

---

## Contents

<b>Applications of Representation Theory to Harmonic Analysis of Lie Groups (and Vice Versa)</b>	
<i>Michael Cowling</i> .....	1
1 Basic Facts of Harmonic Analysis on Semisimple Groups and Symmetric Spaces .....	2
1.1 Structure of Semisimple Lie Algebras .....	2
1.2 Decompositions of Semisimple Lie Groups .....	4
1.3 Parabolic Subgroups .....	5
1.4 Spaces of Homogeneous Functions on $G$ .....	6
1.5 The Plancherel Formula .....	8
2 The Equations of Mathematical Physics on Symmetric Spaces .....	10
2.1 Spherical Analysis on Symmetric Spaces .....	10
2.2 Harmonic Analysis on Semisimple Groups and Symmetric Spaces .....	12
2.3 Regularity of the Laplace–Beltrami Operator .....	16
2.4 Approaches to the Heat Equation .....	18
2.5 Estimates for the Heat and Laplace Equations .....	18
2.6 Approaches to the Wave and Schrödinger Equations .....	20
2.7 Further Results .....	21
3 The Vanishing of Matrix Coefficients .....	22
3.1 Some Examples in Representation Theory .....	22
3.2 Matrix Coefficients of Representations of Semisimple Groups ..	24
3.3 The Kunze–Stein Phenomenon .....	27
3.4 Property $T$ .....	28
3.5 The Generalised Ramanujan–Selberg Property .....	29
4 More General Semisimple Groups .....	31
4.1 Graph Theory and its Riemannian Connection .....	31
4.2 Cayley Graphs .....	32
4.3 An Example Involving Cayley Graphs .....	33
4.4 The Field of $p$ -adic Numbers .....	34
4.5 Lattices in Vector Spaces over Local Fields .....	35
4.6 Adèles .....	36

## VIII Contents

4.7 Further Results . . . . .	37
5 Carnot–Carathéodory Geometry and Group Representations . . . . .	38
5.1 A Decomposition for Real Rank One Groups . . . . .	38
5.2 The Conformal Group of the Sphere in $\mathbb{R}^n$ . . . . .	38
5.3 The Groups $SU(1, n + 1)$ and $Sp(1, n + 1)$ . . . . .	41
References . . . . .	46

## Ramifications of the Geometric Langlands Program

<i>Edward Frenkel</i> . . . . .	51
Introduction . . . . .	51
1 The Unramified Global Langlands Correspondence . . . . .	56
2 Classical Local Langlands Correspondence . . . . .	61
2.1 Langlands Parameters . . . . .	61
2.2 The Local Langlands Correspondence for $GL_n$ . . . . .	62
2.3 Generalization to Other Reductive Groups . . . . .	63
3 Geometric Local Langlands Correspondence over $\mathbb{C}$ . . . . .	64
3.1 Geometric Langlands Parameters . . . . .	64
3.2 Representations of the Loop Group . . . . .	65
3.3 From Functions to Sheaves . . . . .	66
3.4 A Toy Model . . . . .	68
3.5 Back to Loop Groups . . . . .	70
4 Center and Opers . . . . .	71
4.1 Center of an Abelian Category . . . . .	71
4.2 Opers . . . . .	73
4.3 Canonical Representatives . . . . .	75
4.4 Description of the Center . . . . .	76
5 Opers vs. Local Systems . . . . .	77
6 Harish–Chandra Categories . . . . .	81
6.1 Spaces of $K$ -Invariant Vectors . . . . .	81
6.2 Equivariant Modules . . . . .	82
6.3 Categorical Hecke Algebras . . . . .	83
7 Local Langlands Correspondence: Unramified Case . . . . .	85
7.1 Unramified Representations of $G(F)$ . . . . .	85
7.2 Unramified Categories $\widehat{\mathfrak{g}}_{\kappa_c}$ -Modules . . . . .	87
7.3 Categories of $G[[t]]$ -Equivariant Modules . . . . .	88
7.4 The Action of the Spherical Hecke Algebra . . . . .	90
7.5 Categories of Representations and $\mathcal{D}$ -Modules . . . . .	92
7.6 Equivalences Between Categories of Modules . . . . .	96
7.7 Generalization to other Dominant Integral Weights . . . . .	98
8 Local Langlands Correspondence: Tamely Ramified Case . . . . .	99
8.1 Tamely Ramified Representations . . . . .	99
8.2 Categories Admitting $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish–Chandra Modules . . . . .	103
8.3 Conjectural Description of the Categories of $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish–Chandra Modules . . . . .	105
8.4 Connection between the Classical and the Geometric Settings . . . . .	109

8.5	Evidence for the Conjecture . . . . .	115
9	Ramified Global Langlands Correspondence . . . . .	117
9.1	The Classical Setting . . . . .	117
9.2	The Unramified Case, Revisited . . . . .	120
9.3	Classical Langlands Correspondence with Ramification . . . . .	122
9.4	Geometric Langlands Correspondence in the Tamely Ramified Case . . . . .	122
9.5	Connections with Regular Singularities . . . . .	126
9.6	Irregular Connections . . . . .	130
	References . . . . .	132

### **Equivariant Derived Category and Representation of Real Semisimple Lie Groups**

	<i>Masaki Kashiwara</i> . . . . .	137
1	Introduction . . . . .	137
1.1	Harish-Chandra Correspondence . . . . .	138
1.2	Beilinson-Bernstein Correspondence . . . . .	140
1.3	Riemann-Hilbert Correspondence . . . . .	141
1.4	Matsuki Correspondence . . . . .	142
1.5	Construction of Representations of $G_{\mathbb{R}}$ . . . . .	143
1.6	Integral Transforms . . . . .	146
1.7	Commutativity of Fig. 1 . . . . .	147
1.8	Example . . . . .	148
1.9	Organization of the Note . . . . .	151
2	Derived Categories of Quasi-abelian Categories . . . . .	152
2.1	Quasi-abelian Categories . . . . .	152
2.2	Derived Categories . . . . .	154
2.3	$t$ -Structure . . . . .	156
3	Quasi-equivariant $D$ -Modules . . . . .	158
3.1	Definition . . . . .	158
3.2	Derived Categories . . . . .	162
3.3	Sumihiro's Result . . . . .	163
3.4	Pull-back Functors . . . . .	167
3.5	Push-forward Functors . . . . .	168
3.6	External and Internal Tensor Products . . . . .	170
3.7	Semi-outer Hom . . . . .	171
3.8	Relations of Push-forward and Pull-back Functors . . . . .	172
3.9	Flag Manifold Case . . . . .	175
4	Equivariant Derived Category . . . . .	176
4.1	Introduction . . . . .	176
4.2	Sheaf Case . . . . .	176
4.3	Induction Functor . . . . .	179
4.4	Constructible Sheaves . . . . .	179
4.5	$D$ -module Case . . . . .	180
4.6	Equivariant Riemann-Hilbert Correspondence . . . . .	181

5	Holomorphic Solution Spaces	182
5.1	Introduction	182
5.2	Countable Sheaves	183
5.3	$C^\infty$ -Solutions	185
5.4	Definition of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}$	186
5.5	DFN Version	189
5.6	Functorial Properties of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}$	190
5.7	Relation with the de Rham Functor	192
6	Whitney Functor	194
6.1	Whitney Functor	194
6.2	The Functor $\mathbf{R}\mathrm{Hom}_{\mathcal{D}_X}^{\mathrm{top}}(\bullet, \bullet \overset{\mathrm{w}}{\otimes} \mathcal{O}_{X^{\mathrm{an}}})$	195
6.3	Elliptic Case	196
7	Twisted Sheaves	197
7.1	Twisting Data	197
7.2	Twisted Sheaf	198
7.3	Morphism of Twisting Data	199
7.4	Tensor Product	200
7.5	Inverse and Direct Images	200
7.6	Twisted Modules	201
7.7	Equivariant Twisting Data	201
7.8	Character Local System	202
7.9	Twisted Equivariance	202
7.10	Twisting Data Associated with Principal Bundles	203
7.11	Twisting ( $D$ -module Case)	204
7.12	Ring of Twisted Differential Operators	205
7.13	Equivariance of Twisted Sheaves and Twisted $D$ -modules	207
7.14	Riemann-Hilbert Correspondence	207
8	Integral Transforms	208
8.1	Convolutions	208
8.2	Integral Transform Formula	209
9	Application to the Representation Theory	210
9.1	Notations	210
9.2	Beilinson-Bernstein Correspondence	212
9.3	Quasi-equivariant $D$ -modules on the Symmetric Space	214
9.4	Matsuki Correspondence	216
9.5	Construction of Representations	217
9.6	Integral Transformation Formula	219
10	Vanishing Theorems	221
10.1	Preliminary	221
10.2	Calculation (I)	222
10.3	Calculation (II)	224
10.4	Vanishing Theorem	226
	References	229
	<b>List of Notations</b>	231
	<b>Index</b>	233

**Amenability and Margulis Super-Rigidity**

<i>Alain Valette</i> .....	235
1 Introduction .....	235
2 Amenability for Locally Compact Groups .....	236
2.1 Definition, Examples, and First Characterizations .....	236
2.2 Stability Properties .....	239
2.3 Lattices in Locally Compact Groups .....	240
2.4 Reiter's Property ( $P_1$ ) .....	241
2.5 Reiter's Property ( $P_2$ ) .....	242
2.6 Amenability in Riemannian Geometry .....	244
3 Measurable Ergodic Theory .....	244
3.1 Definitions and Examples .....	244
3.2 Moore's Ergodicity Theorem .....	247
3.3 The Howe-Moore Vanishing Theorem .....	249
4 Margulis' Super-rigidity Theorem .....	252
4.1 Statement .....	252
4.2 Mostow Rigidity .....	252
4.3 Ideas to Prove Super-rigidity, $k = \mathbf{R}$ .....	253
4.4 Proof of Furstenberg's Proposition 4.1 - Use of Amenability .....	255
4.5 Margulis' Arithmeticity Theorem .....	256
References .....	257

**Unitary Representations and Complex Analysis**

<i>David A. Vogan, Jr</i> .....	259
1 Introduction .....	259
2 Compact Groups and the Borel-Weil Theorem .....	264
3 Examples for $SL(2, \mathbb{R})$ .....	272
4 Harish-Chandra Modules and Globalization .....	274
5 Real Parabolic Induction and the Globalization Functors .....	284
6 Examples of Complex Homogeneous Spaces .....	294
7 Dolbeault Cohomology and Maximal Globalizations .....	302
8 Compact Supports and Minimal Globalizations .....	318
9 Invariant Bilinear Forms and Maps between Representations .....	327
10 Open Questions .....	341
References .....	343

**Quantum Computing and Entanglement for Mathematicians**

<i>Nolan R. Wallach</i> .....	345
1 The Basics .....	346
1.1 Basic Quantum Mechanics .....	346
1.2 Bits .....	348
1.3 Qubits .....	349
References .....	350
2 Quantum Algorithms .....	351
2.1 Quantum Parallelism .....	351

## XII Contents

2.2	The Tensor Product Structure of $n$ -qubit Space .....	352
2.3	Grover's Algorithm .....	353
2.4	The Quantum Fourier Transform .....	354
	References .....	355
3	Factorization and Error Correction .....	355
3.1	The Complexity of the Quantum Fourier Transform .....	356
3.2	Reduction of Factorization to Period Search .....	359
3.3	Error Correction .....	360
	References .....	362
4	Entanglement .....	362
4.1	Measures of Entanglement .....	363
4.2	Three Qubits .....	365
4.3	Measures of Entanglement for Two and Three Qubits .....	367
	References .....	368
5	Four and More Qubits .....	369
5.1	Four Qubits .....	369
5.2	Some Hilbert Series of Measures of Entanglement .....	374
5.3	A Measure of Entanglement for $n$ Qubits .....	374
	References .....	376

Representation Theory and Complex Analysis

Lectures given at the C.I.M.E. Summer School held in  
Venice, Italy, June 10-17, 2004

Cowling, M.; Frenkel, E.; Kashiwara, M.; Valette, A.;

Vogan, D.A.; Wallach, N.R. - Casadio Tarabusi, E.;

D'Agnolo, A.; Picardello, M.A. (Eds.)

2008, XII, 389 p., Softcover

ISBN: 978-3-540-76891-3