

## Decomposition Frameworks

In this chapter, we introduce and illustrate several principles employed in the formulation of domain decomposition methods for an elliptic equation. In our discussion, we focus on a *two* subdomain decomposition of the domain of the elliptic equation, into overlapping or non-overlapping subdomains, and introduce the notion of a *hybrid formulation* of the elliptic equation. A hybrid formulation is a *coupled* system of elliptic equations which is *equivalent* to the original elliptic equation, with unknowns representing the true solution on each subdomain. Such formulations provide a natural framework for the construction of *divide and conquer* methods for an elliptic equation. Using a hybrid formulation, we heuristically illustrate how novel divide and conquer *iterative* methods, *non-matching grid* discretizations and *heterogeneous* approximations can be constructed for an elliptic equation.

We illustrate four alternative hybrid formulations for an elliptic equation. Each will be described for a decomposition of the domain into *two* subdomains, either overlapping or non-overlapping. We shall describe the following:

- Schwarz formulation.
- Steklov-Poincaré formulation.
- Lagrange multiplier formulation.
- Least squares-control formulation.

For each hybrid formulation, we illustrate how iterative methods, non-matching grid discretizations and heterogeneous approximations can be formulated for the elliptic equation based on its two subdomain decomposition. In Chap. 1.1, we introduce notation and heuristically describe the structure of a hybrid formulation. Chap. 1.2 describes a two subdomain Schwarz hybrid formulation, based on overlapping subdomains. Chap. 1.3 describes the Steklov-Poincaré formulation, based on two non-overlapping subdomains. The Lagrange multiplier formulation described in Chap. 1.4 applies only for a self adjoint and coercive elliptic equation, and it employs two non-overlapping subdomains. Chap. 1.5 describes the least squares-control formulation for a two subdomain overlapping or non-overlapping decomposition.

## 1.1 Hybrid Formulations

Given a subdomain decomposition, a hybrid formulation of an elliptic equation is an *equivalent* coupled system of elliptic equations involving unknowns on each subdomain. In this section, we introduce notation on an elliptic equation and heuristically describe the structure of its two subdomain hybrid formulation. We outline how divide and conquer iterative methods, non-matching grid discretizations, and heterogeneous approximations can be constructed for an elliptic equation, using an hybrid formulation of it. Four commonly used hybrid formulations are described in Chap. 1.2 through Chap. 1.5.

### 1.1.1 Elliptic Equation

We shall consider the following 2nd order elliptic equation:

$$\begin{cases} Lu \equiv -\nabla \cdot (a(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u + c(x)u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

for  $\Omega \subset \mathbb{R}^d$ . The coefficient  $a(x)$  will be assumed to satisfy:

$$0 < a_0 \leq a(x), \quad \forall x \in \Omega,$$

while  $\mathbf{b}(x)$  and  $c(x) \geq 0$  will be assumed to be smooth, and  $f(x) \in L^2(\Omega)$ . Additional restrictions will be imposed on the coefficients as required.

### 1.1.2 Weak Formulation

A weak formulation of (1.1) is typically obtained by multiplying it by a sufficiently smooth test function  $v(x)$  and integrating the diffusion term by parts on  $\Omega$ . It will seek  $u \in H_0^1(\Omega)$  satisfying:

$$\begin{cases} \mathcal{A}(u, v) = F(v), & \forall v \in H_0^1(\Omega), \text{ where} \\ \mathcal{A}(u, v) \equiv \int_{\Omega} (a(x)\nabla u \cdot \nabla v + (\mathbf{b}(x) \cdot \nabla u)v + c(x)uv) dx \\ F(v) \equiv \int_{\Omega} f v dx, \end{cases} \quad (1.2)$$

where the Sobolev space  $H_0^1(\Omega)$  is formally defined as below [NE, LI4, JO2]:

$$H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\},$$

while the space  $H^1(\Omega)$  is defined as:

$$\begin{cases} H^1(\Omega) \equiv \{v \in L^2(\Omega) : \|v\|_{1,\Omega}^2 < \infty\}, & \text{where} \\ \|v\|_{1,\Omega}^2 \equiv \int_{\Omega} (v^2 + |\nabla v|^2) dx, \end{cases}$$

for  $\nabla v \equiv \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_d}\right)$ . The bilinear form  $\mathcal{A}(\cdot, \cdot)$  will be *coercive* if:

$$\mathcal{A}(u, u) \geq \alpha \|u\|_{1,\Omega}^2, \quad \forall u \in H_0^1(\Omega),$$

for some  $\alpha > 0$  independent of  $u$ . Coercivity of  $\mathcal{A}(\cdot, \cdot)$  is guaranteed to hold by the Poincaré-Freidrichs inequality, see [NE].

### 1.1.3 Discretization

A finite element discretization of (1.1) is obtained by Galerkin approximation of (1.2). Let  $\mathcal{T}_h(\Omega)$  denote a triangulation of  $\Omega$  with elements of size  $h$  and let  $V_h$  denote the space of continuous piecewise linear finite element functions on  $\mathcal{T}_h(\Omega)$ , see [ST14, CI2, JO2, BR28, BR]. If  $\{\phi_1, \dots, \phi_n\}$  forms a basis for  $V_h \cap H_0^1(\Omega)$ , then the finite element discretization of (1.1) will yield the system:

$$A \mathbf{u} = \mathbf{f},$$

where  $A_{ij} = \mathcal{A}(\phi_i, \phi_j)$  for  $1 \leq i, j \leq n$  and  $\mathbf{f}_i = F(\phi_i)$  for  $1 \leq i \leq n$ .

### 1.1.4 Subdomain Decompositions

We shall employ the following notation, see Fig. 1.1.

**Definition 1.1.** A collection of two open subregions  $\Omega_i \subset \Omega$  for  $i = 1, 2$  will be referred to as a non-overlapping decomposition of  $\Omega$  if the following hold:

$$\begin{cases} \overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}, \\ \Omega_1 \cap \Omega_2 = \emptyset. \end{cases}$$

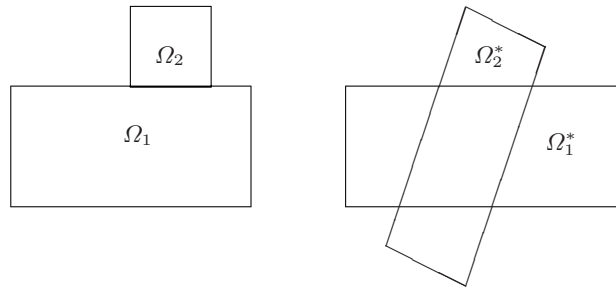
Boundaries of the subdomains will be denoted  $\partial\Omega_i$  and their interior and exterior segments by  $B^{(i)} \equiv \partial\Omega_i \cap \Omega$  and  $B_{[i]} \equiv \partial\Omega_i \cap \partial\Omega$ , respectively. We will denote the common interface by  $B \equiv \partial\Omega_1 \cap \partial\Omega_2$ .

**Definition 1.2.** A collection of two open subregions  $\Omega_i^* \subset \Omega$  for  $i = 1, 2$  will be referred to as an overlapping decomposition of  $\Omega$  if the following holds:

$$\Omega_1^* \cup \Omega_2^* = \Omega.$$

Boundaries of the subdomains will be denoted  $B_i \equiv \partial\Omega_i^*$  and their interior and exterior segments by  $B^{(i)} \equiv \partial\Omega_i^* \cap \Omega$  and  $B_{[i]} \equiv \partial\Omega_i^* \cap \partial\Omega$ , respectively.

Non-overlapping subdomains    Overlapping subdomains



**Fig. 1.1.** Two subdomain decompositions

*Remark 1.3.* In applications, a decomposition of  $\Omega$  into subdomains can be chosen based either on the geometry of  $\Omega$  or on the regularity of the solution  $u$  (if known). An overlapping subdomain  $\Omega_i^*$  can, if desired, be constructed from a nonoverlapping subdomain  $\Omega_i$  by extending it to include all points in  $\Omega$  within a distance  $\beta > 0$  of  $\Omega_i$ , yielding *uniform* overlap.

### 1.1.5 Partition of Unity

A partition of unity subordinate to the overlapping subdomains  $\Omega_1^*$  and  $\Omega_2^*$  consists of smooth functions  $\chi_1(x)$  and  $\chi_2(x)$  satisfying:

$$\begin{cases} \chi_i(x) \geq 0, & \text{in } \overline{\Omega_i^*} \\ \chi_i(x) = 0, & \text{in } \Omega \setminus \overline{\Omega_i^*} \\ \chi_1(x) + \chi_2(x) = 1, & \text{in } \overline{\Omega}. \end{cases} \quad (1.3)$$

Each  $\chi_i(\cdot)$  may be non-zero on  $B_{[i]}$ . In applications, each  $\chi_i(x)$  may be required to satisfy a bound of the form  $|\nabla \chi_i(x)| \leq C h_0^{-1}$ , where  $h_0$  denotes the diameter of each subdomain  $\Omega_i^*$ .

Heuristically, a continuous partition of unity subordinate to  $\Omega_1^*$  and  $\Omega_2^*$  can be computed as follows. Let  $d_i(x)$  denote the distance function:

$$d_i(x) = \begin{cases} \text{dist}(x, B^{(i)}), & \text{if } x \in \overline{\Omega_i^*} \\ 0, & \text{if } x \notin \overline{\Omega_i^*}, \end{cases} \quad (1.4)$$

where  $B^{(i)} \equiv (\partial\Omega_i^* \cap \Omega)$ . Then, formally define:

$$\chi_i(x) \equiv \frac{d_i(x)}{d_1(x) + d_2(x)}, \quad \text{for } 1 \leq i \leq 2. \quad (1.5)$$

By construction, each  $d_i(x)$  will be *continuous*, nonnegative, with support in  $\overline{\Omega_i^*}$ , and satisfy the desired properties. To obtain a smooth function  $\chi_i(x)$ , each  $d_i(x)$  may first be *mollified*, see [ST9].

*Remark 1.4.* Given a *non-overlapping* decomposition  $\Omega_1$  and  $\Omega_2$  of  $\Omega$ , we shall sometimes employ a *discontinuous* partition of unity satisfying:

$$\begin{cases} \chi_i(x) \geq 0, & \text{in } \overline{\Omega_i} \\ \chi_i(x) = 0, & \text{in } \Omega \setminus \overline{\Omega_i} \\ \chi_1(x) + \chi_2(x) = 1, & \text{in } \Omega. \end{cases} \quad (1.6)$$

Each  $\chi_i(x)$  will be discontinuous across  $B = \partial\Omega_1 \cap \partial\Omega_2$ . Such a partition of unity may be constructed using  $d_i(x) = 1$  on  $\overline{\Omega_i}$  in (1.5).

### 1.1.6 Hybrid Formulation

Let  $\Omega_1$  and  $\Omega_2$  (or  $\Omega_1^*$  and  $\Omega_2^*$ ) form a decomposition of a domain  $\Omega$ . Then, a hybrid formulation of (1.1), is a coupled system of partial differential equations

*equivalent* to (1.1), with one unknown function  $w_i(x)$ , representing the local solution, on each subdomain  $\Omega_i$  (or  $\Omega_i^*$ ). Two requirements must be satisfied. *First*, the restriction  $u_i(x)$  of the true solution  $u(x)$  of (1.1) to each subdomain  $\Omega_i$  (or  $\Omega_i^*$ ) must solve the hybrid system, i.e.,  $(u_1(x), u_2(x))$  must solve the hybrid formulation. *Second*, the hybrid formulation must be *well posed* as a coupled system, i.e., its solution  $(w_1(x), w_2(x))$  must exist and be unique, and furthermore, it must depend continuously on the data.

The first requirement ensures that the hybrid formulation is *consistent* with the original problem (1.1), yielding  $w_i(x) = u_i(x)$  for  $i = 1, 2$ . The second requirement ensures that the hybrid formulation is *stable* and uniquely solvable. The latter is essential for the stability of a numerical approximation of the hybrid formulation. Once the hybrid system is solved, the solution  $u(x)$  of (1.1) can be expressed in terms of the local solutions  $w_i(x)$  as:

$$u(x) = \chi_1(x) w_1(x) + \chi_2(x) w_2(x),$$

using a partition of unity  $\chi_1(x)$  and  $\chi_2(x)$  appropriate for the subdomains.

Typically, a hybrid formulation consists of a *local problem* posed on each individual subdomain, along with *matching conditions* that couple the local problems. In some hybrid formulations, a global functional may be employed, whose optima is sought, or new variables may be introduced to couple the local problems. Such coupling must ensure *consistency* and *well posedness*.

**Local Problems.** On each subdomain  $\Omega_i$  (or  $\Omega_i^*$ ), a hybrid formulation will require  $w_i(x)$  to solve the original partial differential equation (1.1):

$$\begin{cases} Lw_i &= f_i, & \text{on } \Omega_i \text{ (or } \Omega_i^*) \\ T_i(w_i, \gamma) &= g_i, & \text{on } B^{(i)} \\ w_i &= 0, & \text{on } B_{[i]} \end{cases} \quad \text{for } i = 1, 2, \quad (1.7)$$

where  $T_i(w_i, \gamma)$  denotes a boundary operator which enforces either Dirichlet, Neumann or Robin boundary conditions on  $B^{(i)}$ :

$$T_i(w_i, \gamma) = \begin{cases} w_i, & \text{for Dirichlet boundary conditions} \\ \mathbf{n}_i \cdot (a(x) \nabla w_i) & \text{for Neumann boundary conditions} \\ \mathbf{n}_i \cdot (a(x) \nabla w_i) + \gamma w_i & \text{for Robin boundary conditions.} \end{cases} \quad (1.8)$$

Here  $\mathbf{n}_i$  denotes the unit exterior normal to  $B^{(i)}$  and  $\gamma(\cdot)$  denotes a coefficient function in the Robin boundary condition. Typically,  $f_i(x)$  is  $f(x)$  restricted to  $\Omega_i$  (or  $\Omega_i^*$ ). The choice of the boundary operator  $T_i(w_i, \gamma)$  may differ with each hybrid formulation. The boundary data  $g_i(\cdot)$  typically corresponds to  $T_i(\cdot)$  applied to the solution on the adjacent domain, however, it may also be a control or a Lagrange multiplier function which *couples* the local problems.

**Matching Conditions.** Matching conditions *couple* the different local problems (1.7) by choosing  $g_i(\cdot)$  to ensure that the hybrid formulation is equivalent to (1.1). Typically, matching conditions are equations satisfied by the

true solution  $u(x)$  restricted to the interfaces or regions of overlap between adjacent subdomains. For an elliptic equation, these may be either *algebraic* equations, such as the requirement of continuity of the local solutions  $u_i(x)$  and  $u_j(x)$  across *adjacent* subdomains:

$$\begin{cases} u_i - u_j = 0, & \text{on } \partial\Omega_i \cap \partial\Omega_j, \text{ non-overlapping case} \\ u_i - u_j = 0, & \text{on } \partial\Omega_i^* \cap \Omega_j^*, \text{ overlapping case} \end{cases}$$

or they may be *differential* constraints, such as continuity of the local *fluxes*:

$$\begin{cases} \mathbf{n}_i \cdot (a(x)\nabla u_i) + \mathbf{n}_j \cdot (a(x)\nabla u_j) = 0, & \text{on } \partial\Omega_i \cap \partial\Omega_j, \text{ non-overlapping case} \\ \mathbf{n}_i \cdot (a(x)\nabla u_i) - \mathbf{n}_i \cdot (a(x)\nabla u_j) = 0, & \text{on } \partial\Omega_i^* \cap \Omega_j^*, \text{ overlapping case} \end{cases}$$

where  $\mathbf{n}_i$  denotes the unit exterior normal to  $\partial\Omega_i$ . Such equations specify  $g_i(\cdot)$ . Other differential constraints may also be employed using linear combinations of the above algebraic and differential constraints. Matching conditions may be enforced either directly, as in the preceding constraints, or indirectly through the use of intermediary variables such as Lagrange multipliers. In the latter case, the hybrid formulation may be derived as a saddle point problem (Chap. 1.4 or Chap. 10) of an associated constrained optimization problem.

We shall express general matching conditions in the form:

$$H_i(w_1, w_2, g_1, g_2) = 0, \quad \text{for } 1 \leq i \leq 2, \quad (1.9)$$

for suitably chosen operators  $H_i(\cdot)$  on the interface  $B^{(i)}$ .

**Reconstruction of the Global Solution.** Once a hybrid formulation consisting of local equations of the form (1.7) for  $1 \leq i \leq 2$  together with equations of the form (1.9) has been formulated and solved, the global solution  $u(\cdot)$  may be represented in the form:

$$u(x) = \chi_1(x) w_1(x) + \chi_2(x) w_2(x), \quad (1.10)$$

where  $\chi_i(x)$  is a (possibly discontinuous) partition of unity subordinate to the subdomains  $\overline{\Omega}_1$  and  $\overline{\Omega}_2$  (or  $\Omega_1^*$  and  $\Omega_2^*$ ).

**Well Posedness of the Hybrid Formulation.** To ensure that the hybrid formulation is solvable and that it may be approximated numerically by stable schemes, we require that the hybrid formulation be well posed [SM7, EV], satisfying, for  $C > 0$  independent of the data, the bound:

$$(\|w_1\| + \|w_2\|) \leq C (\|f_1\| + \|f_2\| + \|g_1\| + \|g_2\|),$$

where  $\|\cdot\|$  and  $|||\cdot|||$  are appropriately chosen norms for the solution and data, as suggested by elliptic regularity theory [GI].

**Iterative Methods.** Domain decomposition iterative algorithms can be formulated for solving (1.1) by directly applying traditional *relaxation*, *descent* or

*saddle point* algorithms to a hybrid formulation. For instance, each unknown  $w_i$  may be updated sequentially using a *relaxation* procedure. Given current approximations of  $w_1, w_2, g_1, g_2$  update for  $w_i$  by solving:

$$\begin{cases} Lw_i &= f_i, \text{ on } \Omega_i \text{ (or } \Omega_i^*) \\ T_i(w_i, \gamma) &= g_i, \text{ on } B^{(i)} \\ w_i &= 0, \text{ on } B_{[i]}, \end{cases}$$

replacing  $T_i(w_i, \gamma) = g_i$  by either of the equations:

$$H_j(w_1, w_2, g_1, g_2) = 0, \quad j = 1, 2,$$

using the current iterates on the other subdomains. Alternatively, a descent or saddle point algorithm can be employed.

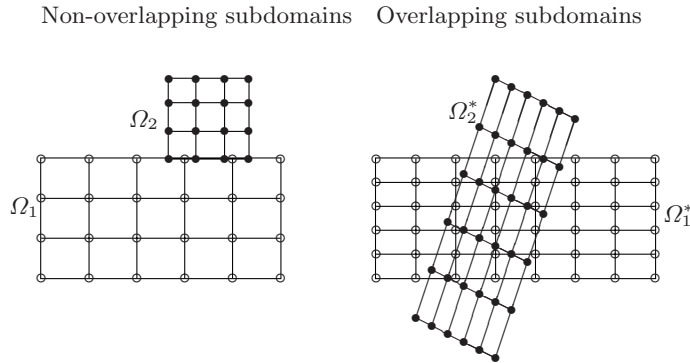
**Discretization on a Nonmatching Grid.** In various applications, it may be of interest to independently triangulate different subregions  $\Omega_i$  (or  $\Omega_i^*$ ) with grids suited to the geometry of each subdomain. The resulting grids, however, may not match on the regions of intersection between the subdomains, and are referred to as nonmatching grids, see Fig. 1.2. On such non-matching grids, a global discretization of (1.1) may be sought by directly discretizing the hybrid formulation, namely, the the local problems and the matching conditions.

Heuristically, the construction of a global discretization of equation (1.1) on a non-matching triangulation on  $\Omega_i$  (or  $\Omega_i^*$ ), will involve the following steps.

- Let  $\mathcal{T}_{h_i}(\Omega)$  (or  $\mathcal{T}_{h_i}(\Omega_i^*)$ ) denote independent triangulations of  $\Omega_i$  (or  $\Omega_i^*$ ) with local grid sizes  $h_i$ , see Fig. 1.2. These grids need not match on the region of intersection or overlap between the subdomains.
- Each local problem in the hybrid formulation can be discretized as:

$$\begin{cases} A_{h_i} \mathbf{w}_{h_i} &= \mathbf{f}_{h_i}, \text{ on } \Omega_{h_i} \text{ (or } \Omega_{h_i}^*) \\ T_{h_i}(\mathbf{w}_{h_i}, \gamma_{h_i}) &= \mathbf{g}_{h_i}, \text{ on } B^{(i)} \\ \mathbf{w}_{h_i} &= \mathbf{0}, \text{ on } B_{[i]}. \end{cases}$$

Each local discretization should be a stable scheme.



**Fig. 1.2.** Nonmatching grids

- The matching conditions should also be discretized:

$$H_i^h(\mathbf{w}_{h_1}, \mathbf{w}_{h_2}, \mathbf{g}_{h_1}, \mathbf{g}_{h_2}) = 0, \quad 1 \leq i \leq 2.$$

To ensure the stability and consistency of the global discretization of the hybrid formulation, care must be exercised in discretizing the matching conditions across the subdomain grids.

Such issues are described in Chap. 1.2 through 1.5, and in Chap. 11.

**Heterogeneous Approximation.** A partial differential equation is said to be *heterogeneous* if its *type* changes from one region to another. An example is *Tricomi's* equation [JO]:

$$u_{x_1 x_1} - x_1 u_{x_2 x_2} = f(x_1, x_2),$$

which is of hyperbolic type for  $x_1 > 0$  and of elliptic type for  $x_1 < 0$ . In various applications, efficient computational methods may be available for the local problems involved in an heterogeneous partial differential equation. In such cases, it may be of interest to approximate a partial differential equation of *heterogeneous character* by a partial differential equation of *heterogeneous type*. We refer to such models as heterogeneous approximations.

Our discussion will be restricted to an elliptic-hyperbolic heterogeneous approximation of a singularly perturbed elliptic equation of heterogeneous character. We shall consider an advection dominated equation:

$$\begin{cases} -\epsilon \Delta u + \mathbf{b}(x) \cdot \nabla u + c(x) u = f(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.11)$$

where  $0 < \epsilon \ll 1$  is a small perturbation parameter. Depending  $f(x)$ , there may be a subdomain  $\Omega_1$  (or  $\Omega_1^*$ ) on which:

$$\epsilon |\Delta u| \ll |\mathbf{b}(x) \cdot \nabla u + c(x) u|, \quad \text{for } x \in \Omega_1 \text{ (or } \Omega_1^*).$$

On  $\Omega_1$  (or  $\Omega_1^*$ ), the restriction of elliptic equation  $Lu = f$  to the subdomain, will be of hyperbolic character, approximately satisfying  $L_1 u = f$ , where:

$$\begin{cases} Lu \equiv \epsilon L_0 u + L_1 u \\ L_0 u \equiv -\Delta u \\ L_1 u \equiv \mathbf{b}(x) \cdot \nabla u + c(x) u. \end{cases}$$

If  $\Omega_2$  (or  $\Omega_2^*$ ) denotes a complementary (*layer*) region, then equation (1.11) will be approximately of elliptic character in  $\Omega_2$  (or  $\Omega_2^*$ ).

Motivated by singular perturbation methodology [LA5, KE5, OM], it may be computationally advantageous to approximate elliptic equation (1.11) by an *heterogeneous approximation* involving an equation of mixed hyperbolic and elliptic character. To obtain an heterogeneous approximation of (1.11),



we may approximate its hybrid formulation based on  $\Omega_i$  (or  $\Omega_i^*$ ) for  $1 \leq i \leq 2$ . For instance, we may approximate (1.7) by:

$$\begin{cases} \tilde{L}_i v_i &= f_i, & \text{on } \Omega_i \text{ (or } \Omega_i^*), \\ \tilde{T}_i(v_i, \gamma) &= \tilde{g}_i, & \text{on } \tilde{B}^{(i)}, \\ v_i &= 0, & \text{on } \tilde{B}_{[i]}, \end{cases} \quad \text{for } i = 1, 2$$

with  $v_i(x) \approx w_i(x)$ , and we may approximate (1.9) by:

$$\tilde{H}_i(v_1, v_2, \tilde{g}_1, \tilde{g}_2) = 0, \quad \text{for } i = 1, 2$$

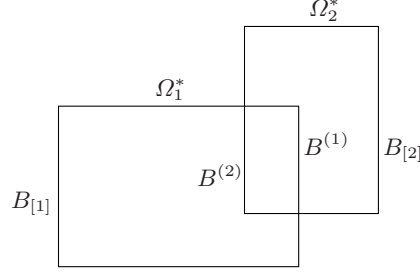
where  $\tilde{L}_i$ ,  $\tilde{T}_i$  and  $\tilde{H}_i(\cdot)$  are *heuristic* local approximations of  $L_i$ ,  $T_i$  and  $H_i(\cdot)$ , obtained by formally omitting  $\epsilon \Delta u$  on  $\Omega_1$  (or  $\Omega_1^*$ ). We refer the reader to Chap. 1.2 through Chap. 1.5 and Chap. 12 for specific examples.

*Remark 1.5.* Care must be exercised in the selection of approximations since each local problem must be well posed, and the global coupled system must also be well posed. For instance, if we define  $\tilde{L}_1 u = L_1 u$  on  $\Omega_1$  (or  $\Omega_1^*$ ) then the local problem will be hyperbolic, and we must replace Dirichlet boundary conditions on  $B^{(1)}$  and  $B_{[1]}$  by *inflow* boundary conditions. Similarly, if we choose  $\tilde{L}_2 u = Lu$  on  $\Omega_2$  (or  $\Omega_2^*$ ) then the local problem on  $\Omega_2$  (or  $\Omega_2^*$ ) will be elliptic and Dirichlet boundary or flux boundary conditions can be employed on  $B^{(2)}$  and  $B_{[2]}$ . Often, approximate matching conditions for a heterogeneous problem can also be derived *heuristically* by a *vanishing viscosity* approach, see Chap. 1.3 and Chap. 12.

## 1.2 Schwarz Framework

The framework that we refer to as the *Schwarz hybrid* formulation is based on the earliest known domain decomposition method, formulated by H. A. Schwarz [SC5] in 1870. Schwarz formulated an iterative method, now referred to as the *Schwarz alternating method*, which solves Laplace's equation on an irregular domain that is the union of regular regions (such as rectangular and circular regions). Although Schwarz's motivation was to study the existence of harmonic functions on irregular regions, the *hybrid formulation* underlying Schwarz's iterative method, applies to a wider class of elliptic equations, and it enables the formulation of other divide and conquer approximations.

In this section, we describe the hybrid formulation underlying the Schwarz alternating method for a two subdomain *overlapping* decomposition of  $\Omega$ . We let  $\Omega_1^*$  and  $\Omega_2^*$  denote the overlapping subdomains, and let  $B^{(i)} = \partial\Omega_i^* \cap \Omega$  and  $B_{[i]} = \partial\Omega_i^* \cap \partial\Omega$  denote the interior and exterior boundary segments of  $\Omega_i^*$ , respectively, see Fig. 1.3. Using the hybrid formulation, we illustrate the formulation of *iterative* methods, *non-matching* grid discretizations, and *heterogeneous* approximations for elliptic equation (1.1).



**Fig. 1.3.** Boundary segments for an overlapping decomposition

### 1.2.1 Motivation

To derive the *hybrid formulation* underlying Schwarz's method, let  $u(x)$  denote the solution of (1.1). Define  $w_i(x) = u(x)$  on  $\Omega_i^*$  for  $1 \leq i \leq 2$ . Then, by construction  $Lw_i = f$  in  $\Omega_i^*$ . Furthermore, the continuity of  $u$  will yield matching of  $w_1$  and  $w_2$  on  $\Omega_1^* \cap \Omega_2^*$ . It will therefore hold that:

$$\begin{cases} Lw_1 = f, & \text{in } \Omega_1^* \\ w_1 = w_2, & \text{on } B^{(1)} \\ w_1 = 0, & \text{on } B_{[1]} \end{cases} \quad \text{and} \quad \begin{cases} Lw_2 = f, & \text{in } \Omega_2^* \\ w_2 = w_1, & \text{on } B^{(2)} \\ w_2 = 0, & \text{on } B_{[2]}. \end{cases}$$

Importantly, if the above coupled, decomposed system for  $w_1(x)$  and  $w_2(x)$ , is *well posed*, then by solving it, the original solution can be recovered with  $u(x) = w_i(x)$  on  $\Omega_i^*$  for  $i = 1, 2$ . We have the following uniqueness result.

**Theorem 1.6.** *Suppose the following assumptions hold.*

1. Let  $c(x) \geq 0$  and  $\nabla \cdot \mathbf{b}(x) \leq 0$ .
2. Let  $u(x)$  denote a sufficiently smooth solution of equation (1.1).
3. Let  $w_1(x)$  and  $w_2(x)$  be sufficiently smooth solutions of the following system of coupled elliptic equations:

$$\begin{cases} Lw_1 = f, & \text{in } \Omega_1^* \\ w_1 = 0, & \text{on } B_{[1]} \\ w_1 = w_2, & \text{on } B^{(1)} \end{cases} \quad \text{and} \quad \begin{cases} Lw_2 = f, & \text{in } \Omega_2^* \\ w_2 = 0, & \text{on } B_{[2]} \\ w_2 = w_1, & \text{on } B^{(2)}. \end{cases} \quad (1.12)$$

Then the following result will hold:

$$u(x) = \begin{cases} w_1(x), & \text{on } \overline{\Omega}_1^* \\ w_2(x), & \text{on } \overline{\Omega}_2^*. \end{cases}$$

*Proof.* If  $u(x)$  is a solution of equation (1.1) and  $w_1(x) \equiv u(x)$  in  $\Omega_1^*$  and  $w_2(x) \equiv u(x)$  in  $\Omega_2^*$ , then  $w_1(x)$  and  $w_2(x)$  will satisfy (1.12) by construction.

To prove the converse, suppose that  $w_1(x)$  and  $w_2(x)$  satisfy (1.12). We will first show that  $w_1(x) = w_2(x)$  on  $\Omega_1^* \cap \Omega_2^*$ . To this end, note that  $w_1(x) - w_2(x)$

has zero boundary conditions on  $\partial(\Omega_1^* \cap \Omega_2^*)$ . Additionally, by construction  $w_1(x) - w_2(x)$  will be  $L$ -harmonic. By uniqueness of  $L$ -harmonic functions for  $c(x) \geq 0$  and  $\nabla \cdot \mathbf{b}(x) \leq 0$ , it will follow that  $w_1(x) - w_2(x) = 0$  in  $\Omega_1^* \cap \Omega_2^*$ . This yields that  $w_1(x) = w_2(x)$  on  $\Omega_1^* \cap \Omega_2^*$ . Now let  $\chi_1(x)$  and  $\chi_2(x)$  denote a sufficiently smooth partition of unity subordinate to the cover  $\Omega_1^*$  and  $\Omega_2^*$ . If we define  $u(x) = \chi_1(x)w_1(x) + \chi_2(x)w_2(x)$ , then  $u(x)$  will satisfy (1.1), since  $w_1 = w_2$  in  $\Omega_1^* \cap \Omega_2^*$  and since  $Lw_i = f$  in  $\Omega_i^*$ .  $\square$

*Remark 1.7.* The above result suggests that given a partition of unity  $\chi_1(x)$  and  $\chi_2(x)$  subordinate to  $\Omega_1^*$  and  $\Omega_2^*$ , respectively, a solution to elliptic equation (1.1) may be obtained by solving (1.12) and defining:

$$u(x) = \chi_1(x)w_1(x) + \chi_2(x)w_2(x).$$

This yields an equivalence between (1.1) and (1.12).

*Remark 1.8.* The preceding theorem yields equivalence between sufficiently smooth solutions to (1.1) and (1.12). It is, however, not a result on the well posedness (stability) of formulation (1.12) under perturbations of its data. The latter requires that the perturbed system:

$$\begin{cases} L\tilde{w}_1 = \tilde{f}_1, & \text{in } \Omega_1^* \\ \tilde{w}_1 = 0, & \text{on } B_{[1]} \\ \tilde{w}_1 = w_2 + \tilde{r}_1, & \text{on } B^{(1)} \end{cases} \quad \text{and} \quad \begin{cases} L\tilde{w}_2 = \tilde{f}_2, & \text{in } \Omega_2^* \\ \tilde{w}_2 = 0, & \text{on } B_{[2]} \\ \tilde{w}_2 = w_1 + \tilde{r}_2, & \text{on } B^{(2)} \end{cases} \quad (1.13)$$

be uniquely solvable and satisfy a bound of the form:

$$(\|\tilde{w}_1\| + \|\tilde{w}_2\|) \leq C \left( \|\tilde{f}_1\| + \|\tilde{f}_2\| + \|\tilde{r}_1\| + \|\tilde{r}_2\| \right),$$

in appropriate norms. See Chap. 15 for maximum norm well posedness.

### 1.2.2 Iterative Methods

The iterative method proposed by H. A. Schwarz is a very popular method for the solution of elliptic partial differential equations, see [SO, MO2, BA2] and [MI, MA37, DR11, LI6, LI7, BR18]. It is robustly convergent for a large class of elliptic equations, and can be motivated heuristically using the block structure of (1.12). If  $w_i^{(k)}$  denotes the  $k$ 'th iterate on subdomain  $\Omega_i^*$ , it can be updated by solving the block equation of (1.12) posed on subdomain  $\Omega_i^*$  with boundary conditions  $w_1 = w_2$  on  $B^{(1)}$  or  $w_2 = w_1$  on  $B^{(2)}$  approximated by the current iterate on its adjacent subdomain:

$$\begin{cases} Lw_1^{(k+1)} = f, & \text{in } \Omega_1^* \\ w_1^{(k+1)} = w_2^{(k)}, & \text{on } B^{(1)} \\ w_1^{(k+1)} = 0, & \text{on } B_{[1]} \end{cases} \quad \text{and} \quad \begin{cases} Lw_2^{(k+1)} = f, & \text{in } \Omega_2^* \\ w_2^{(k+1)} = w_1^{(k+1)}, & \text{on } B^{(2)} \\ w_2^{(k+1)} = 0, & \text{on } B_{[2]}. \end{cases}$$

The resulting algorithm is the *Schwarz alternating method*. It is *sequential* in nature and summarized below.

**Algorithm 1.2.1** (*Schwarz Alternating Method*)

Let  $v^{(0)}$  denote the starting global approximate solution.

1. For  $k = 0, 1, \dots$ , until convergence do:

2. Solve for  $w_1^{(k+1)}$  as follows:

$$\begin{cases} Lw_1^{(k+1)} = f_1, & \text{in } \Omega_1^* \\ w_1^{(k+1)} = v^{(k)}, & \text{on } B^{(1)} \\ w_1^{(k+1)} = g, & \text{on } B_{[1]}, \end{cases}$$

Define  $v^{(k+1/2)}$  as follows:

$$v^{(k+1/2)} \equiv \begin{cases} w_1^{(k+1)}, & \text{on } \Omega_1^* \\ v^{(k)}, & \text{on } \Omega \setminus \Omega_1^*. \end{cases}$$

3. Solve for  $w_2^{(k+1)}$  as follows:

$$\begin{cases} Lw_2^{(k+1)} = f_2, & \text{in } \Omega_2^* \\ w_2^{(k+1)} = g, & \text{on } B_{[2]} \\ w_2^{(k+1)} = v^{(k+1/2)}, & \text{on } B^{(2)} \end{cases}$$

Define  $v^{(k+1)}$  as follows:

$$v^{(k+1)} \equiv \begin{cases} w_2^{(k+1)}, & \text{on } \Omega_2^* \\ v^{(k+1/2)}, & \text{on } \Omega \setminus \Omega_2^*. \end{cases}$$

4. Endfor

Output:  $v^{(k)}$

*Remark 1.9.* The iterates  $v^{(k+\frac{1}{2})}$  and  $v^{(k+1)}$  in the preceding algorithm are continuous *extensions* of the subdomain solutions  $w_1^{(k+1)}$  and  $w_2^{(k+1)}$ , to the entire domain  $\Omega$ . Under suitable assumptions on the coefficients of the elliptic equation and overlap amongst the subdomains  $\Omega_i^*$ , the iterates  $v^{(k)}$  converge geometrically to the true solution  $u$  of (1.1), see Chap. 2.5 when  $\mathbf{b}(x) = \mathbf{0}$ .

The preceding Schwarz algorithm is sequential in nature, requiring the solution of one subdomain problem prior to another. Below, we describe an unaccelerated parallel Schwarz algorithm which requires the concurrent solution of subdomain problems. It is motivated by a popular parallel method, referred to as the additive Schwarz algorithm [DR11], which is employed typically as a preconditioner. The algorithm we describe is based on a partition of unity  $\chi_1(x)$  and  $\chi_2(x)$  subordinate to the overlapping subdomains  $\Omega_1^*$  and  $\Omega_2^*$ , respectively, see [DR11, CA19, MA33, FR8, TA5]. Let  $w_i^{(k)}$  denote the  $k$ 'th iterate on  $\Omega_i^*$  for  $1 \leq i \leq 2$ . Then, new iterates are computed as follows.

**Algorithm 1.2.2** (*Parallel Partition of Unity Schwarz Method*)

Let  $w_1^{(0)}, w_2^{(0)}$  denote starting local approximate solutions.

1. For  $k = 0, 1, \dots$ , until convergence do:
2. For  $i = 1, 2$  determine  $w_i^{(k+1)}$  in parallel:

$$\begin{cases} Lw_i^{(k+1)} = f, & \text{in } \Omega_i^* \\ w_i^{(k+1)} = \chi_1(x) w_1^{(k)}(x) + \chi_2(x) w_2^{(k)}(x), & \text{on } B^{(i)} \\ w_i^{(k+1)} = 0, & \text{on } B_{[i]}, \end{cases}$$

3. Endfor
4. Endfor

Output:  $(w_1^{(k)}, w_2^{(k)})$

If  $c(x) \geq c_0 > 0$  and there is sufficient overlap, the iterates  $v^{(k)}$  defined by:

$$v^{(k)} \equiv \chi_1(x) w_1^{(k)}(x) + \chi_2(x) w_2^{(k)}(x),$$

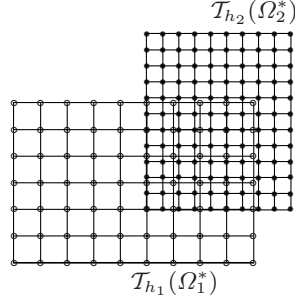
will converge geometrically to the solution  $u$  of (1.1), see Chap. 15.

*Remark 1.10.* In practice, given a discretization of (1.1), discrete versions of the above algorithms must be applied. Matrix versions of Schwarz algorithms are described in Chap. 2. There the multisubdomain case is considered, and coarse space correction is introduced, which is essential for robust convergence. In Chap. 2 it is observed that the matrix version of the Schwarz alternating method corresponds to a generalization (due to overlap) of the traditional block Gauss-Seidel iterative method. The *additive* Schwarz method [DR11] is also introduced there, corresponding to a generalized block Jacobi method.

### 1.2.3 Global Discretization

An advantage of the hybrid formulation (1.12) is that novel discretizations of (1.1) may be obtained by discretizing (1.12). Each subdomain  $\Omega_i^*$  may be independently triangulated, resulting in a possibly *non-matching grid*, see Fig. 1.4. Furthermore, each local problem may be discretized using traditional techniques suited to the local geometry and properties of the solution. The resulting solution, however, may be *nonconforming* along the internal boundaries  $B^{(i)}$  of the subdomains, and care must be exercised in discretizing the *matching* conditions to ensure that the global discretization is stable.

Below, we outline the construction of a global finite difference discretization of (1.12) based on a two subdomain decomposition of  $\Omega$ , as in Fig. 1.4, using finite difference schemes on the subdomains. For details, see Chap. 11. We triangulate each subdomain  $\Omega_i^*$  for  $1 \leq i \leq 2$  by a grid  $\mathcal{T}_{h_i}(\Omega_i^*)$  of size  $h_i$  as in Fig. 1.4. The local triangulation can be suited to the geometry and regularity of the solution on  $\Omega_i^*$ . On each subdomain, we block partition the



**Fig. 1.4.** Nonmatching overset grids

local discrete solution  $\mathbf{w}_{h_i}$  on  $\mathcal{T}_{h_i}(\Omega_i^*)$  as:

$$\mathbf{w}_{h_i} = \left( \mathbf{w}_I^{(i)}, \mathbf{w}_{B^{(i)}}^{(i)}, \mathbf{w}_{B_{[i]}}^{(i)} \right)^T, \quad \text{for } i = 1, 2$$

corresponding to the grid points in the interior and the boundary segments  $B^{(i)}$  and  $B_{[i]}$ , respectively. Let  $n_i$ ,  $m_i$  and  $l_i$  denote the number of grid points of triangulation  $\mathcal{T}_{h_i}(\Omega_i^*)$  in the interior of  $\Omega_i^*$ , on  $B^{(i)}$  and  $B_{[i]}$ , respectively. By assumption on the boundary values of  $\mathbf{w}_{h_i}$  on  $B_{[i]}$ , it will hold that  $\mathbf{w}_{B_{[i]}}^{(i)} = \mathbf{0}$ . Next, for  $i = 1, 2$  discretize the elliptic equation  $Lw_i = f_i$  on  $\Omega_i^*$  by employing a stable scheme on  $\mathcal{T}_{h_i}(\Omega_i^*)$  and denote the discretization as:

$$A_{II}^{(i)} \mathbf{w}_I^{(i)} + A_{IB^{(i)}}^{(i)} \mathbf{w}_{B^{(i)}}^{(i)} = \mathbf{f}_{h_i}, \quad \text{for } 1 \leq i \leq 2.$$

Next, on each boundary segment  $B^{(i)}$ , discretize the *inter-subdomain* matching conditions  $w_1 = w_2$  on  $B^{(1)}$  and  $w_2 = w_1$  on  $B^{(2)}$  by applying appropriate *interpolation* stencils or by discretizing its weak form. If interpolation stencils are employed, then the value  $w_{h_1}(x)$  at a grid point  $x$  on  $B_{h_1}^{(1)}$  may be expressed as a weighted average of nodal values of  $w_{h_2}(\cdot)$  on the grid points of  $\Omega_{h_2}^*$ . We denote the discretized matching conditions as:

$$\mathbf{w}_{B^{(1)}}^{(1)} = I_{h_2}^{h_1} \mathbf{w}_{h_2} \quad \text{and} \quad \mathbf{w}_{B^{(2)}}^{(2)} = I_{h_1}^{h_2} \mathbf{w}_{h_1}.$$

Here  $I_{h_2}^{h_1}$  will denote a matrix of size  $m_1 \times (n_2 + m_2 + l_2)$  and  $I_{h_1}^{h_2}$  will denote a matrix of size  $m_2 \times (n_1 + m_1 + l_1)$ . If the local grids match on each segment  $B_{[i]}$ , then this discretization step would be trivial. However, for nonmatching grids care must be exercised to ensure stability of the global scheme.

The global discretization now will have the following block matrix form:

$$\begin{cases} A_{II}^{(1)} \mathbf{w}_I^{(1)} + A_{IB^{(1)}}^{(1)} \mathbf{w}_{B^{(1)}}^{(1)} = \mathbf{f}_{h_1}, \\ \mathbf{w}_{B^{(1)}}^{(1)} = I_{h_2}^{h_1} \mathbf{w}_{h_2} \\ A_{II}^{(2)} \mathbf{w}_I^{(2)} + A_{IB^{(2)}}^{(2)} \mathbf{w}_{B^{(2)}}^{(2)} = \mathbf{f}_{h_2}, \\ \mathbf{w}_{B^{(2)}}^{(2)} = I_{h_1}^{h_2} \mathbf{w}_{h_1}. \end{cases} \quad (1.14)$$

This algebraic system can be solved by the Schwarz alternating method.

*Remark 1.11.* If  $c(x) \geq c_0 > 0$  and the local discretizations satisfy a discrete maximum principle, if the inter-grid interpolations  $I_{h_1}^{h_2}$  and  $I_{h_2}^{h_1}$  are convex weights, and if the overlap is sufficiently large so that a certain contraction property holds, see Chap. 11, then the above discretization can be shown to be stable and convergent of optimal order in the maximum norm.

### 1.2.4 Heterogeneous Approximation

A heterogeneous approximation of a partial differential equation is a model system of partial differential equations in which the problems posed on different subdomains are not all of the same *type*. Such approximations may be useful if there is a reduction in computational costs resulting from the use of a heterogeneous model. Here, we illustrate the construction of an elliptic-hyperbolic approximation of an advection dominated elliptic equation:

$$\begin{cases} L^\epsilon u \equiv -\epsilon \Delta u + \mathbf{b}(x) \cdot \nabla u + c(x) u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.15)$$

where  $0 < \epsilon \ll 1$  is a perturbation parameter. In this case, depending on the solution  $u$ , the singularly perturbed elliptic equation may be approximately of *hyperbolic* character on some subregions and of *elliptic* character elsewhere, motivating a heterogeneous approximation.

Suppose the overlapping subdomain  $\Omega_1^*$  can be chosen such that:

$$|\epsilon \Delta u(x)| \ll |\mathbf{b}(x) \cdot \nabla u(x) + c(x) u(x)| \quad \text{for } x \in \overline{\Omega_1^*}.$$

Then, on  $\Omega_1^*$  the term  $L^\epsilon u$  may be approximated by  $L_0 u$  defined by:

$$L_0 u \equiv \mathbf{b}(x) \cdot \nabla u + c(x) u.$$

Motivated by singular perturbation theory [LA5, KE5], a global heterogeneous approximation of the singularly perturbed equation (1.15) may be sought by replacing the elliptic equation  $L^\epsilon w_1 = f_1$  on  $\Omega_1^*$  by the hyperbolic equation  $L_0 w_1 = f_1$  within the Schwarz hybrid formulation (1.12).

To ensure well posedness of the local subproblems, however, the Dirichlet boundary value problem on  $\Omega_1^*$  must be replaced by suitable *inflow* boundary conditions, due to the hyperbolic nature of  $L_0 w_1 = f_1$ :

$$\begin{cases} L_0 w_1 = f_1, & \text{in } \Omega_1^* \\ w_1 = 0, & \text{on } B_{[1],in}, \\ w_1 = w_2, & \text{on } B_{in}^{(1)}, \end{cases}$$

where, the inflow boundary segments are defined by:

$$\begin{cases} B_{[1],in} \equiv \{x \in B_{[1]} : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\} \\ B_{in}^{(1)} \equiv \{x \in B^{(1)} : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \end{cases}$$

where  $\mathbf{n}(x)$  denotes the exterior unit normal to  $\partial\Omega_1^*$  at  $x$ . The resulting global heterogeneous approximation will be:

$$\left\{ \begin{array}{l} L_0 w_1 = f_1, \quad \text{in } \Omega_1^* \\ w_1 = 0, \quad \text{on } B_{[1],in} \\ w_1 = w_2, \quad \text{on } B_{in}^{(1)} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} Lw_2 = f_2, \quad \text{in } \Omega_2^* \\ w_2 = 0, \quad \text{on } B_{[2]} \\ w_2 = w_1, \quad \text{on } B^{(2)}. \end{array} \right. \quad (1.16)$$

This heterogeneous system can be discretized, and the resulting algebraic system can be solved by the Schwarz alternating method, see Chap. 12.

*Remark 1.12.* Well posedness of this heterogeneous system, as well as bounds on the error resulting from such approximation are discussed in Chap. 15.

### 1.3 Steklov-Poincaré Framework

The hybrid formulation that we refer to as the Steklov-Poincaré framework is motivated by a principle in physics referred as a *transmission condition*, employed in the study of electric fields in conductors [PO, ST8, LE12, AG, QU5]. The underlying principle states that across any interface within a conducting medium, the electric *potential* as well as the *flux* of electric current must match, i.e., be continuous. The mathematical version of this principle suggests a hybrid formulation for a 2nd order elliptic equation given a two subdomain *non-overlapping* decomposition of its domain, separated by an interface.

#### 1.3.1 Motivation

Consider elliptic equation (1.1) posed on  $\Omega$ :

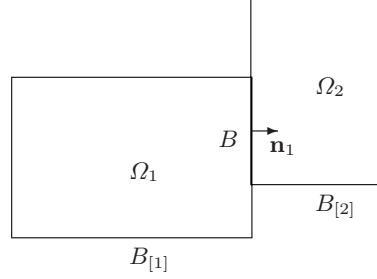
$$\left\{ \begin{array}{l} Lu \equiv -\nabla \cdot (a(x) \nabla u) + \mathbf{b}(x) \cdot \nabla u + c(x) u = f, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega, \end{array} \right. \quad (1.17)$$

Let  $\Omega_1, \Omega_2$  denote a non-overlapping decomposition of  $\Omega$ , as in Fig. 1.5, with interface  $B = \partial\Omega_1 \cap \partial\Omega_2$  separating the two subdomains and  $B_{[i]} \equiv \partial\Omega_i \cap \partial\Omega$ . Let  $\mathbf{n}_i(x)$  denote the unit outward normal vector to  $\partial\Omega_i$  at the point  $x \in B$ . For  $i = 1, 2$ , denote the solution on each subdomain  $\Omega_i$  by  $w_i(x) \equiv u(x)$ . Then, the following *transmission conditions*, which are derived later in this section, will hold on the interface  $B$  for smooth solutions:

$$\left\{ \begin{array}{l} w_1 = w_2, \quad \text{on } B \\ \mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1) = \mathbf{n}_1 \cdot (a \nabla w_2 - \mathbf{b} w_2), \quad \text{on } B. \end{array} \right. \quad (1.18)$$

The first condition requires the subdomain solutions  $w_1$  and  $w_2$  to match on  $B$ , while the second condition requires the local fluxes  $\mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1)$  and  $\mathbf{n}_1 \cdot (a \nabla w_2 - \mathbf{b} w_2)$  associated with  $w_1$  and  $w_2$  to also match on  $B$ .





**Fig. 1.5.** A two subdomain non-overlapping decomposition

Combining the transmission conditions with the elliptic equation on each subdomain, yields the following *hybrid formulation* equivalent to (1.1):

$$\left\{ \begin{array}{ll} Lw_1 = f, & \text{in } \Omega_1 \\ w_1 - w_2 = 0, & \text{on } B \\ w_1 = 0, & \text{on } B_{[1]} \\ Lw_2 = f, & \text{in } \Omega_2 \\ \mathbf{n}_1 \cdot (a\nabla w_1 - \mathbf{b} w_1) + \mathbf{n}_2 \cdot (a\nabla w_2 - \mathbf{b} w_2) = 0, & \text{on } B \\ w_2 = 0, & \text{on } B_{[2]}. \end{array} \right.$$

In this section, we shall outline how this hybrid formulation can be employed to formulate novel domain decomposition iterative methods, discretization methods and heterogeneous approximations for (1.1).

*Remark 1.13.* If the coefficient  $\mathbf{b}(x)$  in elliptic equation (1.1) is continuous, then the flux boundary condition may also be equivalently stated as:

$$\mathbf{n}_1 \cdot (a\nabla w_1) + \mathbf{n}_2 \cdot (a\nabla w_2) = 0, \quad \text{on } B,$$

by taking linear combinations of (1.18), since  $w_1(x) = w_2(x)$  on  $B$  and since  $\mathbf{n}_1(x) = -\mathbf{n}_2(x)$  on  $B$ . In particular, the following equivalent flux transmission condition is *preferred* in several domain decomposition methods:

$$\mathbf{n}_1 \cdot \left( a\nabla w_1 - \frac{1}{2} \mathbf{b} w_1 \right) + \mathbf{n}_2 \cdot \left( a\nabla w_2 - \frac{1}{2} \mathbf{b} w_2 \right) = 0, \quad \text{on } B,$$

for continuous  $\mathbf{b}(x)$ , see [QU6, GA14, AC7, RA3].

Equivalence of the Steklov-Poincaré hybrid formulation is shown next.

**Theorem 1.14.** *Suppose the following assumptions hold.*

1. *Let  $L u$  be defined by (1.1) with smooth coefficient  $\mathbf{b}(x)$  and solution  $u$ .*
2. *Let  $w_1(x)$  and  $w_2(x)$  be smooth solutions of the following coupled system of partial differential equations:*

$$\left\{ \begin{array}{ll} Lw_1 = f, & \text{in } \Omega_1 \\ w_1 = 0, & \text{on } B_{[1]} \\ w_1 = w_2, & \text{on } B \\ Lw_2 = f, & \text{in } \Omega_2 \\ w_2 = 0, & \text{on } B_{[2]} \\ \mathbf{n}_1 \cdot (a \nabla w_2 - \mathbf{b} w_2) = \mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1), & \text{on } B. \end{array} \right. \quad (1.19)$$

*Then, the following result will hold.*

$$\left\{ \begin{array}{l} w_1(x) = u(x), \text{ on } \overline{\Omega}_1 \\ w_2(x) = u(x), \text{ on } \overline{\Omega}_2. \end{array} \right.$$

*Proof.* Suppose  $u$  is a smooth solution to (1.1) and  $w_i \equiv u$  on  $\overline{\Omega}_i$ , we will verify that  $(w_1, w_2)$  solves (1.19). By construction,  $Lw_i = f$  in  $\Omega_i$  and  $w_i = 0$  on  $B_{[i]}$ . By continuity of  $u$  (or an application of the trace theorem), we obtain that  $w_1 = w_2$  on  $B$ . To verify that the local *fluxes* match on  $B$ , employ the following weak formulation of (1.1), and express each integral on  $\Omega$  as a sum of integrals on  $\Omega_1$  and  $\Omega_2$ , to obtain:

$$\sum_{i=1}^2 \int_{\Omega_i} (a \nabla w_i \cdot \nabla v - w_i \nabla \cdot (\mathbf{b} v) + c w_i v) dx = \sum_{i=1}^2 \int_{\Omega_i} f v dx,$$

for  $v \in C_0^\infty(\Omega)$ . If  $v$  is chosen to be of compact support in  $\Omega$  and not identically zero on  $B$ , then integration by parts yields:

$$\left\{ \begin{array}{l} \sum_{i=1}^2 \int_{\Omega_i} -\nabla \cdot (a \nabla w_i) v + (\mathbf{b} \cdot \nabla w_i) v + c w_i v dx \\ - \int_B \mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1 - a \nabla w_2 + \mathbf{b} w_2) v ds_x = \sum_{i=1}^2 \int_{\Omega_i} f v dx, \end{array} \right.$$

for  $v \in C_0^\infty(\Omega)$ . Substituting that  $Lw_i = f$  on  $\Omega_i$ , it follows that:

$$\int_B \mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1 - a \nabla w_2 + \mathbf{b} w_2) v ds_x = 0, \quad \forall v \in C_0^\infty(\Omega),$$

yielding the result that  $\mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1) = \mathbf{n}_1 \cdot (a \nabla w_2 + \mathbf{b} w_2)$  on  $B$ . The converse can be verified analogously.  $\square$

*Remark 1.15.* The above result only demonstrates the equivalence of solutions to both systems. It does not guarantee well posedness of hybrid formulation (1.19). This may be demonstrated using elliptic regularity theory in

appropriately chosen norms (however, we shall omit this). When system (1.19) is well posed, given a solution  $(w_1, w_2)$  to (1.19), we may define:

$$u \equiv \begin{cases} w_1 & \text{in } \overline{\Omega}_1 \\ w_2 & \text{in } \overline{\Omega}_2, \end{cases}$$

thus yielding a solution  $u$  to (1.1).

We now introduce an operator, referred to as a Steklov-Poincaré operator, which represents hybrid formulation (1.19) more compactly.

**Definition 1.16.** *Given sufficiently smooth Dirichlet boundary data  $g(\cdot)$  on the interface  $B$ , we define a Steklov-Poincaré operator  $\mathcal{S}(g, f_1, f_2)$  as follows:*

$$\mathcal{S}(g, f_1, f_2) \equiv \mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1) - \mathbf{n}_1 \cdot (a \nabla w_2 - \mathbf{b} w_2),$$

where  $w_1(\cdot)$  and  $w_2(\cdot)$  are solutions to the following problems:

$$\begin{cases} Lw_1 = f_1, & \text{in } \Omega_1 \\ w_1 = 0, & \text{on } B_{[1]} \\ w_1 = g, & \text{on } B, \end{cases} \text{ and } \begin{cases} Lw_2 = f_2, & \text{in } \Omega_2 \\ w_2 = 0 & \text{on } B_{[2]} \\ w_2 = g, & \text{on } B. \end{cases} \quad (1.20)$$

If the local forcing terms  $f_1(\cdot)$  and  $f_2(\cdot)$  are nonzero, then the action of the Steklov-Poincaré operator  $\mathcal{S}(g, f_1, f_2)$  on  $g(\cdot)$  will be *affine* linear. It will map the Dirichlet data  $g(\cdot)$  on  $B$  to the *jump* in the local fluxes (Neumann data) across interface  $B$  using (1.20). Importantly, if an interface function  $g(\cdot)$  can be found which yields *zero* jump in the flux across  $B$ , i.e.

$$\mathcal{S}(g, f_1, f_2) = 0, \quad (1.21)$$

then, corresponding to this choice of interface data  $g(\cdot)$ , the local solutions  $w_1(\cdot)$  and  $w_2(\cdot)$  to (1.20) will satisfy:

$$\begin{cases} w_1 = w_2 & (= g), & \text{on } B \\ \mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1) = \mathbf{n}_1 \cdot (a \nabla w_2 - \mathbf{b} w_2), & \text{on } B, \end{cases}$$

so that  $(w_1, w_2)$  will solve (1.19). As a result, the search for a solution  $(w_1, w_2)$  to problem (1.19) may be *reduced* to the search for interface data  $g(\cdot)$  which solves the Steklov-Poincaré problem (1.21). For such interface data  $g(\cdot)$ , the local solutions  $(w_1, w_2)$  to (1.20) will yield the solution to (1.19) with  $g(x) = u(x)$  on  $B$ . When a weak formulation is used, if  $X$  denotes the space of Dirichlet data on  $B$ , the flux or Neumann data will belong to its dual space  $X'$ , where  $X = H_{00}^{1/2}(B)$  for a standard subdomain decomposition and  $X = H^{1/2}(B)$  for an immersed subdomain decomposition.

*Remark 1.17.* For computational purposes, the Steklov-Poincaré operator  $\mathcal{S}$  may be expressed as the sum of two subdomain operators:

$$\mathcal{S}(g, f_1, f_2) \equiv \mathcal{S}^{(1)}(g, f_1) + \mathcal{S}^{(2)}(g, f_2),$$

where

$$\begin{cases} \mathcal{S}^{(1)}(g, f_1) \equiv \mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1) \\ \mathcal{S}^{(2)}(g, f_2) \equiv \mathbf{n}_2 \cdot (a \nabla w_2 - \mathbf{b} w_2), \end{cases}$$

for  $w_1$  and  $w_2$  defined by (1.20). By definition, each operator  $\mathcal{S}^{(i)}$  will require only subdomain information and will be affine linear.

*Remark 1.18.* Both  $\mathcal{S}^{(1)}$  and  $\mathcal{S}^{(2)}$  map the *Dirichlet* interface data  $g(\cdot)$  prescribed on  $B$  to the corresponding *Neumann* flux data  $\mathbf{n}_1 \cdot (a \nabla w_1 - \mathbf{b} w_1)$  and  $\mathbf{n}_2 \cdot (a \nabla w_2 - \mathbf{b} w_2)$  on  $B$ , respectively, obtained by solution of the local problems (1.20). As a result, the maps  $\mathcal{S}^{(i)}$  are commonly referred to as local *Dirichlet to Neumann* maps. These Dirichlet to Neumann maps are *not* differential operators since the solutions  $w_i$  to (1.20) have representations as integral operators acting on the data  $g$ . They are referred to as *pseudo-differential operators*, and for the correct choice of Dirichlet interface data  $g(\cdot)$  on  $B$ , the jump in the Neumann data on  $B$  will be zero for the local solutions.

In the rest of this section, we outline how iterative methods, global discretizations and heterogeneous approximations can be constructed for the original problem (1.1) using the Steklov-Poincaré formulation (1.19).

### 1.3.2 Iterative Methods

The block structure of the Steklov-Poincaré system (1.19) suggests various iterative algorithms for its solution. For instance, if  $w_1^{(k)}$  and  $w_2^{(k)}$  denote the  $k$ 'th iterates on subdomains  $\Omega_1$  and  $\Omega_2$ , respectively, then the system of equations posed on subdomain  $\Omega_i$  in (1.19) can be solved to yield updates  $w_i^{(k+1)}$  for the local solutions, with boundary conditions chosen using preceding iterates. The resulting iterative algorithm sequentially enforces either the *continuity* or *flux* transmission boundary conditions on  $B$ , and is referred to as a Dirichlet-Neumann algorithm as it requires the solution of Dirichlet and Neumann boundary value problems. In the following, suppose that  $\mathbf{b}(x) = \mathbf{0}$  in  $\Omega$ , and let  $0 < \theta < 1$  denote a *relaxation* parameter required to ensure convergence [BJ9, BR11, FU, MA29].

**Algorithm 1.3.1** (*Dirichlet-Neumann Algorithm*)

Let  $v_2^{(0)}$  (where  $v_2^{(0)} \equiv w_2^{(0)}$  on  $B$ ) denote a starting guess.

1. For  $k = 0, 1, \dots$ , until convergence do:
2. Solve for  $w_1^{(k+1)}$  as follows:

$$\begin{cases} Lw_1^{(k+1)} = f_1, & \text{in } \Omega_1 \\ w_1^{(k+1)} = v_2^{(k)}, & \text{on } B \\ w_1^{(k+1)} = 0, & \text{on } B_{[1]}, \end{cases}$$

3. Solve for  $w_2^{(k+1)}$  as follows:

$$\begin{cases} Lw_2^{(k+1)} = f_2, & \text{in } \Omega_2 \\ w_2^{(k+1)} = 0, & \text{on } B_{[2]} \\ \mathbf{n}_2 \left( a \nabla w_2^{(k+1)} \right) = \mathbf{n}_2 \left( a \nabla w_1^{(k+1)} \right), & \text{on } B. \end{cases}$$

4. Update:  $v_2^{(k+1)} = \theta w_2^{(k+1)} + (1 - \theta)v_2^{(k)}$  on  $B$ .
5. Endfor

Output:  $(w_1^{(k)}, w_2^{(k)})$

*Remark 1.19.* In step 2, the local solution  $w_1^{(k+1)}$  matches  $v_2^{(k)}$  on  $B$  (however, the local fluxes may not match on  $B$ ). This step requires the solution of an elliptic equation on  $\Omega_1$  with Dirichlet conditions on  $B_{[1]}$  and  $B$ . In step 3, the flux of  $w_2^{(k+1)}$  matches the flux of  $w_1^{(k+1)}$  on  $B$  (though  $w_2^{(k+1)}$  may not match  $w_1^{(k+1)}$  on  $B$ ). This step requires the solution of an elliptic equation on  $\Omega_2$  with Dirichlet conditions on  $B_{[2]}$  and Neumann conditions on  $B$ . A matrix formulation of this algorithm is given in Chap. 3.

*Remark 1.20.* Under restrictions on the coefficients (such as  $\mathbf{b}(x) \equiv 0$  and  $c(x) \geq 0$ ), and additional restrictions on the parameter  $0 < \theta < 1$ , the iterates  $w_i^{(k)}$  in the Dirichlet-Neumann algorithm will converge geometrically to the true local solution  $w_i$  of (1.19) as  $k \rightarrow \infty$ , see [FU, MA29].

The preceding Dirichlet-Neumann algorithm has sequential steps. Various algorithms have been proposed which solve subdomain problems in parallel, see [BO7, DE3, DR18, MA14, DO13, QU6, GA14, AC7, RA3]. Multidomain matrix versions of such algorithms are described in Chap. 3. Below, we describe a two fractional step algorithm, each step requiring the solution of subdomain problems in parallel [DO13, DO18, YA2]. We assume  $\mathbf{b}(x) = \mathbf{0}$ .

**Algorithm 1.3.2** (*A Parallel Dirichlet-Neumann Algorithm*)

Let  $w_1^{(0)}$  and  $w_2^{(0)}$  denote a starting guess on each subdomain.

Let  $0 < \theta, \delta, \beta, \alpha < 1$  denote relaxation parameters.

1. For  $k = 0, 1, \dots$ , until convergence do:
  2. Compute 
$$\begin{cases} \boldsymbol{\mu}^{(k+\frac{1}{2})} = \theta \mathbf{n}_1 \cdot (a \nabla w_1^{(k)}) + (1 - \theta) \mathbf{n}_1 \cdot (a \nabla w_2^{(k)}), & \text{on } B \\ \mathbf{g}^{(k+\frac{1}{2})} = \delta w_1^{(k)} + (1 - \delta) w_2^{(k)}, & \text{on } B. \end{cases}$$
  3. In parallel solve for  $w_1^{(k+\frac{1}{2})}$  and  $w_2^{(k+\frac{1}{2})}$ 

$$\begin{cases} Lw_1^{(k+\frac{1}{2})} = f, & \text{in } \Omega_1 \\ w_1^{(k+\frac{1}{2})} = 0, & \text{on } B_{[1]} \\ \mathbf{n}_1 \cdot (a \nabla w_1^{(k+\frac{1}{2})}) = \boldsymbol{\mu}^{(k+\frac{1}{2})}, & \text{on } B, \end{cases} \text{ and } \begin{cases} Lw_2^{(k+\frac{1}{2})} = f, & \text{in } \Omega_2 \\ w_2^{(k+\frac{1}{2})} = 0, & \text{on } B_{[2]} \\ w_2^{(k+\frac{1}{2})} = \mathbf{g}^{(k+\frac{1}{2})}, & \text{on } B, \end{cases}$$
  4. Compute 
$$\begin{cases} \boldsymbol{\mu}^{(k+1)} = \beta \mathbf{n}_2 \cdot (a \nabla w_1^{(k+\frac{1}{2})}) + (1 - \beta) \mathbf{n}_2 \cdot (a \nabla w_2^{(k+\frac{1}{2})}), & \text{on } B \\ \mathbf{g}^{(k+1)} = \alpha w_1^{(k+\frac{1}{2})} + (1 - \alpha) w_2^{(k+\frac{1}{2})}, & \text{on } B. \end{cases}$$
  5. In parallel solve for  $w_1^{(k+1)}$  and  $w_2^{(k+1)}$ 

$$\begin{cases} Lw_1^{(k+1)} = f, & \text{in } \Omega_1 \\ w_1^{(k+1)} = 0, & \text{on } B_{[1]} \\ w_1^{(k+1)} = \mathbf{g}^{(k+1)}, & \text{on } B, \end{cases} \text{ and } \begin{cases} Lw_2^{(k+1)} = f, & \text{in } \Omega_2 \\ w_2^{(k+1)} = 0, & \text{on } B_{[2]} \\ \mathbf{n}_2 \cdot (a \nabla w_2^{(k+1)}) = \boldsymbol{\mu}^{(k+1)}, & \text{on } B, \end{cases}$$
  6. Endfor

Output:  $(w_1^{(k)}, w_2^{(k)})$

*Remark 1.21.* Under appropriate restrictions on the coefficients  $a(x)$  and  $c(x)$ , and the relaxation parameters  $\theta, \delta, \beta, \alpha$ , this parallel algorithm will converge geometrically [YA2]. For related parallel algorithms, see [DO13, DO18].

When the advection coefficient  $\mathbf{b}(x) \neq 0$ , a parallel algorithm, referred to as a Robin-Robin algorithm can also be used [QU6, GA14, AC7, RA3]. Let:

$$\Phi_i(w) \equiv \mathbf{n}_i \cdot \left( a(x) \nabla w - \frac{1}{2} \mathbf{b}(x) w \right) + z_i(x) w,$$

denote a local Robin boundary operator on  $B$  for  $i = 1, 2$  for an appropriately chosen bounded interface function  $z_i(x) > 0$ . For convenience,  $\tilde{i}$  will denote a complementary index to  $i$  (namely,  $\tilde{i} = 2$  when  $i = 1$  and  $\tilde{i} = 1$  when  $i = 2$ ). Then, the Robin-Robin algorithm has the following form.

**Algorithm 1.3.3** (*A Robin-Robin Algorithm*)

Let  $w_1^{(0)}$  and  $w_2^{(0)}$  denote a starting guess on each subdomain

Let  $0 < \theta < 1$  denote a relaxation parameter

1. For  $k = 0, 1, \dots$ , until convergence do:
2. For  $i = 1, 2$  in parallel solve:

$$\begin{cases} Lw_i^{(k+1)} = f_i, & \text{in } \Omega_i \\ w_i^{(k+1)} = 0, & \text{on } B_{[i]} \\ \Phi_i(w_i^{(k+1)}) = \theta \Phi_i(w_i^{(k)}) + (1 - \theta) \Phi_i(w_i^{(k)}), & \text{on } B \end{cases}$$

3. Endfor

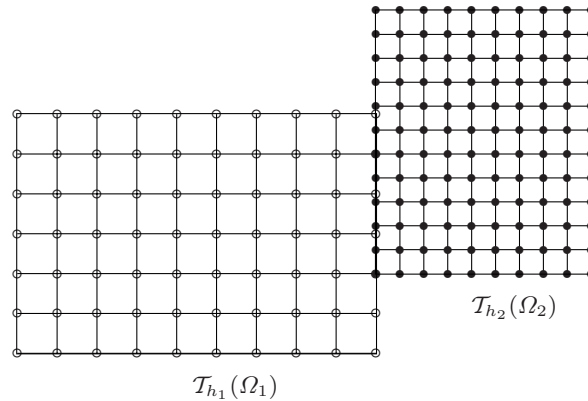
4. Endfor

Output:  $(w_1^{(k)}, w_2^{(k)})$

*Remark 1.22.* When  $(c(x) - \frac{1}{2} \nabla \cdot \mathbf{b}(x)) \geq \beta > 0$ , the Robin-Robin iterates will converge geometrically, for a suitable choice of relaxation parameter  $0 < \theta < 1$  and  $z_i(x) > 0$ , see [QU6, GA14, AC7, RA3].

**1.3.3 Global Discretization**

Hybrid formulation (1.19) can be used to construct a global discretization of (1.1). Such discretizations have not been studied extensively, however, see [AG, AG2, DO4] and in the context of spectral methods, see [MA4, PH]. A potential advantage of discretizing (1.19) is that each subdomain  $\Omega_i$  can be independently triangulated, see Fig. 1.6, by methods suited to the local geometry and regularity of the local solution, and each subproblem may be discretized independently. However, care must be exercised in discretizing the transmission conditions so that the resulting global discretization is stable. Below, we *heuristically* outline the general stages that would be involved in discretizing (1.19) using finite element methods.



**Fig. 1.6.** Nonmatching local grids

On each subdomain  $\Omega_i$ , generate a grid  $\mathcal{T}_{h_i}(\Omega_i)$  of size  $h_i$  suited to the local geometry and solution. If the resulting local grids do not match along  $B$ , as in Fig. 1.6, they will be referred to as nonmatching grids. On each subdomain  $\Omega_i$ , employ a traditional method to discretize the following Neumann problem:

$$\begin{cases} Lw_i = f, & \text{in } \Omega_i \\ w_i = 0, & \text{on } B_{[i]} \\ \mathbf{n}_i \cdot (a\nabla w_i - \mathbf{b} w_i) = g_i, & \text{on } B, \end{cases}$$

where  $\mathbf{n}_i$  denotes the exterior unit normal to  $\partial\Omega_i$  and the flux data  $g_i$  is to be chosen when the transmission conditions are applied. Employing block matrix notation, denote the resulting local discretization by:

$$\begin{bmatrix} A_{II}^{(i)} & A_{IB}^{(i)} \\ A_{BI}^{(i)} & A_{BB}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_B^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{h_i} \\ \mathbf{g}_{h_i} \end{bmatrix},$$

where  $\mathbf{w}_I^{(i)}$  denotes the interior unknowns on  $\Omega_{h_i}$  and  $\mathbf{w}_B^{(i)}$  denotes the boundary unknowns on  $B$  associated with the discrete solution on  $\mathcal{T}_{h_i}(\Omega_i)$ . Separately discretize the two transmission conditions on  $B$ :

$$\begin{cases} w_1 = w_2, & \text{on } B \\ \mathbf{n}_1 \cdot (a\nabla w_1 - \mathbf{b} w_1) = \mathbf{n}_1 \cdot (a\nabla w_2 - \mathbf{b} w_2), & \text{on } B. \end{cases}$$

Since the grid functions  $(\mathbf{w}_I^{(i)}, \mathbf{w}_B^{(i)})$  may be nonmatching on  $B$ , care must be exercised to ensure well posedness and stability of this discretization.

Below, we indicate how each transmission condition can be discretized by a “mortar” element type method. Let  $n_i$  and  $m_i$  denote the number of unknowns in  $\mathbf{w}_I^{(i)}$  and  $\mathbf{w}_B^{(i)}$  respectively. Then the continuity equation  $w_1 = w_2$  on  $B$  may be discretized by a Petrov-Galerkin approximation of its weak form:

$$\int_B (w_1 - w_2) v \, ds_x = 0, \quad v \in X_h(B),$$

where  $X_h(B)$  denotes some appropriately chosen subspace of  $L^2(B)$ . In a mortar element discretization,  $X_h(B)$  is typically chosen as a finite element space defined on a triangulation of  $B$  inherited from either triangulation  $\mathcal{T}_{h_1}(\Omega_1)$  or  $\mathcal{T}_{h_2}(\Omega_2)$ . Examples of such spaces are described in Chap. 11. For definiteness suppose  $X_h(B) = X_{h_1}(B)$  is chosen to be of dimension  $m_1$  based on the triangulation of  $B$  inherited from  $\mathcal{T}_{h_1}(\Omega_1)$ . Then, the discretized continuity transmission condition will have the following matrix form:

$$M_{11}\mathbf{w}_B^{(1)} = M_{12}\mathbf{w}_B^{(2)},$$

where  $M_{11}$  and  $M_{12}$  are  $m_1 \times m_1$  and  $m_1 \times m_2$  mass matrices, respectively.

The flux transmission condition on  $B$  may be similarly discretized:

$$\int_B (\mathbf{n}_1 \cdot (a\nabla w_1 - \mathbf{b} w_1) - \mathbf{n}_1 \cdot (a\nabla w_2 - \mathbf{b} w_2)) \mu \, ds_x = 0, \quad \forall \mu \in Y_h(B),$$



where it is sufficient to choose  $Y_h(B) \subset H_0^1(B)$ . Again,  $Y_h(B)$  may be chosen as a finite element space defined on the triangulation of  $B$  inherited from either triangulation  $\Omega_{h_1}$  or  $\Omega_{h_2}$ . However, to ensure that the total number of equations equals the total number of unknowns in the global system, it will be preferable that  $Y_h(B)$  be chosen using the complementary triangulation. In the above example, since  $X_h(B) = X_{h_1}(B)$  is of dimension  $m_1$ , we choose  $Y_h(B) = Y_{h_2}(B)$  of dimension  $m_2$  based on triangulation  $\Omega_{h_2}$ . This will yield  $m_2$  constraints, which we denote as:

$$M_{21} \left( A_{BI}^{(1)} \mathbf{w}_I^{(1)} + A_{BB}^{(1)} \mathbf{w}_B^{(1)} - \mathbf{f}_B^{(1)} \right) = -M_{22} \left( A_{BI}^{(2)} \mathbf{w}_I^{(2)} + A_{BB}^{(2)} \mathbf{w}_B^{(2)} - \mathbf{f}_B^{(2)} \right),$$

where  $M_{21}$  and  $M_{22}$  are  $m_2 \times m_1$  and  $m_2 \times m_2$  matrices, respectively. The interface forcing terms  $\mathbf{f}_B^{(i)}$  have been added to account for the approximation resulting from integration by parts. The actual choice of subspaces  $X_{h_1}(B)$  and  $Y_{h_2}(B)$  will be *critical* to the stability of the resulting global discretization:

$$\left\{ \begin{array}{l} A_{II}^{(1)} \mathbf{w}_I^{(1)} + A_{IB}^{(1)} \mathbf{w}_B^{(1)} = \mathbf{f}_{h_1} \\ M_{11} \mathbf{w}_B^{(1)} = M_{12} \mathbf{w}_B^{(2)} \\ A_{II}^{(2)} \mathbf{w}_I^{(2)} + A_{IB}^{(2)} \mathbf{w}_B^{(2)} = \mathbf{f}_{h_2} \\ M_{21} \left( A_{BI}^{(1)} \mathbf{w}_I^{(1)} + A_{BB}^{(1)} \mathbf{w}_B^{(1)} - \mathbf{f}_B^{(1)} \right) = -M_{22} \left( A_{BI}^{(2)} \mathbf{w}_I^{(2)} + A_{BB}^{(2)} \mathbf{w}_B^{(2)} - \mathbf{f}_B^{(2)} \right). \end{array} \right.$$

General theoretical results on the stability of such discretizations of (1.19) are not known to the author, and this scheme was heuristically considered only for its intrinsic interest.

*Remark 1.23.* If the grids  $\mathcal{T}_{h_1}(\Omega_1)$  and  $\mathcal{T}_{h_2}(\Omega_2)$  match on  $B$ , then  $m_1 = m_2$ . We would then obtain  $M_{11} = M_{12}$ , both square and nonsingular, yielding:

$$\mathbf{w}_B^{(1)} = \mathbf{w}_B^{(2)}.$$

Similarly,  $M_{21} = M_{22}$  will be square and nonsingular yielding:

$$\left( A_{BI}^{(1)} \mathbf{w}_I^{(1)} + A_{BB}^{(1)} \mathbf{w}_B^{(1)} - \mathbf{f}_B^{(1)} \right) = - \left( A_{BI}^{(2)} \mathbf{w}_I^{(2)} + A_{BB}^{(2)} \mathbf{w}_B^{(2)} - \mathbf{f}_B^{(2)} \right).$$

The resulting global discretization will then correspond to the standard finite element discretization of (1.1).

### 1.3.4 Heterogeneous Approximations

A heterogeneous approximation of a partial differential equation is a coupled system of partial differential equations which approximates the given equation, in which the approximating partial differential equations are not of the same *type* in different subregions [GA15, QU6]. In the following, motivated

by classical singular perturbation approximations [KE5, LA5], we heuristically outline how an elliptic-hyperbolic heterogeneous approximation can be constructed for the following singularly perturbed elliptic equation:

$$\begin{cases} L^\epsilon u \equiv -\epsilon \Delta u + \mathbf{b}(x) \cdot \nabla u + c(x) u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (1.22)$$

where  $0 < \epsilon \ll 1$  is a perturbation parameter. The Steklov-Poincaré hybrid formulation (1.19) will be employed to heuristically approximate (1.22).

Suppose  $\Omega_1$  and  $\Omega_2$  form a non-overlapping decomposition of  $\Omega$  such that:

$$\epsilon |\Delta u| \ll |\mathbf{b} \cdot \nabla u + c u|, \quad \text{on } \overline{\Omega}_1.$$

Then, on subdomain  $\Omega_1$ , we may approximate  $L^\epsilon u = f$  by  $L_0 u = f$ , where  $L_0 u \equiv \mathbf{b}(x) \cdot \nabla u + c(x) u$ . Formally, a global heterogeneous approximation of (1.22) may be obtained by substituting the preceding approximation in the hybrid formulation corresponding to (1.22), yielding:

$$\begin{cases} L_0 w_1 = f, & \text{in } \Omega_1 \\ w_1 = 0, & \text{on } B_{[1]} \\ w_1 = w_2, & \text{on } B \\ L^\epsilon w_2 = f, & \text{in } \Omega_2 \\ w_2 = 0, & \text{on } B_{[2]} \\ \mathbf{n}_1 \cdot (\epsilon \nabla w_2 - \mathbf{b} w_2) = \mathbf{n}_1 \cdot (\epsilon \nabla w_1 - \mathbf{b} w_1), & \text{on } B, \end{cases}$$

However, retaining the Dirichlet boundary conditions on  $B$  and  $B_{[1]}$  for  $w_1(\cdot)$  will yield an *ill-posed* problem for  $w_1(\cdot)$ , since  $L_0 w_1$  is hyperbolic on  $\Omega_1$ . Indeed, denote the *inflow* and *outflow* boundary segments on  $B$  and  $B_{[1]}$  by:

$$\begin{cases} B_{in} & \equiv \{x \in B : \mathbf{n}_1 \cdot \mathbf{b}(x) < 0\} \\ B_{out} & \equiv \{x \in B : \mathbf{n}_1 \cdot \mathbf{b}(x) > 0\} \\ B_{[1],in} & \equiv \{x \in B_{[1]} : \mathbf{n}_1 \cdot \mathbf{b}(x) < 0\}. \end{cases}$$

Since  $L_0 w_1 = f$  is hyperbolic, specification of Dirichlet or Neumann boundary conditions on the entire boundary  $\partial\Omega_1$  will yield a locally ill posed problem. Fortunately, replacing the Dirichlet conditions by *inflow* conditions, resolves this local ill-posedness on  $\Omega_1$ , see [GA15, QU6].

Thus, the boundary conditions  $w_1 = 0$  on  $B_{[1]}$  and  $w_1 = w_2$  on  $B$  can be replaced by inflow boundary conditions  $w_1 = 0$  on  $B_{[1],in}$  and  $w_1 = w_2$  on  $B_{in}$ , respectively. To deduce the remaining transmission boundary conditions in the heterogeneous approximation, a subdomain vanishing viscosity approach may be employed as in [GA15]. Accordingly, the elliptic equation  $L^\epsilon u = f$  may be approximated by the discontinuous coefficient elliptic problem:

$$\begin{cases} L^{\epsilon,\eta} v = f, & \text{on } \Omega \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $L^{\epsilon, \eta} v \equiv -\nabla \cdot (a(x, \eta) \nabla v) + \mathbf{b}(x) \cdot \nabla v + c(x) v$  and  $a(x, \eta)$  is defined by:

$$a(x, \eta) \equiv \begin{cases} \eta & \text{for } x \in \Omega_1 \\ \epsilon & \text{for } x \in \Omega_2. \end{cases}$$

For  $\epsilon > 0$  and  $\eta > 0$ , the problem will be elliptic and the traditional transmission conditions should hold:

$$\begin{cases} w_1 = w_2, & \text{on } B \\ \mathbf{n}_1 \cdot (\eta \nabla w_1 - \mathbf{b} w_1) = \mathbf{n}_1 \cdot (\epsilon \nabla w_2 - \mathbf{b} w_2), & \text{on } B. \end{cases}$$

However, letting  $\eta \rightarrow 0^+$ , and imposing the *inflow* condition on  $B_{in}$  yields:

$$\begin{cases} w_1 = w_2, & \text{on } B_{in} \\ -\mathbf{n}_1 \cdot \mathbf{b} w_1 = \mathbf{n}_1 \cdot (\epsilon \nabla w_2 - \mathbf{b} w_2), & \text{on } B. \end{cases}$$

When  $\mathbf{b}(x)$  is continuous, the substitution that  $w_1 = w_2$  on  $B_{in}$  will yield the following additional simplifications:

$$\begin{cases} w_1 = w_2, & \text{on } B_{in} \\ 0 = \mathbf{n}_1 \cdot \epsilon \nabla w_2, & \text{on } B_{in} \\ -\mathbf{n}_1 \cdot \mathbf{b} w_1 = \mathbf{n}_1 \cdot (\epsilon \nabla w_2 - \mathbf{b} w_2), & \text{on } B_{out}. \end{cases}$$

As a result, heuristically, the global system of partial differential equations satisfied by the weak limit of the solutions  $v^{\epsilon, \eta}$  as  $\eta \rightarrow 0$  will be:

$$\begin{cases} L_0 w_1 = f, & \text{in } \Omega_1 \\ w_1 = 0, & \text{on } B_{[1], in} \\ w_1 = w_2, & \text{on } B_{in} \\ L^\epsilon w_2 = f, & \text{in } \Omega_2 \\ \mathbf{n}_2 \cdot (\epsilon \nabla w_2 - \mathbf{b} w_2) = -\mathbf{n}_2 \cdot \mathbf{b} w_1, & \text{on } B_{out}, \\ \mathbf{n}_2 \cdot \nabla w_2 = 0, & \text{on } B_{in}, \\ w_2 = 0, & \text{on } B_{[2]}. \end{cases}$$

Dirichlet-Neumann iterative methods can be formulated to solve the above heterogeneous approximation to (1.22), see [GA15, QU6] and Chap. 12.

*Remark 1.24.* For rigorous results on the well posedness of the preceding heterogeneous system, readers are referred to [GA15].

## 1.4 Lagrange Multiplier Framework

The framework we refer to as the Lagrange multiplier formulation [GL, GL7], underlies a variety of *non-overlapping* domain decomposition methods. It is employed in the FETI (Finite Element Tearing and Interconnection) method

(a constrained optimization based parallel iterative method [FA16, FA15]), the mortar element method (a method for discretizing elliptic equations on nonmatching grids [MA4, BE22, BE18, BE6, BE4, WO4, WO5]), and in non-overlapping Schwarz iterative methods [LI8, GL8]. In this section, we illustrate its application to formulate *iterative* algorithms, *non-matching grid* discretizations and *heterogeneous* approximations.

The Lagrange multiplier framework is applicable only when there is an *optimization* principle associated with the elliptic equation. Thus, the solution  $u$  must optimize some *energy functional*  $J(\cdot)$ . For such a property to hold, the elliptic equation (1.1) must be *self adjoint* and *coercive*, requiring that  $\mathbf{b}(x) = 0$  and  $c(x) \geq 0$ . Accordingly, in this section we shall consider:

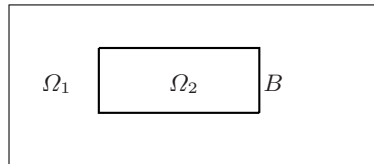
$$\begin{cases} Lu \equiv -\nabla \cdot (a(x) \nabla u) + c(x) u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.23)$$

with  $c(x) \geq 0$ . It is well known that the solution  $u$  minimizes an energy  $J(\cdot)$ , see (1.24) and (1.25) within  $H_0^1(\Omega)$ . Given any non-overlapping subdomain decomposition of  $\Omega$ , we will show that the optimization problem (1.24) can be reformulated as a *constrained optimization* problem based on the subdomains. The Lagrange multiplier hybrid formulation will be the saddle point problem associated with this constrained minimization problem.

#### 1.4.1 Motivation

Let  $\Omega_1$  and  $\Omega_2$  form a *non-overlapping* decomposition of the domain  $\Omega$  of elliptic equation (1.23), see Fig. 1.7. Using this decomposition of  $\Omega$ , we may decompose the energy functional  $J(\cdot)$  associated with (1.23) as a sum of energy contributions  $J_i(\cdot)$  from each subdomain  $\Omega_i$ . The resulting sum of local energies will be well defined even if the local displacement functions are discontinuous across the interface  $B = \partial\Omega_1 \cap \partial\Omega_2$ . It is thus an *extended* energy functional.

A constrained minimization problem equivalent to the minimization of  $J(\cdot)$  can be obtained by minimizing this extended energy functional, subject to the constraint that the local displacements *match* on the interface  $B$ . The Lagrange multiplier hybrid formulation is the saddle point problem associated with this constrained minimization problem. We outline the steps below.



**Fig. 1.7.** An immersed non-overlapping decomposition

**Minimization Formulation.** It is well known, see [ST14, CI2, JO2, BR28], that the solution  $u$  to (1.23) minimizes the energy  $J(\cdot)$  associated with (1.23):

$$J(u) = \min_{w \in X} J(w), \quad (1.24)$$

where

$$\begin{cases} J(w) & \equiv \frac{1}{2} \mathcal{A}(w, w) - F(w), \\ \mathcal{A}(v, w) & \equiv \int_{\Omega} (a \nabla v \cdot \nabla w + c v w) dx, \quad \text{for } v, w \in X \\ F(w) & \equiv \int_{\Omega} f w dx, \quad \text{for } w \in X, \\ X & \equiv H_0^1(\Omega). \end{cases} \quad (1.25)$$

**Constrained Minimization Formulation.** Let  $\{\Omega_i\}_{i=1}^2$  be a non-overlapping decomposition of  $\Omega$ . Suppose  $w_i \equiv w$  on  $\overline{\Omega}_i$  for  $1 \leq i \leq 2$ . We may express the energy  $J(w) = J_{\mathcal{E}}(w_1, w_2) \equiv J_1(w_1) + J_2(w_2)$ , where:

$$\begin{cases} J_{\mathcal{E}}(w_1, w_2) \equiv J_1(w_1) + J_2(w_2), & \text{for } w_i \in X_i \\ J_i(w_i) & \equiv \frac{1}{2} \mathcal{A}_i(w_i, w_i) - F_i(w_i), & \text{for } w_i \in X_i \\ \mathcal{A}_i(v_i, w_i) & \equiv \int_{\Omega_i} (\nabla v_i \cdot a \nabla w_i + c v_i w_i) dx, & \text{for } v_i, w_i \in X_i \\ F_i(w_i) & \equiv \int_{\Omega_i} f w_i dx, & \text{for } w_i \in X_i, \\ X_i & \equiv \{v \in H^1(\Omega_i) : v = 0 \text{ on } B_{[i]}\}. \end{cases}$$

Here  $J_{\mathcal{E}}(w_1, w_2)$  is defined even when  $w_1 \neq w_2$  on  $B$ . To obtain a *constrained* minimization problem equivalent to (1.24), we minimize  $J_{\mathcal{E}}(v_1, v_2)$  within the larger (extended) class of functions  $X_1 \times X_2$  defined above, but subject to the *weak* constraint that the subdomain functions *match* on  $B$ :

$$m((v_1, v_2), \mu) \equiv \int_B (v_1 - v_2) \mu ds_x = 0, \quad \forall \mu \in Y,$$

where  $Y \equiv H_{00}^{-1/2}(B)$  (the dual space of  $H_{00}^{1/2}(B)$ ). Problem (1.24) will thus be formally equivalent to the following constrained minimization problem:

$$J_1(w_1) + J_2(w_2) = \min_{(v_1, v_2) \in \mathcal{K}} J_1(v_1) + J_2(v_2), \quad (1.26)$$

where

$$\mathcal{K} \equiv \{(v_1, v_2) \in X_1 \times X_2 : m((v_1, v_2), \mu) = 0, \quad \forall \mu \in Y\}.$$

**Saddle Point Formulation.** By optimization theory, see [CI4] and Chap. 10, the solution  $(w_1, w_2)$  to the constrained minimization problem (1.26) can be expressed as components in the saddle point  $((w_1, w_2), \mu)$  of an associated *Lagrangian* functional  $\mathcal{L}(\cdot, \cdot)$ , where  $\mu \in Y$  denotes an artificially introduced variable referred to as a Lagrange multiplier. We define the Lagrangian function for  $((v_1, v_2), \eta) \in X_1 \times X_2 \times Y$  as:

$$\mathcal{L}((v_1, v_2), \eta) \equiv J_1(v_1) + J_2(v_2) + m((v_1, v_2), \eta). \quad (1.27)$$

At the saddle point  $((w_1, w_2), \mu) \in X_1 \times X_2 \times Y$  of  $\mathcal{L}(\cdot)$ , we obtain:

$$\mathcal{L}((w_1, w_2), \eta) \leq \mathcal{L}((w_1, w_2), \mu) \leq \mathcal{L}((v_1, v_2), \mu) \quad (1.28)$$

for any choice of  $(v_1, v_2) \in X_1 \times X_2$  and  $\eta \in Y$ . Requiring the first order variation at the saddle point  $((w_1, w_2), \mu)$  to be zero yields:

$$\begin{cases} \sum_{i=1}^2 \mathcal{A}_i(w_i, v_i) + m((v_1, v_2), \mu) = \sum_{i=1}^2 F_i(v_i), & \text{for } v_i \in X_i \\ m((w_1, w_2), \eta) = 0, & \text{for } \eta \in Y. \end{cases} \quad (1.29)$$

The above system is referred to as a saddle point problem.

**Hybrid Formulation.** If we integrate the weak form (1.29) by parts, we can express it in terms of partial differential equations involving  $w_1(\cdot)$ ,  $w_2(\cdot)$  and the Lagrange multiplier variable  $\mu(\cdot)$  as follows. We seek  $(w_1, w_2, \mu)$  satisfying:

$$\begin{cases} Lw_1 = f, & \text{in } \Omega_1 \\ w_1 = 0, & \text{on } B_{[1]} \\ \mathbf{n}_1 \cdot (a \nabla w_1) = -\mu, & \text{on } B \\ Lw_2 = f, & \text{in } \Omega_2 \\ w_2 = 0, & \text{on } B_{[2]} \\ \mathbf{n}_2 \cdot (a \nabla w_2) = \mu, & \text{on } B \\ w_1 = w_2, & \text{on } B \end{cases} \quad (1.30)$$

where  $B_{[i]} \equiv \partial\Omega_i \cap \partial\Omega$  is the exterior boundary and  $\mathbf{n}_i$  is the unit exterior normal to  $\partial\Omega_i$  for  $i = 1, 2$ . For each choice of Neumann data  $\mu(\cdot)$ , each subdomain problem for  $w_i(\cdot)$  will be *uniquely* solvable provided  $B_{[i]} \neq \emptyset$ . We must choose the Lagrange multiplier  $\mu(\cdot)$  (representing the flux on  $B$ ) so that  $w_1 = w_2$  on  $B$ . The next result indicates the equivalence of (1.30) to (1.23).

**Theorem 1.25.** *Suppose the following assumptions hold.*

1. *Let  $u$  be a solution to (1.23).*
2. *Let  $(w_1, w_2, \mu)$  be a solution to the hybrid formulation (1.30).*

*Then  $u(x) = w_1(x)$  in  $\overline{\Omega}_1$  and  $u(x) = w_2(x)$  in  $\overline{\Omega}_2$ .*

*Proof.* The equivalence follows since (1.23) is equivalent to (1.19), and since (1.30) is equivalent to (1.19) for the substitution  $\mu = \mathbf{n}_2 \cdot (a \nabla u)$  on  $B$ .  $\square$

*Remark 1.26.* The preceding result only asserts the equivalence between solutions of (1.23) and (1.30). It does not demonstrate the well posedness of (1.30). The latter can be demonstrated for (1.30) by employing general results on the well posedness of the saddle point problem (1.29) associated with it [GI3].

### 1.4.2 Iterative Methods

Since the Lagrange multiplier  $\mu(\cdot)$  determines  $w_1(\cdot)$  and  $w_2(\cdot)$  in (1.30), an iterative method for solving (1.23) can be obtained by applying a saddle point iterative algorithm such as Uzawa's method, see Chap. 10, to update the Lagrange multiplier function  $\mu(\cdot)$ , as described below.

**Algorithm 1.4.1** (*Uzawa's Method*)

Let  $\mu^{(0)}$  denote a starting guess with chosen step size  $\tau > 0$ .

1. For  $k = 0, 1, \dots$  until convergence do:
2. Determine  $w_1^{(k+1)}$  and  $w_2^{(k+1)}$  in parallel:

$$\left\{ \begin{array}{ll} -\nabla \cdot (a \nabla w_1^{(k+1)}) + c w_1^{(k+1)} = f, & \text{in } \Omega_1 \\ w_1^{(k+1)} = 0, & \text{on } B_{[1]} \\ \mathbf{n}_1 \cdot (a \nabla w_1^{(k+1)}) = -\mu^{(k)}, & \text{on } B, \\ -\nabla \cdot (a \nabla w_2^{(k+1)}) + c w_2^{(k+1)} = f, & \text{in } \Omega_2 \\ w_2^{(k+1)} = 0, & \text{on } B_{[2]} \\ \mathbf{n}_2 \cdot (a \nabla w_2^{(k+1)}) = \mu^{(k)}, & \text{on } B. \end{array} \right.$$

3. Update  $\mu^{(k+1)}$  as follows:

$$\mu^{(k+1)}(x) = \mu^{(k)}(x) + \tau \left( w_1^{(k+1)}(x) - w_2^{(k+1)}(x) \right), \quad \text{for } x \in B.$$

4. Endfor

Output:  $(w_1^{(k)}, w_2^{(k)})$

*Remark 1.27.* The map  $\mu^{(k)} \rightarrow (w_1^{(k)} - w_2^{(k)})$  will be compact, and thus the iterates will converge geometrically to the true solution for sufficiently small  $\tau > 0$ . Discrete versions of Uzawa's algorithm are described in Chap. 10.

*Remark 1.28.* The FETI method [FA16, FA15], see Chap. 4, is also based on updating the Lagrange multiplier  $\mu$ . However, it generalizes the preceding saddle point iterative algorithm to the multisubdomain case, where the rate of convergence may deteriorate with increasing number of subdomains, and where the local problems may be singular.

An *alternative* hybrid formulation equivalent to (1.30) can be obtained by replacing the Lagrangian functional  $\mathcal{L}(\cdot, \cdot)$  by an *augmented* Lagrangian  $\mathcal{L}_\delta(\cdot, \cdot)$ , where an additional non-negative functional is added to the original Lagrangian functional with a coefficient  $\delta > 0$ , see [GL7, GL8]:

$$\mathcal{L}_\delta((v_1, v_2), \mu) \equiv J_1(v_1) + J_2(v_2) + m((v_1, v_2), \mu) + \frac{\delta}{2} \|v_1 - v_2\|_{L^2(B)}^2.$$

The augmented term  $\frac{\delta}{2} \|v_1 - v_2\|_{L^2(B)}^2$  will be zero when the constraint  $v_1 = v_2$  is satisfied on  $B$ . As a result, both formulations will be equivalent, and the saddle point of the augmented Lagrangian will also yield the desired solution. Applying an *alternating directions implicit* (ADI) method to determine the saddle point of the augmented Lagrangian functional, will yield the following algorithm, referred to as the *non-overlapping Schwarz* method [LI8, GL8].

**Algorithm 1.4.2** (*Non-Overlapping Schwarz Method*)

Let  $w_1^{(0)}, w_2^{(0)}$  denote starting guesses.

Let  $\delta > 0$  be a chosen parameter.

1. For  $k = 0, 1, \dots$  until convergence do:
2.   Solve in parallel:

$$\left\{ \begin{array}{ll} -\nabla \cdot (a \nabla w_1^{(k+1)}) + c w_1^{(k+1)} = f, & \text{in } \Omega_1 \\ w_1^{(k+1)} = 0, & \text{on } B_{[1]} \\ \mathbf{n}_1 \cdot (a \nabla w_1^{(k+1)}) + \delta w_1^{(k+1)} = \mathbf{n}_1 \cdot (a \nabla w_2^{(k)}) + \delta w_2^{(k)}, & \text{on } B, \\ -\nabla \cdot (a \nabla w_2^{(k+1)}) + c w_2^{(k+1)} = f, & \text{in } \Omega_2 \\ w_2^{(k+1)} = 0, & \text{on } B_{[2]} \\ \mathbf{n}_2 \cdot (a \nabla w_2^{(k+1)}) + \delta w_2^{(k+1)} = \mathbf{n}_2 \cdot (a \nabla w_1^{(k)}) + \delta w_1^{(k)}, & \text{on } B. \end{array} \right.$$

3. Endfor

Output:  $(w_1^{(k)}, w_2^{(k)})$

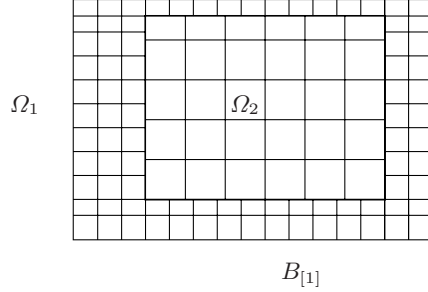
*Remark 1.29.* In practice, a careful choice of parameter  $\delta > 0$  will be necessary for optimal convergence [LI8, GL8].

### 1.4.3 Global Discretization

In principle, a discretization of (1.23) can be obtained by discretizing (1.30). Each subdomain can be triangulated independently without requiring the local triangulations to match on  $B$ . However, to ensure that the resulting discretization yields a constrained minimization problem, it is advantageous to employ a Galerkin approximation of the saddle point problem (1.29). An extensive literature exists on such nonmatching grid discretization techniques, see [MA4, BE22, DO4, BE4, WO4, WO5]. The resulting discretization is referred to as a mortar element method, see also Chap. 11.

Triangulate each subdomain  $\Omega_i$  by a grid  $\mathcal{T}_{h_i}(\Omega_i)$  of size  $h_i$  suited to the local geometry and solution for  $1 \leq i \leq 2$ , see Fig. 1.8. Let  $X_{h_i} \subset X_i$  denote a traditional finite element space defined on the triangulation  $\mathcal{T}_{h_i}(\Omega_i)$ . Select a triangulation of interface  $B$  inherited either from  $\mathcal{T}_{h_1}(\Omega_1)$  or  $\mathcal{T}_{h_2}(\Omega_2)$ . For definiteness, suppose that  $\mathcal{T}_{h_1}(\Omega_1)$  is chosen. Construct a finite element space  $Y_{h_1}(B) \subset L^2(B) \subset Y$  consisting of piecewise polynomial functions defined on





**Fig. 1.8.** Non-overlapping nonmatching grids

the triangulation of  $B$  inherited from  $\mathcal{T}_{h_1}(\Omega_1)$ . The dimension of  $Y_{h_1}$  should equal the dimension of  $X_{h_1} \cap H_0^1(B)$ . See Chap. 11 for multiplier spaces  $Y_{h_1}(B)$ .

Discretization of the saddle point formulation (1.29) using the subspaces  $X_{h_1} \times X_{h_2} \times Y_{h_1}(B)$  will yield a linear system of the form:

$$\begin{bmatrix} A^{(1)} & 0 & M^{(1)T} \\ 0 & A^{(2)} & -M^{(2)T} \\ M^{(1)} & -M^{(2)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_{h_1} \\ \mathbf{w}_{h_2} \\ \boldsymbol{\mu}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{h_1} \\ \mathbf{f}_{h_2} \\ \mathbf{0} \end{bmatrix},$$

where:

$$\begin{cases} \mathcal{A}_i(w_{h_i}, w_{h_i}) &= \mathbf{w}_{h_i}^T A^{(i)} \mathbf{w}_{h_i}, & \text{for } 1 \leq i \leq 2 \\ F(w_{h_i}) &= \mathbf{w}_{h_i}^T \mathbf{f}_{h_i}, & \text{for } 1 \leq i \leq 2 \\ m((w_{h_1}, w_{h_2}), \mu_h) &= \boldsymbol{\mu}_h^T (M^{(1)} \mathbf{w}_{h_1} - M^{(2)} \mathbf{w}_{h_2}). \end{cases}$$

Here we have used  $w_{h_i}$  and  $\mu_h$  to denote finite element functions and  $\mathbf{w}_{h_i}$  and  $\boldsymbol{\mu}_h$  as their vector representations with respect to some fixed basis.

If each nodal vector  $\mathbf{w}_{h_i}$  is block partitioned as  $\mathbf{w}_{h_i} = (\mathbf{w}_I^{(i)}, \mathbf{w}_B^{(i)})^T$  corresponding to the unknowns in the interior of each subdomain and on the interface  $B$ , then matrices  $A^{(i)}$  and  $M^{(i)}$  will have the block structure:

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{IB}^{(i)} \\ A_{IB}^{(i)T} & A_{BB}^{(i)} \end{bmatrix} \quad \text{and} \quad M^{(i)} = \begin{bmatrix} 0 & M_B^{(i)} \end{bmatrix}, \quad \text{for } 1 \leq i \leq 2$$

where  $\mathbf{w}_I^{(i)}$  and  $\mathbf{w}_B^{(i)}$  are of size  $n_i$  and  $m_i$ . Substituting, we obtain:

$$\begin{bmatrix} A_{II}^{(1)} & A_{IB}^{(1)} & 0 & 0 & 0 \\ A_{IB}^{(1)T} & A_{BB}^{(1)} & 0 & 0 & M_B^{(1)T} \\ 0 & 0 & A_{II}^{(2)} & A_{IB}^{(2)} & 0 \\ 0 & 0 & A_{IB}^{(2)T} & A_{BB}^{(2)} & -M_B^{(2)T} \\ 0 & M_B^{(1)} & 0 & -M_B^{(2)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(1)} \\ \mathbf{w}_B^{(1)} \\ \mathbf{w}_I^{(2)} \\ \mathbf{w}_B^{(2)} \\ \boldsymbol{\mu}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I^{(1)} \\ \mathbf{f}_B^{(1)} \\ \mathbf{f}_I^{(2)} \\ \mathbf{f}_B^{(2)} \\ \mathbf{0} \end{bmatrix}.$$

If the dimension of the space  $Y_{h_1}(B)$  is  $m_1$ , then matrix  $M_B^{(1)}$  will be square and *invertible* of size  $m_1$ . In this case, we may parameterize the solution space of the interface constraints as:

$$\mathbf{w}_B^{(1)} \equiv R_{12} \mathbf{w}_B^{(2)} \quad \text{where} \quad R_{12} \equiv M_B^{(1)^{-1}} M_B^{(2)}.$$

The local unknowns can then be represented as  $\mathbf{w}_I^{(1)}$ ,  $\mathbf{w}_B^{(1)} = R_{12} \mathbf{w}_B^{(2)}$ ,  $\mathbf{w}_I^{(2)}$ , and  $\mathbf{w}_B^{(2)}$ . Substituting this representation into the discrete energy  $J_{h_1}(\mathbf{w}_I^{(1)}, R_{12} \mathbf{w}_B^{(2)}) + J_{h_2}(\mathbf{w}_I^{(2)}, \mathbf{w}_B^{(2)})$  and applying first order stationarity conditions for its minimum yields the following linear system:

$$\begin{bmatrix} A_{II}^{(1)} & 0 & A_{IB}^{(1)} R_{12} \\ 0 & A_{II}^{(2)} & A_{IB}^{(2)} \\ R_{12}^T A_{IB}^{(1)T} & A_{IB}^{(2)T} & R_{12}^T A_{BB}^{(1)} R_{12} + A_{BB}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(1)} \\ \mathbf{w}_I^{(2)} \\ \mathbf{w}_B^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I^{(1)} \\ \mathbf{f}_I^{(2)} \\ R_{12}^T \mathbf{f}_B^{(1)} + \mathbf{f}_B^{(2)} \end{bmatrix}.$$

If both grids match, then  $R_{12} = I$  and the above discretization reduces to the traditional conforming finite element discretization of (1.23).

Mortar element spaces  $Y_{h_i}(B)$  are described in Chap. 11. They include piecewise polynomial functions which are continuous across elements as well as piecewise polynomial functions which are discontinuous across elements [MA4, BE22, BE18, BE6, BE4]. In the latter case, a basis for  $Y_{h_i}(B)$  can be constructed so that matrix  $M_B^{(i)}$  is diagonal [WO4, WO5]. The resulting global discretization will be stable and convergent of optimal order.

#### 1.4.4 Heterogeneous Approximations

When elliptic equation (1.23) is *singularly perturbed*, its Lagrange multiplier formulation (1.30) can be employed to *heuristically* study an heterogeneous approximation of it. Below, we illustrate two alternative approximations of the following singularly perturbed, self adjoint elliptic equation [KE5]:

$$\begin{cases} -\nabla \cdot (\epsilon \nabla u) + c(x) u = f(x), & \text{in } \Omega \\ u = g(x), & \text{on } \partial\Omega, \end{cases} \quad (1.31)$$

where  $0 < \epsilon \ll 1$  is a small perturbation parameter and  $c(x) \geq c_0 > 0$ . Suppose  $\Omega_1$  and  $\Omega_2$  form a nonoverlapping decomposition of  $\Omega$ , such that:

$$|\epsilon \Delta u| \ll |c(x) u|, \quad \text{for } x \in \overline{\Omega}_1.$$

Then,  $\Omega_2$  must enclose the boundary layer region of the solution.

To obtain an heterogeneous approximation of (1.31), we heuristically apply the subdomain vanishing viscosity method as in [GA15]:

$$\begin{cases} -\nabla \cdot (a_{\epsilon, \eta}(x) \nabla u) + c(x) u = f(x), & \text{in } \Omega \\ u = g(x), & \text{on } \partial\Omega, \end{cases} \quad (1.32)$$

where

$$a_{\epsilon,\eta}(x) \equiv \begin{cases} \eta & \text{for } x \in \Omega_1 \\ \epsilon & \text{for } x \in \Omega_2. \end{cases}$$

For  $\epsilon > 0$  and  $\eta > 0$ , the above problem is elliptic and coercive. However, as  $\eta \rightarrow 0^+$ , *formally* the limiting system (1.30) becomes:

$$\left\{ \begin{array}{ll} c(x) w_1 = f(x), & \text{in } \Omega_1 \\ w_1 = g(x), & \text{on } B_{[1]} \\ 0 = \mu, & \text{on } B \\ -\epsilon \Delta w_2 + c(x) w_2 = f(x), & \text{in } \Omega_2 \\ w_2 = g(x), & \text{on } B_{[2]} \\ \epsilon \frac{\partial w_2}{\partial n} = \mu, & \text{on } B \\ w_1 = w_2, & \text{on } B. \end{array} \right.$$

Two alternative approximations may be constructed. Either the *transmission* condition  $w_1 = w_2$  or  $\epsilon \frac{\partial w_2}{\partial n} = 0$  can be enforced, but *not* both, since  $w_1(\cdot)$  formally satisfies a *zeroth order* equation in  $\Omega_1$ . Since  $c(x) \geq c_0 > 0$ , the limiting equation on  $\Omega_1$  for  $w_1(x)$  can be solved to formally yield:

$$w_1(x) = \frac{f(x)}{c(x)}, \quad \text{on } \Omega_1.$$

If  $B_{[1]} \neq \emptyset$  and the boundary data  $g(x)$  is not compatible with the formal solution  $\frac{f(x)}{c(x)}$ , i.e., if  $g(x) \neq \frac{f(x)}{c(x)}$  on  $B_{[1]}$ , then the local solution may be ill posed, indicating a poor choice of subdomain  $\Omega_1$ .

If a *continuous* (or  $H^1(\cdot)$ ) solution is sought, then continuity of the local solutions must be enforced and the flux transmission condition needs to be *omitted*, yielding the following system:

$$\left\{ \begin{array}{ll} c(x) w_1 = f(x), & \text{in } \Omega_1 \\ w_1 = g(x), & \text{on } B_{[1]} \\ -\epsilon \Delta w_2 + c(x) w_2 = f(x), & \text{in } \Omega_2 \\ w_2 = w_1, & \text{on } B \\ w_2 = g(x), & \text{on } B_{[2]}. \end{array} \right.$$

If a *discontinuous* approximation is sought, then the continuity transmission condition can be *omitted*, and the flux transmission condition can be enforced, yielding the alternative system:

$$\left\{ \begin{array}{ll} c(x) w_1 = f(x), & \text{in } \Omega_1 \\ w_1 = g(x), & \text{on } B_{[1]} \\ -\epsilon \Delta w_2 + c(x) w_2 = f(x), & \text{in } \Omega_2 \\ \epsilon \frac{\partial w_2}{\partial n} = 0, & \text{on } B \\ w_2 = g(x), & \text{on } B_{[2]}. \end{array} \right.$$

In this case, the subproblems for  $w_1$  and  $w_2$  are formally *decoupled*. In both cases, the limiting solutions may not minimize the energy functional  $J_{\epsilon,\eta}(\cdot)$  associated with (1.32) as  $\eta \rightarrow 0^+$ .

*Remark 1.30.* Since (1.30) is equivalent to (1.19), rigorous results on the well posedness of the above approximation may be deduced from [GA15].

*Remark 1.31.* Similar heuristics may be applied to construct an approximation of the singularly perturbed *anisotropic* elliptic equation using (1.30):

$$\begin{cases} -\epsilon u_{x_1 x_1} - u_{x_2 x_2} - u_{x_3 x_3} + c(x) u = f(x), & \text{in } \Omega \\ u = g(x), & \text{on } \partial\Omega, \end{cases}$$

for which the limiting problem is a degenerate elliptic equation. In this case, both transmission conditions can be retained in the limiting problem.

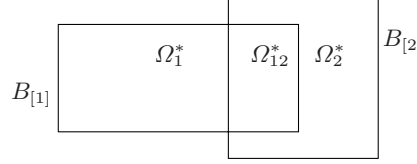
## 1.5 Least Squares-Control Framework

The least squares-control method [LI2, GL] is a general optimization method, which has various applications to partial differential equations. It results in a *constrained least squares* problem, and is based on the minimization of a *square norm* objective functional, subject to *constraints*. In domain decomposition applications, see [AT, GL13, GU3, GU2], the square norm functional typically measures the difference between the subdomain solutions on the regions of overlap or intersection between the subdomains, while the constraints require the local solutions to solve the original partial differential equation on each subdomain, with appropriate boundary conditions. Since the boundary data on each subdomain boundary is *unknown*, it is regarded as a *control* function which parameterizes the local solution. The control boundary data must be determined to minimize the square norm function, hence the name least squares-control. Importantly, an optimization principle need not be associated with the underlying partial differential equation.

In this section, we describe the hybrid formulation associated with the least squares-control method for the following elliptic equation:

$$\begin{cases} Lu \equiv -\nabla \cdot (a(x) \nabla u) + \mathbf{b}(x) \cdot \nabla u + c(x) u = f(x), & \text{in } \Omega \\ u = 0, & \text{in } \partial\Omega, \end{cases} \quad (1.33)$$

in which the domain  $\Omega$  is decomposed into *two* subdomains. The subdomains can be *overlapping* or *non-overlapping*, but we focus on the overlapping case. We illustrate the formulation of *iterative* methods, *non-matching grid* discretizations, and *heterogeneous* approximations for (1.33).



**Fig. 1.9.** An overlapping decomposition

### 1.5.1 Motivation

Let  $\Omega_1^*$  and  $\Omega_2^*$  form an overlapping decomposition of  $\Omega$ , with  $\Omega_{12}^* = \Omega_1^* \cap \Omega_2^*$ , see Fig. 1.9. Let  $B^{(i)} = \partial\Omega_i^* \cap \Omega$  and  $B_{[i]} = \partial\Omega_i^* \cap \partial\Omega$  denote the interior and exterior segments, respectively, of the subdomain boundaries, and let  $\mathbf{n}_i$  denote the unit exterior normal to  $\partial\Omega_i^*$ . On each subdomain  $\Omega_i^*$  for  $1 \leq i \leq 2$ , we let  $w_i$  denote the approximation of the solution  $u$  to (1.33) on  $\Omega_i^*$ , and let  $g_i$  denote the local Neumann data associated with  $w_i$  on  $B^{(i)}$ .

If  $w_i(\cdot) = u(\cdot)$  on  $\Omega_i^*$  and  $g_i(\cdot) = \mathbf{n}_i \cdot (a(x)\nabla u)$  on  $B^{(i)}$ , then  $w_i$  will satisfy:

$$\begin{cases} Lw_i = f, & \text{in } \Omega_i^* \\ w_i = 0, & \text{on } B_{[i]} \\ \mathbf{n}_i \cdot (a\nabla w_i) = g_i, & \text{on } B^{(i)}. \end{cases}$$

Furthermore, since  $w_1$  and  $w_2$  will *match* on  $\Omega_{12}^*$ , i.e.,  $w_1 = w_2$ , on  $\Omega_{12}^*$ , it will hold that  $\|w_1 - w_2\|_{L^2(\Omega_{12}^*)}^2 = 0$  and  $|w_1 - w_2|_{H^1(\Omega_{12}^*)}^2 = 0$ . Motivated by this, define the following square norm functional  $J(\cdot)$ :

$$J(v_1, v_2) \equiv \frac{\gamma_1}{2} \int_{\Omega_{12}^*} (v_1 - v_2)^2 dx + \frac{\gamma_2}{2} \int_{\Omega_{12}^*} |\nabla(v_1 - v_2)|^2 dx. \quad (1.34)$$

Typically  $(\gamma_1 = 1, \gamma_2 = 0)$ , but other choices are possible. Then, it will hold:

$$J(w_1, w_2) = 0,$$

for the true subdomain solutions.

The preceding observation suggests the following *constrained minimization* problem equivalent to (1.33). Determine  $(w_1, w_2)$  which minimizes  $J(\cdot)$  (with minimum value zero), within a class  $\mathcal{K}$ :

$$J(w_1, w_2) = \min_{(v_1, v_2) \in \mathcal{K}} J(v_1, v_2), \quad (1.35)$$

where  $\mathcal{K}$  is defined by the constraints:

$$\mathcal{K} \equiv \left\{ (v_1, v_2) : \begin{array}{ll} Lv_i = f, & \text{in } \Omega_i^* \\ \mathbf{n}_i \cdot (a\nabla v_i) = g_i, & \text{on } B^{(i)} \\ v_i = 0, & \text{on } B_{[i]} \end{array} \text{ for } 1 \leq i \leq 2 \right\}. \quad (1.36)$$

Instead of Neumann conditions on  $B^{(i)}$ , we may alternatively pose Robin or Dirichlet conditions. However, in the non-overlapping case, we cannot pose Dirichlet conditions on  $B^{(i)}$ , since the functional  $J(\cdot)$  typically measures the difference between the Dirichlet data. To avoid cumbersome notation, we often omit explicit inclusion of  $g_i$  as an argument in the definition of  $J(\cdot, \cdot)$  and  $\mathcal{K}$ . In a strict sense, we must replace  $v_i$  by  $(v_i, g_i)$ . Hopefully, such omission should be clear from the context.

The following equivalence will hold.

**Theorem 1.32.** *Suppose the following assumptions hold.*

1. *Let the solution  $u$  of (1.33) exist and be smooth.*
2. *Let  $(w_1, w_2)$  minimize (1.35) subject to the constraints (1.36).*

*Then at the minimum:*

$$J(w_1, w_2) = \min_{(v_1, v_2) \in \mathcal{K}} J(v_1, v_2),$$

*it will hold that:*

$$\begin{cases} w_1 = u, & \text{on } \Omega_1^* \\ w_2 = u, & \text{on } \Omega_2^*. \end{cases}$$

*Proof.* Suppose  $u$  is the solution to (1.33) and  $w_i \equiv u$  on  $\Omega_i^*$  for  $1 \leq i \leq 2$ . Then,  $(w_1, w_2)$  will satisfy all the required constraints (1.36). Furthermore:

$$w_1 - w_2 = u - u = 0, \quad \text{in } \Omega_{12}^*,$$

yields that  $J(w_1, w_2) = 0$  and minimizes  $J(\cdot, \cdot) \geq 0$ .

Conversely, suppose a solution to (1.35) exists, subject to constraints (1.36) and minimizes  $J(v_1, v_2)$ . Then this minimum value must be *zero*, since for  $u_i \equiv u$  in  $\Omega_i^*$  for  $1 \leq i \leq 2$  it will hold that  $(u_1, u_2) \in \mathcal{K}$  and  $J(u_1, u_2) = 0$ . Thus, using the definition of  $J(\cdot, \cdot)$  and that  $J(w_1, w_2) = 0$ , we obtain that  $w_1 = w_2$  on  $\Omega_{12}^*$ . Let  $\chi_1(x)$  and  $\chi_2(x)$  form a partition of unity subordinate to the cover  $\Omega_1^*$  and  $\Omega_2^*$ . Then it is easily verified that  $\chi_1(x)w_1(x) + \chi_2(x)w_2(x)$  solves (1.33), since  $Lw_i = f$  in  $\Omega_i^*$  and since  $w_1 = w_2$  in  $\Omega_{12}^*$ . Thus, by the uniqueness of solutions to (1.33) it follows that:

$$u(x) \equiv \chi_1(x)w_1(x) + \chi_2(x)w_2(x).$$

The desired result follows using  $w_1 = w_2$  on  $\Omega_{12}^*$ .  $\square$

*Remark 1.33.* The preceding result only demonstrates an equivalence between the solutions of (1.33) and (1.35). It does not guarantee the well posedness of (1.35) under perturbation of data. Such a result, however, will hold under appropriate assumptions (such as  $\mathbf{b} = \mathbf{0}$ , coercivity of (1.33)) given sufficient overlap between the subdomains.

*Remark 1.34.* Well posedness of the constrained minimization problem (1.35) will depend on the definition of  $J(\cdot)$ . For instance, when the elliptic equation (1.33) is self adjoint and coercive,  $J(v_1, v_2) = \frac{1}{2} \|v_1 - v_2\|_{H^1(\Omega_{12}^*)}^2$  can be shown to yield a well posed saddle point problem [GL, AT], where the term  $J(v_1, v_2)$  is coercive in the constraint space  $\mathcal{K}$ . More generally, an augmented Lagrangian formulation [GL7] may be employed to regularize (1.35).

As mentioned earlier, the constraint set  $\mathcal{K}$  in (1.36) can be *parameterized* in terms of the Dirichlet, Neumann or Robin data  $g_i$  specified on each boundary segment  $B^{(i)}$ , for  $1 \leq i \leq 2$ . For instance, when Neumann boundary conditions are imposed on each  $B^{(i)}$ , define an *affine* linear mapping  $\mathcal{E}_i$  as follows:

$$\mathcal{E}_i g_i \equiv v_i, \quad \text{where} \quad \begin{cases} L v_i = f, & \text{in } \Omega_i^* \\ \mathbf{n}_i \cdot (a \nabla v_i) = g_i, & \text{on } B^{(i)} \\ v_i = 0, & \text{on } B_{[i]}. \end{cases}$$

Then, the constraint set  $\mathcal{K}$  can be represented as:

$$\mathcal{K} \equiv \{(\mathcal{E}_1 g_1, \mathcal{E}_2 g_2) : \text{ for } g_i \in X_i, \quad 1 \leq i \leq 2\},$$

where  $g_1$  and  $g_2$  are regarded as *control* data. For Neumann conditions, the function space  $X_i$  for the boundary data for  $g_i$  is typically chosen for each  $1 \leq i \leq 2$  as  $X_i = (H_{00}^{1/2}(B^{(i)}))'$  or  $X_i = H^{-1/2}(B^{(i)})$ . This parameterization enables the reformulation of this *constrained* minimization problem (1.35) as an *unconstrained* minimization problem. Define a function  $H(\cdot)$ :

$$H(g_1, g_2) \equiv J(\mathcal{E}_1 g_1, \mathcal{E}_2 g_2). \quad (1.37)$$

Then, the unconstrained minimum  $(g_1^*, g_2^*)$  of  $H(\cdot, \cdot)$ :

$$H(g_1^*, g_2^*) = \min_{(g_1, g_2)} H(g_1, g_2), \quad (1.38)$$

will yield the *constrained* minimum of  $J(\cdot, \cdot)$  as  $(w_1, w_2) = (\mathcal{E}_1 g_1^*, \mathcal{E}_2 g_2^*)$ . Thus, once  $g_1^*$  and  $g_2^*$  have been determined by minimizing  $H(\cdot, \cdot)$ , the desired local solutions will satisfy  $w_i \equiv \mathcal{E}_i g_i^*$  for  $1 \leq i \leq 2$ . Such unconstrained minimization does not require Lagrange multipliers.

The unknown control data  $g_1$  and  $g_2$  can be determined by solving the system of equations which result from the application of first order stationarity conditions  $\delta H = 0$  at the minimum of  $H(\cdot)$ . We shall omit the derivation of these equations, except to note that the calculus of variations may be applied to (1.38), or such equations may be derived by heuristic analogy with the associated discrete saddle point problem, as described in Chap. 6.

The resulting first order stationarity equations will be of the form:

$$\delta H(g_1, g_2) = 0 \Leftrightarrow \begin{cases} v_1(x) = 0, & \text{for } x \in B^{(1)} \\ v_2(x) = 0, & \text{for } x \in B^{(2)} \end{cases}$$

where  $v_1(x)$  and  $v_2(x)$  are defined in terms of  $g_1(x)$  and  $g_2(x)$  as follows. Solve:

$$\begin{cases} -\nabla \cdot (a \nabla w_i) + \mathbf{b} \cdot \nabla w_i + c w_i = f(x), & \text{in } \Omega_i^* \\ w_i = 0, & \text{on } B_{[i]} \\ \mathbf{n}_i \cdot (a \nabla w_i) = g_i(x), & \text{on } B^{(i)} \end{cases} \quad \text{for } i = 1, 2$$

for  $w_1(x)$  and  $w_2(x)$  using  $g_1(x)$  and  $g_2(x)$ . Next, compute:

$$r(x) \equiv \begin{cases} w_1(x) - w_2(x), & \text{for } x \in \Omega_{12}^* \\ 0, & \text{for } x \notin \Omega_{12}^*. \end{cases}$$

Then,  $v_1(x)$  and  $v_2$  are defined as the solutions to:

$$\begin{cases} -\nabla \cdot (a \nabla v_i) - \nabla \cdot (\mathbf{b} v_i) + c v_i = r(x), & \text{in } \Omega_i^* \\ v_i = 0, & \text{on } B_{[i]} \\ \mathbf{n}_i \cdot (a \nabla v_i + \mathbf{b} v_i) = 0, & \text{on } B^{(i)} \end{cases} \quad \text{for } 1 \leq i \leq 2.$$

The control data  $g_1(x)$  and  $g_2(x)$  must be chosen to ensure that  $v_i(x) = 0$  on  $B^{(i)}$  for  $i = 1, 2$ . Later, we shall outline a gradient method to determine  $g_1$  and  $g_2$  iteratively. When (1.35) is discretized, an explicit matrix representation can be derived for  $H(\cdot)$  and its gradient, see Chap. 6. In this case, a preconditioned CG method can be employed to solve the resulting linear system.

*Remark 1.35.* If  $\Omega$  is decomposed into *non-overlapping* subdomains  $\Omega_1$  and  $\Omega_2$  with common interface  $B = \partial\Omega_1 \cap \partial\Omega_2$ , a least squares-control formulation may be constructed as follows [GU3, GU2]. Seek  $(w_1, w_2)$  which minimizes:

$$J(w_1, w_2) = \min_{(v_1, v_2) \in \mathcal{K}} J(v_1, v_2),$$

where

$$J(v_1, v_2) \equiv \frac{1}{2} \|v_1 - v_2\|_{L^2(B)}^2,$$

and  $\mathcal{K}$  consists of all  $(v_1, v_2)$  satisfying the following constraints:

$$\begin{cases} Lv_1 = f(x), & \text{in } \Omega_1 \\ v_1 = 0, & \text{on } B_{[1]} \\ \mathbf{n}_1 \cdot (a \nabla v_1) = \mu(x), & \text{on } B \\ Lv_2 = f(x), & \text{in } \Omega_2 \\ v_2 = 0, & \text{on } B_{[2]} \\ \mathbf{n}_2 \cdot (a \nabla v_2) = -\mu(x), & \text{on } B. \end{cases}$$

Here  $\mu(x)$  is a flux variable on the interface  $B$  (which can be eliminated). The above constraints will ensure that the original elliptic equation is solved on



each subdomain, and that the Neumann fluxes of the two subdomain solutions match on  $B$ . In this case, the feasible set  $\mathcal{K}$  can be parameterized in terms of the flux  $\mu(x) = \mathbf{n}_1 \cdot (a \nabla v_1)$  on  $B$ . In applications, an alternative choice of objective functional  $J(v_1, v_2) \equiv \frac{1}{2} \|v_1 - v_2\|_{H_{00}^{1/2}(B)}^2$  may also be employed, where  $H_{00}^{1/2}(B)$  denotes a fractional Sobolev norm (defined in Chap. 3).

### 1.5.2 Iterative Methods

The solution to (1.33) can be determined iteratively, by formally applying a steepest descent method to the unconstrained minimization problem (1.38), with sufficiently small step size  $\tau > 0$ . Such an algorithm can be derived formally using calculus of variations, or by analogy with the discrete version of this algorithm described in Chap. 6.

**Algorithm 1.5.1** (*Gradient Least Squares-Control Algorithm*)

Let  $g_1^{(0)}(x)$  and  $g_2^{(0)}(x)$  denote starting guesses and  $\tau > 0$  a fixed step size.

1. For  $k = 0, 1, \dots$  until convergence do:
2. For  $i = 1, 2$  in parallel solve:

$$\begin{cases} -\nabla \cdot (a \nabla v_i) + \mathbf{b} \cdot \nabla v_i + c v_i = f(x), & \text{in } \Omega_i^* \\ v_i = 0, & \text{on } B_{[i]} \\ \mathbf{n}_i \cdot (a \nabla v_i) = g_i^{(k)}(x), & \text{on } B^{(i)}. \end{cases}$$

3. Endfor
4. Compute:

$$r(x) \equiv \begin{cases} v_1(x) - v_2(x), & \text{for } x \in \Omega_{12}^* \\ 0, & \text{for } x \notin \Omega_{12}^* \end{cases}$$

5. For  $i = 1, 2$  in parallel solve the adjoint problems:

$$\begin{cases} -\nabla \cdot (a \nabla w_i) - \nabla \cdot (\mathbf{b} w_i) + c w_i = r(x), & \text{in } \Omega_i^* \\ w_i = 0, & \text{on } B_{[i]} \\ \mathbf{n}_i \cdot (a \nabla w_i + \mathbf{b} w_i) = 0, & \text{on } B^{(i)}. \end{cases}$$

6. Endfor
7. Update:

$$\begin{cases} g_1^{(k+1)}(x) = g_1^{(k)}(x) - \tau w_1(x), & \text{for } x \in B^{(1)} \\ g_2^{(k+1)}(x) = g_2^{(k)}(x) + \tau w_2(x), & \text{for } x \in B^{(2)}. \end{cases}$$

8. Endfor

Output:  $(g_1^{(k)}, g_2^{(k)})$

Alternative divide and conquer iterative algorithms can be formulated for (1.33) using its saddle point formulation. However, the resulting algorithm may require more computational resources. For instance, suppose that:

$$J(v_1, v_2) = \frac{1}{2} \|v_1 - v_2\|_{L^2(\Omega_{12}^*)}^2,$$

and that Neumann boundary conditions are imposed on  $B^{(i)}$ . Then, as described in Chap. 10, a constrained minimization problem such as (1.35) with (1.36), can be equivalently formulated as a saddle point problem, and saddle point iterative algorithms can be formulated to solve it.

Indeed, if  $\lambda_1$  and  $\lambda_2$  denote the Lagrange multipliers, then the saddle point problem associated with (1.35) would formally be of the form:

$$\begin{cases} \chi_{\Omega_{12}} (w_1 - w_2) + L_1^* \lambda_1 = 0, \\ -\chi_{\Omega_{12}} (w_1 - w_2) + L_2^* \lambda_2 = 0, \\ L_1 \tilde{w}_1 = f_1, \\ L_2 \tilde{w}_2 = f_2. \end{cases} \quad (1.39)$$

Here  $L_i \tilde{w}_i = f_i$  formally denotes the operator equation associated with  $L w_i = f$  in  $\Omega_i^*$  with Neumann conditions  $\mathbf{n}_i \cdot (a \nabla w_i) - g_i = 0$  on  $B^{(i)}$  and homogeneous Dirichlet boundary conditions  $w_i = 0$  on  $B_{[i]}$ , with  $\tilde{w}_i = (w_i, g_i)$ . The operator  $L_i^*$  formally denotes the adjoint of  $L_i$ . Here,  $\chi_{\Omega_{12}^*}(x)$  denotes the characteristic (indicator) function of  $\Omega_{12}^*$ . We omit elaborating on such a saddle point problem here, except to note that, it may be obtained by *heuristic* analogy with the discrete saddle point problems described in Chap. 10. The  $\lambda_i(x)$  corresponds to Lagrange multiplier functions, see [GL, AT]. In this saddle point problem, the Lagrange multiplier variables will not be unique, and an augmented Lagrangian formulation would be preferable.

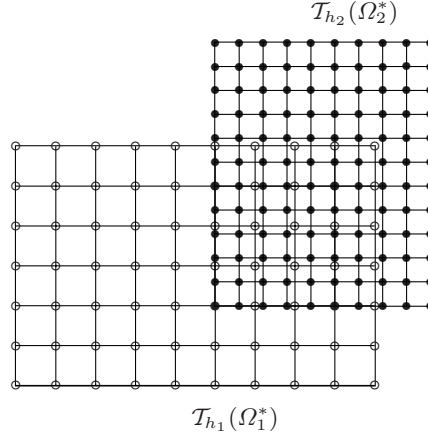
### 1.5.3 Global Discretization

Hybrid formulation (1.35) or (1.38) can, in principle, be employed to discretize (1.33) on a nonmatching grid such as in Fig. 1.10. Such discretizations have not been considered in the literature, however, a *heuristic* discussion of such a discretization is outlined here for its intrinsic interest, employing formulation (1.38). We employ finite element discretizations on the subdomains.

A nonmatching grid discretization of (1.38) will require discretizing  $J(\cdot)$ :

$$J(v_1, v_2) = \frac{1}{2} \|v_1 - v_2\|_{H^1(\Omega_{12}^*)}^2,$$

and this will involve two overlapping non-matching grids. In the following, we heuristically outline a mortar element discretization of  $J(v_1, v_2)$  on  $\Omega_{12}^*$ , and employ this to construct a global non-matching grid discretization of (1.33), with Dirichlet boundary controls on each subdomain boundary  $B^{(i)}$ . Each subdomain problem will involve only a conforming grid.



**Fig. 1.10.** Overlapping nonmatching grids

*Remark 1.36.* If  $J(v_1, v_2)$  is replaced by  $J_B(v_1, v_2) \equiv \frac{1}{2} \|v_1 - v_2\|_B^2$  where  $B = \partial\Omega_1 \cap \partial\Omega_2$  and  $\Omega_i^*$  is an extension of a non-overlapping decomposition  $\Omega_i$ , such a discretization would be considerably simpler.

**Local Triangulation.** For  $1 \leq i \leq 2$  triangulate each subdomain  $\Omega_i^*$  by a grid  $\mathcal{T}_{h_i}(\Omega_i^*)$  according to the local geometry and regularity of the solution, see Fig. 1.10. We shall assume that at least one of the local grids triangulates the region of overlap  $\Omega_{12}^*$ . For definiteness assume that triangulation  $\mathcal{T}_{h_1}(\Omega_1^*)$  triangulates  $\Omega_{12}^*$ . Let  $n_i$  and  $m_i$  denote the number of nodes of grid  $\mathcal{T}_{h_i}(\Omega_i^*)$  in the interior of  $\Omega_i^*$  and on  $B^{(i)}$ , respectively. Additionally, let  $l_i$  denote the number of nodes of triangulation  $\mathcal{T}_{h_i}(\Omega_i^*)$  in  $\overline{\Omega_{12}^*}$ .

**Local Discretizations.** For  $1 \leq i \leq 2$ , employ Dirichlet boundary conditions on  $B^{(i)}$  in (1.36) and discretize the resulting local problems using a finite element space  $X_{h_i} \subset X_i$  based on triangulation  $\mathcal{T}_{h_i}(\Omega_i^*)$ :

$$X_i \equiv \{v_i \in H^1(\Omega_i^*) : v_i = 0 \text{ on } B_{[i]}\}.$$

Block partition the unknowns  $\mathbf{w}_{h_i} = (\mathbf{w}_I^{(i)}, \mathbf{w}_B^{(i)})^T$  according to the interior unknowns and the unknowns on the boundary  $B^{(i)}$  respectively. Denote the block partitioned linear system for the discretized Dirichlet problem as:

$$\begin{cases} A_{II}^{(i)} \mathbf{w}_I^{(i)} + A_{IB}^{(i)} \mathbf{w}_B^{(i)} = \mathbf{f}_I^{(i)}, \\ \mathbf{w}_B^{(i)} = \mathbf{g}_B^{(i)}. \end{cases}$$

**Weak Matching on  $\Omega_{12}^*$ .** Choose a finite element space:

$$Y_h(\Omega_{12}^*) \subset L^2(\Omega_{12}^*)$$

based on the triangulation of  $\Omega_{12}^*$  inherited from  $\mathcal{T}_{h_1}(\Omega_1^*)$ , of dimension  $l_1$ . Define the weak matching condition on  $\Omega_{12}^*$  as:

$$\int_{\Omega_{12}^*} (w_{h_1} - w_{h_2}) \mu_{h_1} dx = 0, \quad \text{for } \mu_{h_1} \in Y_{h_1}(\Omega_{12}^*),$$

enforced using the subspace  $Y_{h_1}(\Omega_{12}^*)$ . Denote its matrix form as:

$$M_{11} \mathbf{w}_{h_1} - M_{12} \mathbf{w}_{h_2} = \mathbf{0},$$

where  $M_{11}$  is invertible of size  $l_1$ . Define an oblique projection  $P_1 \equiv M_{11}^{-1} M_{12}$ .

**Discrete Functional  $J(\cdot, \cdot)$ .** Let  $A^{(12)}$  be the stiffness matrix associated with  $J(\cdot)$  on the triangulation  $\mathcal{T}_{h_1}(\Omega_{12}^*)$ . The quadratic functional  $J(\cdot)$  can be discretized using  $A^{(12)}$  and the projection  $P_1$  as follows:

$$\begin{cases} J(v_{h_1}, v_{h_2}) \equiv \frac{1}{2} \|v_{h_1} - v_{h_2}\|_{H^1(\Omega_{12}^*)}^2 \\ \quad \approx \frac{1}{2} (\mathbf{v}_{h_1} - P_1 \mathbf{v}_{h_2})^T R_{12}^T A^{(12)} R_{12} (\mathbf{v}_{h_1} - P_1 \mathbf{v}_{h_2}) \\ \quad \equiv J_h(\mathbf{v}_{h_1}, \mathbf{v}_{h_2}). \end{cases}$$

Here  $R_{12}$  is a restriction map onto the nodes of  $\overline{\Omega}_{12}^*$  from  $\Omega_1^*$ , see Chap. 6. The reduced functional  $H_h(\cdot)$  can be discretized using:

$$H_h(\mathbf{g}_{h_1}, \mathbf{g}_{h_2}) \equiv J_h(\mathbf{v}_{h_1}, \mathbf{v}_{h_2}),$$

where

$$\mathbf{v}_{h_i} = \begin{bmatrix} A_{II}^{(i)-1} (\mathbf{f}_I^{(i)} - A_{IB}^{(i)} \mathbf{g}_B^{(i)}) \\ \mathbf{g}_B^{(i)} \end{bmatrix} \quad \text{for } 1 \leq i \leq 2.$$

**Stationarity Condition.** The first order derivative conditions for the minimum of  $H_h(\cdot)$  will yield the following equations for  $(\mathbf{g}_B^{(1)}, \mathbf{g}_B^{(2)})$ :

$$\begin{bmatrix} E_1^T R_{12}^T A^{(12)} R_{12} E_1 & -E_1^T R_{12}^T A^{(12)} R_{12} P_1 E_2 \\ -E_2^T P_1^T R_{12}^T A^{(12)} R_{12} E_1 & E_2^T P_1^T R_{12}^T A^{(12)} R_{12} P_1 E_2 \end{bmatrix} \begin{bmatrix} \mathbf{g}_B^{(1)} \\ \mathbf{g}_B^{(2)} \end{bmatrix} = \begin{bmatrix} \gamma_B^{(1)} \\ \gamma_B^{(2)} \end{bmatrix} \quad (1.40)$$

where

$$\begin{cases} \gamma_B^{(1)} \equiv E_1^T R_{12}^T A^{(12)} R_{12} (-\boldsymbol{\mu}_I^{(1)} + P_1 \boldsymbol{\mu}_I^{(2)}), \\ \gamma_B^{(2)} \equiv E_2^T P_1^T R_{12}^T A^{(12)} R_{12} (-\boldsymbol{\mu}_I^{(1)} + P_1 \boldsymbol{\mu}_I^{(2)}), \\ E_i \equiv \begin{bmatrix} -A_{II}^{(i)-1} A_{IB}^{(i)} \\ I \end{bmatrix}, \\ \boldsymbol{\mu}_I^{(i)} \equiv \begin{bmatrix} A_{II}^{(i)-1} \mathbf{f}_I^{(i)} \\ \mathbf{0} \end{bmatrix}, \\ \mathbf{w}_I^{(i)} = A_{II}^{(i)-1} (\mathbf{f}_I^{(i)} - A_{IB}^{(i)} \mathbf{g}_B^{(i)}), \end{cases} \quad \text{for } i = 1, 2.$$

Thus, a non-matching grid discretization of (1.33) based on the subdomains involves solving system (1.40) for the control boundary data  $\mathbf{g}_B^{(1)}$  and  $\mathbf{g}_B^{(2)}$ . Subsequently, the subdomain solution  $\mathbf{w}_I^{(i)}$  can be determined as:

$$\mathbf{w}_I^{(i)} = A_{II}^{(i)-1} \left( \mathbf{f}_I^{(i)} - A_{IB}^{(i)} \mathbf{g}_I^{(i)} \right), \quad \text{for } 1 \leq i \leq 2.$$

*Remark 1.37.* General results on the stability and convergence properties of such discretizations are not known. However, when both local grids match on  $\Omega_{12}^*$ , projection  $P_1 = I$  and the global discretization will be equivalent to a traditional discretization of (1.33) on the global triangulation.

#### 1.5.4 Heterogeneous Approximations

The least square-control formulation (1.35) provides a flexible framework for constructing heterogeneous approximations of general systems of partial differential equations of heterogeneous character [AT, GL13]. We illustrate here how an elliptic-hyperbolic approximation can be constructed for the following singularly perturbed elliptic equation:

$$\begin{cases} L^\epsilon u \equiv -\epsilon \Delta u + \mathbf{b}(x) \cdot \nabla u + c(x) u = f, & \text{on } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.41)$$

where  $0 < \epsilon \ll 1$  is a perturbation parameter. Suppose  $\Omega_1^*$  and  $\Omega_2^*$  form an overlapping covering of  $\Omega$  such that:

$$|\epsilon \Delta u| \ll |\mathbf{b}(x) \cdot \nabla u + c(x) u|, \quad \text{in } \Omega_1^*.$$

We may then heuristically approximate  $L^\epsilon u = f$  in  $\Omega_1^*$  by  $L_0 u = f$  where  $L_0 u \equiv \mathbf{b}(x) \cdot \nabla u + c(x) u$ . To construct an elliptic-hyperbolic approximation of (1.41), replace the elliptic problem  $L^\epsilon v_1 = f$  on  $\Omega_1^*$  by the hyperbolic problem  $L_0 v_1 = f$  within the least squares-control formulation (1.35) of (1.41). The resulting heterogeneous problem will seek  $(w_1, w_2)$  which minimizes:

$$\hat{J}(w_1, w_2) = \min_{(v_1, v_2) \in \hat{\mathcal{K}}} \hat{J}(v_1, v_2),$$

where

$$\hat{J}(v_1, v_2) \equiv \frac{1}{2} \|v_1 - v_2\|_{L^2(\Omega_{12}^*)}^2,$$

and  $\hat{\mathcal{K}}$  consists of  $(v_1, v_2)$  which satisfy the constraints:

$$\begin{cases} L_0 v_1 = f, & \text{on } \Omega_1^* \\ v_1 = g_1, & \text{on } B_{in}^{(1)} \\ v_1 = 0, & \text{on } B_{[1],in} \end{cases} \quad \text{and} \quad \begin{cases} L^\epsilon v_2 = f, & \text{on } \Omega_2^* \\ v_2 = g_2, & \text{on } B^{(2)} \\ v_2 = 0, & \text{on } B_{[2]}. \end{cases} \quad (1.42)$$

Here the *inflow* boundary segments of  $B^{(1)}$  and  $B_{[1]}$  are defined by:

$$\begin{cases} B_{in}^{(1)} & \equiv \{x \in B^{(1)} : \mathbf{n}_1(x) \cdot \mathbf{b}(x) < 0\} \\ B_{[1],in} & \equiv \{x \in B_{[1]} : \mathbf{n}_1(x) \cdot \mathbf{b}(x) < 0\}, \end{cases}$$

where  $\mathbf{n}_1(x)$  is the unit outward normal to  $B_1$  at  $x$ .

*Remark 1.38.* The admissible set  $\hat{\mathcal{K}}$  may be parameterized in terms of the local boundary data. An equivalent unconstrained minimization problem may then be obtained analogous to (1.37) and (1.38). See also Chap. 12.

*Remark 1.39.* The solution  $(w_1, w_2)$  to the above heterogeneous model may not match on  $\Omega_{12}^*$  and the minimum value of  $\hat{J}(\cdot)$  within the class  $\hat{\mathcal{K}}$  may no longer be zero. A continuous global solution, however, may be obtained by employing a partition of unity  $\chi_1(x)$  and  $\chi_2(x)$  subordinate to the cover  $\Omega_1^*$  and  $\Omega_2^*$  and by defining:

$$w(x) \equiv \chi_1(x) w_1(x) + \chi_2(x) w_2(x).$$

*Remark 1.40.* Rigorous results are not known on the well posedness of the above heterogeneous model. The above procedure has been generalized and employed to construct heterogeneous approximations to the Boltzmann, Navier-Stokes and Euler equations [AT, GL13].

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