

---

# Contents

<b>Introduction</b> .....	1
---------------------------	---

---

## Part I Basic Hodge Theory

---

<b>1 Compact Kähler Manifolds</b> .....	11
1.1 Classical Hodge Theory .....	11
1.1.1 Harmonic Theory .....	11
1.1.2 The Hodge Decomposition .....	15
1.1.3 Hodge Structures in Cohomology and Homology .....	17
1.2 The Lefschetz Decomposition .....	20
1.2.1 Representation Theory of $SL(2, \mathbb{R})$ .....	20
1.2.2 Primitive Cohomology .....	24
1.3 Applications .....	28
<b>2 Pure Hodge Structures</b> .....	33
2.1 Hodge Structures .....	33
2.1.1 Basic Definitions .....	33
2.1.2 Polarized Hodge Structures .....	38
2.2 Mumford-Tate Groups of Hodge Structures .....	40
2.3 Hodge Filtration and Hodge Complexes .....	43
2.3.1 Hodge to De Rham Spectral Sequence .....	43
2.3.2 Strong Hodge Decompositions .....	45
2.3.3 Hodge Complexes and Hodge Complexes of Sheaves ...	49
2.4 Refined Fundamental Classes .....	51
2.5 Almost Kähler $V$ -Manifolds .....	56
<b>3 Abstract Aspects of Mixed Hodge Structures</b> .....	61
3.1 Introduction to Mixed Hodge Structures: Formal Aspects .....	62
3.2 Comparison of Filtrations .....	66
3.3 Mixed Hodge Structures and Mixed Hodge Complexes .....	69

3.4	The Mixed Cone .....	76
3.5	Extensions of Mixed Hodge Structures .....	79
3.5.1	Mixed Hodge Extensions .....	79
3.5.2	Iterated Extensions and Absolute Hodge Cohomology ..	83

---

## Part II Mixed Hodge structures on Cohomology Groups

---

<b>4</b>	<b>Smooth Varieties</b> .....	89
4.1	Main Result .....	89
4.2	Residue Maps .....	92
4.3	Associated Mixed Hodge Complexes of Sheaves .....	96
4.4	Logarithmic Structures .....	99
4.5	Independence of the Compactification and Further Complements .....	101
4.5.1	Invariance .....	101
4.5.2	Restrictions for the Hodge Numbers .....	102
4.5.3	Theorem of the Fixed Part and Applications .....	103
4.5.4	Application to Lefschetz Pencils .....	105
<b>5</b>	<b>Singular Varieties</b> .....	109
5.1	Simplicial and Cubical Sets .....	109
5.1.1	Basic Definitions .....	109
5.1.2	Sheaves on Semi-simplicial Spaces and Their Cohomology .....	114
5.1.3	Cohomological Descent and Resolutions .....	117
5.2	Construction of Cubical Hyperresolutions .....	119
5.3	Mixed Hodge Theory for Singular Varieties .....	124
5.3.1	The Basic Construction .....	124
5.3.2	Mixed Hodge Theory of Proper Modifications. ....	128
5.3.3	Restriction on the Hodge Numbers. ....	130
5.4	Cup Product and the Künneth Formula. ....	133
5.5	Relative Cohomology .....	135
5.5.1	Construction of the Mixed Hodge Structure .....	135
5.5.2	Cohomology with Compact Support .....	137
<b>6</b>	<b>Singular Varieties: Complementary Results</b> .....	141
6.1	The Leray Filtration .....	141
6.2	Deleted Neighbourhoods of Algebraic Sets .....	144
6.2.1	Mixed Hodge Complexes .....	144
6.2.2	Products and Deleted Neighbourhoods .....	146
6.2.3	Semi-purity of the Link .....	150
6.3	Cup and Cap Products, and Duality .....	152
6.3.1	Duality for Cohomology with Compact Supports .....	152
6.3.2	The Extra-Ordinary Cup Product .....	156

<b>7</b>	<b>Applications to Algebraic Cycles and to Singularities</b>	161
7.1	The Hodge Conjectures	161
7.1.1	Versions for Smooth Projective Varieties	161
7.1.2	The Hodge Conjecture and the Intermediate Jacobian	164
7.1.3	A Version for Singular Varieties	166
7.2	Deligne Cohomology	168
7.2.1	Basic Properties	168
7.2.2	Cycle Classes for Deligne Cohomology	172
7.3	The Filtered De Rham Complex And Applications	173
7.3.1	The Filtered De Rham Complex	173
7.3.2	Application to Vanishing Theorems	178
7.3.3	Applications to Du Bois Singularities	183

---

## Part III Mixed Hodge Structures on Homotopy Groups

---

<b>8</b>	<b>Hodge Theory and Iterated Integrals</b>	191
8.1	Some Basic Results from Homotopy Theory	192
8.2	Formulation of the Main Results	196
8.3	Loop Space Cohomology and the Homotopy De Rham Theorem	199
8.3.1	Iterated Integrals	199
8.3.2	Chen's Version of the De Rham Theorem	201
8.3.3	The Bar Construction	202
8.3.4	Iterated Integrals of 1-Forms	204
8.4	The Homotopy De Rham Theorem for the Fundamental Group	205
8.5	Mixed Hodge Structure on the Fundamental Group	208
8.6	The Sullivan Construction	211
8.7	Mixed Hodge Structures on the Higher Homotopy Groups	213
<b>9</b>	<b>Hodge Theory and Minimal Models</b>	219
9.1	Minimal Models of Differential Graded Algebras	220
9.2	Postnikov Towers and Minimal Models; the Simply Connected Case	222
9.3	Mixed Hodge Structures on the Minimal Model	224
9.4	Formality of Compact Kähler Manifolds	230
9.4.1	The 1-Minimal Model	230
9.4.2	The De Rham Fundamental Group	232
9.4.3	Formality	234

---

**Part IV Hodge Structures and Local Systems**


---

<b>10 Variations of Hodge Structure</b>	239
10.1 Preliminaries: Local Systems over Complex Manifolds	239
10.2 Abstract Variations of Hodge Structure	241
10.3 Big Monodromy Groups, an Application	245
10.4 Variations of Hodge Structures Coming From Smooth Families	248
<b>11 Degenerations of Hodge Structures</b>	253
11.1 Local Systems Acquiring Singularities	253
11.1.1 Connections with Logarithmic Poles	253
11.1.2 The Riemann-Hilbert Correspondence (I)	256
11.2 The Limit Mixed Hodge Structure on Nearby Cycle Spaces	259
11.2.1 Asymptotics for Variations of Hodge Structure over a Punctured Disk	259
11.2.2 Geometric Set-Up and Preliminary Reductions	260
11.2.3 The Nearby and Vanishing Cycle Functor	262
11.2.4 The Relative Logarithmic de Rham Complex and Quasi-unipotency of the Monodromy	263
11.2.5 The Complex Monodromy Weight Filtration and the Hodge Filtration	268
11.2.6 The Rational Structure	271
11.2.7 The Mixed Hodge Structure on the Limit	272
11.3 Geometric Consequences for Degenerations	274
11.3.1 Monodromy, Specialization and Wang Sequence	274
11.3.2 The Monodromy and Local Invariant Cycle Theorems	279
11.4 Examples	285
<b>12 Applications of Asymptotic Hodge theory</b>	289
12.1 Applications to Singularities	289
12.1.1 Localizing Nearby Cycles	289
12.1.2 A Mixed Hodge Structure on the Cohomology of Milnor Fibres	291
12.1.3 The Spectrum of Singularities	293
12.2 An Application to Cycles: Grothendieck's Induction Principle	295
<b>13 Perverse Sheaves and <math>D</math>-Modules</b>	301
13.1 Verdier Duality	301
13.1.1 Dimension	301
13.1.2 The Dualizing Complex	302
13.1.3 Statement of Verdier Duality	304
13.1.4 Extraordinary Pull Back	305
13.2 Perverse Complexes	306
13.2.1 Intersection Homology and Cohomology	306

13.2.2	Constructible and Perverse Complexes . . . . .	308
13.2.3	An Example: Nearby and Vanishing Cycles . . . . .	312
13.3	Introduction to $D$ -Modules . . . . .	313
13.3.1	Integrable Connections and $D$ -Modules . . . . .	313
13.3.2	From Left to Right and Vice Versa . . . . .	315
13.3.3	Derived Categories of $D$ -modules . . . . .	316
13.3.4	Inverse and Direct Images . . . . .	317
13.3.5	An Example: the Gauss-Manin System . . . . .	320
13.4	Coherent $D$ -Modules . . . . .	320
13.4.1	Basic Definitions . . . . .	321
13.4.2	Good Filtrations and Characteristic Varieties . . . . .	323
13.4.3	Behaviour under Direct and Inverse Images . . . . .	325
13.5	Filtered $D$ -modules . . . . .	327
13.5.1	Derived Categories . . . . .	327
13.5.2	Duality . . . . .	328
13.5.3	Functoriality . . . . .	328
13.6	Holonomic $D$ -Modules . . . . .	329
13.6.1	Symplectic Geometry . . . . .	329
13.6.2	Basics on Holonomic $D$ -Modules . . . . .	331
13.6.3	The Riemann-Hilbert Correspondence (II) . . . . .	332
<b>14</b>	<b>Mixed Hodge Modules . . . . .</b>	<b>337</b>
14.1	An Axiomatic Introduction . . . . .	338
14.1.1	The Axioms . . . . .	338
14.1.2	First Consequences of the Axioms . . . . .	340
14.1.3	Spectral Sequences . . . . .	343
14.1.4	Intersection Cohomology . . . . .	345
14.1.5	Refined Fundamental Classes . . . . .	347
14.2	The Kashiwara-Malgrange Filtration . . . . .	347
14.2.1	Motivation . . . . .	347
14.2.2	The Rational $V$ -Filtration . . . . .	349
14.3	Polarizable Hodge Modules . . . . .	353
14.3.1	Hodge Modules . . . . .	353
14.3.2	Polarizations . . . . .	357
14.3.3	Lefschetz Operators and the Decomposition Theorem . . . . .	359
14.4	Mixed Hodge Modules . . . . .	362
14.4.1	Variations of Mixed Hodge Structure . . . . .	362
14.4.2	Defining Mixed Hodge Modules . . . . .	365
14.4.3	About the Axioms . . . . .	366
14.4.4	Application: Vanishing Theorems . . . . .	367
14.4.5	The Motivic Hodge Character and Motivic Chern Classes . . . . .	368

---

**Part V Appendices**


---

<b>A</b>	<b>Homological Algebra</b>	375
A.1	Additive and Abelian Categories	375
A.1.1	Pre-Abelian Categories	376
A.1.2	Additive Categories	377
A.2	Derived Categories	380
A.2.1	The Homotopy Category	380
A.2.2	The Derived Category	382
A.2.3	Injective and Projective Resolutions	386
A.2.4	Derived Functors	388
A.2.5	Properties of the Ext-functor	391
A.2.6	Yoneda Extensions	391
A.3	Spectral Sequences and Filtrations	394
A.3.1	Filtrations	394
A.3.2	Spectral Sequences and Exact Couples	397
A.3.3	Filtrations Induce Spectral Sequences	398
A.3.4	Derived Functors and Spectral Sequences	401
<b>B</b>	<b>Algebraic and Differential Topology</b>	405
B.1	Singular (Co)homology and Borel-Moore Homology	405
B.1.1	Basic Definitions and Tools	405
B.1.2	Pairings and Products	409
B.2	Sheaf Cohomology	410
B.2.1	The Godement Resolution and Cohomology	410
B.2.2	Cohomology and Supports	412
B.2.3	Čech Cohomology	414
B.2.4	De Rham Theorems	416
B.2.5	Direct and Inverse Images	417
B.2.6	Sheaf Cohomology and Closed Subspaces	420
B.2.7	Mapping Cones and Cylinders	421
B.2.8	Duality Theorems on Manifolds	422
B.2.9	Orientations and Fundamental Classes	424
B.3	Local Systems and Their Cohomology	427
B.3.1	Local Systems and Locally Constant Sheaves	428
B.3.2	Homology and Cohomology	429
B.3.3	Local Systems and Flat Connections	430
<b>C</b>	<b>Stratified Spaces and Singularities</b>	433
C.1	Stratified Spaces	433
C.1.1	Pseudomanifolds	433
C.1.2	Whitney Stratifications	434
C.2	Fibrations, and the Topology of Singularities	437
C.2.1	The Milnor Fibration	437

C.2.2	Topology of One-parameter Degenerations . . . . .	438
C.2.3	An Example: Lefschetz Pencils . . . . .	441
<b>References</b>	. . . . .	445
<b>Index of Notations</b>	. . . . .	457
<b>Index</b>	. . . . .	461



<http://www.springer.com/978-3-540-77015-2>

Mixed Hodge Structures

Peters, C.; Steenbrink, J.H.M.

2008, XIV, 470 p. 6 illus., Hardcover

ISBN: 978-3-540-77015-2