \in \mathrm{Op}_{^L G}(D^\times)$ and homomorphisms (equivalently, characters)

$$\mathrm{Fun}\,\mathrm{Op}_{^L G}(D^\times) \to \mathbb{C},$$

corresponding to evaluating a function at χ. Hence points of $\mathrm{Op}_{^L G}(D^\times)$ parametrize **central characters** $Z_{\kappa_c}(\widehat{\mathfrak{g}}) \to \mathbb{C}$.

Given a $^L G$-oper $\chi \in \mathrm{Op}_{^L G}(D^\times)$, define the category

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$$

as a full subcategory of $\widehat{\mathfrak{g}}_{\kappa_c}$-mod whose objects are $\widehat{\mathfrak{g}}$-modules of critical level (hence $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$-modules) on which the center $Z_{\kappa_c}(\widehat{\mathfrak{g}}) \subset \widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ acts according to the central character corresponding to χ. More generally, for any closed algebraic subvariety $Y \subset \mathrm{Op}_{^L G}(D^\times)$ (not necessarily a point), we have an ideal

$$I_Y \subset \mathrm{Fun}\,\mathrm{Op}_{^L G}(D^\times) \simeq Z_{\kappa_c}(\widehat{\mathfrak{g}})$$

of those functions that vanish on Y. We then have a full subcategory $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_Y$ of $\widehat{\mathfrak{g}}_{\kappa_c}$-mod whose objects are $\widehat{\mathfrak{g}}$-modules of critical level on which I_Y acts by 0. This category is an example of a "base change" of the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod with respect to the morphism $Y \to \mathrm{Op}_{^L G}(D^\times)$. It is easy to generalize this definition to an arbitrary affine scheme Y equipped with a morphism $Y \to \mathrm{Op}_{^L G}(D^\times)$.[9]

Since the algebra $\mathrm{Op}_{^L G}(D^\times)$ acts on the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod, one can say that the category $\widehat{\mathfrak{g}}_\kappa$-mod "fibers" over the space $\mathrm{Op}_{^L G}(D^\times)$, in such a way that the fiber-category corresponding to $\chi \in \mathrm{Op}_{^L G}(D^\times)$ is the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$.[10]

Recall that the group $G(\!(t)\!)$ acts on $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ and on the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod. One can show (see [BD1], Remark 3.7.11(iii)) that the action of $G(\!(t)\!)$ on $Z_{\kappa_c}(\widehat{\mathfrak{g}}) \subset \widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ is trivial. Therefore the subcategories $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ (and, more generally, $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_Y$) are stable under the action of $G(\!(t)\!)$. Thus, the group $G(\!(t)\!)$ acts "along the fibers" of the "fibration" $\widehat{\mathfrak{g}}_{\kappa_c}$-mod $\to \mathrm{Op}_{^L G}(D^\times)$ (see [FG2], Sect. 20, for more details).

The fibration $\widehat{\mathfrak{g}}_{\kappa_c}$-mod $\to \mathrm{Op}_{^L G}(D^\times)$ almost gives us the desired local Langlands correspondence for loop groups. But there is one important difference: we asked that the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod fiber over the space $\mathrm{Loc}_{^L G}(D^\times)$ of local systems on $D^\times$. We have shown, however, that $\widehat{\mathfrak{g}}_{\kappa_c}$-mod fibers over the space $\mathrm{Op}_{^L G}(D^\times)$ of $^L G$-opers.

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9 The corresponding base changed categories $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_Y$ may then be "glued" together, which allows us to define the base changed category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_Y$ for any scheme Y mapping to $\mathrm{Op}_{^L G}(D^\times)$.

10 The precise notion of an abelian category fibering over a scheme is spelled out in [Ga3].

What is the difference between the two spaces? While a ${}^L G$-local system is a pair $(\mathcal{F}, \nabla)$, where $\mathcal{F}$ is an ${}^L G$-bundle and ∇ is a connection on $\mathcal{F}$, an ${}^L G$-oper is a triple $(\mathcal{F}, \nabla, \mathcal{F}_{{}^L B})$, where $\mathcal{F}$ and ∇ are as before, and $\mathcal{F}_{{}^L B}$ is an additional piece of structure, namely, a reduction of $\mathcal{F}$ to a (fixed) Borel subgroup ${}^L B \subset {}^L G$ satisfying the transversality condition explained in Section 4.2. Thus, for any curve X we clearly have a forgetful map

$$\mathrm{Op}_{{}^L G}(X) \to \mathrm{Loc}_{{}^L G}(X).$$

The fiber of this map over $(\mathcal{F}, \nabla) \in \mathrm{Loc}_{{}^L G}(X)$ consists of all ${}^L B$-reductions of $\mathcal{F}$ satisfying the transversality condition with respect to ∇.

For a general X it may well be that this map is not surjective, i.e., that the fiber of this map over a particular local system $(\mathcal{F}, \nabla)$ is empty. For example, if X is a projective curve and ${}^L G$ is a group of adjoint type, then there is a unique ${}^L G$-bundle $\mathcal{F}_{{}^L G}$ such that the fiber over $(\mathcal{F}_{{}^L G}, \nabla)$ is non-empty, as we saw in Section 1.

The situation is quite different when $X = D^\times$. In this case any ${}^L G$-bundle $\mathcal{F}$ may be trivialized. A connection ∇ therefore may be represented as a first order operator $\partial_t + A(t), A(t) \in {}^L\mathfrak{g}(\!(t)\!)$. However, the trivialization of $\mathcal{F}$ is not unique; two trivializations differ by an element of ${}^L G(\!(t)\!)$. Therefore the set of equivalence classes of pairs $(\mathcal{F}, \nabla)$ is identified with the quotient (5.1).

Suppose now that $(\mathcal{F}, \nabla)$ carries an oper reduction $\mathcal{F}_{{}^L B}$. Then we consider only those trivializations of $\mathcal{F}$ which come from trivializations of $\mathcal{F}_{{}^L B}$. There are fewer of those, since two trivializations now differ by an element of ${}^L B(\!(t)\!)$ rather than ${}^L G(\!(t)\!)$. Due to the oper transversality condition, the connection ∇ must have a special form with respect to any of those trivializations, namely,

$$\nabla = \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t),$$

where each $\psi_i(t) \neq 0$ and $\mathbf{v}(t) \in {}^L\mathfrak{b}(\!(t)\!)$ (see Section 4.2). Thus, we obtain a concrete realization of the space of opers as a space of gauge equivalence classes

(5.2)

$$\mathrm{Op}_{{}^L G}(D^\times) = \left\{ \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t),\ \psi_i \neq 0, \mathbf{v}(t) \in {}^L\mathfrak{b}(\!(t)\!) \right\} \Big/ \ {}^L B(\!(t)\!).$$

Now the map

$$\alpha : \mathrm{Op}_{{}^L G}(D^\times) \to \mathrm{Loc}_{{}^L G}(D^\times)$$

simply takes a ${}^L B(\!(t)\!)$-equivalence class of operators of the form (5.2) to its ${}^L G(\!(t)\!)$-equivalence class.

Unlike the case of projective curves X discussed above, we expect that the map α is **surjective** for any simple Lie group ${}^L G$. In the case of $G = SL_n$ this follows from the results of P. Deligne [De1], and we conjecture it to be true in general.

Conjecture 1. *The map α is surjective for any simple Lie group ${}^L G$.*

Now we find ourselves in the following situation: we *expect* that there exists a category $\mathcal{C}$ fibering over the space $\operatorname{Loc}_{{}^L G}(D^\times)$ of "true" local Langlands parameters, equipped with a fiberwise action of the loop group $G((t))$. The fiber categories $\mathcal{C}_\sigma$ corresponding to various $\sigma \in \operatorname{Loc}_{{}^L G}(D^\times)$ should satisfy various, not yet specified, properties. This should be the ultimate form of the local Langlands correspondence. On the other hand, we have *constructed* a category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod which fibers over a close cousin of the space $\operatorname{Loc}_{{}^L G}(D^\times)$, namely, the space $\operatorname{Op}_{{}^L G}(D^\times)$ of ${}^L G$-opers, and is equipped with a fiberwise action of the loop group $G((t))$.

What should be the relationship between the two?

The idea of [FG2] is that the second fibration is a "base change" of the first one, that is we have a Cartesian diagram

$$
\begin{array}{ccc}
\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod} & \longrightarrow & \mathcal{C} \\
\Big\downarrow & & \Big\downarrow \\
\operatorname{Op}_{{}^L G}(D^\times) & \xrightarrow{\alpha} & \operatorname{Loc}_{{}^L G}(D^\times)
\end{array}
\tag{5.3}
$$

that commutes with the action of $G((t))$ along the fibers of the two vertical maps. In other words,

$$
\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod} \simeq \mathcal{C} \underset{\operatorname{Loc}_{{}^L G}(D^\times)}{\times} \operatorname{Op}_{{}^L G}(D^\times).
$$

At present, we do not have a definition of $\mathcal{C}$, and therefore we cannot make this isomorphism precise. But we will use it as our guiding principle. We will now discuss various corollaries of this conjecture and various pieces of evidence that make us believe that it is true.

In particular, let us fix a Langlands parameter $\sigma \in \operatorname{Loc}_{{}^L G}(D^\times)$ that is in the image of the map α (according to Conjecture 1, all Langlands parameters are). Let χ be a ${}^L G$-oper in the preimage of σ, $\alpha^{-1}(\sigma)$. Then, according to the above conjecture, the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ is equivalent to the "would be" Langlands category $\mathcal{C}_\sigma$ attached to σ. Hence we may take $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ as the **definition** of $\mathcal{C}_\sigma$.

The caveat is, of course, that we need to ensure that this definition is independent of the choice of χ in $\alpha^{-1}(\sigma)$. This means that for any two ${}^L G$-opers, χ and χ', in the preimage of σ, the corresponding categories, $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi'}$, should be equivalent to each other, and this equivalence should commute with the action of the loop group $G((t))$. Moreover, we should expect that these equivalences are compatible with each other as we move along the fiber $\alpha^{-1}(\sigma)$. We will not try to make this condition more precise here (however, we will explain below in Conjecture 4 what this means for regular opers).

Even putting the questions of compatibility aside, we arrive at the following rather non-trivial conjecture (see [FG2]).

Conjecture 2. *Suppose that* $\chi, \chi' \in \mathrm{Op}_{^LG}(D^\times)$ *are such that* $\alpha(\chi) = \alpha(\chi')$*, i.e., that the flat* ^LG*-bundles on* $D^\times$ *underlying the* ^LG*-opers* χ *and* χ' *are isomorphic to each other. Then there is an equivalence between the categories* $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ *and* $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi'}$ *which commutes with the actions of the group* $G(\!(t)\!)$ *on the two categories.*

Thus, motivated by our quest for the local Langlands correspondence, we have found an unexpected symmetry in the structure of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}$ of $\widehat{\mathfrak{g}}$-modules of critical level.

6 Harish–Chandra Categories

As explained in Section 3, the local Langlands correspondence for the loop group $G(\!(t)\!)$ should be viewed as a categorification of the local Langlands correspondence for the group $G(F)$, where F is a local non-archimedian field. This means that the categories $\mathcal{C}_\sigma$, equipped with an action of $G(\!(t)\!)$, that we are trying to attach to the Langlands parameters $\sigma \in \mathrm{Loc}_{^LG}(D^\times)$ should be viewed as categorifications of the smooth representations of $G(F)$ on complex vector spaces attached to the corresponding local Langlands parameters discussed in Section 2.3. Here we use the term "categorification" to indicate that we expect the Grothendieck groups of the categories $\mathcal{C}_\sigma$ to "look like" irreducible smooth representations of $G(F)$. We begin by taking a closer look at the structure of these representations.

6.1 Spaces of K-Invariant Vectors

It is known that an irreducible smooth representation (R, π) of $G(F)$ is automatically **admissible**, in the sense that for any open compact subgroup K, such as the Nth congruence subgroup K_N defined in Section 2.1, the space $R^{\pi(K)}$ of K-invariant vectors in R is finite-dimensional. Thus, while most of the irreducible smooth representations (R, π) of $G(F)$ are infinite-dimensional, they are filtered by the finite-dimensional subspaces $R^{\pi(K)}$ of K-invariant vectors, where K are smaller and smaller open compact subgroups. The space $R^{\pi(K)}$ does not carry an action of $G(F)$, but it carries an action of the **Hecke algebra** $H(G(F), K)$.

By definition, $H(G(F), K)$ is the space of compactly supported K bi-invariant functions on $G(F)$. It is given an algebra structure with respect to the **convolution product**

$$(f_1 \star f_2)(g) = \int_{G(F)} f_1(gh^{-1}) f_2(h)\, dh, \tag{6.1}$$

where dh is the Haar measure on $G(F)$ normalized in such a way that the volume of the subgroup $K_0 = G(\mathcal{O})$ is equal to 1 (here $\mathcal{O}$ is the ring of integers of F; e.g., for $F = \mathbb{F}_q((t))$ we have $\mathcal{O} = \mathbb{F}_q[[t]]$). The algebra $H(G(F), K)$ acts on the space $R^{\pi(K)}$ by the formula

$$(6.2) \qquad f \star v = \int_{G(F)} f_1(gh^{-1})(\pi(h) \cdot v) \, dh, \qquad v \in R^{\pi(K)}.$$

Studying the spaces of K-invariant vectors and their $\mathcal{H}(G(F), K)$-module structure gives us an effective tool for analyzing representations of the group $G(F)$, where $F = \mathbb{F}_q((t))$.

Can we find a similar structure in the categorical local Langlands correspondence for loop groups?

6.2 Equivariant Modules

In the categorical setting a representation (R, π) of the group $G(F)$ is replaced by a category equipped with an action of $G((t))$, such as $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$. The open compact subgroups of $G(F)$ have obvious analogues for the loop group $G((t))$ (although they are, of course, not compact with respect to the usual topology on $G((t))$). For instance, we have the "maximal compact subgroup" $K_0 = G[[t]]$, or, more generally, the Nth congruence subgroup K_N, whose elements are congruent to 1 modulo $t^N \mathbb{C}[[t]]$. Another important example is the analogue of the **Iwahori subgroup**. This is the subgroup of $G[[t]]$, which we denote by I, whose elements $g(t)$ have the property that their value at 0, that is $g(0)$, belong to a fixed Borel subgroup $B \subset G$.

Now, for a subgroup $K \subset G((t))$ of this type, an analogue of a K-invariant vector in the categorical setting is an object of our category, i.e., a smooth $\widehat{\mathfrak{g}}_{\kappa_c}$-module (M, ρ), where $\rho : \widehat{\mathfrak{g}}_{\kappa_c} \to \operatorname{End} M$, which is stable under the action of K. Recall from Section 3.5 that for any $g \in G((t))$ we have a new $\widehat{\mathfrak{g}}_{\kappa_c}$-module (M, ρ_g), where $\rho_g(x) = \rho(\operatorname{Ad}_g(x))$. We say that (M, ρ) is stable under K, or that (M, ρ) is **weakly K-equivariant**, if there is a compatible system of isomorphisms between (M, ρ) and (M, ρ_k) for all $k \in K$. More precisely, this means that for each $k \in K$ there exists a linear map $T_k^M : M \to M$ such that

$$T_k^M \rho(x) (T_k^M)^{-1} = \rho(\operatorname{Ad}_k(x))$$

for all $x \in \widehat{\mathfrak{g}}_{\kappa_c}$, and we have

$$T_1^M = \operatorname{Id}_M, \qquad T_{k_1}^M T_{k_2}^M = T_{k_1 k_2}^M.$$

Thus, M becomes a representation of the group K.[11] Consider the corresponding representation of the Lie algebra $\mathfrak{k} = \operatorname{Lie} K$ on M. Let us assume that the embedding $\mathfrak{k} \hookrightarrow \mathfrak{g}((t))$ lifts to $\mathfrak{k} \hookrightarrow \widehat{\mathfrak{g}}_{\kappa_c}$ (i.e., that the central extension cocycle is trivial on $\mathfrak{k}$). This is true, for instance, for any subgroup contained

[11] In general, it is reasonable to modify the last condition to allow for a non-trivial two-cocycle and hence a non-trivial central extension of K; however, in the case of interest K does not have any non-trivial central extensions.

in $K_0 = G[[t]]$, or its conjugate. Then we also have a representation of $\mathfrak{k}$ on M obtained by restriction of ρ. In general, the two representations do not have to coincide. If they do coincide, then the module M is called **strongly K-equivariant**, or simply **K-equivariant**.

The pair $(\widehat{\mathfrak{g}}_{\kappa_c}, K)$ is an example of **Harish-Chandra pair**, that is a pair $(\mathfrak{g}, H)$ consisting of a Lie algebra $\mathfrak{g}$ and a Lie group H whose Lie algebra is contained in $\mathfrak{g}$. The K-equivariant $\widehat{\mathfrak{g}}_{\kappa_c}$-modules are therefore called $(\widehat{\mathfrak{g}}_{\kappa_c}, K)$ **Harish-Chandra modules**. These are (smooth) $\widehat{\mathfrak{g}}_{\kappa_c}$-modules on which the action of the Lie algebra $\operatorname{Lie} K \subset \widehat{\mathfrak{g}}_{\kappa_c}$ may be exponentiated to an action of K (we will assume that K is connected). We denote by $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^K$ and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$ the full subcategories of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}$ and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, respectively, whose objects are $(\widehat{\mathfrak{g}}_{\kappa_c}, K)$ Harish-Chandra modules.

We will stipulate that the analogues of K-invariant vectors in the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ are $(\widehat{\mathfrak{g}}_{\kappa_c}, K)$ Harish-Chandra modules. Thus, while the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ should be viewed as analogues of smooth irreducible representations (R, π) of the group $G(F)$, the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$ are analogues of the spaces of K-invariant vectors $R^{\pi(K)}$.

Next, we discuss the categorical analogue of the Hecke algebra $H(G(F), K)$.

6.3 Categorical Hecke Algebras

We recall that $H(G(F), K)$ is the algebra of compactly supported K bi-invariant functions on $G(F)$. We realize it as the algebra of left K-invariant compactly supported functions on $G(F)/K$. In Section 3.4 we have already discussed the question of categorification of the algebra of functions on a homogeneous space like $G(F)/K$. Our conclusion was that the categorical analogue of this algebra, when $G(F)$ is replaced by the complex loop group $G(\!(t)\!)$, is the category of $\mathcal{D}$-modules on $G(\!(t)\!)/K$. More precisely, this quotient has the structure of an ind-scheme which is a direct limit of finite-dimensional algebraic varieties with respect to closed embeddings. The appropriate notion of (right) $\mathcal{D}$-modules on such ind-schemes is formulated in [BD1] (see also [FG1, FG2]). As the categorical analogue of the algebra of left K-invariant functions on $G(F)/K$, we take the category $\mathcal{H}(G(\!(t)\!), K)$ of K-equivariant $\mathcal{D}$-modules on the ind-scheme $G(\!(t)\!)/K$ (with respect to the left action of K on $G(\!(t)\!)/K$). We call it the **categorical Hecke algebra** associated to K.

It is easy to define the convolution of two objects of $\mathcal{H}(G(\!(t)\!), K)$ by imitating formula (6.1). Namely, we interpret this formula as a composition of the operations of pulling back and integrating functions. Then we apply the same operations to $\mathcal{D}$-modules, thinking of the integral as push-forward. However, here one encounters two problems. The first problem is that for a general group K the morphisms involved will not be proper, and so we have to choose between the $*$- and !-push-forward. This problem does not arise, however, if K is such that $I \subset K \subset G[[t]]$, which will be our main case of interest. The second, and more serious, issue is that in general the push-forward is not an exact functor, and so the convolution of two $\mathcal{D}$-modules will not be a $\mathcal{D}$-module, but a complex, more precisely, an object of the corresponding K-equivariant (bounded) derived category $D^b(G((t))/K)^K$ of $\mathcal{D}$-modules on $G((t))/K$. We will not spell out the exact definition of this category here, referring the interested reader to [BD1] and [FG2]. The exception is the case of the subgroup $K_0 = G[[t]]$, when the convolution functor is exact and so we may restrict ourselves to the abelian category of K_0-equivariant $\mathcal{D}$-modules on $G((t))/K_0$.

Now the category $D^b(G((t))/K)^K$ has a monoidal structure, and as such it acts on the derived category of $(\widehat{\mathfrak{g}}_{\kappa_c}, K)$ Harish-Chandra modules (again, we refer the reader to [BD1, FG2] for the precise definition). In the special case when $K = K_0$, we may restrict ourselves to the corresponding abelian categories. This action should be viewed as the categorical analogue of the action of $H(G(F), K)$ on the space $R^{\pi(K)}$ of K-invariant vectors discussed above.

Our ultimate goal is understanding the "local Langlands categories" $\mathcal{C}_\sigma$ associated to the "local Langlands parameters $\sigma \in \mathrm{Loc}_{^L G}(D^\times)$. We now have a candidate for the category $\mathcal{C}_\sigma$, namely, the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, where $\sigma = \alpha(\chi)$. Therefore $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ should be viewed as a categorification of a smooth representation (R, π) of $G(F)$. The corresponding category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$ of $(\widehat{\mathfrak{g}}_{\kappa_c}, K)$ Harish-Chandra modules should therefore be viewed as a categorification of $R^{\pi(K)}$. This category (or, more precisely, its derived category) is acted upon by the categorical Hecke algebra $\mathcal{H}(G((t)), K)$. We summarize this analogy in the following table.

Classical Theory	**Geometric Theory**
Representation of $G(F)$ on a vector space R	Representation of $G((t))$ on a category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$
A vector in R	An object of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$
The subspace $R^{\pi(K)}$ of K-invariant vectors of R	The subcategory $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$ of $(\widehat{\mathfrak{g}}_{\kappa_c}, K)$ Harish-Chandra modules
Hecke algebra $H(G(F), K)$ acts on $R^{\pi(K)}$	Categorical Hecke algebra $\mathcal{H}(G((t)), K)$ acts on $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$

Now we may test our proposal for the local Langlands correspondence by studying the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^K$ of Harish-Chandra modules and comparing their structure to the structure of the spaces $R^{\pi(K)}$ of K-invariant vectors of smooth representations of $G(F)$ in the known cases. Another possibility is to test Conjecture 2 when applied to the categories of Harish-Chandra modules.

In the next section we consider the case of the "maximal compact subgroup" $K_0 = G[[t]]$ and find perfect agreement with the classical results about unramified representations of $G(F)$. We then take up the more complicated case of the Iwahori subgroup I. There we also find the conjectures and results of [FG2] to be consistent with the known results about representations of $G(F)$ with Iwahori fixed vectors.

7 Local Langlands Correspondence: Unramified Case

We first take up the case of the "maximal compact subgroup" $K_0 = G[[t]]$ of $G((t))$ and consider the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ which contain non-trivial K_0-equivariant objects.

7.1 Unramified Representations of $G(F)$

These categories are analogues of smooth representations of the group $G(F)$, where F is a local non-archimedian field (such as $\mathbb{F}_q((t))$) that contain non-zero K_0-invariant vectors. Such representations are called **unramified**. The classification of the irreducible unramified representations of $G(F)$ is the simplest case of the local Langlands correspondence discussed in Sections 2.2 and 2.3. Namely, we have a bijection between the sets of equivalence classes of the following objects:

$$\boxed{\begin{array}{c}\text{unramified admissible}\\ \text{homomorphisms } W'_F \to {}^L G\end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c}\text{irreducible unramified}\\ \text{representations of } G(F)\end{array}} \tag{7.1}$$

where W'_F is the Weil-Deligne group introduced in Section 2.1.

By definition, unramified homomorphisms $W'_F \longrightarrow {}^L G$ are those which factor through the quotient

$$W'_F \to W_F \to \mathbb{Z}$$

(see Section 2.1 for the definitions of these groups and homomorphisms). It is admissible if its image in ${}^L G$ consists of semi-simple elements. Therefore the set on the left hand side of (7.1) is just the set of conjugacy classes of semi-simple elements of ${}^L G$. Thus, the above bijection may be reinterpreted as follows:

$$\boxed{\begin{array}{c}\text{semi-simple conjugacy}\\ \text{classes in } {}^L G\end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c}\text{irreducible unramified}\\ \text{representations of } G(F)\end{array}} \tag{7.2}$$

To construct this bijection, we look at the Hecke algebra $H(G(F), K_0)$. According to the Satake isomorphism [Sat], in the interpretation of Langlands [L], this algebra is commutative and isomorphic to the representation ring of the Langlands dual group ${}^L G$:

$$H(G(F), K_0) \simeq \operatorname{Rep} {}^L G. \tag{7.3}$$

We recall that $\operatorname{Rep} {}^L G$ consists of finite linear combinations $\sum_i a_i [V_i]$, where the V_i are finite-dimensional representations of ${}^L G$ (without loss of generality we may assume that they are irreducible) and $a_i \in \mathbb{C}$, with respect to the multiplication

$$[V] \cdot [W] = [V \otimes W].$$

Since $\operatorname{Rep} {}^L G$ is commutative, its irreducible modules are all one-dimensional. They correspond to characters $\operatorname{Rep} {}^L G \to \mathbb{C}$. We have a bijection

$$\boxed{\begin{array}{c}\text{semi-simple conjugacy} \\ \text{classes in } {}^L G\end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c}\text{characters} \\ \text{of } \operatorname{Rep} {}^L G\end{array}} \tag{7.4}$$

where the character ϕ_γ corresponding to the conjugacy class γ is given by the formula[12]

$$\phi_\gamma : [V] \mapsto \operatorname{Tr}(\gamma, V).$$

Now, if (R, π) is a representation of $G(F)$, then the space $R^{\pi(K_0)}$ of K_0-invariant vectors in V is a module over $H(G(F), K_0)$. It is easy to show that this sets up a one-to-one correspondence between equivalence classes of irreducible unramified representations of $G(F)$ and irreducible $H(G(F), K_0)$-modules. Combining this with the bijection (7.4) and the isomorphism (7.3), we obtain the sought-after bijections (7.1) and (7.2).

In particular, we find that, because the Hecke algebra $H(G(F), K_0)$ is commutative, the space $R^{\pi(K_0)}$ of K_0-invariants of an irreducible representation, which is an irreducible $H(G(F), K_0)$-module, is either zero or one-dimensional. If it is one-dimensional, then $H(G(F), K_0)$ acts on it by the character ϕ_γ for some γ:

$$H_V \star v = \operatorname{Tr}(\gamma, V) v, \qquad v \in R^{\pi(K_0)}, [V] \in \operatorname{Rep} {}^L G, \tag{7.5}$$

where H_V is the element of $H(G(F), K_0)$ corresponding to $[V]$ under the isomorphism (7.3) (see formula (6.2) for the definition of the convolution action).

We now discuss the categorical analogues of these statements.

[12] It is customary to multiply the right hand side of this formula, for irreducible representation V, by a scalar depending on q and the highest weight of V, but this is not essential for our discussion.

7.2 Unramified Categories $\widehat{\mathfrak{g}}_{\kappa_c}$-Modules

In the categorical setting, the role of an irreducible representation (R,π) of $G(F)$ is played by the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ for some $\chi \in \mathrm{Op}_{^LG}(D^\times)$. The analogue of an unramified representation is a category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ which contains non-zero $(\widehat{\mathfrak{g}}_{\kappa_c}, G[[t]])$ Harish-Chandra modules. This leads us to the following question: for what $\chi \in \mathrm{Op}_{^LG}(D^\times)$ does the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ contain non-zero $(\widehat{\mathfrak{g}}_{\kappa_c}, G[[t]])$ Harish-Chandra modules?

We saw in the previous section that (R,π) is unramified if and only if it corresponds to an unramified Langlands parameter, which is a homomorphism $W'_F \to {}^LG$ that factors through $W'_F \to \mathbb{Z}$. Recall that in the geometric setting the Langlands parameters are ${}^LG$-local systems on $D^\times$. The analogues of unramified homomorphisms $W'_F \to {}^LG$ are those local systems on $D^\times$ which extend to the disc D, in other words, have no singularity at the origin $0 \in D$. Note that there is a unique, up to isomorphism local system on D. Indeed, suppose that we are given a regular connection on a ${}^LG$-bundle $\mathcal{F}$ on D. Let us trivialize the fiber $\mathcal{F}_0$ of $\mathcal{F}$ at $0 \in D$. Then, because D is contractible, the connection identifies $\mathcal{F}$ with the trivial bundle on D. Under this identification the connection itself becomes trivial, i.e., represented by the operator $\nabla = \partial_t$.

Therefore all regular ${}^LG$-local systems (i.e., those which extend to D) correspond to a single point of the set $\mathrm{Loc}_{^LG}(D^\times)$, namely, the equivalence class of the trivial local system σ_0.[13] From the point of view of the realization of $\mathrm{Loc}_{^LG}(D^\times)$ as the quotient (3.2) this simply means that there is a unique ${}^LG((t))$ gauge equivalence class containing all regular connections of the form $\partial_t + A(t)$, where $A(t) \in {}^L\mathfrak{g}[[t]]$.

The gauge equivalence class of regular connections is the unique local Langlands parameter that we may view as unramified in the geometric setting. Therefore, by analogy with the unramified Langlands correspondence for $G(F)$, we expect that the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ contains non-zero $(\widehat{\mathfrak{g}}_{\kappa_c}, G[[t]])$ Harish-Chandra modules if and only if the ${}^LG$-oper $\chi \in \mathrm{Op}_{^LG}(D^\times)$ is ${}^LG((t))$ gauge equivalent to the trivial connection, or, in other words, χ belongs to the fiber $\alpha^{-1}(\sigma_0)$ over σ_0.

What does this fiber look like? Let P^+ be the set of dominant integral weights of G (equivalently, dominant integral coweights of ${}^LG$). In [FG2] we defined, for each $\lambda \in P^+$, the space $\mathrm{Op}^\lambda_{^LG}$ of ${}^LB[[t]]$-equivalence classes of operators of the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} t^{\langle \check{\alpha}_i, \lambda \rangle} \psi_i(t) f_i + \mathbf{v}(t), \tag{7.6}$$

[13] Note however that the trivial ${}^LG$-local system on D has a non-trivial group of automorphisms, namely, the group ${}^LG$ itself (it may be realized as the group of automorphisms of the fiber at $0 \in D$). Therefore if we think of $\mathrm{Loc}_{^LG}(D^\times)$ as a stack rather than as a set, then the trivial local system corresponds to a substack $\mathrm{pt}/{}^LG$.

where $\psi_i(t) \in \mathbb{C}[[t]], \psi_i(0) \neq 0$, $\mathbf{v}(t) \in {}^L\mathfrak{b}[[t]]$.

Lemma 2. *Suppose that the local system underlying an oper* $\chi \in \operatorname{Op}_{^LG}(D^\times)$ *is trivial. Then* χ *belongs to the disjoint union of the subsets* $\operatorname{Op}^\lambda_{^LG} \subset \operatorname{Op}_{^LG}(D^\times), \lambda \in P^+$.

Proof. It is clear from the definition that any oper in $\operatorname{Op}^\lambda_{^LG}$ is regular on the disc D and is therefore ${}^LG(\!(t)\!)$ gauge equivalent to the trivial connection.

Now suppose that we have an oper $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{^LB})$ such that the underlying ${}^LG$-local system is trivial. Then ∇ is ${}^LG(\!(t)\!)$ gauge equivalent to a regular connection, that is one of the form $\partial_t + A(t)$, where $A(t) \in {}^L\mathfrak{g}[[t]]$. We have the decomposition ${}^LG(\!(t)\!) = {}^LG[[t]]{}^LB(\!(t)\!)$. The gauge action of ${}^LG[[t]]$ clearly preserves the space of regular connections. Therefore if an oper connection ∇ is ${}^LG(\!(t)\!)$ gauge equivalent to a regular connection, then its ${}^LB(\!(t)\!)$ gauge class already must contain a regular connection. The oper condition then implies that this gauge class contains a connection operator of the form (7.6) for some dominant integral weight λ of ${}^LG$. Therefore $\chi \in \operatorname{Op}^\lambda_{^LG}$. $\square$

Thus, we see that the set of opers corresponding to the (unique) unramified Langlands parameter is the disjoint union $\bigsqcup_{\lambda \in P^+} \operatorname{Op}^\lambda_{^LG}$. We call such opers "unramified". The following result then confirms our expectation that the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ is "unramified", that is contains non-zero $G[[t]]$-equivariant objects, if and only if χ is unramified (see [FG3] for a proof).

Lemma 3. *The category* $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ *contains a non-zero* $(\widehat{\mathfrak{g}}_{\kappa_c}, G[[t]])$ *Harish-Chandra module if and only if*

$$\chi \in \bigsqcup_{\lambda \in P^+} \operatorname{Op}^\lambda_{^LG}. \tag{7.7}$$

The next question is to describe the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi^{G[[t]]}$ of $(\widehat{\mathfrak{g}}_{\kappa_c}, G[[t]])$ modules for $\chi \in \operatorname{Op}^\lambda_{^LG}$.

7.3 Categories of $G[[t]]$-Equivariant Modules

Let us recall from Section 7.1 that the space of K_0-invariant vectors in an unramified irreducible representation of $G(F)$ is always one-dimensional. We have proposed that the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi^{G[[t]]}$ should be viewed as a categorical analogue of this space. Therefore we expect it to be the simplest possible abelian category: the category of $\mathbb{C}$-vector spaces. Here we assume that χ belongs to the union of the spaces $\operatorname{Op}^\lambda_{^LG}$, where $\lambda \in P^+$, for otherwise the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi^{G[[t]]}$ would be trivial (zero object is the only object).

In this subsection we will prove, following [FG1] (see also [BD1]), that our expectation is in fact correct provided that $\lambda = 0$, in which case $\operatorname{Op}^0_{^LG} = \operatorname{Op}_{^LG}(D)$, and so

$$\chi \in \operatorname{Op}_{^LG}(D) \subset \operatorname{Op}_{^LG}(D^\times).$$

We will also conjecture that this is true for $\chi \in \mathrm{Op}^{\lambda}_{{}^L G}$ for all $\lambda \in P^+$.

Recall the vacuum module $\mathbb{V}_0 = V_{\kappa_c}(\mathfrak{g})$. According to [FF3, F3], we have

$$\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_0 \simeq \mathrm{Fun}\, \mathrm{Op}_{{}^L G}(D). \tag{7.8}$$

Let $\chi \in \mathrm{Op}_{{}^L G}(D) \subset \mathrm{Op}_{{}^L G}(D^\times)$. Then χ defines a character of the algebra $\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_0$. Let $\mathbb{V}_0(\chi)$ be the quotient of $\mathbb{V}_0$ by the kernel of this character. Then we have the following result.

Theorem 3. *Let $\chi \in \mathrm{Op}_{{}^L G}(D) \subset \mathrm{Op}_{{}^L G}(D^\times)$. The category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{G[[t]]}$ is equivalent to the category of vector spaces: its unique, up to isomorphism, irreducible object is $\mathbb{V}_0(\chi)$ and any other object is isomorphic to the direct sum of copies of $\mathbb{V}_0(\chi)$.*

This theorem provides the first piece of evidence for Conjecture 2: we see that the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{G[[t]]}$ are equivalent to each other for all $\chi \in \mathrm{Op}_{{}^L G}(D)$.

It is more convenient to consider, instead of an individual regular ${}^L G$-oper χ, the entire family $\mathrm{Op}^0_{{}^L G} = \mathrm{Op}_{{}^L G}(D)$ of regular opers on the disc D. Let $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$ be the full subcategory of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}$ whose objects are $\widehat{\mathfrak{g}}_{\kappa_c}$-modules on which the action of the center $Z(\widehat{\mathfrak{g}})$ factors through the homomorphism

$$Z(\widehat{\mathfrak{g}}) \simeq \mathrm{Fun}\, \mathrm{Op}_{{}^L G}(D^\times) \to \mathrm{Fun}\, \mathrm{Op}_{{}^L G}(D).$$

Note that the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$ is an example of a category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_V$ introduced in Section 5, in the case when $V = \mathrm{Op}_{{}^L G}(D)$.

Let $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]}$ be the corresponding $G[[t]]$-equivariant category. It is instructive to think of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$ and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]}$ as categories fibered over $\mathrm{Op}_{{}^L G}(D)$, with the fibers over $\chi \in \mathrm{Op}_{{}^L G}(D)$ being $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{G[[t]]}$, respectively.

We will now describe the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]}$. This description will in particular imply Theorem 3.

In order to simplify our formulas, in what follows we will use the following notation for $\mathrm{Fun}\, \mathrm{Op}_{{}^L G}(D)$:

$$\mathfrak{z} = \mathfrak{z}(\widehat{\mathfrak{g}}) = \mathrm{Fun}\, \mathrm{Op}_{{}^L G}(D).$$

Let $\mathfrak{z}$-mod be the category of modules over the commutative algebra $\mathfrak{z}$. Equivalently, this is the category of quasicoherent sheaves on the space $\mathrm{Op}_{{}^L G}(D)$.

By definition, any object of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]}$ is a $\mathfrak{z}$-module. Introduce the functors

$$\begin{aligned} \mathsf{F} &: \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]} \to \mathfrak{z}\text{-mod}, \qquad && M \mapsto \mathrm{Hom}_{\widehat{\mathfrak{g}}_{\kappa_c}}(\mathbb{V}_0, M), \\ \mathsf{G} &: \mathfrak{z}\text{-mod} \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]}, \qquad && \mathcal{F} \mapsto \mathbb{V}_0 \underset{\mathfrak{z}}{\otimes} \mathcal{F}. \end{aligned}$$

The following theorem has been proved in [FG1], Theorem 6.3 (important results in this direction were obtained earlier in [BD1]).

Theorem 4. *The functors* F *and* G *are mutually inverse equivalences of categories*

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]} \simeq \mathfrak{z}\text{-mod}\,. \tag{7.9}$$

This immediately implies Theorem 3. Indeed, for each $\chi \in \mathrm{Op}_{{}^L G}(D)$ the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi}^{G[[t]]}$ is the full subcategory of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}^{G[[t]]}$ which are annihilated, as $\mathfrak{z}$-modules, by the maximal ideal I_χ of χ. By Theorem 4, this category is equivalent to the category of $\mathfrak{z}$-modules annihilated by I_χ. But this is the category of $\mathfrak{z}$-modules supported (scheme-theoretically) at the point χ, which is equivalent to the category of vector spaces.

7.4 The Action of the Spherical Hecke Algebra

In Section 7.1 we discussed irreducible unramified representations of the group $G(F)$, where F is a local non-archimedian field. We have seen that such representations are parameterized by conjugacy classes of the Langlands dual group ${}^L G$. Given such a conjugacy class γ, we have an irreducible unramified representation (R_γ, π_γ), which contains a one-dimensional subspace $(R_\gamma)^{\pi_\gamma(K_0)}$ of K_0-invariant vectors. The spherical Hecke algebra $H(G(F), K_0)$, which is isomorphic to $\mathrm{Rep}\, {}^L G$ via the Satake isomorphism, acts on this space by a character ϕ_γ, see formula (7.5).

In the geometric setting, we have argued that for any $\chi \in \mathrm{Op}_{{}^L G}(D)$ the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, equipped with an action of the loop group $G(\!(t)\!)$, should be viewed as a categorification of (R_γ, π_γ). Furthermore, its subcategory $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi}^{G[[t]]}$ of $(\widehat{\mathfrak{g}}_{\kappa_c}, G[[t]])$ Harish-Chandra modules should be viewed as a categorification of the one-dimensional space $(R_\gamma)^{\pi_\gamma(K_0)}$. According to Theorem 3, the latter category is equivalent to the category of vector spaces, which is consistent with our expectations.

We now discuss the categorical analogue of the action of the spherical Hecke algebra.

As explained in Section 6.3, the categorical analogue of the spherical Hecke algebra is the category of $G[[t]]$-equivariant $\mathcal{D}$-modules on the **affine Grassmannian** $\mathrm{Gr} = G(\!(t)\!)/G[[t]]$. We refer the reader to [BD1, FG2] for the precise definition of Gr and this category. There is an important property that is satisfied in the unramified case: the convolution functors with these $\mathcal{D}$-modules are exact, which means that we do not need to consider the derived category; the abelian category of such $\mathcal{D}$-modules will do. Let us denote this abelian category by $\mathcal{H}(G(\!(t)\!), G[[t]])$.

According to the results of [MV], this category carries a natural structure of tensor category, which is equivalent to the tensor category $\mathcal{R}ep\, {}^L G$ of representations of ${}^L G$. This should be viewed as a categorical analogue of the

Satake isomorphism. Thus, for each object V of $\mathcal{R}ep\,{}^L G$ we have an object of $\mathcal{H}(G(\!(t)\!), G[[t]])$ which we denote by $\mathcal{H}_V$. What should be the analogue of the Hecke eigenvector property (7.5)?

As we explained in Section 6.3, the category $\mathcal{H}(G(\!(t)\!), G[[t]])$ naturally acts on the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{G[[t]]}$, and this action should be viewed as a categorical analogue of the action of $H(G(F), K_0)$ on $(R_\gamma)^{\pi_\gamma(K_0)}$.

Now, by Theorem 3, any object of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{G[[t]]}$ is a direct sum of copies of $\mathbb{V}_0(\chi)$. Therefore it is sufficient to describe the action of $\mathcal{H}(G(\!(t)\!), G[[t]])$ on $\mathbb{V}_0(\chi)$. This action is described by the following statement, which follows from [BD1]: there exists a family of isomorphisms

$$\alpha_V : \mathcal{H}_V \star \mathbb{V}_0(\chi) \xrightarrow{\sim} \underline{V} \otimes \mathbb{V}_0(\chi), \qquad V \in \mathcal{R}ep\,{}^L G, \tag{7.10}$$

where $\underline{V}$ is the vector space underlying the representation V. Moreover, these isomorphisms are compatible with the tensor product structure on $\mathcal{H}_V$ (given by the convolution) and on $\underline{V}$ (given by tensor product of vector spaces).

In view of Theorem 3, this is not surprising. Indeed, it follows from the definition that $\mathcal{H}_V \star \mathbb{V}_0(\chi)$ is again an object of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{G[[t]]}$. Therefore it must be isomorphic to $U_V \otimes_{\mathbb{C}} \mathbb{V}_0(\chi)$, where U_V is a vector space. But then we obtain a functor $\mathcal{H}(G(\!(t)\!), G[[t]]) \to \mathcal{V}ect, \mathcal{H}_V \mapsto U_V$. It follows from the construction that this is a tensor functor. Therefore the standard Tannakian formalism implies that U_V is isomorphic to $\underline{V}$.

The isomorphisms (7.10) should be viewed as the categorical analogues of the Hecke eigenvector conditions (7.5). The difference is that while in (7.5) the action of elements of the Hecke algebra on a K_0-invariant vector in R_γ amounts to multiplication by a scalar, the action of an object of the Hecke category $\mathcal{H}(G(\!(t)\!), G[[t]])$ on the $G[[t]]$-equivariant object $\mathbb{V}_0(\chi)$ of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ amounts to multiplication by a *vector space*, namely, the vector space underlying the corresponding representation of ${}^L G$. It is natural to call a module satisfying this property a **Hecke eigenmodule**. Thus, we obtain that $\mathbb{V}_0(\chi)$ is a Hecke eigenmodule. This is in agreement with our expectation that the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{G[[t]]}$ is a categorical version of the space of K_0-invariant vectors in R_γ.

One ingredient that is missing in the geometric case is the conjugacy class γ of ${}^L G$. We recall that in the classical Langlands correspondence this was the image of the Frobenius element of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, which does not have an analogue in the geometric setting where our ground field is $\mathbb{C}$, which is algebraically closed. So while unramified local systems in the classical case are parameterized by the conjugacy classes γ, there is only one, up to an isomorphism, unramified local system in the geometric case. However, this local system has a large group of automorphisms, namely, ${}^L G$ itself. One can argue that what replaces γ in the geometric setting is the action of this group ${}^L G$ by automorphisms of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, which we will discuss in the next two sections.

7.5 Categories of Representations and $\mathcal{D}$-Modules

When we discussed the procedure of categorification of representations in Section 3.5, we saw that there are two possible scenarios for constructing categories equipped with an action of the loop group $G((t))$. In the first one we consider categories of $\mathcal{D}$-modules on the ind-schemes $G((t))/K$, where K is a "compact" subgroup of $G((t))$, such as $G[[t]]$ or the Iwahori subgroup. In the second one we consider categories of representations $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$. So far we have focused exclusively on the second scenario, but it is instructive to also discuss categories of the first type.

In the toy model considered in Section 3.4 we discussed the category of $\mathfrak{g}$-modules with fixed central character and the category of $\mathcal{D}$-modules on the flag variety G/B. We have argued that both could be viewed as categorifications of the representation of the group $G(\mathbb{F}_q)$ on the space of functions on $(G/B)(\mathbb{F}_q)$. These categories are equivalent, according to the Beilinson-Bernstein theory, with the functor of global sections connecting the two. Could something like this be true in the case of affine Kac-Moody algebras as well?

The affine Grassmannian $\mathrm{Gr} = G((t))/G[[t]]$ may be viewed as the simplest possible analogue of the flag variety G/B for the loop group $G((t))$. Consider the category of $\mathcal{D}$-modules on $G((t))/G[[t]]$ (see [BD1, FG2] for the precise definition). We have a functor of global sections from this category to the category of $\mathfrak{g}((t))$-modules. In order to obtain $\widehat{\mathfrak{g}}_{\kappa_c}$-modules, we need to take instead the category $\mathcal{D}_{\kappa_c}\text{-mod}$ of $\mathcal{D}$-modules twisted by a line bundle $\mathcal{L}_{\kappa_c}$. This is the unique line bundle $\mathcal{L}_{\kappa_c}$ on Gr which carries an action of $\widehat{\mathfrak{g}}_{\kappa_c}$ (such that the central element $\mathbf{1}$ is mapped to the identity) lifting the natural action of $\mathfrak{g}((t))$ on Gr. Then for any object $\mathcal{M}$ of $\mathcal{D}_{\kappa_c}\text{-mod}$, the space of global sections $\Gamma(\mathrm{Gr}, \mathcal{M})$ is a $\widehat{\mathfrak{g}}_{\kappa_c}$-module. Moreover, it is known (see [BD1, FG1]) that $\Gamma(\mathrm{Gr}, \mathcal{M})$ is in fact an object of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$. Therefore we have a functor of global sections

$$\Gamma : \mathcal{D}_{\kappa_c}\text{-mod} \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}} .$$

We note that the categories $\mathcal{D}\text{-mod}$ and $\mathcal{D}_{\kappa_c}\text{-mod}$ are equivalent under the functor $\mathcal{M} \mapsto \mathcal{M} \otimes \mathcal{L}_{\kappa_c}$. But the corresponding global sections functors are very different.

However, unlike in the Beilinson-Bernstein scenario, the functor Γ cannot possibly be an equivalence of categories. There are two reasons for this. First of all, the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$ has a large center, namely, the algebra $\mathfrak{z} = \mathrm{Fun}\,\mathrm{Op}_{^L G}(D)$, while the center of the category $\mathcal{D}_{\kappa_c}\text{-mod}$ is trivial.[14] The second, and more serious, reason is that the category $\mathcal{D}_{\kappa_c}\text{-mod}$ carries an

[14] Recall that we are under the assumption that G is a connected simply-connected algebraic group, and in this case Gr has one connected component. In general, the center of the category $\mathcal{D}_{\kappa_c}\text{-mod}$ has a basis enumerated by the connected components of Gr and is isomorphic to the group algebra of the finite group $\pi_1(G)$.

additional symmetry, namely, an action of the tensor category $\mathcal{R}ep^L G$ of representations of the Langlands dual group ${}^L G$, and this action trivializes under the functor Γ as we explain presently.

Over $\mathrm{Op}_{^L G}(D)$ there exists a canonical principal ${}^L G$-bundle, which we will denote by $\mathcal{P}$. By definition, the fiber of $\mathcal{P}$ at $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{^L B}) \in \mathrm{Op}_{^L G}(D)$ is $\mathcal{F}_0$, the fiber at $0 \in D$ of the ${}^L G$-bundle $\mathcal{F}$ underlying χ. For an object $V \in \mathcal{R}ep\,{}^L G$ let us denote by $\mathcal{V}$ the associated vector bundle over $\mathrm{Op}_{^L G}(D)$, i.e.,

$$\mathcal{V} = \mathcal{P} \underset{^L G}{\times} V.$$

Next, consider the category $\mathcal{D}_{\kappa_c}\text{-mod}^{G[[t]]}$ of $G[[t]]$-equivariant $\mathcal{D}_{\kappa_c}$-modules on Gr. It is equivalent to the category

$$\mathcal{D}\text{-mod}^{G[[t]]} = \mathcal{H}(G((t)), G[[t]])$$

considered above. This is a tensor category, with respect to the convolution functor, which is equivalent to the category $\mathcal{R}ep\,{}^L G$. We will use the same notation $\mathcal{H}_V$ for the object of $\mathcal{D}_{\kappa_c}\text{-mod}^{G[[t]]}$ corresponding to $V \in \mathcal{R}ep\,{}^L G$. The category $\mathcal{D}_{\kappa_c}\text{-mod}^{G[[t]]}$ acts on $\mathcal{D}_{\kappa_c}\text{-mod}$ by convolution functors

$$\mathcal{M} \mapsto \mathcal{H}_V \star \mathcal{M}$$

which are exact. This amounts to a tensor action of the category $\mathcal{R}ep^L G$ on $\mathcal{D}_{\kappa_c}\text{-mod}$.

Now, A. Beilinson and V. Drinfeld have proved in [BD1] that there are functorial isomorphisms

$$\Gamma(\mathrm{Gr}, \mathcal{H}_V \star \mathcal{M}) \simeq \Gamma(\mathrm{Gr}, \mathcal{M}) \underset{\mathfrak{z}}{\otimes} \mathcal{V}, \qquad V \in \mathcal{R}ep\,{}^L G,$$

compatible with the tensor structure. Thus, we see that there are non-isomorphic objects of $\mathcal{D}_{\kappa_c}\text{-mod}$, which the functor Γ sends to isomorphic objects of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$. Therefore the category $\mathcal{D}_{\kappa_c}\text{-mod}$ and the functor Γ need to be modified in order to have a chance to obtain a category equivalent to $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$.

In [FG2] it was shown how to modify the category $\mathcal{D}_{\kappa_c}\text{-mod}$, by simultaneously "adding" to it $\mathfrak{z}$ as a center, and "dividing" it by the above $\mathcal{R}ep\,{}^L G$-action. As the result, we obtain a candidate for a category that can be equivalent to $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$. This is the category of **Hecke eigenmodules** on Gr, denoted by $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$.

By definition, an object of $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$ is an object of $\mathcal{D}_{\kappa_c}\text{-mod}$, equipped with an action of the algebra $\mathfrak{z}$ by endomorphisms and a system of isomorphisms

$$\alpha_V : \mathcal{H}_V \star \mathcal{M} \xrightarrow{\sim} \mathcal{V} \underset{\mathfrak{z}}{\otimes} \mathcal{M}, \qquad V \in \mathcal{R}ep\,{}^L G,$$

compatible with the tensor structure.

The above functor Γ naturally gives rise to a functor

$$\Gamma^{\text{Hecke}} : \mathcal{D}_{\kappa_c}^{\text{Hecke}}\text{-mod}_{\text{reg}} \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{reg}} . \tag{7.11}$$

This is in fact a general property. Suppose for simplicity that we have an abelian category $\mathcal{C}$ which is acted upon by the tensor category $\mathcal{R}ep\, H$, where H is an algebraic group; we denote this functor by $\mathcal{M} \mapsto \mathcal{M} \star V, V \in \mathcal{R}ep\, H$. Let $\mathcal{C}^{\text{Hecke}}$ be the category whose objects are collections $(\mathcal{M}, \{\alpha_V\}_{V \in \mathcal{R}ep\, H})$, where $\mathcal{M} \in \mathcal{C}$ and $\{\alpha_V\}$ is a compatible system of isomorphisms

$$\alpha_V : \mathcal{M} \star V \xrightarrow{\sim} \underline{V} \underset{\mathbb{C}}{\otimes} \mathcal{M}, \qquad V \in \mathcal{R}ep\, H,$$

where $\underline{V}$ is the vector space underlying V. One may think of $\mathcal{C}^{\text{Hecke}}$ as the "de-equivariantized" category $\mathcal{C}$ with respect to the action of H. It carries a natural action of the group H: for $h \in H$, we have

$$h \cdot (\mathcal{M}, \{\alpha_V\}_{V \in \mathcal{R}ep\, H}) = (\mathcal{M}, \{(h \otimes \text{id}_{\mathcal{M}}) \circ \alpha_V\}_{V \in \mathcal{R}ep\, H}).$$

In other words, $\mathcal{M}$ remains unchanged, but the isomorphisms α_V get composed with h.

The category $\mathcal{C}$ may in turn be reconstructed as the category of H-equivariant objects of $\mathcal{C}^{\text{Hecke}}$ with respect to this action, see [Ga3].

Suppose that we have a functor $\mathsf{G} : \mathcal{C} \to \mathcal{C}'$, such that we have functorial isomorphisms

$$\mathsf{G}(\mathcal{M} \star V) \simeq \mathsf{G}(\mathcal{M}) \underset{\mathbb{C}}{\otimes} \underline{V}, \qquad V \in \mathcal{R}ep\, H, \tag{7.12}$$

compatible with the tensor structure. Then, according to [AG], there exists a functor $\mathsf{G}^{\text{Hecke}} : \mathcal{C}^{\text{Hecke}} \to \mathcal{C}'$ such that $\mathsf{G} \simeq \mathsf{G}^{\text{Hecke}} \circ \text{Ind}$, where the functor $\text{Ind} : \mathcal{C} \to \mathcal{C}^{\text{Hecke}}$ sends $\mathcal{M}$ to $\mathcal{M} \star \mathcal{O}_H$, where $\mathcal{O}_H$ is the regular representation of H. The functor $\mathsf{G}^{\text{Hecke}}$ may be explicitly described as follows: the isomorphisms α_V and (7.12) give rise to an action of the algebra $\mathcal{O}_H$ on $\mathsf{G}(\mathcal{M})$, and $\mathsf{G}^{\text{Hecke}}(\mathcal{M})$ is obtained by taking the fiber of $\mathsf{G}(\mathcal{M})$ at $1 \in H$.

We take $\mathcal{C} = \mathcal{D}_{\kappa_c}\text{-mod}$, $\mathcal{C}' = \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{reg}}$, and $\mathsf{G} = \Gamma$. The only difference is that now we are working over the base $\text{Op}_{^LG}(D)$, which we have to take into account. Thus, we obtain a functor (7.11) (see [FG2, FG4] for more details). Moreover, the left action of the group $G(\!(t)\!)$ on Gr gives rise to its action on the category $\mathcal{D}_{\kappa_c}^{\text{Hecke}}\text{-mod}_{\text{reg}}$, and the functor Γ^{Hecke} intertwines this action with the action of $G(\!(t)\!)$ on $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{reg}}$.

The following was conjectured in [FG2]:

Conjecture 3. *The functor Γ^{Hecke} in formula* (7.11) *defines an equivalence of the categories $\mathcal{D}_{\kappa_c}^{\text{Hecke}}\text{-mod}_{\text{reg}}$ and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{reg}}$.*

It was proved in [FG2] that the functor Γ^{Hecke}, when extended to the derived categories, is fully faithful. Furthermore, it was proved in [FG4] that it sets up an equivalence of the corresponding I^0-equivariant categories, where $I^0 = [I, I]$ is the radical of the Iwahori subgroup.

Let us specialize Conjecture 3 to a point $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{^L B}) \in \mathrm{Op}_{^L G}(D)$. Then on the right hand side we consider the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$, and on the left hand side we consider the category $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-mod$_\chi$. Its objects consist of a $\mathcal{D}_{\kappa_c}$-module $\mathcal{M}$ and a collection of isomorphisms

$$\alpha_V : \mathcal{H}_V \star \mathcal{M} \xrightarrow{\sim} V_{\mathcal{F}_0} \otimes \mathcal{M}, \qquad V \in \mathcal{R}ep\, {}^L G. \tag{7.13}$$

Here $V_{\mathcal{F}_0}$ is the twist of the representation V by the ${}^L G$-torsor $\mathcal{F}_0$. These isomorphisms have to be compatible with the tensor structure on the category $\mathcal{H}(G(\!(t)\!), G[[t]])$.

Conjecture 3 implies that there is a canonical equivalence of categories

$$\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_\chi \simeq \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi . \tag{7.14}$$

It is this conjectural equivalence that should be viewed as an analogue of the Beilinson-Bernstein equivalence.

From this point of view, one can think of each of the categories $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-mod$_\chi$ as the second incarnation of the sought-after Langlands category $\mathcal{C}_{\sigma_0}$ corresponding to the trivial ${}^L G$-local system.

Now we give another explanation why it is natural to view the category $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-mod$_\chi$ as a categorification of an unramified representation of the group $G(F)$. First of all, observe that these categories are all equivalent to each other and to the category $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-mod, whose objects are $\mathcal{D}_{\kappa_c}$-modules $\mathcal{M}$ together with a collection of isomorphisms

$$\alpha_V : \mathcal{H}_V \star \mathcal{M} \xrightarrow{\sim} \underline{V} \otimes \mathcal{M}, \qquad V \in \mathcal{R}ep\, {}^L G. \tag{7.15}$$

Comparing formulas (7.13) and (7.15), we see that there is an equivalence

$$\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_\chi \simeq \mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod},$$

for each choice of trivialization of the ${}^L G$-torsor $\mathcal{F}_0$ (the fiber at $0 \in D$ of the principal ${}^L G$-bundle $\mathcal{F}$ on D underlying the oper χ).

Now recall from Section 7.1 that to each semi-simple conjugacy class γ in ${}^L G$ corresponds an irreducible unramified representation (R_γ, π_γ) of $G(F)$ via the Satake correspondence (7.2). It is known that there is a non-degenerate pairing

$$\langle , \rangle : R_\gamma \times R_{\gamma^{-1}} \to \mathbb{C},$$

in other words, $R_{\gamma^{-1}}$ is the representation of $G(F)$ which is contragredient to R_γ (it may be realized in the space of smooth vectors in the dual space to R_γ).

Let $v \in R_{\gamma^{-1}}$ be a non-zero vector such that $K_0 v = v$ (this vector is unique up to a scalar). It then satisfies the Hecke eigenvector property (7.5) (in which we need to replace γ by γ^{-1}). This allows us to embed R_γ into the space of locally constant right K_0-invariant functions on $G(F)$ (equivalently, functions on $G(F)/K_0$), by using matrix coefficients, as follows:

$$u \in R_\gamma \mapsto f_u, \qquad f_u(g) = \langle u, gv \rangle.$$

The Hecke eigenvector property (7.5) implies that the functions f_u are right K_0-invariant and satisfy the condition

$$f \star H_V = \mathrm{Tr}(\gamma^{-1}, V) f, \tag{7.16}$$

where $\star$ denotes the convolution product (6.1). Let $C(G(F)/K_0)_\gamma$ be the space of locally constant functions on $G(F)/K_0$ satisfying (7.16). It carries a representation of $G(F)$ induced by its left action on $G(F)/K_0$. We have constructed an injective map $R_\gamma \to C(G(R)/G(R))_\gamma$, and one can show that for generic γ it is an isomorphism.

Thus, we obtain a realization of an irreducible unramified representation of $G(F)$ in the space of functions on the quotient $G(F)/K_0$ satisfying the Hecke eigenfunction condition (7.16). The Hecke eigenmodule condition (7.15) may be viewed as a categorical analogue of (7.16). Therefore the category $\mathcal{D}_{\kappa_c}^{\mathrm{Hecke}}$-mod of twisted $\mathcal{D}$-modules on $\mathrm{Gr} = G(\!(t)\!)/K_0$ satisfying the Hecke eigenmodule condition (7.15), equipped with a $G(\!(t)\!)$-action appears to be a natural categorification of the irreducible unramified representations of $G(F)$.

7.6 Equivalences Between Categories of Modules

All opers in $\mathrm{Op}_{^LG}(D)$ correspond to one and the same ^LG-local system, namely, the trivial local system. Therefore, according to Conjecture 2, we expect that the categories $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ are equivalent to each other. More precisely, for each isomorphism between the underlying local systems of any two opers in $\mathrm{Op}_{^LG}(D)$ we wish to have an equivalence of the corresponding categories, and these equivalences should be compatible with respect to the operation of composition of these isomorphisms.

Let us spell this out in detail. Let $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{^LB})$ and $\chi' = (\mathcal{F}', \nabla', \mathcal{F}'_{^LB})$ be two opers in $\mathrm{Op}_{^LG}(D)$. Then an isomorphism between the underlying local systems $(\mathcal{F}, \nabla) \xrightarrow{\sim} (\mathcal{F}', \nabla')$ is the same as an isomorphism $\mathcal{F}_0 \xrightarrow{\sim} \mathcal{F}'_0$ between the ^LG-torsors $\mathcal{F}_0$ and $\mathcal{F}'_0$, which are the fibers of the ^LG-bundles $\mathcal{F}$ and $\mathcal{F}'$, respectively, at $0 \in D$. Let us denote this set of isomorphisms by $\mathrm{Isom}_{\chi,\chi'}$. Then we have

$$\mathrm{Isom}_{\chi,\chi'} = \mathcal{F}_0 \underset{^LG}{\times} {}^LG \underset{^LG}{\times} \mathcal{F}'_0,$$

where we twist ^LG by $\mathcal{F}_0$ with respect to the left action and by $\mathcal{F}'_0$ with respect to the right action. In particular,

$$\mathrm{Isom}_{\chi,\chi} = {}^LG_{\mathcal{F}_0} = \mathcal{F}_0 \underset{^LG}{\times} \mathrm{Ad}\, {}^LG$$

is just the group of automorphisms of $\mathcal{F}_0$.

It is instructive to combine the sets $\mathrm{Isom}_{\chi,\chi'}$ into a groupoid Isom over $\mathrm{Op}_{^LG}(D)$. Thus, by definition Isom consists of triples (χ, χ', ϕ), where $\chi, \chi' \in \mathrm{Op}_{^LG}(D)$ and $\phi \in \mathrm{Isom}_{\chi,\chi}$ is an isomorphism of the underlying local systems. The two morphisms $\mathrm{Isom} \to \mathrm{Op}_{^LG}(D)$ correspond to sending such a triple to χ and χ'. The identity morphism $\mathrm{Op}_{^LG}(D) \to \mathrm{Isom}$ sends χ to $(\chi, \chi, \mathrm{Id})$, and the composition morphism

$$\mathrm{Isom} \underset{\mathrm{Op}_{^LG}(D)}{\times} \mathrm{Isom} \to \mathrm{Isom}$$

corresponds to composing two isomorphisms.

Conjecture 2 has the following more precise formulation for regular opers:

Conjecture 4. *For each* $\phi \in \mathrm{Isom}_{\chi,\chi'}$ *there exists an equivalence*

$$E_\phi : \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi'},$$

which intertwines the actions of $G((t))$ *on the two categories, such that* $E_{\mathrm{Id}} = \mathrm{Id}$ *and there exist isomorphisms* $\beta_{\phi,\phi'} : E_{\phi\circ\phi'} \simeq E_\phi \circ E_{\phi'}$ *satisfying*

$$\beta_{\phi\circ\phi',\phi''}\beta_{\phi,\phi'} = \beta_{\phi,\phi'\circ\phi''}\beta_{\phi',\phi''}$$

for all isomorphisms ϕ, ϕ', ϕ''*, whenever they may be composed in the appropriate order.*

In other words, the groupoid Isom *over* $\mathrm{Op}_{^LG}(D)$ *acts on the category* $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$ *fibered over* $\mathrm{Op}_{^LG}(D)$*, preserving the action of* $G((t))$ *along the fibers.*

In particular, this conjecture implies that the group ${}^LG_{\mathcal{F}_0}$ acts on the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ for any $\chi \in \mathrm{Op}_{^LG}(D)$.

Now we observe that Conjecture 3 implies Conjecture 4. Indeed, by Conjecture 3, there is a canonical equivalence of categories (7.14),

$$\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_\chi \simeq \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi .$$

It follows immediately from the definition of the category $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_\chi$ (namely, formula (7.13)) that for each isomorphism $\phi \in \mathrm{Isom}_{\chi,\chi'}$, i.e., an isomorphism of the ${}^LG$-torsors $\mathcal{F}_0$ and $\mathcal{F}'_0$ underlying the opers χ and χ', there is a canonical equivalence

$$\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_\chi \simeq \mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_{\chi'} .$$

Therefore we obtain the sought-after equivalence $E_\phi : \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi'}$. Furthermore, it is clear that these equivalences satisfy the conditions of Conjecture 4. In particular, they intertwine the actions of $G((t))$, which affects the $\mathcal{D}$-module $\mathcal{M}$ underlying an object of $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}\text{-mod}_\chi$, but does not affect the isomorphisms α_V.

Equivalently, we can express this by saying that the groupoid Isom naturally acts on the category $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-$\mathrm{mod}_{\mathrm{reg}}$. By Conjecture 3, this gives rise to an action of Isom on $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_{\mathrm{reg}}$.

In particular, we construct an action of the group $({}^L G)_{\mathcal{F}_0}$, the twist of ${}^L G$ by the ${}^L G$-torsor $\mathcal{F}_0$ underlying a particular oper χ, on the category $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-$\mathrm{mod}_\chi$. Indeed, each element $g \in ({}^L G)_{\mathcal{F}_0}$ acts on the $\mathcal{F}_0$-twist $V_{\mathcal{F}_0}$ of any finite-dimensional representation V of ${}^L G$. Given an object $(\mathcal{M}, (\alpha_V))$ of $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-$\mathrm{mod}_{\chi'}$, we construct a new object, namely, $(\mathcal{M}, ((g \otimes \mathrm{Id}_{\mathcal{M}}) \circ \alpha_V))$. Thus, we do not change the $\mathcal{D}$-module $\mathcal{M}$, but we change the isomorphisms α_V appearing in the Hecke eigenmodule condition (7.13) by composing them with the action of g on $V_{\mathcal{F}_0}$. According to Conjecture 3, the category $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-$\mathrm{mod}_\chi$ is equivalent to $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi$. Therefore this gives rise to an action of the group $({}^L G)_{\mathcal{F}_0}$ on $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi$. But this action is much more difficult to describe in terms of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules.

7.7 Generalization to other Dominant Integral Weights

We have extensively studied above the categories $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi$ and $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi^{G[[t]]}$ associated to regular opers $\chi \in \mathrm{Op}_{{}^L G}(D)$. However, according to Lemma 2, the (set-theoretic) fiber of the map $\alpha : \mathrm{Op}_{{}^L G}(D^\times) \to \mathrm{Loc}_{{}^L G}(D^\times)$ over the trivial local system σ_0 is the disjoint union of the subsets $\mathrm{Op}^\lambda_{{}^L G}, \lambda \in P^+$. Here we discuss briefly the categories $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi$ and $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi^{G[[t]]}$ for $\chi \in \mathrm{Op}^\lambda_{{}^L G}$, where $\lambda \neq 0$.

Consider the Weyl module $\mathbb{V}_\lambda$ with highest weight λ,

$$\mathbb{V}_\lambda = U(\widehat{\mathfrak{g}}_{\kappa_c}) \underset{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1})}{\otimes} V_\lambda.$$

According to [FG6], we have

$$\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_\lambda \simeq \mathrm{Fun}\, \mathrm{Op}^\lambda_{{}^L G}\,. \tag{7.17}$$

Let $\chi \in \mathrm{Op}^\lambda_{{}^L G} \subset \mathrm{Op}_{{}^L G}(D^\times)$. Then χ defines a character of the algebra $\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_\lambda$. Let $\mathbb{V}_\lambda(\chi)$ be the quotient of $\mathbb{V}_\lambda$ by the kernel of this character. The following conjecture of [FG6] is an analogue of Theorem 3:

Conjecture 5. *Let $\chi \in \mathrm{Op}^\lambda_{{}^L G} \subset \mathrm{Op}_{{}^L G}(D^\times)$. Then the category $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi^{G[[t]]}$ is equivalent to the category of vector spaces: its unique, up to isomorphism, irreducible object is $\mathbb{V}_\lambda(\chi)$ and any other object is isomorphic to the direct sum of copies of $\mathbb{V}_\lambda(\chi)$.*

Note that this is consistent with Conjecture 2, which tells us that the categories $\widehat{\mathfrak{g}}_{\kappa_c}$-$\mathrm{mod}_\chi^{G[[t]]}$ should be equivalent to each other for all opers which are gauge equivalent to the trivial local system on D.

8 Local Langlands Correspondence: Tamely Ramified Case

In the previous section we have considered categorical analogues of the irreducible unramified representations of a reductive group $G(F)$ over a local non-archimedian field F. We recall that these are the representations containing non-zero vectors fixed by the maximal compact subgroup $K_0 \subset G(F)$. The corresponding Langlands parameters are unramified admissible homomorphisms from the Weil-Deligne group W'_F to ${}^L G$, i.e., those which factor through the quotient

$$W'_F \to W_F \to \mathbb{Z},$$

and whose image in ${}^L G$ is semi-simple. Such homomorphisms are parameterized by semi-simple conjugacy classes in ${}^L G$.

We have seen that the categorical analogues of unramified representations of $G(F)$ are the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ (equipped with an action of the loop group $G(\!(t)\!)$), where χ is a ${}^L G$-oper on $D^\times$ whose underlying ${}^L G$-local system is trivial. These categories can be called unramified in the sense that they contain non-zero $G[[t]]$-equivariant objects. The corresponding Langlands parameter is the trivial ${}^L G$-local system σ_0 on $D^\times$, which should be viewed as an analogue of an unramified homomorphism $W'_F \to {}^L G$. However, the local system σ_0 is realized by many different opers, and this introduces an additional complication into our picture: at the end of the day we need to show that the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, where χ is of the above type, are equivalent to each other. In particular, Conjecture 4, which describes what we expect to happen when $\chi \in \mathrm{Op}_{{}^L G}(D)$.

The next natural step is to consider categorical analogues of representations of $G(F)$ that contain vectors invariant under the Iwahori subgroup $I \subset G[[t]]$, the preimage of a fixed Borel subgroup $B \subset G$ under the evaluation homomorphism $G[[t]] \to G$. We begin this section by recalling a classification of these representations, due to D. Kazhdan and G. Lusztig [KL] and V. Ginzburg [CG]. We then discuss the categorical analogues of these representations following [FG2]–[FG5] and the intricate interplay between the classical and the geometric pictures.

8.1 Tamely Ramified Representations

The Langlands parameters corresponding to irreducible representations of $G(F)$ with I-invariant vectors are **tamely ramified** homomorphisms $W'_F \to {}^L G$. Recall from Section 2.1 that $W'_F = W_F \ltimes \mathbb{C}$. A homomorphism $W'_F \to {}^L G$ is called tamely ramified if it factors through the quotient

$$W'_F \to \mathbb{Z} \ltimes \mathbb{C}.$$

According to the relation (2.1), the group $\mathbb{Z} \ltimes \mathbb{C}$ is generated by two elements $F = 1 \in \mathbb{Z}$ (Frobenius) and $M = 1 \in \mathbb{C}$ (monodromy) satisfying the relation

$$FMF^{-1} = qM. \tag{8.1}$$

Under an admissible tamely ramified homomorphism the generator F goes to a semi-simple element $\gamma \in {}^LG$ and the generator M goes to a unipotent element $N \in {}^LG$. According to formula (8.1), they have to satisfy the relation

$$\gamma N \gamma^{-1} = N^q. \tag{8.2}$$

Alternatively, we may write $N = \exp(u)$, where u is a nilpotent element of ${}^L\mathfrak{g}$. Then this relation becomes

$$\gamma u \gamma^{-1} = qu.$$

Thus, we have the following bijection between the sets of equivalence classes

(8.3)

$$\boxed{\begin{array}{c}\text{tamely ramified admissible}\\ \text{homomorphisms } W'_F \to {}^LG\end{array}} \iff \boxed{\begin{array}{c}\text{pairs } \gamma \in {}^LG, \text{ semi-simple,}\\ u \in {}^L\mathfrak{g}, \text{ nilpotent}, \gamma u \gamma^{-1} = qu\end{array}}$$

In both cases equivalence relation amounts to conjugation by an element of ${}^LG$.

Now to each Langlands parameter of this type we wish to attach an irreducible representation of $G(F)$ which contains non-zero I-invariant vectors. It turns out that if $G = GL_n$ there is indeed a bijection, proved in [BZ], between the sets of equivalence classes of the following objects:

(8.4)

$$\boxed{\begin{array}{c}\text{tamely ramified admissible}\\ \text{homomorphisms } W'_F \to GL_n\end{array}} \iff \boxed{\begin{array}{c}\text{irreducible representations}\\ (R,\pi) \text{ of } GL_n(F), R^{\pi(I)} \neq 0\end{array}}$$

However, such a bijection is no longer true for other reductive groups: two new phenomena appear, which we discuss presently.

The first one is the appearance of L-**packets**. One no longer expects to be able to assign to a particular admissible homomorphism $W'_F \to {}^LG$ a single irreducible smooth representations of $G(F)$. Instead, a finite collection of such representations (more precisely, a collection of equivalence classes of representations) is assigned, called an L-packet. In order to distinguish representations in a given L-packet, one needs to introduce an additional parameter. We will see how this is done in the case at hand shortly. However, and this is the second subtlety alluded to above, it turns out that not all irreducible representations of $G(F)$ within the L-packet associated to a given tamely ramified homomorphism $W'_F \to {}^LG$ contain non-zero I-invariant vectors. Fortunately, there is a certain property of the extra parameter used to distinguish representations inside the L-packet that tells us whether the corresponding representation of $G(F)$ has I-invariant vectors.

In the case of tamely ramified homomorphisms $W'_F \to {}^LG$ this extra parameter is an irreducible representation ρ of the finite group $C(\gamma, u)$ of components of the simultaneous centralizer of γ and u in ${}^LG$, on which the center of ${}^L G$ acts trivially (see [Lu1]). In the case of $G = GL_n$ these centralizers are always connected, and so this parameter never appears. But for other reductive groups G this group of components is often non-trivial. The simplest example is when ${}^L G = G_2$ and u is a subprincipal nilpotent element of the Lie algebra ${}^L\mathfrak{g}$.[15] In this case for some γ satisfying $\gamma u \gamma^{-1} = qu$ the group of components $C(\gamma, u)$ is the symmetric group S_3, which has three irreducible representations (up to equivalence). Each of them corresponds to a particular member of the L-packet associated with the tamely ramified homomorphism $W'_F \to {}^L G$ defined by (γ, u). Thus, the L-packet consists of three (equivalence classes of) irreducible smooth representations of $G(F)$. However, not all of them contain non-zero I-invariant vectors.

The representations ρ of the finite group $C(\gamma, u)$ which correspond to representations of $G(F)$ with I-invariant vectors are distinguished by the following property. Consider the **Springer fiber** Sp_u. We recall that

$$\mathrm{Sp}_u = \{\mathfrak{b}' \in {}^L G/{}^L B \mid u \in \mathfrak{b}'\}. \tag{8.5}$$

The group $C(\gamma, u)$ acts on the homology of the variety Sp_u^γ of γ-fixed points of Sp_u. A representation ρ of $C(\gamma, u)$ corresponds to a representation of $G(F)$ with non-zero I-invariant vectors if and only if ρ occurs in the homology of Sp_u^γ, $H_\bullet(\mathrm{Sp}_u^\gamma)$.

In the case of G_2 the Springer fiber Sp_u of the subprincipal element u is a union of four projective lines connected with each other as in the Dynkin diagram of D_4. For some γ the set Sp_u^γ is the union of a projective line (corresponding to the central vertex in the Dynkin diagram of D_4) and three points (each in one of the remaining three projective lines). The corresponding group $C(\gamma, u) = S_3$ on Sp_u^γ acts trivially on the projective line and by permutation of the three points. Therefore the trivial and the two-dimensional representations of S_3 occur in $H_\bullet(\mathrm{Sp}_u^\gamma)$, but the sign representation does not. The irreducible representations of $G(F)$ corresponding to the first two contain non-zero I-invariant vectors, whereas the one corresponding to the sign representation of S_3 does not.

The ultimate form of the local Langlands correspondence for representations of $G(F)$ with I-invariant vectors is then as follows (here we assume, as in [KL, CG]), that the group G is split and has connected center):

(8.6)

$$\boxed{\begin{array}{c}\text{triples } (\gamma, u, \rho), \gamma u \gamma^{-1} = qu, \\ \rho \in \mathcal{R}ep\, C(\gamma, u) \text{ occurs in } H_\bullet(\mathrm{Sp}_u^\gamma, \mathbb{C})\end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c}\text{irreducible representations} \\ (R, \pi) \text{ of } G(F), R^{\pi(I)} \neq 0\end{array}}$$

Again, this should be understood as a bijection between two sets of equivalence classes of the objects listed. This bijection is due to [KL] (see also [CG]). It was conjectured by Deligne and Langlands, with a subsequent modification (addition of ρ) made by Lusztig.

[15] The term "subprincipal" means that the adjoint orbit of this element has codimension 2 in the nilpotent cone.

How to set up this bijection? The idea is to replace irreducible representations of $G(F)$ appearing on the right hand side of (8.6) with irreducible modules over the corresponding Hecke algebra $H(G(F), I)$. Recall from Section 6.1 that this is the algebra of compactly supported I bi-invariant functions on $G(F)$, with respect to convolution. It naturally acts on the space of I-invariant vectors of any smooth representation of $G(F)$ (see formula (6.2)). Thus, we obtain a functor from the category of smooth representations of $G(F)$ to the category of $H(G(F), I)$. According to a theorem of A. Borel [B1], it induces a bijection between the set of equivalence classes of irreducible representations of $G(F)$ with non-zero I-invariant vectors and the set of equivalence classes of irreducible $H(G(F), I)$-modules.

The algebra $H(G(F), I)$ is known as the **affine Hecke algebra** and has the standard description in terms of generators and relations. However, for our purposes we need another description, due to [KL, CG], which identifies it with the equivariant K-theory of the **Steinberg variety**

$$\mathrm{St} = \widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \widetilde{\mathcal{N}},$$

where $\mathcal{N} \subset {}^L\mathfrak{g}$ is the nilpotent cone and $\widetilde{\mathcal{N}}$ is the **Springer resolution**

$$\widetilde{\mathcal{N}} = \{x \in \mathcal{N}, \mathfrak{b}' \in {}^LG/{}^LB \mid x \in \mathfrak{b}'\}.$$

Thus, a point of St is a triple consisting of a nilpotent element of ${}^L\mathfrak{g}$ and two Borel subalgebras containing it. The group ${}^LG \times \mathbb{C}^\times$ naturally acts on St, with ${}^LG$ conjugating members of the triple and $\mathbb{C}^\times$ acting by multiplication on the nilpotent elements,

$$a \cdot (x, \mathfrak{b}', \mathfrak{b}'') = (a^{-1}x, \mathfrak{b}', \mathfrak{b}''). \tag{8.7}$$

According to a theorem of [KL, CG], there is an isomorphism

$$H(G(F), I) \simeq K^{{}^LG \times \mathbb{C}^\times}(\mathrm{St}). \tag{8.8}$$

The right hand side is the ${}^LG \times \mathbb{C}^\times$-equivariant K-theory of St. It is an algebra with respect to a natural operation of convolution (see [CG] for details). It is also a free module over its center, isomorphic to

$$K^{{}^LG \times \mathbb{C}^\times}(\mathrm{pt}) = \mathrm{Rep}\, {}^LG \otimes \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}].$$

Under the isomorphism (8.8) the element $\mathbf{q}$ goes to the standard parameter $\mathbf{q}$ of the affine Hecke algebra $H(G(F), I)$ (here we consider $H(G(F), I)$ as a $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$-module).

Now, the algebra $K^{{}^LG \times \mathbb{C}^\times}(\mathrm{St})$, and hence the algebra $H(G(F), I)$, has a natural family of modules which are parameterized precisely by the conjugacy classes of pairs (γ, u) as above. On these modules $H(G(F), I)$ acts via a central character corresponding to a point in $\mathrm{Spec}\, \mathrm{Rep}\, {}^LG \underset{\mathbb{C}}{\otimes} \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$, which is just a pair (γ, q), where γ is a semi-simple conjugacy class in ${}^L G$ and $q \in \mathbb{C}^\times$. In our situation q is the cardinality of the residue field of F (hence a power of a prime), but in what follows we will allow a larger range of possible values of q: all non-zero complex numbers except for the roots of unity. Consider the quotient of $H(G(F), I)$ by the central character defined by (γ, u). This is just the algebra $K^{{}^L G \times \mathbb{C}^\times}(\mathrm{St})$, specialized at (γ, q). We denote it by $K^{{}^L G \times \mathbb{C}^\times}(\mathrm{St})_{(\gamma,q)}$.

Now for a nilpotent element $u \in \mathcal{N}$ consider the Springer fiber Sp_u. The condition that $\gamma u \gamma^{-1} = qu$ means that u, and hence Sp_u, is stabilized by the action of $(\gamma, q) \in {}^L G \times \mathbb{C}^\times$ (see formula (8.7)). Let A be the smallest algebraic subgroup of ${}^L G \times \mathbb{C}^\times$ containing (γ, q). The algebra $K^{{}^L G \times \mathbb{C}^\times}(\mathrm{St})_{(\gamma,q)}$ naturally acts on the equivariant K-theory $K^A(\mathrm{Sp}_u)$ specialized at (γ, q),

$$K^A(\mathrm{Sp}_u)_{(\gamma,q)} = K^A(\mathrm{Sp}_u) \underset{\mathrm{Rep}\, A}{\otimes} \mathbb{C}_{(\gamma,q)}.$$

It is known that $K^A(\mathrm{Sp}_u)_{(\gamma,q)}$ is isomorphic to the homology $H_\bullet(\mathrm{Sp}_u^\gamma)$ of the γ-fixed subset of Sp_u (see [KL, CG]). Thus, we obtain that $K^A(\mathrm{Sp}_u)_{(\gamma,q)}$ is a module over $H(G(F), I)$.

Unfortunately, these $H(G(F), I)$-modules are not irreducible in general, and one needs to work harder to describe the irreducible modules over $H(G(F), I)$. For $G = GL_n$ one can show that each of these modules has a unique irreducible quotient, and this way one recovers the bijection (8.4). But for a general groups G the finite groups $C(\gamma, u)$ come into play. Namely, the group $C(\gamma, u)$ acts on $K^A(\mathrm{Sp}_u)_{(\gamma,q)}$, and this action commutes with the action of $K^{{}^L G \times \mathbb{C}^\times}(\mathrm{St})_{(\gamma,q)}$. Therefore we have a decomposition

$$K^A(\mathrm{Sp}_u)_{(\gamma,q)} = \bigoplus_{\rho \in \mathrm{Irrep}\, C(\gamma,u)} \rho \otimes K^A(\mathrm{Sp}_u)_{(\gamma,q,\rho)},$$

of $K^A(\mathrm{Sp}_u)_{(\gamma,q)}$ as a representation of $C(\gamma, u) \times H(G(F), I)$. One shows (see [KL, CG] for details) that each $H(G(F), I)$-module $K^A(\mathrm{Sp}_u)_{(\gamma,q,\rho)}$ has a unique irreducible quotient, and this way one obtains a parameterization of irreducible modules by the triples appearing in the left hand side of (8.6). Therefore we obtain that the same set is in bijection with the right hand side of (8.6). This is how the tame local Langlands correspondence (8.6), also known as the Deligne–Langlands conjecture, is proved.

8.2 Categories Admitting $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra Modules

We now wish to find categorical analogues of the above results in the framework of the categorical Langlands correspondence for loop groups.

As we explained in Section 6.2, in the categorical setting a representation of $G(F)$ is replaced by a category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ equipped with an action of $G(\!(t)\!)$, and the space of I-invariant vectors is replaced by the subcategory of $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra modules in $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$. Hence the analogue of the question which representations of $G(F)$ admit non-zero I-invariant vectors becomes the following question: for what χ does the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ contain non-zero $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra modules?

To answer this question, we introduce the space $\mathrm{Op}^{\mathrm{RS}}_{{}^L G}(D)$ of **opers with regular singularity**. By definition (see [BD1], Sect. 3.8.8), an element of this space is an ${}^L N[[t]]$-conjugacy class of operators of the form

$$\nabla = \partial_t + t^{-1}\left(p_{-1} + \mathbf{v}(t)\right), \tag{8.9}$$

where $\mathbf{v}(t) \in {}^L\mathfrak{b}[[t]]$. One can show that a natural map $\mathrm{Op}^{\mathrm{RS}}_{{}^L G}(D) \to \mathrm{Op}_{{}^L G}(D^\times)$ is an embedding.

Following [BD1], we associate to an oper with regular singularity its **residue**. For an operator (8.9) the residue is by definition equal to $p_{-1} + \mathbf{v}(0)$. Clearly, under gauge transformations by an element $x(t)$ of ${}^L N[[t]]$ the residue gets conjugated by $x(0) \in N$. Therefore its projection onto

$${}^L\mathfrak{g}/{}^L G = \mathrm{Spec}(\mathrm{Fun}\, {}^L\mathfrak{g})^{{}^L G} = \mathrm{Spec}(\mathrm{Fun}\, {}^L\mathfrak{h})^W = \mathfrak{h}^*/W$$

is well-defined.

Given $\mu \in \mathfrak{h}^*$, we write $\varpi(\mu)$ for the projection of μ onto $\mathfrak{h}^*/W$. Finally, let P be the set of integral (not necessarily dominant) weights of $\mathfrak{g}$, viewed as a subset of $\mathfrak{h}^*$. The next result follows from [F3, FG2].

Lemma 4. *The category* $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ *contains a non-zero* $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ *Harish-Chandra module if and only if*

$$\chi \in \bigsqcup_{\nu \in P/W} \mathrm{Op}^{\mathrm{RS}}_{{}^L G}(D)_{\varpi(\nu)}. \tag{8.10}$$

Thus, the opers χ for which the corresponding category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ contain non-trivial I-equivariant objects are precisely the points of the subscheme (8.10) of $\mathrm{Op}_{{}^L G}(D^\times)$. The next question is what are the corresponding ${}^L G$-local systems.

Let $\mathrm{Loc}^{\mathrm{RS,uni}}_{{}^L G} \subset \mathrm{Loc}_{{}^L G}(D^\times)$ be the locus of ${}^L G$-local systems on $D^\times$ with regular singularity and unipotent monodromy. Such a local system is determined, up to an isomorphism, by the conjugacy class of its monodromy (see, e.g., [BV], Sect. 8). Therefore $\mathrm{Loc}^{\mathrm{RS,uni}}_{{}^L G}$ is an algebraic stack isomorphic to $\mathcal{N}/{}^L G$. The following result is proved in a way similar to the proof of Lemma 2.

Lemma 5. *If the local system underlying an oper* $\chi \in \mathrm{Op}_{{}^L G}(D^\times)$ *belongs to* $\mathrm{Loc}^{\mathrm{RS,uni}}_{{}^L G}$*, then* χ *belongs to the subset* (8.10) *of* $\mathrm{Op}_{{}^L G}(D^\times)$.

Indeed, the subscheme (8.10) is precisely the (set-theoretic) preimage of $\mathrm{Loc}^{\mathrm{RS,uni}}_{{}^L G} \subset \mathrm{Loc}_{{}^L G}(D^\times)$ under the map $\alpha : \mathrm{Op}_{{}^L G}(D^\times) \to \mathrm{Loc}_{{}^L G}(D^\times)$.

This hardly comes as a surprise. Indeed, by analogy with the classical Langlands correspondence we expect that the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ containing non-trivial I-equivariant objects correspond to the Langlands parameters which are the geometric counterparts of tamely ramified homomorphisms $W'_F \to {}^LG$. The most obvious candidates for those are precisely the ${}^LG$-local systems on $D^\times$ with regular singularity and unipotent monodromy. For this reason we will call such local systems **tamely ramified**.

Let us summarize: suppose that σ is a tamely ramified ${}^LG$-local system on $D^\times$, and let χ be a ${}^LG$-oper that is in the gauge equivalence class of σ. Then χ belongs to the subscheme (8.10), and the corresponding category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ contains non-zero I-equivariant objects, by Lemma 4. Let $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^I$ be the corresponding category of I-equivariant (or, equivalently, $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra) modules. Note that according to Conjecture 2, the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ (resp., $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^I$) should be equivalent to each other for all χ which are gauge equivalent to each other as ${}^LG$-local systems.

In the next section, following [FG2], we will give a conjectural description of the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^I$ for $\chi \in \mathrm{Op}^{\mathrm{RS}}_{{}^LG}(D)_{\varpi(-\rho)}$ in terms of the category of coherent sheaves on the Springer fiber corresponding to the residue of χ. This description in particular implies that at least the derived categories of these categories are equivalent to each other for the opers corresponding to the same local system. We have a similar conjecture for $\chi \in \mathrm{Op}^{\mathrm{RS}}_{{}^LG}(D)_{\varpi(\nu)}$ for other $\nu \in P$, which the reader may easily reconstruct from our discussion of the case $\nu = -\rho$.

8.3 Conjectural Description of the Categories of $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra Modules

Let us consider one of the connected components of the subscheme (8.10), namely, $\mathrm{Op}^{\mathrm{RS}}_{{}^LG}(D)_{\varpi(-\rho)}$. Here it will be convenient to use a different realization of this space, as the space $\mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$ of **nilpotent opers** introduced in [FG2]. By definition, an element of this space is an ${}^LN[[t]]$-gauge equivalence class of operators of the form

$$\nabla = \partial_t + p_{-1} + \mathbf{v}(t) + \frac{v}{t}, \tag{8.11}$$

where $\mathbf{v}(t) \in {}^L\mathfrak{b}[[t]]$ and $v \in {}^L\mathfrak{n}$. It is shown in [FG2] that $\mathrm{Op}^{\mathrm{nilp}}_{{}^LG} \simeq \mathrm{Op}^{\mathrm{RS}}_{{}^LG}(D)_{\varpi(-\rho)}$. In particular, $\mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$ is a subspace of $\mathrm{Op}_{{}^LG}(D^\times)$.

We have the (secondary) residue map

$$\mathrm{Res} : \mathrm{Op}^{\mathrm{nilp}}_{{}^LG} \to {}^L\mathfrak{n}_{\mathcal{F}_{{}^LB,0}} = \mathcal{F}_{{}^LB,0} \underset{{}^LB}{\times} {}^L\mathfrak{n},$$

sending a gauge equivalence class of operators (8.11) to v. By abuse of notation, we will denote the corresponding map

$$\mathrm{Op}^{\mathrm{nilp}}_{{}^LG} \to {}^L\mathfrak{n}/{}^LB = \widetilde{\mathcal{N}}/{}^LG$$

also by Res.

For any $\chi \in \mathrm{Op}^{\mathrm{nilp}}_{{}^L G}$ the ${}^L G$-gauge equivalence class of the corresponding connection is a tamely ramified ${}^L G$-local system on $D^\times$. Moreover, its monodromy conjugacy class is equal to $\exp(2\pi i \operatorname{Res}(\chi))$.

We wish to describe the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^I_\chi$ of $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra modules with the central character $\chi \in \mathrm{Op}^{\mathrm{nilp}}_{{}^L G}$. However, here we face the first major complication as compared to the unramified case. While in the ramified case we worked with the abelian category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{G[[t]]}_\chi$, this does not seem to be possible in the tamely ramified case. So from now on we will work with the appropriate derived category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^I$. By definition, this is the full subcategory of the bounded derived category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)$ whose objects are complexes with cohomologies in $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^I_\chi$.

Roughly speaking, the conjecture of [FG2] is that $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^I$ is equivalent to $D^b(\mathrm{QCoh}(\mathrm{Sp}_{\mathrm{Res}(\chi)}))$, where $\mathrm{QCoh}(\mathrm{Sp}_{\mathrm{Res}(\chi)})$ is the category of quasicoherent sheaves on the Springer fiber of $\mathrm{Res}(\chi)$. However, we need to make some adjustments to this statement. These adjustments are needed to arrive at a "nice" statement, Conjecture 6 below. We now explain what these adjustments are the reasons behind them.

The first adjustment is that we need to consider a slightly larger category of representations than $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^I$. Namely, we wish to include extensions of I-equivariant $\widehat{\mathfrak{g}}_{\kappa_c}$-modules which are not necessarily I-equivariant, but only I^0-equivariant, where $I^0 = [I, I]$. To explain this more precisely, let us choose a Cartan subgroup $H \subset B \subset I$ and the corresponding Lie subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \operatorname{Lie} I$. We then have an isomorphism $I = H \ltimes I^0$. An I-equivariant $\widehat{\mathfrak{g}}_{\kappa_c}$-module is the same as a module on which $\mathfrak{h}$ acts diagonally with eigenvalues given by integral weights and the Lie algebra $\operatorname{Lie} I^0$ acts locally nilpotently. However, there may exist extensions between such modules on which the action of $\mathfrak{h}$ is no longer semi-simple. Such modules are called I-**monodromic**. More precisely, an I-monodromic $\widehat{\mathfrak{g}}_{\kappa_c}$-module is a module that admits an increasing filtration whose consecutive quotients are I-equivariant. It is natural to include such modules in our category. However, it is easy to show that an I-monodromic object of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ is the same as an I^0-equivariant object of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ for any $\chi \in \mathrm{Op}^{\mathrm{nilp}}_{{}^L G}$ (see [FG2]). Therefore instead of I-monodromic modules we will use I^0-equivariant modules. Denote by $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^{I^0}$ the the full subcategory of $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)$ whose objects are complexes with the cohomologies in $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{I^0}_\chi$.

The second adjustment has to do with the non-flatness of the Springer resolution $\widetilde{\mathcal{N}} \to \mathcal{N}$. By definition, the Springer fiber Sp_u is the fiber product $\widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \mathrm{pt}$, where pt is the point $u \in \mathcal{N}$. This means that the structure sheaf of Sp_u is given by

$$\mathcal{O}_{\mathrm{Sp}_u} = \mathcal{O}_{\widetilde{\mathcal{N}}} \underset{\mathcal{O}_{\mathcal{N}}}{\otimes} \mathbb{C}. \tag{8.12}$$

However, because the morphism $\widetilde{\mathcal{N}} \to \mathcal{N}$ is not flat, this tensor product functor is not left exact, and there are non-trivial higher derived tensor products (the Tor's). Our (conjectural) equivalence is not going to be an exact functor: it sends a general object of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I^0}$ not to an object of the category of quasicoherent sheaves, but to a complex of sheaves, or, more precisely, an object of the corresponding derived category. Hence we are forced to work with derived categories, and so the higher derived tensor products need to be taken into account.

To understand better the consequences of this non-exactness, let us consider the following model example. Suppose that we have established an equivalence between the derived category $D^b(\mathrm{QCoh}(\widetilde{\mathcal{N}}))$ and another derived category $D^b(\mathcal{C})$. In particular, this means that both categories carry an action of the algebra $\mathrm{Fun}\,\mathcal{N}$ (recall that $\mathcal{N}$ is an affine algebraic variety). Let us suppose that the action of $\mathrm{Fun}\,\mathcal{N}$ on $D^b(\mathcal{C})$ comes from its action on the abelian category $\mathcal{C}$. Thus, $\mathcal{C}$ fibers over $\mathcal{N}$, and let $\mathcal{C}_u$ the fiber category corresponding to $u \in \mathcal{N}$. This is the full subcategory of $\mathcal{C}$ whose objects are objects of $\mathcal{C}$ on which the ideal of u in $\mathrm{Fun}\,\mathcal{N}$ acts by 0.[16] What is the category $D^b(\mathcal{C}_u)$ equivalent to?

It is tempting to say that it is equivalent to $D^b(\mathrm{QCoh}(\mathrm{Sp}_u)$. However, this does not follow from the equivalence of $D^b(\mathrm{QCoh}(\mathcal{N}))$ and $D^b(\mathcal{C})$ because of the tensor product (8.12) having non-trivial higher derived functors. The correct answer is that $D^b(\mathcal{C}_u)$ is equivalent to the category $D^b(\mathrm{QCoh}(\mathrm{Sp}_u^{\mathrm{DG}})$, where $\mathrm{Sp}_u^{\mathrm{DG}}$ is the "DG fiber" of $\widetilde{\mathcal{N}} \to \mathcal{N}$ at u. By definition, a quasicoherent sheaf on $\mathrm{Sp}_u^{\mathrm{DG}}$ is a DG module over the DG algebra

$$\mathcal{O}_{\mathrm{Sp}_u^{\mathrm{DG}}} = \mathcal{O}_{\widetilde{\mathcal{N}}} \overset{L}{\underset{\mathcal{O}_{\mathcal{N}}}{\otimes}} \mathbb{C}_u, \tag{8.13}$$

where we now take the full derived functor of tensor product. Thus, the category $D^b(\mathrm{QCoh}(\mathrm{Sp}_u^{\mathrm{DG}}))$ may be thought of as the derived category of quasicoherent sheaves on the "DG scheme" $\mathrm{Sp}_u^{\mathrm{DG}}$ (see [CK] for a precise definition of DG scheme).

Finally, the last adjustment is that we should consider the non-reduced Springer fibers. This means that instead of the Springer resolution $\widetilde{\mathcal{N}}$ we should consider the "thickened" Springer resolution

$$\widetilde{\widetilde{\mathcal{N}}} = {}^L\widetilde{\mathfrak{g}} \underset{{}^L\mathfrak{g}}{\times} \mathcal{N},$$

where ${}^L\widetilde{\mathfrak{g}}$ is the so-called **Grothendieck alteration**,

$${}^L\widetilde{\mathfrak{g}} = \{x \in {}^L\mathfrak{g}, \mathfrak{b}' \in {}^LG/{}^LB \mid x \in \mathfrak{b}'\}.$$

The variety $\widetilde{\widetilde{\mathcal{N}}}$ is non-reduced, and the underlying reduced variety is the Springer resolution $\widetilde{\mathcal{N}}$. For instance, the fiber of $\widetilde{\mathcal{N}}$ over a regular element

[16] The relationship between $\mathcal{C}$ and $\mathcal{C}_u$ is similar to the relationship between $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}$ and and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, where $\chi \in \mathrm{Op}_{{}^LG}(D^\times)$.

in $\mathcal{N}$ consists of a single point, but the corresponding fiber of $\widetilde{\widetilde{\mathcal{N}}}$ is the spectrum of the Artinian ring $h_0 = \operatorname{Fun} {}^L\mathfrak{h}/(\operatorname{Fun} {}^L\mathfrak{h})^W_+$. Here $(\operatorname{Fun} {}^L\mathfrak{h})^W_+$ is the ideal in $\operatorname{Fun} {}^L\mathfrak{h}$ generated by the augmentation ideal of the subalgebra of W-invariants. Thus, $\operatorname{Spec} h_0$ is the scheme-theoretic fiber of $\varpi : {}^L\mathfrak{h} \to {}^L\mathfrak{h}/W$ at 0. It turns out that in order to describe the category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^{I^0}$ we need to use the "thickened" Springer resolution.

Let us summarize: in order to construct the sought-after equivalence of categories we take, instead of individual Springer fibers, the whole Springer resolution, and we further replace it by the "thickened" Springer resolution $\widetilde{\widetilde{\mathcal{N}}}$ defined above. In this version we will be able to formulate our equivalence in such a way that we avoid DG schemes.

This means that instead of considering the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ for individual nilpotent opers χ, we should consider the "universal" category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{nilp}}$ which is the "family version" of all of these categories. By definition, the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{nilp}}$ is the full subcategory of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}$ whose objects have the property that the action of $Z(\widehat{\mathfrak{g}}) = \operatorname{Fun} \operatorname{Op}_{^LG}(D)$ on them factors through the quotient $\operatorname{Fun} \operatorname{Op}_{^LG}(D) \to \operatorname{Fun} \operatorname{Op}^{\text{nilp}}_{^LG}$. Thus, the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{nilp}}$ is similar to the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\text{reg}}$ that we have considered above. While the former fibers over $\operatorname{Op}^{\text{nilp}}_{^LG}$, the latter fibers over $\operatorname{Op}_{^LG}(D)$. The individual categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ are now realized as fibers of these categories over particular opers χ.

Our naive idea was that for each $\chi \in \operatorname{Op}^{\text{nilp}}_{^LG}$ the category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^{I^0}$ is equivalent to $\operatorname{QCoh}(\operatorname{Sp}_{\operatorname{Res}(\chi)})$. We would like to formulate now a "family version" of such an equivalence. To this end we form the fiber product

$$^L\widetilde{\widetilde{\mathfrak{n}}} = {}^L\widetilde{\mathfrak{g}} \underset{^L\mathfrak{g}}{\times} {}^L\mathfrak{n}.$$

It turns out that this fiber product does not suffer from the problem of the individual Springer fibers, as the following lemma shows:

Lemma 6 ([FG2],Lemma 6.4). *The derived tensor product*

$$\operatorname{Fun} {}^L\widetilde{\mathfrak{g}} \overset{L}{\underset{\operatorname{Fun} {}^L\mathfrak{g}}{\otimes}} \operatorname{Fun} {}^L\mathfrak{n}$$

is concentrated in cohomological dimension 0.

The variety $^L\widetilde{\widetilde{\mathfrak{n}}}$ may be thought of as the family of (non-reduced) Springer fibers parameterized by ${}^L\mathfrak{n} \subset {}^L\mathfrak{g}$. It is important to note that it is singular, reducible and non-reduced. For example, if $\mathfrak{g} = \mathfrak{sl}_2$, it has two components, one of which is $\mathbb{P}^1$ (the Springer fiber at 0) and the other is the doubled affine line (i.e., $\operatorname{Spec} \mathbb{C}[x,y]/(y^2)$).

We note that the corresponding reduced scheme is

$$^L\widetilde{\mathfrak{n}} = \widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} {}^L\mathfrak{n}. \tag{8.14}$$

However, the derived tensor product corresponding to (8.14) is not concentrated in cohomological dimension 0, and this is the reason why we prefer to use ${}^L\widetilde{\widetilde{\mathfrak{n}}}$ rather than ${}^L\widetilde{\mathfrak{n}}$.

Now we set

$$\mathrm{MOp}^{\mathrm{nilp}}_{{}^LG} = \mathrm{Op}^{\mathrm{nilp}}_{{}^LG} \underset{{}^L\mathfrak{n}/{}^LB}{\times} {}^L\widetilde{\widetilde{\mathfrak{n}}}/{}^LB,$$

where we use the residue morphism $\mathrm{Res} : \mathrm{Op}^{\mathrm{nilp}}_{{}^LG} \to {}^L\mathfrak{n}/{}^LB$. Thus, informally $\mathrm{MOp}^{\mathrm{nilp}}_{{}^LG}$ may be thought as the family over $\mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$ whose fiber over $\chi \in \mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$ is the (non-reduced) Springer fiber of $\mathrm{Res}(\chi)$.

The space $\mathrm{MOp}^{\mathrm{nilp}}_{{}^LG}$ is the space of **Miura opers** whose underlying opers are nilpotent, introduced in [FG2].

We also introduce the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{I^0}_{\mathrm{nilp}}$ which is a full subcategory of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}}$ whose objects are I^0-equivariant. Let $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^{I^0}$ be the corresponding derived category.

Now we can formulate the Main Conjecture of [FG2]:

Conjecture 6. *There is an equivalence of categories*

$$D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^{I^0} \simeq D^b(\mathrm{QCoh}(\mathrm{MOp}^{\mathrm{nilp}}_{{}^LG})) \tag{8.15}$$

which is compatible with the action of the algebra $\mathrm{Fun}\,\mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$ *on both categories.*

Note that the action of $\mathrm{Fun}\,\mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$ on the first category comes from the action of the center $Z(\widehat{\mathfrak{g}})$, and on the second category it comes from the fact that $\mathrm{MOp}^{\mathrm{nilp}}_{{}^LG}$ is a scheme over $\mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$.

Another important remark is that the equivalence (8.15) does not preserve the t-structures on the two categories. In other words, (8.15) is expected in general to map objects of the abelian category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{I^0}_{\mathrm{nilp}}$ to complexes in $D^b(\mathrm{QCoh}(\mathrm{MOp}^{\mathrm{nilp}}_{{}^LG}))$, and vice versa.

There are similar conjectures for the categories corresponding to the spaces $\mathrm{Op}^{\mathrm{nilp},\lambda}_{{}^LG}$ of nilpotent opers with dominant integral weights $\lambda \in P^+$.

In the next section we will discuss the connection between Conjecture 6 and the classical tamely ramified Langlands correspondence. We then present some evidence for this conjecture.

8.4 Connection between the Classical and the Geometric Settings

Let us discuss the connection between the equivalence (8.15) and the realization of representations of affine Hecke algebras in terms of K-theory of the Springer fibers. As we have explained, we would like to view the category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^{I^0}$ for $\chi \in \mathrm{Op}^{\mathrm{nilp}}_{{}^LG}$ as, roughly, a categorification of the space $R^{\pi(I)}$ of I-invariant vectors in an irreducible representation (R,π) of

$G(F)$. Therefore, we expect that the Grothendieck group of the category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^{I^0}$ is somehow related to the space $R^{\pi(I)}$.

Let us try to specialize the statement of Conjecture 6 to a particular oper

$$\chi = (\mathcal{F}, \nabla, \mathcal{F}_{^L B}) \in \mathrm{Op}^{\mathrm{nilp}}_{^L G}.$$

Let $\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_{\mathrm{Res}(\chi)}$ be the DG fiber of $\mathrm{MOp}^{\mathrm{nilp}}_{^L G}$ over χ. By definition (see Section 8.3), the residue $\mathrm{Res}(\chi)$ of χ is a vector in the twist of ${}^L\mathfrak{n}$ by the ${}^L B$-torsor $\mathcal{F}_{^L B,0}$. It follows that $\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_{\mathrm{Res}(\chi)}$ is the DG fiber over $\mathrm{Res}(\chi)$ of the $\mathcal{F}_{^L B,0}$-twist of the Grothendieck alteration.

If we trivialize $\mathcal{F}_{^L B,0}$, then $u = \mathrm{Res}(\chi)$ becomes an element of ${}^L\mathfrak{n}$. By definition, the (non-reduced) DG Springer fiber $\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_u$ is the DG fiber of the Grothendieck alteration ${}^L\widetilde{\mathfrak{g}} \to {}^L\mathfrak{g}$ at u. In other words, the corresponding structure sheaf is the DG algebra

$$\mathcal{O}_{\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_u} = \mathcal{O}_{^L\widetilde{\mathfrak{g}}} \overset{L}{\underset{\mathcal{O}_{^L\mathfrak{g}}}{\otimes}} \mathbb{C}_u$$

(compare with formula (8.13)).

To see what these DG fibers look like, let $u = 0$. Then the naive Springer fiber is just the flag variety ${}^L G/{}^L B$ (it is reduced in this case), and $\mathcal{O}_{\widetilde{\mathrm{Sp}}_0}$ is the structure sheaf of ${}^L G/{}^L B$. But the sheaf $\mathcal{O}_{\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_0}$ is a sheaf of DG algebras, which is quasi-isomorphic to the complex of differential forms on ${}^L G/{}^L B$, with the zero differential. In other words, $\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_0$ may be viewed as a "$\mathbb{Z}$-graded manifold" such that the corresponding supermanifold, obtained by replacing the $\mathbb{Z}$-grading by the corresponding $\mathbb{Z}/2\mathbb{Z}$-grading, is $\Pi T({}^L G/{}^L B)$, the tangent bundle to ${}^L G/{}^L B$ with the parity of the fibers changed from even to odd.

We expect that the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{I^0}_{\mathrm{nilp}}$ is flat over $\mathrm{Op}^{\mathrm{nilp}}_{^L G}$. Therefore, specializing Conjecture 6 to a particular oper $\chi \in \mathrm{Op}^{\mathrm{nilp}}_{^L G}$, we obtain as a corollary an equivalence of categories

$$D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^{I^0} \simeq D^b(\mathrm{QCoh}(\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_{\mathrm{Res}(\chi)})). \tag{8.16}$$

This bodes well with Conjecture 2 saying that the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi_1}$ and $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi_2}$ (and hence $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi_1})^{I^0}$ and $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi_2})^{I^0}$) should be equivalent if the underlying local systems of the opers χ_1 and χ_2 are isomorphic. For nilpotent opers χ_1 and χ_2 this is so if and only if their monodromies are conjugate to each other. Since their monodromies are obtained by exponentiating their residues, this is equivalent to saying that the residues, $\mathrm{Res}(\chi_1)$ and $\mathrm{Res}(\chi_2)$, are conjugate with respect to the $\mathcal{F}_{^L B,0}$-twist of ${}^L G$. But in this case the DG Springer fibers corresponding to χ_1 and χ_2 are also isomorphic, and so $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi_1})^{I^0}$ and $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi_2})^{I^0}$ are equivalent to each other, by (8.16).

The Grothendieck group of the category $D^b(\mathrm{QCoh}(\widetilde{\mathrm{Sp}}_u^{\mathrm{DG}}))$, where u is any nilpotent element, is the same as the Grothendieck group of $\mathrm{QCoh}(\mathrm{Sp}_u)$. In other words, the Grothendieck group does not "know" about the DG or the non-reduced structure of $\widetilde{\mathrm{Sp}}_u^{\mathrm{DG}}$. Hence it is nothing but the algebraic K-theory $K(\mathrm{Sp}_u)$. As we explained at the end of Section 8.1, equivariant variants of this algebraic K-theory realize the "standard modules" over the affine Hecke algebra $H(G(F), I)$. Moreover, the spaces of I-invariant vectors $R^{\pi(I)}$ as above, which are naturally modules over the affine Hecke algebra, may be realized as subquotients of $K(\mathrm{Sp}_u)$. This indicates that the equivalences (8.16) and (8.15) are compatible with the classical results.

However, at first glance there are some important differences between the classical and the categorical pictures, which we now discuss in more detail.

In the construction of $H(G(F), I)$-modules outlined in Section 8.1 we had to pick a semi-simple element γ of ${}^LG$ such that $\gamma u \gamma^{-1} = qu$, where q is the number of elements in the residue field of F. Then we consider the specialized A-equivariant K-theory $K^A(\mathrm{Sp}_u)_{(\gamma,q)}$ where A is the the smallest algebraic subgroup of ${}^LG \times \mathbb{C}^\times$ containing (γ, q). This gives $K(\mathrm{Sp}_u)$ the structure of an $H(G(F), I)$-module. But this module carries a residual symmetry with respect to the group $C(\gamma, u)$ of components of the centralizer of γ and u in ${}^LG$, which commutes with the action of $H(G(F), I)$. Hence we consider the $H(G(F), I)$-module

$$K^A(\mathrm{Sp}_u)_{(\gamma,q,\rho)} = \mathrm{Hom}_{C(\gamma,u)}(\rho, K(\mathrm{Sp}_u)),$$

corresponding to an irreducible representation ρ of $C(\gamma, u)$. Finally, each of these components has a unique irreducible quotient, and this is an irreducible representation of $H(G(F), I)$ which is realized on the space $R^{\pi(I)}$, where (R, π) is an irreducible representation of $G(F)$ corresponding to (γ, u, ρ) under the bijection (8.6). How is this intricate structure reflected in the categorical setting?

Our category $D^b(\mathrm{QCoh}(\widetilde{\mathrm{Sp}}_u^{\mathrm{DG}}))$, where $u = \mathrm{Res}(\chi)$, is a particular categorification of the (non-equivariant) K-theory $K(\mathrm{Sp}_u)$. Note that in the classical local Langlands correspondence (8.6) the element u of the triple (γ, u, ρ) is interpreted as the logarithm of the monodromy of the corresponding representation of the Weil-Deligne group W'_F. This is in agreement with the interpretation of $\mathrm{Res}(\chi)$ as the logarithm of the monodromy of the ${}^LG$-local system on $D^\times$ corresponding to χ, which plays the role of the local Langlands parameter for the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ (up to the inessential factor $2\pi i$).

But what about the other parameters, γ and ρ? And why does our category correspond to the non-equivariant K-theory of the Springer fiber, and not the equivariant K-theory, as in the classical setting?

The element γ corresponding to the Frobenius in W'_F does not seem to have an analogue in the geometric setting. We have already seen this above in the unramified case: while in the classical setting unramified local Langlands parameters are the semi-simple conjugacy classes γ in ${}^LG$, in the geometric setting we have only one unramified local Langlands parameter, namely, the trivial local system.

To understand better what's going on here, we revisit the unramified case. Recall that the spherical Hecke algebra $H(G(F),K_0)$ is isomorphic to the representation ring $\operatorname{Rep}{}^L G$. The one-dimensional space of K_0-invariants in an irreducible unramified representation (R,π) of $G(F)$ realizes a one-dimensional representation of $H(G(F),K_0)$, i.e., a homomorphism $\operatorname{Rep}{}^L G \to \mathbb{C}$. The unramified Langlands parameter γ of (R,π), which is a semi-simple conjugacy class in ${}^L G$, is the point in $\operatorname{Spec}(\operatorname{Rep}{}^L G)$ corresponding to this homomorphism. What is a categorical analogue of this homomorphism? The categorification of $\operatorname{Rep}{}^L G$ is the category $\mathcal{R}ep\,{}^L G$. The product structure on $\operatorname{Rep}{}^L G$ is reflected in the structure of tensor category on $\mathcal{R}ep\,{}^L G$. On the other hand, the categorification of the algebra $\mathbb{C}$ is the category $\mathcal{V}ect$ of vector spaces. Therefore a categorical analogue of a homomorphism $\operatorname{Rep}{}^L G \to \mathbb{C}$ is a functor $\mathcal{R}ep\,{}^L G \to \mathcal{V}ect$ respecting the tensor structures on both categories. Such functors are called the fiber functors. The fiber functors form a category of their own, which is equivalent to the category of ${}^L G$-torsors. Thus, any two fiber functors are isomorphic, but not canonically. In particular, the group of automorphisms of each fiber functor is isomorphic to ${}^L G$. (Incidentally, this is how ${}^L G$ is reconstructed from a fiber functor in the Tannakian formalism.) Thus, we see that while in the categorical world we do not have analogues of semi-simple conjugacy classes γ (the points of $\operatorname{Spec}(\operatorname{Rep}{}^L G)$), their role is in some sense played by the group of automorphisms of a fiber functor.

This is reflected in the fact that while in the categorical setting we have a unique unramified Langlands parameter, namely, the trivial ${}^L G$-local system σ_0 on $D^\times$, this local system has a non-trivial group of automorphisms, namely, ${}^L G$. We therefore expect that the group ${}^L G$ should act by automorphisms of the Langlands category $\mathcal{C}_{\sigma_0}$ corresponding to σ_0, and this action should commute with the action of the loop group $G(\!(t)\!)$ on $\mathcal{C}_{\sigma_0}$. It is this action of ${}^L G$ that is meant to compensate for the lack of unramified Langlands parameters, as compared to the classical setting.

We have argued in Section 7 that the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, where $\chi = (\mathcal{F},\nabla,\mathcal{F}_{{}^L B}) \in \operatorname{Op}_{{}^L G}(D)$, is a candidate for the Langlands category $\mathcal{C}_{\sigma_0}$. Therefore we expect that the group ${}^L G$ (more precisely, its twist ${}^L G_{\mathcal{F}}$) acts on the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$. In Section 7.6 we showed how to obtain this action using the conjectural equivalence between $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$ and the category $\mathcal{D}^{\text{Hecke}}_{\kappa_c}\text{-mod}_\chi$ of Hecke eigenmodules on the affine Grassmannian Gr (see Conjecture 3). The category $\mathcal{D}^{\text{Hecke}}_{\kappa_c}\text{-mod}_\chi$ was defined in Section 7.5 as a "de-equivariantization" of the category $\mathcal{D}_{\kappa_c}\text{-mod}$ of twisted $\mathcal{D}$-modules on Gr with respect to the monoidal action of the category $\mathcal{R}ep\,{}^L G$.

Now comes a crucial observation which will be useful for understanding the way things work in the tamely ramified case: the category $\mathcal{R}ep\,{}^L G$ may be interpreted as the category of ${}^L G$-equivariant quasicoherent sheaves on the variety $\mathrm{pt} = \operatorname{Spec}\mathbb{C}$. In other words, $\mathcal{R}ep\,{}^L G$ may be interpreted as the category of quasicoherent sheaves on the stack $\mathrm{pt}/{}^L G$. The existence of monoidal action of the category $\mathcal{R}ep\,{}^L G$ on $\mathcal{D}_{\kappa_c}$-mod should be viewed as the statement that the category $\mathcal{D}_{\kappa_c}$-mod "lives" over the stack $\mathrm{pt}/{}^L G$. The statement of Conjecture 3 may then be interpreted as saying that

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi \simeq \mathcal{D}_{\kappa_c}\text{-mod} \underset{\mathrm{pt}/{}^L G}{\times} \mathrm{pt}\,.$$

In other words, if $\mathcal{C}$ is the conjectural Langlands category fibering over the stack $\mathrm{Loc}_{{}^L G}(D^\times)$ of all ${}^L G$-local systems on $D^\times$, then

$$\mathcal{D}_{\kappa_c}\text{-mod} \simeq \mathcal{C} \underset{\mathrm{Loc}_{{}^L G}(D^\times)}{\times} \mathrm{pt}/{}^L G,$$

whereas

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi \simeq \mathcal{C} \underset{\mathrm{Loc}_{{}^L G}(D^\times)}{\times} \mathrm{pt},$$

where the morphism $\mathrm{pt} \to \mathrm{Loc}_{{}^L G}(D^\times)$ corresponds to the oper χ.

Thus, in the categorical setting there are two different ways to think about the trivial local system σ_0: as a point (defined by a particular ${}^L G$-bundle on D with connection, such as a regular oper χ), or as a stack $\mathrm{pt}/{}^L G$. The base change of the Langlands category in the first case gives us a category with an action of ${}^L G$, such as the categories $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ or $\mathcal{D}^{\mathrm{Hecke}}_{\kappa_c}$-mod. The base change in the second case gives us a category with a monoidal action of $\mathcal{R}ep\,{}^L G$, such as the category $\mathcal{D}_{\kappa_c}$-mod. We can go back and forth between the two by applying the procedures of equivariantization and de-equivariantization with respect to ${}^L G$ and $\mathcal{R}ep\,{}^L G$, respectively.

Now we return to the tamely ramified case. The semi-simple element γ appearing in the triple (γ, u, ρ) plays the same role as the unramified Langlands parameter γ. However, now it must satisfy the identity $\gamma u \gamma^{-1} = qu$. Recall that the center Z of $H(G(F), I)$ is isomorphic to $\mathrm{Rep}\,{}^L G$, and so $\mathrm{Spec}\, Z$ is the set of all semi-simple elements in ${}^L G$. For a fixed nilpotent element u the equation $\gamma u \gamma^{-1} = qu$ cuts out a locus C_u in $\mathrm{Spec}\, Z$ corresponding to those central characters which may occur on irreducible $H(G(F), I)$-modules corresponding to u. In the categorical setting (where we set $q = 1$) the analogue of C_u is the centralizer $Z(u)$ of u in ${}^L G$, which is precisely the group $\mathrm{Aut}(\sigma)$ of automorphisms of a tame local system σ on $D^\times$ with monodromy $\exp(2\pi i u)$. On general grounds we expect that the group $\mathrm{Aut}(\sigma)$ acts on the Langlands category $\mathcal{C}_\sigma$, just as we expect the group ${}^L G$ of automorphisms of the trivial local system σ_0 to act on the category $\mathcal{C}_{\sigma_0}$. It is this action that replaces the parameter γ in the geometric setting.

In the classical setting we also have one more parameter, ρ. Let us recall that ρ is a representation of the group $C(\gamma, u)$ of connected components of the centralizer $Z(\gamma, u)$ of γ and u. But the group $Z(\gamma, u)$ is a subgroup of $Z(u)$, which becomes the group $\mathrm{Aut}(\sigma)$ in the geometric setting. Therefore one can argue that the parameter ρ is also absorbed into the action of $\mathrm{Aut}(\sigma)$ on the category $\mathcal{C}_\sigma$.

If we have an action of $\mathrm{Aut}(\sigma)$ on the category $\mathcal{C}_\sigma$, or on one of its many incarnations $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi, \chi \in \mathrm{Op}^{\mathrm{nilp}}_{^LG}$, it means that these categories must be "de-equivariantized", just like the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi, \chi \in \mathrm{Op}_{^LG}(D)$, in the unramified case. This is the reason why in the equivalence (8.16) (and in Conjecture 6) we have the non-equivariant categories of quasicoherent sheaves (whose Grothendieck groups correspond to the non-equivariant K-theory of the Springer fibers).

However, there is also an equivariant version of these categories. Consider the substack of tamely ramified local systems in $\mathrm{Loc}_{^LG}(D^\times)$ introduced in Section 8.2. Since a tamely ramified local system is completely determined by the logarithm of its (unipotent) monodromy, this substack is isomorphic to $\mathcal{N}/{}^LG$. This substack plays the role of the substack $\mathrm{pt}/{}^LG$ corresponding to the trivial local system. Let us set

$$\mathcal{C}_{\mathrm{tame}} = \mathcal{C} \underset{\mathrm{Loc}_{^LG}(D^\times)}{\times} \mathcal{N}/{}^LG.$$

Then, according to our general conjecture expressed by the Cartesian diagram (5.3), we expect to have

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}} \simeq \mathcal{C}_{\mathrm{tame}} \underset{\mathcal{N}/{}^LG}{\times} \mathrm{Op}^{\mathrm{nilp}}_{^LG}. \tag{8.17}$$

Let $D^b(\mathcal{C}_{\mathrm{tame}})^{I^0}$ be the I^0-equivariant derived category corresponding to $\mathcal{C}_{\mathrm{tame}}$. Combining (8.17) with Conjecture 6, and noting that

$$\mathrm{MOp}^{\mathrm{nilp}}_{^LG} \simeq \mathrm{Op}^{\mathrm{nilp}}_{^LG} \underset{\mathcal{N}/{}^LG}{\times} \widetilde{\widetilde{\mathcal{N}}}/{}^LG,$$

we obtain the following conjecture (see [FG2]):

$$D^b(\mathcal{C}_{\mathrm{tame}})^{I^0} \simeq D^b(\mathrm{QCoh}(\widetilde{\widetilde{\mathcal{N}}}/{}^LG)). \tag{8.18}$$

The category on the right hand side may be interpreted as the derived category of ${}^LG$-equivariant quasicoherent sheaves on the "thickened" Springer resolution $\widetilde{\widetilde{\mathcal{N}}}$.

Together, the conjectural equivalences (8.16) and (8.18) should be thought of as the categorical versions of the realizations of modules over the affine Hecke algebra in the K-theory of the Springer fibers.

One corollary of the equivalence (8.16) is the following: the classes of irreducible objects of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I_0}$ in the Grothendieck group of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I_0}$ give rise to a basis in the algebraic K-theory $K(\mathrm{Sp}_u)$, where $u = \mathrm{Res}(\chi)$. Presumably, this basis is closely related to the bases in (equivariant version of) this K-theory constructed by G. Lusztig in [Lu2] (from the perspective of unrestricted $\mathfrak{g}$-modules in positive characteristic).

8.5 Evidence for the Conjecture

We now describe some evidence for Conjecture 6. It consists of the following four groups of results:

- Interpretation of the Wakimoto modules as $\widehat{\mathfrak{g}}_{\kappa_c}$-modules corresponding to the skyscraper sheaves on $\mathrm{MOp}^{\mathrm{nilp}}_{^L G}$;
- Connection to R. Bezrukavnikov's theory;
- Proof of the equivalence of certain quotient categories of $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^{I^0}$ and $D^b(\mathrm{QCoh}(\mathrm{MOp}^{\mathrm{nilp}}_{^L G}))$, [FG2].
- Proof of the restriction of the equivalence (8.15) to regular opers, [FG4].

We start with the discussion of Wakimoto modules.

Suppose that we have proved the equivalence of categories (8.15). Then each quasicoherent sheaf on $\mathrm{MOp}^{\mathrm{nilp}}_{^L G}$ should correspond to an object of the derived category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^{I^0}$. The simplest quasicoherent sheaves on $\mathrm{MOp}^{\mathrm{nilp}}_{^L G}$ are the **skyscraper sheaves** supported at the $\mathbb{C}$-points of $\mathrm{MOp}^{\mathrm{nilp}}_{^L G}$. It follows from the definition that a $\mathbb{C}$-point of $\mathrm{MOp}^{\mathrm{nilp}}_{^L G}$, which is the same as a $\mathbb{C}$-point of the reduced scheme MOp^0_G, is a pair $(\chi, \mathfrak{b}')$, where $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{^L B})$ is a nilpotent $^L G$-oper in $\mathrm{Op}^{\mathrm{nilp}}_{^L G}$ and $\mathfrak{b}'$ is a point of the Springer fiber corresponding to $\mathrm{Res}(\chi)$, which is the variety of Borel subalgebras in $^L\mathfrak{g}_{\mathcal{F}_0}$ that contain $\mathrm{Res}(\chi)$. Thus, if Conjecture 6 is true, we should have a family of objects of the category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^{I^0}$ parameterized by these data. What are these objects?

The answer is that these are the **Wakimoto modules**. These modules were originally introduced by M. Wakimoto [Wak] for $\mathfrak{g} = \mathfrak{sl}_2$ and by B. Feigin and myself in general in [FF1, FF2] (see also [F3]). We recall from [F3] that Wakimoto modules of critical level are parameterized by the space $\mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ of connections on the $^L H$-bundle $\Omega^{-\rho}$ over $D^\times$. This is the push-forward of the $\mathbb{C}^\times$-bundle corresponding to the canonical line bundle Ω with respect to the homomorphism $\rho : \mathbb{C}^\times \to {}^L H$. Let us denote the Wakimoto module corresponding to $\overline{\nabla} \in \mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ by $W_{\overline{\nabla}}$. According to [F3], Theorem 12.6, the center $Z(\widehat{\mathfrak{g}})$ acts on $W_{\overline{\nabla}}$ via the central character $\mu(\overline{\nabla})$, where

$$\mu : \mathrm{Conn}(\Omega^{-\rho})_{D^\times} \to \mathrm{Op}_{^L G}(D^\times)$$

is the **Miura transformation**.

It is not difficult to show that if $\chi \in \mathrm{Op}^{\mathrm{nilp}}_{^L G}$, then $W_{\overline{\nabla}}$ is an object of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^I_\chi$ for any $\overline{\nabla} \in \mu^{-1}(\chi)$. Now, according to the results presented in [FG2], the points of the fiber $\mu^{-1}(\chi)$ of the Miura transformation over χ are in bijection with the points of the Springer fiber $\mathrm{Sp}_{\mathrm{Res}(\chi)}$ corresponding to the nilpotent element $\mathrm{Res}(\chi)$. Therefore to each point of $\mathrm{Sp}_{\mathrm{Res}(\chi)}$ we may assign a Wakimoto module, which is an object of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{I^0}_\chi$ (and hence of the corresponding derived category). In other words, Wakimoto modules are objects of the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^I_{\mathrm{nilp}}$ parameterized by the $\mathbb{C}$-points of $\mathrm{MOp}^{\mathrm{nilp}}_{^LG}$. It is natural to assume that they correspond to the skyscraper sheaves on $\mathrm{MOp}^{\mathrm{nilp}}_{^LG}$ under the equivalence (8.15). This was in fact one of our motivations for this conjecture.

Incidentally, this gives us a glimpse into how the group of automorphisms of the ^LG-local system underlying the oper χ acts on the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$. This group is $Z(\mathrm{Res}(\chi))$, the centralizer of the residue $\mathrm{Res}(\chi)$, and it acts on the Springer fiber $\mathrm{Sp}_{\mathrm{Res}(\chi)}$. Therefore $g \in Z(\mathrm{Res}(\chi))$ sends the skyscraper sheaf supported at a point $p \in \mathrm{Sp}_{\mathrm{Res}(\chi)}$ to the skyscraper sheaf supported at $g \cdot p$. Thus, we expect that g sends the Wakimoto module corresponding to p to the Wakimoto module corresponding to $g \cdot p$.

If the Wakimoto modules indeed correspond to the skyscraper sheaves, then the equivalence (8.15) may be thought of as a kind of "spectral decomposition" of the category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^{I^0}$, with the basic objects being the Wakimoto modules $W_{\overline{\nabla}}$, where $\overline{\nabla}$ runs over the locus in $\mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ which is isomorphic, pointwise, to $\mathrm{MOp}^{\mathrm{nilp}}_{^LG}$ (see [FG5] for more details).

Now we discuss the second piece of evidence, connection with Bezrukanikov's theory.

To motivate it, let us recall that in Section 7.4 we discussed the action of the categorical spherical algebra $\mathcal{H}(G((t)), G[[t]])$ on the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi$, where χ is a regular oper. The affine Hecke algebra $H(G(F), I)$ also has a categorical analogue. Consider the **affine flag variety** $\mathrm{Fl} = G((t))/I$. The categorical Hecke algebra is the category $\mathcal{H}(G((t)), I)$ which is the full subcategory of the derived category of $\mathcal{D}$-modules on $\mathrm{Fl} = G((t))/I$ whose objects are complexes with I-equivariant cohomologies. This category naturally acts on the derived category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi)^I$. What does this action correspond to on the other side of the equivalence (8.15)?

The answer is given by a theorem of R. Bezrukavnikov [Bez2], which may be viewed as a categorification of the isomorphism (8.8):

$$D^b(\mathcal{D}^{\mathrm{Fl}}_{\kappa_c}\text{-mod})^{I^0} \simeq D^b(\mathrm{QCoh}(\widetilde{\mathrm{St}})), \tag{8.19}$$

where $\mathcal{D}^{\mathrm{Fl}}_{\kappa_c}$-mod is the category of twisted $\mathcal{D}$-modules on Fl and $\widetilde{\mathrm{St}}$ is the "thickened" Steinberg variety

$$\widetilde{\mathrm{St}} = \widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \widetilde{\widetilde{\mathcal{N}}} = \widetilde{\mathcal{N}} \underset{^L\mathfrak{g}}{\times} {}^L\widetilde{\mathfrak{g}}.$$

Morally, we expect that the two categories in (8.19) act on the two categories in (8.15) in a compatible way. However, strictly speaking, the left hand side of (8.19) acts like this:

$$D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^I \to D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})^{I^0},$$

and the right hand side of (8.19) acts like this:

$$D^b(\mathrm{QCoh}(\mathrm{MOp}^0_{^LG})) \to D^b(\mathrm{QCoh}(\mathrm{MOp}^{\mathrm{nilp}}_{^LG})).$$

So one needs a more precise statement, which may be found in [Bez2], Sect. 4.2. Alternatively, one can consider the corresponding actions of the affine braid group of ${}^L G$, as in [Bez2].

A special case of this compatibility concerns some special objects of the category $D^b(\mathcal{D}^{\mathrm{Fl}}_{\kappa_c}\text{-mod})^I$, the central sheaves introduced in [Ga1]. They correspond to the central elements of the affine Hecke algebra $H(G(F), I)$. These central elements act as scalars on irreducible $H(G(F), I)$-modules, as well as on the standard modules $K^A(\mathrm{Sp}_u)_{(\gamma,q,\rho)}$ discussed above. We have argued that the categories $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I^0}, \chi \in \mathrm{Op}^{\mathrm{nilp}}_{{}^L G}$, are categorical versions of these representations. Therefore it is natural to expect that its objects are "eigenmodules" with respect to the action of the central sheaves from $D^b(\mathcal{D}^{\mathrm{Fl}}_{\kappa_c}\text{-mod})^I$ (in the sense of Section 7.4). This has indeed been proved in [FG3].

This discussion indicates an intimate connection between the category $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})$ and the category of twisted $\mathcal{D}$-modules on the affine flag variety, which is similar to the connection between $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{reg}}$ and the category of twisted $\mathcal{D}$-modules on the affine Grassmannian which we discussed in Section 7.5. A more precise conjecture relating $D^b(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\mathrm{nilp}})$ and $D^b(\mathcal{D}^{\mathrm{Fl}}_{\kappa_c}\text{-mod})$ was formulated in [FG2] (see the Introduction and Sect. 6), where we refer the reader for more details. This conjecture may be viewed as an analogue of Conjecture 3 for nilpotent opers. As explained in [FG2], this conjecture is supported by the results of [AB, ABG] (see also [Bez1, Bez2]). Together, these results and conjectures provide additional evidence for the equivalence (8.15).

9 Ramified Global Langlands Correspondence

We now discuss the implications of the local Langlands correspondence for the global geometric Langlands correspondence.

We begin by briefly discussing the setting of the classical global Langlands correspondence.

9.1 The Classical Setting

Let X be a smooth projective curve over $\mathbb{F}_q$. Denote by F the field $\mathbb{F}_q(X)$ of rational functions on X. For any closed point x of X we denote by F_x the completion of F at x and by $\mathcal{O}_x$ its ring of integers. If we choose a local coordinate t_x at x (i.e., a rational function on X which vanishes at x to order one), then we obtain isomorphisms $F_x \simeq \mathbb{F}_{q_x}((t_x))$ and $\mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t_x]]$, where $\mathbb{F}_{q_x}$ is the residue field of x; in general, it is a finite extension of $\mathbb{F}_q$ containing $q_x = q^{\mathrm{ord}_x}$ elements.

Thus, we now have a local field attached to each point of X. The ring $\mathbb{A} = \mathbb{A}_F$ of **adèles** of F is by definition the **restricted** product of the fields F_x, where x runs over the set $|X|$ of all closed points of X. The word "restricted" means that we consider only the collections $(f_x)_{x \in |X|}$ of elements of F_x in which $f_x \in \mathcal{O}_x$ for all but finitely many x. The ring $\mathbb{A}$ contains the field F, which is embedded into $\mathbb{A}$ diagonally, by taking the expansions of rational functions on X at all points.

While in the local Langlands correspondence we considered irreducible smooth representations of the group GL_n over a local field, in the global Langlands correspondence we consider irreducible **automorphic representations** of the group $GL_n(\mathbb{A})$. The word "automorphic" means, roughly, that the representation may be realized in a reasonable space of functions on the quotient $GL_n(F) \backslash GL_n(\mathbb{A})$ (on which the group $GL_n(\mathbb{A})$ acts from the right).

On the other side of the correspondence we consider n-dimensional representations of the Galois group $\mathrm{Gal}(\overline{F}/F)$, or, more precisely, the Weil group W_F, which is a subgroup of $\mathrm{Gal}(\overline{F}/F)$ defined in the same way as in the local case.

Roughly speaking, the global Langlands correspondence is a bijection between the set of equivalence classes of n-dimensional representations of W_F and the set of equivalence classes of irreducible automorphic representations of $GL_n(\mathbb{A})$:

$$\boxed{\begin{array}{c} n\text{-dimensional representations} \\ \text{of } W_F \end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c} \text{irreducible automorphic} \\ \text{representations of } GL_n(\mathbb{A}) \end{array}}$$

The precise statement is more subtle. For example, we should consider the so-called ℓ-adic representations of the Weil group (while in the local case we considered the admissible complex representations of the Weil-Deligne group; the reason is that in the local case those are equivalent to the ℓ-adic representations). Moreover, under this correspondence important invariants attached to the objects appearing on both sides (Frobenius eigenvalues on the Galois side and the Hecke eigenvalues on the other side) are supposed to match. We refer the reader to Part I of the review [F6] for more details.

The global Langlands correspondence has been proved for GL_2 in the 80's by V. Drinfeld [Dr1]–[Dr4] and more recently by L. Lafforgue [Laf] for GL_n with an arbitrary n.

Like in the local story, we may also wish to replace the group GL_n by an arbitrary reductive algebraic group defined over F. Then on one side of the global Langlands correspondence we have homomorphisms $\sigma : W_F \to {}^L G$ satisfying some properties (or perhaps, some more refined data, as in [A]). We expect to be able to attach to each σ an **automorphic representation** π of $GL_n(\mathbb{A}_F)$.[17] The word "automorphic" again means, roughly, that the representation may be realized in a reasonable space of functions on the

[17] In this section, by abuse of notation, we will use the same symbol to denote a representation of the group and the vector space underlying this representation.

quotient $GL_n(F)\backslash GL_n(\mathbb{A})$ (on which the group $GL_n(\mathbb{A})$ acts from the right). We will not try to make this precise. In general, we expect not one but several automorphic representations assigned to σ which are the global analogues of the L-packets discussed above (see [A]). Another complication is that the multiplicity of a given irreducible automorphic representation in the space of functions on $GL_n(F)\backslash GL_n(\mathbb{A})$ may be greater than one. We will mostly ignore all of these issues here, as our main interest is in the geometric theory (note also that these issues do not arise if $G = GL_n$).

An irreducible automorphic representation may always be decomposed as the restricted tensor product $\bigotimes'_{x\in X} \pi_x$, where each π_x is an irreducible representation of $G(F_x)$. Moreover, for all by finitely many $x \in X$ the factor π_x is an **unramified** representation of $G(F_x)$: it contains a non-zero vector invariant under the maximal compact subgroup $K_{0,x} = G(\mathcal{O}_x)$ (see Section 7.1). Let us choose such a vector $v_x \in \pi_x$ (it is unique up to a scalar). The word "restricted" means that we consider the span of vectors of the form $\otimes_{x\in X} u_x$, where $u_x \in \pi_x$ and $u_x = v_x$ for all but finitely many $x \in X$.

An important property of the global Langlands correspondence is its compatibility with the local one. We can embed the Weil group W_{F_x} of each of the local fields F_x into the global Weil group W_F. Such an embedding is not unique, but it is well-defined up to conjugation in W_F. Therefore an equivalence class of $\sigma : W_F \to {}^L G$ gives rise to a well-defined equivalence class of $\sigma_x : W_{F_x} \to {}^L G$. We will impose the condition on σ that for all but finitely many $x \in X$ the homomorphism σ_x is unramified (see Section 7.1).

By the local Langlands correspondence, to σ_x one can attach an equivalence class of irreducible smooth representations π_x of $G(F_x)$.[18] Moreover, an unramified σ_x will correspond to an unramified irreducible representation π_x. The compatibility between local and global correspondences is the statement that the automorphic representation of $G(\mathbb{A})$ corresponding to σ should be isomorphic to the restricted tensor product $\bigotimes'_{x\in X} \pi_x$. Schematically, this is represented as follows:

$$\sigma \xleftrightarrow{\text{global}} \pi = \bigotimes_{x\in X}{}' \pi_x$$

$$\sigma_x \xleftrightarrow{\text{local}} \pi_x .$$

In this section we discuss an analogue of this local-to-global principle in the geometric setting and the implications of our local results and conjectures for the global geometric Langlands correspondence. We focus in particular on the unramified and tamely ramified Langlands parameters. At the end of the section we also discuss connections with irregular singularities.

[18] Here we are considering ℓ-adic homomorphisms from the Weil group W_{F_x} to ${}^L G$, and therefore we do not need to pass from the Weil group to the Weil-Deligne group.

9.2 The Unramified Case, Revisited

An important special case is when $\sigma : W_F \to {}^L G$ is everywhere unramified. Then for each $x \in X$ the corresponding homomorphism $\sigma_x : W_{F_x} \to {}^L G$ is unramified, and hence corresponds, as explained in Section 7.1, to a semisimple conjugacy class γ_x in ${}^L G$, which is the image of the Frobenius element under σ_x. This conjugacy class in turn gives rise to an unramified irreducible representation π_x of $G(F_x)$ with a unique, up to a scalar, vector v_x such that $G(\mathcal{O}_x)v_x = v_x$. The spherical Hecke algebra $H(G(F_x), G(\mathcal{O}_x)) \simeq \operatorname{Rep} {}^L G$ acts on this vector according to formula (7.5):

$$H_{V,x} \star v_x = \operatorname{Tr}(\gamma_x, V) v_x, \qquad [V] \in \operatorname{Rep} {}^L G. \tag{9.1}$$

The tensor product $v = \otimes_{x \in X} v_x$ of this vectors is a $G(\mathcal{O})$-invariant vector in $\pi = \bigotimes'_{x \in X} \pi_x$, which, according to the global Langlands conjecture is automorphic. This means that π is realized in the space of functions on $G(F) \backslash G(\mathbb{A}_F)$. In this realization vector v corresponds to a right $G(\mathcal{O})$-invariant function on $G(F) \backslash G(\mathbb{A}_F)$, or equivalently, a function on the double quotient

$$G(F) \backslash G(\mathbb{A}_F) / G(\mathcal{O}). \tag{9.2}$$

Thus, an unramified global Langlands parameter σ gives rise to a function on (9.2). This function is the **automorphic function** corresponding to σ. We denote it by f_π. Since it corresponds to a vector in an irreducible representation π of $G(\mathbb{A}_F)$, the entire representation π may be reconstructed from this function. Thus, we do not lose any information by passing from π to f_π.

Since $v \in \pi$ is an eigenvector of the Hecke operators, according to formula (9.1), we obtain that the function f_π is a **Hecke eigenfunction** on the double quotient (9.2). In fact, the local Hecke algebras $H(G(F_x), G(\mathcal{O}_x))$ act naturally (from the right) on the space of functions on (9.2), and f_π is an eigenfunction of this action. It satisfies the same property (9.1).

To summarize, the unramified global Langlands correspondence in the classical setting may be viewed as a correspondence between unramified homomorphisms $\sigma : W_F \to {}^L G$ and Hecke eigenfunctions on (9.2) (some irreducibility condition on σ needs to be added to make this more precise, but we will ignore this).

What should be the geometric analogue of this correspondence, when X is a complex algebraic curve?

As explained in Section 3.1, the geometric analogue of an unramified homomorphism $W_F \to {}^L G$ is a homomorphism $\pi_1(X) \to {}^L G$, or equivalently, since X is assumed to be compact, a holomorphic ${}^L G$-bundle on X with a holomorphic connection (it automatically gives rise to a flat connection). The global geometric Langlands correspondence should therefore associate to a flat holomorphic ${}^L G$-bundle on X a geometric object on a geometric version of the double quotient (9.2). As we argued in Section 3.3, this should be a $\mathcal{D}$-module on an algebraic variety whose set of points is (9.2).

Now, it is known that (9.2) is in bijection with the set of isomorphism classes of G-bundles on X. This key result is due to A. Weil (see, e.g., [F6], Sect. 3.2). This suggests that (9.2) is the set of points of the moduli space of G-bundles on X. Unfortunately, in general this is not an algebraic variety, but an algebraic stack, which locally looks like the quotient of an algebraic variety by an action of an algebraic group. We denote it by Bun_G. The theory of $\mathcal{D}$-modules has been developed in the setting of algebraic stacks like Bun_G in [BD1], and so we can use it for our purposes. Thus, we would like to attach to a flat holomorphic ${}^L G$-bundle E on X a $\mathcal{D}$-module Aut_E on Bun_G. This $\mathcal{D}$-module should satisfy an analogue of the Hecke eigenfunction condition, which makes it into a **Hecke eigensheaf** with eigenvalue E. This notion is spelled out in [F6], Sect. 6.1 (following [BD1]), where we refer the reader for details.

This brings us to the following question:

How to relate this global correspondence to the local geometric Langlands correspondence discussed above?

As we have already seen in Section 1, the key element in answering this question is a **localization functor** $\Delta_{\kappa_c,x}$ from $(\widehat{\mathfrak{g}}_{\kappa_c,x}, G(\mathcal{O}_x))$-modules to (twisted) $\mathcal{D}$-modules on Bun_G. In Section 1 we have applied this functor to the object $\mathbb{V}_0(\chi_x)$ of the Harish-Chandra category $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^{G(\mathcal{O}_x)}$, where $\chi_x \in \mathrm{Op}_{{}^L G}(D_x)$. For an oper χ_x which extends from D_x to the entire curve X we have obtained this way the Hecke eigensheaf associated to the underlying ${}^L G$-local system (see Theorem 1).

For a ${}^L G$-local system $E = (\mathcal{F}, \nabla)$ on X which does not admit the structure of a regular oper on X, the above construction may be modified as follows (see the discussion in [F6], Sect. 9.6, based on an unpublished work of Beilinson and Drinfeld). In this case one can choose an ${}^L B$-reduction $\mathcal{F}_{{}^L B}$ satisfying the oper condition away from a finite set of points $y_1, \ldots, y_n$ and such that the restriction χ_{y_i} of the corresponding oper χ on $X \backslash \{y_1, \ldots, y_n\}$ to $D_{y_i}^\times$ belongs to $\mathrm{Op}_{{}^L G}^{\lambda_i}(D_{y_i}) \subset \mathrm{Op}_{{}^L G}(D_{y_i}^\times)$ for some $\lambda_i \in P^+$. Then one can construct a Hecke eigensheaf corresponding to E by applying a multi-point version of the localization functor to the tensor product of the quotients $\mathbb{V}_{\lambda_i}(\chi_{y_i})$ of the Weyl modules $\mathbb{V}_{\lambda_i, y_i}$ (see [F6], Sect. 9.6).

The main lesson of this construction is that in the geometric setting the localization functor gives us a powerful tool for converting local Langlands categories, such as $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^{G(\mathcal{O}_x)}$, into global categories of Hecke eigensheaves. The category $\widehat{\mathfrak{g}}_{\kappa_c,x}\text{-mod}_{\chi_x}^{G(\mathcal{O}_x)}$ turns out to be very simple: it has a unique irreducible object, $\mathbb{V}_0(\chi_x)$. That is why it is sufficient to consider its image under the localization functor, which turns out to be the desired Hecke eigensheaf Aut_{E_χ}. For general opers, with ramification, the corresponding local categories are more complicated, as we have seen above, and so are the corresponding categories of Hecke eigensheaves. We will consider examples of these categories in the next section.

9.3 Classical Langlands Correspondence with Ramification

Let us first consider ramified global Langlands correspondence in the classical setting. Suppose that we are given a homomorphism $\sigma : W_F \to {}^L G$ that is ramified at finitely many points $y_1, \dots, y_n$ of X. Then we expect that to such σ corresponds an automorphic representation $\bigotimes'_{x \in X} \pi_x$ (more precisely, an L-packet of representations). Here π_x is still unramified for all $x \in X \backslash \{y_1, \dots, y_n\}$, but is *ramified* at $y_1, \dots, y_n$, i.e., the space of $G(\mathcal{O}_{y_i})$-invariant vectors in π_{y_i} is zero. In particular, consider the special case when each $\sigma_{y_i} : W_{F_{y_i}} \to {}^L G$ is tamely ramified (see Section 8.1 for the definition). Then, according to the results presented in Section 8.1, the corresponding L-packet of representations of $G(F_{y_i})$ contains an irreducible representation π_{y_i} with non-zero invariant vectors with respect to the Iwahori subgroup I_{y_i}. Let us choose such a representation for each point y_i.

Consider the subspace

$$\bigotimes_{i=1}^n \pi_{y_i}^{I_{y_i}} \otimes \bigotimes_{x \neq y_i} v_x \subset \bigotimes_{x \in X} {}' \pi_x, \tag{9.3}$$

where v_x is a $G(\mathcal{O}_x)$-vector in $\pi_x, x \neq y_i, i = 1, \dots, n$. Then, because $\bigotimes'_{x \in X} \pi_x$ is realized in the space of functions on $G(F) \backslash G(\mathbb{A}_F)$, we obtain that the subspace (9.3) is realized in the space of functions on the double quotient

$$G(F) \backslash G(\mathbb{A}_F) / \prod_{i=1}^n I_{y_i} \times \prod_{x \neq y_i} G(\mathcal{O}_x). \tag{9.4}$$

The spherical Hecke algebras $H(G(F_x), G(\mathcal{O}_x)), x \neq y_i$, act on the subspace (9.3), and all elements of (9.3) are eigenfunctions of these algebras (they satisfy formula (9.1)). At the points y_i we have, instead of the action of the commutative spherical Hecke algebra $H(G(F_{y_i}), G(\mathcal{O}_{y_i})$, the action of the non-commutative affine Hecke algebra $H(G(F_{y_i}), I_{y_i})$. Thus, we obtain a subspace of the space of functions on (9.4), which consists of Hecke eigenfunctions with respect to the spherical Hecke algebras $H(G(F_x), G(\mathcal{O}_x)), x \neq y_i$, and which realize a module over $\bigotimes_{i=1}^n H(G(F_{y_i}), I_{y_i})$ (which is irreducible, since each π_{y_i} is irreducible).

This subspace encapsulates the automorphic representation $\bigotimes'_{x \in X} \pi_x$ the way the automorphic function f_π encapsulates an unramified automorphic representation. The difference is that in the unramified case the function f_π spans the one-dimensional space of invariants of the maximal compact subgroup $G(\mathcal{O})$ in $\bigotimes'_{x \in X} \pi_x$, whereas in the tamely ramified case the subspace (9.3) is in general a multi-dimensional vector space.

9.4 Geometric Langlands Correspondence in the Tamely Ramified Case

Now let us see how this plays out in the geometric setting. As we discussed before, the analogue of a homomorphism $\sigma : W_F \to {}^L G$ tamely ramified at the points $y_1, \dots, y_n \in X$ is now a local system $E = (\mathcal{F}, \nabla)$, where $\mathcal{F}$ a ${}^L G$-bundle $\mathcal{F}$ on X with a connection ∇ that has regular singularities at $y_1, \dots, y_n$ and unipotent monodromies around these points. We will call such a local system **tamely ramified** at $y_1, \dots, y_n$. What should the global geometric Langlands correspondence attach to such a local system? It is clear that we need to find a geometric object replacing the finite-dimensional vector space (9.3) realized in the space of functions on (9.4).

Just as (9.2) is the set of points of the moduli stack Bun_G of G-bundles, the double quotient (9.4) is the set of points of the moduli stack $\mathrm{Bun}_{G,(y_i)}$ of G-bundles on X with the **parabolic structures** at $y_i, i = 1, \dots, n$. By definition, a parabolic structure of a G-bundle $\mathcal{P}$ at $y \in X$ is a reduction of the fiber $\mathcal{P}_y$ of $\mathcal{P}$ at y to a Borel subgroup $B \subset G$. Therefore, as before, we obtain that a proper replacement for (9.3) is a category of $\mathcal{D}$-modules on $\mathrm{Bun}_{G,(y_i)}$. As in the unramified case, we have the notion of a Hecke eigensheaf on $\mathrm{Bun}_{G,(y_i)}$. But because the Hecke functors are now defined using the Hecke correspondences over $X \backslash \{y_1, \dots, y_n\}$ (and not over X as before), an "eigenvalue" of the Hecke operators is now an ${}^L G$-local system on $X \backslash \{y_1, \dots, y_n\}$ (rather than on X). Thus, we obtain that the global geometric Langlands correspondence now should assign to a ${}^L G$-local system E on X, tamely ramified at the points $y_1, \dots, y_n$, a **category** $\mathcal{A}ut_E$ of $\mathcal{D}$-modules on $\mathrm{Bun}_{G,(y_i)}$ with the eigenvalue $E|_{X \backslash \{y_1, \dots, y_n\}}$,

$$E \mapsto \mathcal{A}ut_E.$$

We now construct these categories using a generalization of the localization functor we used in the unramified case (see [FG2]). For the sake of notational simplicity, let us assume that our ${}^L G$-local system $E = (\mathcal{F}, \nabla)$ is tamely ramified at a single point $y \in X$. Suppose that this local system on $X \backslash y$ admits the structure of a ${}^L G$-oper $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{{}^L B})$ whose restriction χ_y to the punctured disc $D_y^\times$ belongs to the subspace $\mathrm{Op}^{\mathrm{nilp}}_{{}^L G}(D_y)$ of nilpotent ${}^L G$-opers.

For a simple Lie group G, the moduli stack $\mathrm{Bun}_{G,y}$ has a realization analogous to (1.1):

$$\mathrm{Bun}_{G,y} \simeq G_{\mathrm{out}} \backslash G(\mathcal{K}_y)/I_y.$$

Let $\mathcal{D}_{\kappa_c, I_y}$ be the sheaf of twisted differential operators on $\mathrm{Bun}_{G,y}$ acting on the line bundle corresponding to the critical level (it is the pull-back of the square root of the canonical line bundle $K^{1/2}$ on Bun_G under the natural projection $\mathrm{Bun}_{G,y} \to \mathrm{Bun}_G$). Applying the formalism of the previous section, we obtain a localization functor

$$\Delta_{\kappa_c, I_y} : \widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}^{I_y} \to \mathcal{D}_{\kappa_c, I_y}\text{-mod}\,.$$

However, in order to make contact with the results obtained above we also consider the larger category $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}^{I_y^0}$ of I_y^0-equivariant modules, where $I_y^0 = [I_y, I_y]$.

Set

$$\mathrm{Bun}'_{G,y} = G_{\mathrm{out}} \backslash G(\mathcal{K}_y)/I_y^0,$$

and let $\mathcal{D}_{\kappa_c, I_y^0}$ be the sheaf of twisted differential operators on $\mathrm{Bun}'_{G,y}$ acting on the pull-back of the line bundle $K^{1/2}$ on Bun_G. Let $\mathcal{D}_{\kappa_c, I_y^0}$-mod be the category of $\mathcal{D}_{\kappa_c, I_y^0}$-modules. Applying the general formalism, we obtain a localization functor

$$\Delta_{\kappa_c, I_y^0} : \widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}^{I_y^0} \to \mathcal{D}_{\kappa_c, I_y^0}\text{-mod}. \tag{9.5}$$

We note that a version of the categorical affine Hecke algebra $\mathcal{H}(G(\mathcal{K}_y), I_y)$ discussed in Section 8.5 naturally acts on the derived categories of the above categories, and the functors Δ_{κ_c, I_y} and Δ_{κ_c, I_y^0} intertwine these actions. Equivalently, one can say that this functor intertwines the corresponding actions of the affine braid group associated to ${}^L G$ on the two categories (as in [Bez2]).

We now restrict the functors Δ_{κ_c, I_y} and Δ_{κ_c, I_y^0} to the subcategories $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y}$ and $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y^0}$, respectively. By using the same argument as in [BD1], we obtain the following analogue of Theorem 1.

Theorem 5. *Fix $\chi_y \in \mathrm{Op}_{{}^L G}^{\mathrm{nilp}}(D_y)$ and let M be an object of the category $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y}$ (resp. $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y^0}$). Then*

(1) *$\Delta_{\kappa_c, I_y}(M) = 0$ (resp., $\Delta_{\kappa_c, I_y^0}(M) = 0$) unless χ_y is the restriction of a regular oper $\chi = (\mathcal{F}, \nabla, \mathcal{F}_{{}^L B})$ on $X \backslash y$ to $D_y^\times$.*

(2) *In that case $\Delta_{\kappa_c, y}(M)$ (resp., $\Delta_{\kappa_c, I_y^0}(M)$) is a Hecke eigensheaf with the eigenvalue $E_\chi = (\mathcal{F}, \nabla)$.*

Thus, we obtain that if $\chi_y = \chi|_{D_y^\times}$, then the image of any object of $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y}$ under the functor Δ_{κ_c, I_y} belongs to the category $\mathcal{A}ut_{E_\chi}^{I_y}$ of Hecke eigensheaves on $\mathrm{Bun}_{G,y}$. Now consider the restriction of the functor Δ_{κ_c, I_y^0} to $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y^0}$. As discussed in Section 8.3, the category $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y^0}$ coincides with the corresponding category $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y, m}$ of I_y-monodromic modules. Therefore the image of any object of $\widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y^0}$ under the functor Δ_{κ_c, I_y^0} belongs to the subcategory $\mathcal{D}_{\kappa_c, I_y^0}^m$-mod of $\mathcal{D}_{\kappa_c, I_y^0}$-mod whose objects admit an increasing filtration such that the consecutive quotients are pull-backs of $\mathcal{D}_{\kappa_c, I_y}$-modules from $\mathrm{Bun}_{G,y}$. Such $\mathcal{D}_{\kappa_c, I_y^0}$–modules are called **monodromic**.

Let $\mathcal{A}ut_{E_\chi}^{I_y, m}$ be the subcategory of $\mathcal{D}_{\kappa_c, I_y^0}^m$-mod whose objects are Hecke eigensheaves with eigenvalue E_χ.

Thus, we obtain the functors

$$\Delta_{\kappa_c, I_y} : \widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y} \to \mathcal{A}ut_{E_\chi}^{I_y}, \qquad \Delta_{\kappa_c, I_y^0} : \widehat{\mathfrak{g}}_{\kappa_c, y}\text{-mod}_{\chi_y}^{I_y^0} \to \mathcal{A}ut_{E_\chi}^{I_y, m} \tag{9.6}$$

It is tempting to conjecture (see [FG2]) that these functors are equivalences of categories, at least for generic χ. Suppose that this is true. Then we may identify the *global* categories $\mathcal{A}ut_{E_\chi}^{I_y}$ and $\mathcal{A}ut_{E_\chi}^{I_y, m}$ of Hecke eigensheaves on

Bun_{G,I_y} and Bun'_{G,I_y^0} with the *local* categories $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y}$ and $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0}$, respectively. Therefore we can use our results and conjectures on the local Langlands categories, such as $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0}$, to describe the global categories of Hecke eigensheaves on the moduli stacks of G-bundles on X with parabolic structures.

We have the following conjectural description of the derived category of I_y^0-equivariant modules, $D^b(\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y})^{I_y^0}$ (see formula (8.16)):

$$D^b(\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y})^{I_y^0} \simeq D^b(\mathrm{QCoh}(\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_{\mathrm{Res}(\chi_y)})). \tag{9.7}$$

The corresponding I_y-equivariant version is

$$D^b(\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y})^{I_y} \simeq D^b(\mathrm{QCoh}(\mathrm{Sp}^{\mathrm{DG}}_{\mathrm{Res}(\chi_y)})), \tag{9.8}$$

where we replace the non-reduced DG Springer fiber by the reduced one: it is defined as the DG fiber of the morphism $\widetilde{\mathcal{N}} \to \mathfrak{g}$ over u.

If the functors (9.6) are equivalences, then by combining them with (9.7) and (9.8), we obtain the following conjectural equivalences of categories:

$$D^b(\mathcal{A}ut_{E_\chi}^{I_y}) \simeq D^b(\mathrm{QCoh}(\mathrm{Sp}^{\mathrm{DG}}_{\mathrm{Res}(\chi_y)})), \quad D^b(\mathcal{A}ut_{E_\chi}^{I_y,m}) \simeq D^b(\mathrm{QCoh}(\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_{\mathrm{Res}(\chi_y)})). \tag{9.9}$$

In other words, the derived category of a global Langlands category (monodromic or not) corresponding to a local system tamely ramified at $y \in X$ is equivalent to the derived category of quasicoherent sheaves on the DG Springer fiber of its residue at y (non-reduced or reduced).

Again, these equivalences are supposed to intertwine the natural actions on the above categories of the categorical affine Hecke algebra $\mathcal{H}(G(\mathcal{K}_y), I_y)$ (or, equivalently, the affine braid group associated to ${}^LG$).

The categories appearing in (9.9) actually make sense for an arbitrary ${}^LG$-local system E on X tamely ramified at y. It is therefore tempting to conjecture that these equivalences still hold in general:

$$D^b(\mathcal{A}ut_{E}^{I_y}) \simeq D^b(\mathrm{QCoh}(\mathrm{Sp}^{\mathrm{DG}}_{\mathrm{Res}(E)})), \quad D^b(\mathcal{A}ut_{E}^{I_y,m}) \simeq D^b(\mathrm{QCoh}(\widetilde{\mathrm{Sp}}^{\mathrm{DG}}_{\mathrm{Res}(E)})). \tag{9.10}$$

The corresponding localization functors are constructed as follows: we represent a general local system E on X with tame ramification at y by an oper χ on the complement of finitely many points $y_1, \ldots, y_n$, whose restriction to $D_{y_i}^\times$ belongs to $\mathrm{Op}_{{}^LG}^{\lambda_i}(D_{y_i}) \subset \mathrm{Op}_{{}^LG}(D_{y_i}^\times)$ for some $\lambda_i \in P^+$. Then, in the same way as in the unramified case, we construct localization functors from $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y}$ to $\mathcal{A}ut_E^{I_y}$ and from $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0}$ to $\mathcal{A}ut_E^{I_y,m}$ (here, as before, $\chi_y = \chi|_{D_y^\times}$), and this leads us to the conjectural equivalences (9.10).

The equivalences (9.10) also have family versions in which we allow E to vary. It is analogous to the family version (8.15) of the local equivalences. As in the local case, in a family version we can avoid using DG schemes.

The above construction may be generalized to allow local systems tamely ramified at finitely many points $y_1, \ldots, y_n$. The corresponding Hecke eigensheaves are then $\mathcal{D}$-modules on the moduli stack of G-bundles on X with parabolic structures at $y_1, \ldots, y_n$. Non-trivial examples of these Hecke eigensheaves arise already in genus zero. These sheaves were constructed explicitly in [F1] (see also [F4, F5]), and they are closely related to the Gaudin integrable system.

9.5 Connections with Regular Singularities

So far we have only considered the categories of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules corresponding to ${}^LG$-opers on X which are regular everywhere except at a point $y \in X$ (or perhaps, at several points) and whose restriction to $D_y^\times$ is a nilpotent oper χ_y in $\mathrm{Op}^{\mathrm{nilp}}_{{}^LG}(D_y)$. In other words, χ_y is an oper with regular singularity at y with residue $\varpi(-\rho)$ (where $\varpi : \mathfrak{h}^* \to \mathfrak{h}^*/W$). However, we can easily generalize the localization functor to the categories of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules corresponding to ${}^LG$-opers which have regular singularity at y with *arbitrary* residue.

So suppose we are given an oper $\chi \in \mathrm{Op}^{\mathrm{RS}}_{{}^LG}(D)_{\varpi(-\lambda-\rho)}$ with regular singularity and residue $\varpi(-\lambda-\rho)$, where $\lambda \in \mathfrak{h}^*$. In this case the monodromy of this oper around y is conjugate to

$$M = \exp(2\pi i(\lambda + \rho)) = \exp(2\pi i \lambda).$$

We then have the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I^0}$ of I^0-equivariant $\widehat{\mathfrak{g}}_{\kappa_c}$-modules with central character χ. The case of $\lambda = 0$ is an "extremal" case when the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I^0}$ is most complicated. On the other "extreme" is the case of generic opers χ, corresponding to a generic λ. In this case one can show that the category $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I^0}$ is quite simple: it contains irreducible objects $\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi)$ labeled by the Weyl group of $\mathfrak{g}$, and each object of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I^0}$ is a direct sum of these irreducible modules. Here $\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi)$ is the quotient of the Verma module

$$\mathbb{M}_{w(\lambda+\rho)-\rho} = \mathrm{Ind}_{\widehat{\mathfrak{b}}_+ \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{C}_{w(\lambda+\rho)-\rho}, \qquad w \in W,$$

by the central character corresponding to χ.

For other values of λ the structure of $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi^{I^0}$ is somewhere in-between these two extreme cases.

Recall that we have a localization functor (9.5)

$$\Delta^\lambda_{\kappa_c, I^0_y} : \widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}^{I^0_y} \to \mathcal{D}_{\kappa_c, I^0_y}\text{-mod}\,.$$

from $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I^0_y}$ to a category of $\mathcal{D}$-modules on Bun'_{G,I_y} twisted by the pull-back of the line bundle $K^{1/2}$ on Bun_G. We now restrict this functor to the subcategory $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I^0_y}$ where χ_y is a ${}^LG$-oper on D_y with regular singularity at y and residue $\varpi(-\lambda-\rho)$.

Consider first the case when $\lambda \in \mathfrak{h}^*$ is generic. Suppose that χ_y extends to a regular oper χ on $X \backslash y$. One then shows in the same way as in Theorem 5 that for any object M of $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0}$ the corresponding $\mathcal{D}_{\kappa_c,I_y^0}$-module $\Delta_{\kappa_c,I_y^0}(M)$ is a Hecke eigensheaf with eigenvalue E_χ, which is the ${}^L G$-local system on X with regular singularity at y underlying χ (if χ_y cannot be extended to $X \backslash y$, then $\Delta^\lambda_{\kappa_c,I_y}(M) = 0$, as before). Therefore we obtain a functor

$$\Delta_{\kappa_c,I_y^0} : \widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0} \to \mathcal{A}ut_{E_\chi}^{I_y^0},$$

where $\mathcal{A}ut_{E_\chi}^{I_y^0}$ is the category of Hecke eigensheaves on Bun'_{G,I_y} with eigenvalue E_χ.

Since we have assumed that the residue of the oper χ_y is generic, the monodromy of E_χ around y belongs to a regular semi-simple conjugacy class of ${}^L G$ containing $\exp(2\pi i\lambda)$. In this case the category $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0}$ is particularly simple, as we have discussed above. We expect that the functor Δ_{κ_c,I_y^0} sets up an equivalence between $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0}$ and $\mathcal{A}ut_{E_\chi}^{I_y^0}$.

We can formulate this more neatly as follows. For $M \in {}^L G$ let $\mathcal{B}_M$ be the variety of Borel subgroups containing M. Observe that if M is regular semi-simple, then $\mathcal{B}_M$ is a set of points which is in bijection with W. Therefore our conjecture is that $\mathcal{A}ut_{E_\chi}^{I_y^0}$ is equivalent to the category $\mathrm{QCoh}(\mathcal{B}_M)$ of quasicoherent sheaves on $\mathcal{B}_M$, where M is a representative of the conjugacy class of the monodromy of E_χ.

Consider now an arbitrary ${}^L G$-local system E on X with regular singularity at $y \in X$ whose monodromy around y is regular semi-simple. It is then tempting to conjecture that, at least if E is generic, this category has the same structure as in the case when E has the structure of an oper, i.e., it is equivalent to the category $\mathrm{QCoh}(\mathcal{B}_M)$, where M is a representative of the conjugacy class of the monodromy of E around y.

On the other hand, if the monodromy around y is unipotent, then $\mathcal{B}_M$ is nothing but the Springer fiber Sp_u, where $M = \exp(2\pi i u)$. The corresponding category $\mathcal{A}ut_E^{I_y^0}$ was discussed in Section 9.4 (we expect that it coincides with $\mathcal{A}ut_E^{I_y,m}$). Thus, we see that in both “extreme” cases: unipotent monodromy and regular semi-simple monodromy, our conjectures identify the derived category of $\mathcal{A}ut_E^{I_y^0}$ with the derived category of the category $\mathrm{QCoh}(\mathcal{B}_M)$ (where $\mathcal{B}_M$ should be viewed as a DG scheme $\widetilde{\mathrm{Sp}}_u^{\mathrm{DG}}$ in the unipotent case). One is then led to conjecture, most ambitiously, that for *any* ${}^L G$-local system E on X with regular singularity at $y \in X$ the derived category of $\mathcal{A}ut_E^{I_y^0}$ is equivalent to the derived category of quasicoherent sheaves on a suitable DG version of the scheme $\mathcal{B}_M$, where M is a representative of the conjugacy class of the monodromy of E around y:

$$D^b(\mathcal{A}ut_E^{I_y^0}) \simeq D^b(\mathrm{QCoh}(\mathcal{B}_M^{\mathrm{DG}})).$$

This has an obvious generalization to the case of multiple ramification points, where on the right hand side we take the Cartesian product of the varieties $\mathcal{B}^{\mathrm{DG}}_{M_i}$ corresponding to the monodromies. Thus, we obtain a conjectural realization of the categories of Hecke eigensheaves, whose eigenvalues are local systems with regular singularities, in terms of categories of quasicoherent sheaves.

It is useful to note that the Hecke eigensheaves on Bun'_{G,I_y} obtained above via the localization functors may be viewed as pull-backs of twisted $\mathcal{D}$-modules on Bun_{G,I_y} (or, more generally, extensions of such pull-backs).

More precisely, for each $\lambda \in \mathfrak{h}^*$ we have the sheaf of twisted differential operators on $\mathrm{Bun}_{G,y}$ acting on a "line bundle" $\widetilde{\mathcal{L}}_\lambda$. If λ were an integral weight, this would be an actual line bundle, which is constructed as follows: note that the map $p : \mathrm{Bun}_{G,I_y} \to \mathrm{Bun}_G$, corresponding to forgetting the parabolic structure, is a fibration with the fibers isomorphic to the flag manifold G/B. For each integral weight λ we have the G-equivariant line bundle $\ell_\lambda = G \underset{B}{\times} \mathbb{C}_\lambda$ on G/B. The line bundle $\mathcal{L}_\lambda$ on Bun_{G,I_y} is defined in such a way that its restriction to each fiber of the projection p is isomorphic to ℓ_λ. We then set $\widetilde{\mathcal{L}}_\lambda = \mathcal{L}_\lambda \otimes p^*(K^{1/2})$, where $K^{1/2}$ is the square root of the canonical line bundle on Bun_G corresponding to the critical level. Now, it is well-known (see, e.g., [BB]) that even though the line bundle $\widetilde{\mathcal{L}}_\lambda$ does not exist if λ is not an integral weight, the corresponding sheaf $\mathcal{D}^\lambda_{\kappa_c,I_y}$ of $\widetilde{\mathcal{L}}_\lambda$-twisted differential operators on Bun_{G,I_y} is still well-defined.

Observe that we have an equivalence between the category $\mathcal{D}^\lambda_{\kappa_c,I_y}$-mod and the category of weakly H-equivariant $\mathcal{D}_{\kappa_c,I^0_y}$-module on $\mathrm{Bun}'_{G,y}$ on which $\mathfrak{h}$ acts via the character $\lambda : \mathfrak{h} \to \mathbb{C}$. If $\mathcal{F}$ is an object of $\mathcal{D}^\lambda_{\kappa_c,I_y}$-mod, then the corresponding weakly H-equivariant $\mathcal{D}_{\kappa_c,I^0_y}$-module on $\mathrm{Bun}'_{G,y}$ is $\pi^*(\mathcal{F})$, where $\pi : \mathrm{Bun}'_{G,y} \to \mathrm{Bun}_{G,I_y}$.

Now, it is easy to see that the $\mathcal{D}_{\kappa_c,I^0_y}$-modules on $\mathrm{Bun}'_{G,y}$ obtained by applying the localization functor Δ_{κ_c,I^0_y} to objects of $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}^{I^0_y}_{\chi_y}$ are always weakly H-equivariant. Consider, for example, the case when χ_y is a generic oper with regular singularity at y. Then its residue is equal to $\varpi(-\lambda-\rho)$, where λ is a regular element of $\mathfrak{h}^*$, and so its monodromy is $M = \exp(2\pi i\lambda)$. The corresponding category $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}^{I^0_y}_{\chi_y}$ has objects $\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi_y)$ that we introduced above. The Cartan subalgebra $\mathfrak{h}$ of $\widehat{\mathfrak{g}}_{\kappa_c,y}$ acts on $\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi_y)$ semi-simply with the eigenvalues given by the weights of the form $w(\lambda+\rho)-\rho+\mu$, where μ is an integral weight. In other words,

$$\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi_y) \otimes \mathbb{C}_{-w(\lambda+\rho)+\rho}$$

is I_y-equivariant. Therefore we find that $\Delta_{\kappa_c,I^0_y}(\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi_y))$ is weakly H-equivariant, and the corresponding action of $\mathfrak{h}$ is given by $w(\lambda+\rho)-\rho : \mathfrak{h} \to \mathbb{C}$. Thus, $\Delta_{\kappa_c,I^0_y}(\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi_y))$ is the pull-back of a $\mathcal{D}^{w(\lambda+\rho)-\rho}_{\kappa_c,I_y}$-module on $\mathrm{Bun}_{G,y}$. This $\mathcal{D}^{w(\lambda+\rho)-\rho}_{\kappa_c,I_y}$-module is a Hecke eigensheaf with eigenvalue E_χ provided that $\chi_y = \chi|_{D_y^\times}$, where χ is a regular oper on $X\backslash y$.

Thus, for a given generic oper χ_y we have $|W|$ different Hecke eigensheaves

$$\Delta_{\kappa_c,I_y^0}(\mathbb{M}_{w(\lambda+\rho)-\rho}(\chi_y)), \qquad w \in W,$$

on $\mathrm{Bun}'_{G,y}$. However, each of them is the pull-back of a twisted $\mathcal{D}$-module on $\mathrm{Bun}_{G,y}$ corresponding to a particular twist: namely, by a "line bundle" $\widetilde{\mathcal{L}}_{w(\lambda+\rho)-\rho}$. (Since we have assumed that λ is generic, all of these twists are different; note also that if $\mu = w(\lambda+\rho)-\rho$, then $\exp(2\pi i\mu)$ is in the conjugacy class of the monodromy $\exp(2\pi i\lambda)$.) It is therefore natural to conjecture that there is a unique Hecke eigensheaf on $\mathrm{Bun}_{G,y}$ with eigenvalue E_χ, which is a twisted $\mathcal{D}$-module with the twisting given by $\widetilde{\mathcal{L}}_{w(\lambda+\rho)-\rho}$.

More generally, suppose that E is a local system on X with regular singularity at y and generic regular semi-simple monodromy. Let us choose a representative M of the monodromy which belongs to the Cartan subgroup ${}^L H \subset {}^L G$. Choose $\mu \in \mathfrak{h}^* \simeq {}^L\mathfrak{h}$ to be such that $M = \exp(2\pi i\mu)$. Note that there are exactly $|W|$ such choices up to a shift by an integral weight ν. Let $\mathcal{A}ut_E^{I_y,\mu}$ be the category of Hecke eigensheaves with eigenvalue E in the category of twisted $\mathcal{D}$-modules on Bun_{G,I_y} with the twisting given by $\widetilde{\mathcal{L}}_\mu$. Then we expect that for generic E the category $\mathcal{A}ut_E^{I_y,\mu}$ has a unique irreducible object. Its pull-back to $\mathrm{Bun}'_{G,y}$ is one of the $|W|$ irreducible objects of $\mathcal{A}ut_E^{I_y^0}$. (Note that tensoring with the line bundle $\mathcal{L}_\nu$, where ν is an integral weight, we identify the categories $\mathcal{A}ut_E^{I_y,\mu}$ and $\mathcal{A}ut_E^{I_y,\mu'}$ if $\mu' = \mu+\nu$.)

Similarly, one can describe the Hecke eigensheaves on $\mathrm{Bun}'_{G,y}$ obtained by applying Δ_{κ_c,I_y^0} to the categories $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{I_y^0}$ for other opers χ_y in terms of twisted $\mathcal{D}$-modules on $\mathrm{Bun}_{G,y}$. In the opposite extreme case, when the residue of χ_y is 0 (and so χ_y is a nilpotent oper), this is explained in Section 9.4. (In this case one may choose to consider monodromic $\mathcal{D}$-modules; this is not necessary if λ is generic, because in this case there are no non-trivial extensions.)

Finally, it is natural to ask whether these equivalences for individual local systems may be combined into a family version encompassing all of them. The global geometric Langlands correspondence in the unramified case may be viewed as a kind of non-abelian Fourier-Mukai transform relating the (derived) category of $\mathcal{D}$-modules on Bun_G and the (derived) category of quasicoherent sheaves on $\mathrm{Loc}_{{}^L G}(X)$, the stack of ${}^L G$-local systems on the curve X. Under this correspondence, the skyscraper sheaf supported at a ${}^L G$-local system E is supposed to go to the Hecke eigensheaf Aut_E on Bun_G. Thus, one may think of $\mathrm{Loc}_{{}^L G}(X)$ as a parameter space of a "spectral decomposition" of the derived category of $\mathcal{D}$-modules on Bun_G (see, e.g., [F6], Sect. 6.2, for more details).

The above results and conjectures suggest that one may also view the geometric Langlands correspondence in the tamely ramified case in a similar way. Now the role of $\mathrm{Loc}_{^LG}(X)$ should be played by the stack $\mathrm{Loc}_{^LG,y}(X)$ of parabolic ^LG-local systems with regular singularity at $y \in X$ (or, more generally, multiple points) and unipotent monodromy. This stack classifies triples $(\mathcal{F}, \nabla, \mathcal{F}_{^LB,y})$, where $\mathcal{F}$ is a ^LG-bundle on X, ∇ is a connection on $\mathcal{F}$ with regular singularity at y and unipotent monodromy, and $\mathcal{F}_{^LB,y}$ is a ^LB-reduction of the fiber $\mathcal{F}_y$ of $\mathcal{F}$ at y, which is preserved by ∇. This stack is now a candidate for a parameter space of a "spectral decomposition" of the derived category of $\mathcal{D}$-modules on the moduli stack $\mathrm{Bun}_{G,y}$ of G-bundles with parabolic structure at y.[19]

9.6 Irregular Connections

We now generalize the above results to the case of connections with irregular singularities. Let $\mathcal{F}$ be a ^LG-bundle on X with connection ∇ that is regular everywhere except for a point $y \in X$, where it has a pole of order greater than 1. As before, we assume first that $(\mathcal{F}, \nabla)$ admits the structure of a ^LG-oper on $X \backslash y$, which we denote by χ. Let χ_y be the the restriction of χ to $D_y^\times$. A typical example of such an oper is a ^LG-oper with pole of order $\leq n$ on the disc D_y, which is, by definition (see [BD1], Sect. 3.8.8), an $^LN[[t]]$-conjugacy class of operators of the form

$$\nabla = \partial_t + \frac{1}{t^n} \left(p_{-1} + \mathbf{v}(t) \right), \qquad \mathbf{v}(t) \in {}^L\mathfrak{b}[[t]]. \tag{9.11}$$

We denote the space of such opers by $\mathrm{Op}_{^LG}^{\leq n}(D_y)$.

One can show that for $\chi_y \in \mathrm{Op}_{^LG}^{\leq n}(D_y)$ the category $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^K$ is non-trivial if K is the congruence subgroup $K_{m,y} \subset G(\mathcal{O}_y)$ with $m \geq n$. (We recall that for $m > 0$ we have $K_{m,y} = \exp(\mathfrak{g} \otimes (\mathfrak{m}_y)^m)$, where $\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_y$.) Let us take the category $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{K_{n,y}}$. Then our general formalism gives us a localization functor

$$\Delta_{\kappa_c,K_{n,y}} : \widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{K_{n,y}} \to \mathcal{D}_{\kappa_c,K_{n,y}}\text{-mod},$$

where $\mathcal{D}_{\kappa_c,K_{n,y}}$-mod is the category of critically twisted[20] $\mathcal{D}$-modules on

$$\mathrm{Bun}_{G,y,n} \simeq G_{\mathrm{out}} \backslash G(\mathcal{K}_y)/K_{n,y}.$$

This is the moduli stack of G-bundles on X with a level n structure at $y \in X$ (which is a trivialization of the restriction of the G-bundle to the nth infinitesimal neighborhood of y).

[19] One may also try to extend this "spectral decomposition" to the case of all connections with regular singularities, but here the situation is more subtle, as can already be seen in the abelian case.

[20] this refers to the twisting by the line bundle on $\mathrm{Bun}_{G,y,n}$ obtained by pull-back of the line bundle $K^{1/2}$ on Bun_G, as before

In the same way as above, one shows that the $\mathcal{D}$-modules obtained by applying $\Delta_{\kappa_c,K_{n,y}}$ to objects of $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{K_{n,y}}$ are Hecke eigensheaves with the eigenvalue $E_\chi|_{X\backslash y}$, where E_χ is the ${}^LG$-local system underlying the oper χ. Let $\mathcal{A}ut_{E_\chi}^{K_{n,y}}$ be the category of these eigensheaves. Thus, we really obtain a functor

$$\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{K_{n,y}} \to \mathcal{A}ut_{E_\chi}^{K_{n,y}}.$$

By analogy with the case of regular connections, we expect that this functor is an equivalence of categories.

As before, this functor may be generalized to an arbitrary flat bundle $E = (\mathcal{F}, \nabla)$, where ∇ has singularity at y, by representing it as an oper with mild ramification at additional points $y_1, \ldots, y_m$ on X. Let χ_y be the restriction of this oper to $D_y^\times$. Then it belongs to $\mathrm{Op}_{{}^LG}^{\leq n}(D_y)$ for some n, and we obtain a functor

$$\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{K_{n,y}} \to \mathcal{A}ut_{E}^{K_{n,y}},$$

which we expect to be an equivalence of categories for generic E. This also has an obvious multi-point generalization.

This way we obtain a conjectural description of the categories of Hecke eigensheaves corresponding to (generic) connections on X with arbitrary singularities at finitely many points in terms of categories of Harish-Chandra modules of critical level over $\widehat{\mathfrak{g}}$. However, in the case of regular singularities, we also have an alternative description of these categories: in terms of (derived) categories of quasicoherent sheaves on the varieties $\mathcal{B}_M^{\mathrm{DG}}$. It would be desirable to obtain such a description for irregular connections as well.

Finally, we remark that the above construction has a kind of limiting version where we take the infinite level structure at y, i.e., a trivialization of the restriction of a G-bundle to the disc D_y. Let $\mathrm{Bun}_{G,y,\infty}$ be the moduli stack of G-bundles on X with an infinite level structure at y. Then

$$\mathrm{Bun}_{G,y,\infty} \simeq G_{\mathrm{out}} \backslash G(\mathcal{K}_y).$$

We now have a localization functor

$$\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y} \to \mathcal{A}ut_E^\infty,$$

where E and χ_y are as above, and $\mathcal{A}ut_E^\infty$ is the category of Hecke eigensheaves on $\mathrm{Bun}_{G,y,\infty}$ with eigenvalue $E|_{X\backslash y}$. Thus, instead of the category $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}^{K_{n,y}}$ of Harish-Chandra modules we now have the category $\widehat{\mathfrak{g}}_{\kappa_c,y}\text{-mod}_{\chi_y}$ of all (smooth) $\widehat{\mathfrak{g}}_{\kappa_c,y}$-modules with fixed central character (corresponding to χ).

According to our general local conjecture, this is precisely the local Langlands category associated to the restriction of the local system E to $D_y^\times$ (equipped with an action of the loop group $G(\mathcal{K}_y)$). It is natural to assume that for generic E this functor establishes an equivalence between this category and the category $\mathcal{A}ut_E^\infty$ of Hecke eigensheaves on $\mathrm{Bun}_{G,y,\infty}$ (also equipped with an action of the loop group $G(\mathcal{K}_y)$). This may be thought of as the ultimate form of the local–to–global compatibility in the geometric Langlands Program:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathcal{A}ut_E^\infty \\ E|_{D_y^\times} \arrow[r] & \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_{\chi_y} \arrow[u] \end{tikzcd} .$$

Let us summarize: by using representation theory of affine Kac-Moody algebras at the critical level we have constructed the local Langlands categories corresponding to the local Langlands parameters: ${}^L G$-local systems on the punctured disc. We then applied the technique of localization functors to produce from these local categories, the global categories of Hecke eigensheaves on the moduli stacks of G-bundles on a curve X with parabolic (or level) structures. These global categories correspond to the global Langlands parameters: ${}^L G$-local systems on X with ramification. We have used our results and conjectures on the structure of the local categories to investigate these global categories. We hope that in this way representation theory of affine Kac-Moody algebras may one day fulfill the dream of uncovering the mysteries of the geometric Langlands correspondence.

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