2

Preliminaries

...Having computed the m's several years earlier ..., I recognized them[1] in the Poincaré polynomials while listening to Chevalley's address at the International Congress in 1950. I am grateful to A. J. Coleman for drawing my attention to the relevant work of Racah, which helps to explain the "coincidence"; also, to J. S. Frame for many helpful suggestions...

H. S. M. Coxeter, [Cox51, p.765], 1951

2.1 The Cartan matrix and the Tits form

2.1.1 The generalized and symmetrizable Cartan matrix

Let K be an $n \times n$ matrix with the following properties: [Mo68], [Kac80]

$$\begin{array}{ll} (C1) & k_{ii} = 2 \text{ for } i = 1,..,n, \\ (C2) & -k_{ij} \in \mathbb{Z}_+ = \{0,1,2,...\} \text{ for } i \neq j, \\ (C3) & k_{ij} = 0 \text{ implies } k_{ji} = 0 \text{ for } i,j = 1,...,n. \end{array} \tag{2.1}$$

Such a matrix is called a *generalized Cartan matrix*. A generalized Cartan matrix M is said to be *symmetrizable* if there exists an invertible diagonal matrix U with positive integer coefficients and a symmetric matrix $\mathbf{B}$ such that $M = U\mathbf{B}$.

[1] The numbers m_i mentioned by H. S. M. Coxeter in the epigraph are the exponents of the eigenvalues of the Coxeter transformation, they are called the *exponents* of the Weyl group, see §2.3.2.

Remark 2.1. According to the classical Bourbaki definition a *Cartan matrix* is the matrix satisfying the condition (2.1), such that the non-diagonal entries constitute a certain subset of the set $\{-1, -2, -3, -4\}$, see ([Bo, Ch.6.1.5, Def. 3]). The following inclusions hold between three classes of Cartan matrices:

$$\begin{aligned} &\{M \mid M \text{ is a Cartan matrix by Bourbaki}\} \subset \\ &\qquad \{M \mid M \text{ is a symmetrizable matrix satisfying (2.1)}\} \subset \\ &\qquad\qquad \{M \mid M \text{ is a generalized Cartan matrix}\}. \end{aligned} \tag{2.2}$$

According to V. Kac [Kac80], [Kac82], a matrix (2.1) is referred to a *Cartan matrix*, but in the first edition of Kac's book [Kac93] the matrix satisfying (2.1) is already called a *generalized Cartan matrix*. We are interested in the case of symmetrizable generalized Cartan matrices, so when no confusion is possible, "the Cartan matrix" means "the symmetrizable generalized Cartan matrix". □

Let Γ_0 (resp. Γ_1) be the set of vertices (resp. edges) of a given graph Γ. A *valued graph* or *diagram* (Γ, d) ([DR76], [AuRS95, p. 241]) is a finite set Γ_1 (of edges) rigged with numbers d_{ij} for all pairs $i, j \in \partial\Gamma_1 \subset \Gamma_0$ of the endpoints of the edges in such a way that

$$\begin{aligned} &(D1) \quad d_{ii} = 2 \text{ for } i = 1, .., n, \\ &(D2) \quad d_{ij} \in \mathbb{Z}_+ = \{0, 1, 2, ...\} \text{ for } i \neq j, \\ &(D3) \quad d_{ij} = 0 \ \text{ implies } \ d_{ji} = 0 \text{ for } i, j = 1, ..., n. \end{aligned}$$

The rigging of the edges of Γ_1 is depicted by symbols

$$i \xrightarrow{\hspace{1em}(d_{ij}, d_{ji})\hspace{1em}} j$$

If $d_{ij} = d_{ji} = 1$, we simply write

$$i \underline{\hspace{4em}} j$$

There is, clearly, a one-to-one correspondence between valued graphs and generalized Cartan matrices, see [AuRS95]. The following relation holds:

$$d_{ij} = |k_{ij}| \text{ for } i \neq j,$$

where k_{ij} are elements of the Cartan matrix (2.1), see [AuRS95, p. 241]. The integers d_{ij} of the valued graph are called *weights*, and the corresponding edges are called *weighted edges*. If

$$d_{ij} = d_{ji} = 1, \tag{2.3}$$

we say that the corresponding edge is not weighted. A diagram is called *simply-laced* (resp. *multiply-laced*) if it does not contain (resp. contains) weighted edges.

Remark 2.2. Any generalized Cartan matrix K whose diagram contains no cycles is symmetrizable ([Mo68, §3]). Any generalized Cartan matrix whose diagram is a simply-laced diagram (even with cycles) is symmetrizable because it is symmetric. In particular, $\widetilde{A}_n$ has a symmetrizable Cartan matrix. In this work we consider only diagrams without cycles, so the diagrams we consider correspond to symmetrizable Cartan matrices.

For notation of diagrams with the non-negative Tits form, i.e., extended Dynkin diagrams, see §2.1.6. The notation of $\widetilde{A}_n$ in the context of affine Lie algebras is $A_n^{(1)}$, see Table 2.1, Table 2.2. □

Let $\mathcal{B}$ be the quadratic form associated with matrix $\mathbf{B}$, and let $(\cdot,\cdot)$ be the corresponding symmetric bilinear form. The quadratic form $\mathcal{B}$ is called the *quadratic Tits form.* We have

$$\mathcal{B}(\alpha) = (\alpha, \alpha).$$

In the simply-laced case,

$$K = 2\mathbf{B}$$

with the symmetric Cartan matrix K. So, in the simply-laced case, the Tits form is the Cartan-Tits form. In the multiply-laced case, the symmetrizable matrix K factorizes

$$K = U\mathbf{B}, \tag{2.4}$$

where U is a diagonal matrix with positive integers on the diagonal, and $\mathbf{B}$ is a symmetric matrix. It is easy to see that the matrix U is unique up to a factor.

2.1.2 The Tits form and diagrams $T_{p,q,r}$

Consider the diagram $T_{p,q,r}$ depicted in Fig. 2.1. The *diagram* $T_{p,q,r}$ is defined as the tree graph with three arms of lengths p, q, r having one common vertex. On Fig. 2.1 this common vertex is

$$x_p = y_q = z_r.$$

Proposition 2.3. *Let $\mathcal{B}$ be the quadratic Tits form connected to the diagram $T_{p,q,r}$. Then*

$$2\mathcal{B} = U + (\mu - 1)u_0^2, \tag{2.5}$$

where U is a non-negative quadratic form, and

$$\mu = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}. \tag{2.6}$$

Proof. By eq.(3.3) the quadratic Tits form $\mathcal{B}$ can be expressed in the following form

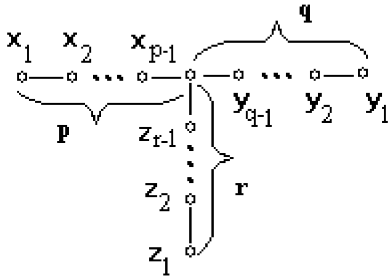

Fig. 2.1. The diagram $T_{p,q,r}$

$$\begin{aligned}\mathcal{B}(z) =& x_1^2 + x_2^2 + \cdots + x_{p-1}^2 - x_1x_2 - \cdots - x_{p-2}x_{p-1} - x_{p-1}u_0 + \\ & y_1^2 + y_2^2 + \cdots + y_{q-1}^2 - y_1y_2 - \cdots - y_{q-2}y_{q-1} - y_{q-1}u_0 + \\ & z_1^2 + z_2^2 + \cdots + z_{r-1}^2 - z_1z_2 - \cdots - z_{r-2}z_{r-1} - z_{r-1}u_0 + u_0^2,\end{aligned} \tag{2.7}$$

where u_0 is the coordinate of the vector z corresponding to the branch point of the diagram. Let

$$\begin{aligned}& V(x_1, ..., x_{p-1}, u_0) = \\ & \qquad x_1^2 + x_2^2 + \cdots + x_{p-1}^2 - x_1x_2 - \cdots - x_{p-2}x_{p-1} - x_{p-1}u_0, \\ & V(y_1, ..., y_{q-1}, u_0) = \\ & \qquad y_1^2 + y_2^2 + \cdots + y_{q-1}^2 - y_1y_2 - \cdots - y_{q-2}y_{q-1} - y_{q-1}u_0, \\ & V(z_1, ..., z_{r-1}, u_0) = \\ & \qquad z_1^2 + z_2^2 + \cdots + z_{r-1}^2 - z_1z_2 - \cdots - z_{r-2}z_{r-1} - z_{r-1}u_0.\end{aligned}$$

Then

$$\mathcal{B}(z) = V(x_1, ..., x_{p-1}) + V(y_1, ...y_{q-1}) + V(z_1, ...z_{r-1}) + u_0^2.$$

It is easy to check that

$$\begin{aligned}2V(x_1, ..., x_{p-1}, u_0) &= \sum_{i=1}^{p-1} \frac{i+1}{i}(x_i - \frac{i}{i+1}x_{i+1})^2 - \frac{p-1}{p}u_0^2, \\ 2V(y_1, ..., y_{q-1}, u_0) &= \sum_{i=1}^{q-1} \frac{i+1}{i}(y_i - \frac{i}{i+1}y_{i+1})^2 - \frac{q-1}{q}u_0^2, \\ 2V(z_1, ..., z_{r-1}, u_0) &= \sum_{i=1}^{r-1} \frac{i+1}{i}(z_i - \frac{i}{i+1}z_{i+1})^2 - \frac{r-1}{r}u_0^2.\end{aligned} \tag{2.8}$$

Denote by $U(x)$, $U(y)$, $U(z)$ the corresponding sums in (2.8):

$$2V(x_1, ..., x_{p-1}, u_0) = U(x) - \frac{p-1}{p}u_0^2,$$
$$2V(y_1, ..., y_{q-1}, u_0) = U(y) - \frac{q-1}{q}u_0^2,$$
$$2V(z_1, ..., z_{r-1}, u_0) = U(z) - \frac{r-1}{r}u_0^2.$$

Thus,

$$2\mathcal{B}(z) = U(x) + U(y) + U(z) - \frac{p-1}{p}u_0^2 - \frac{q-1}{q}u_0^2 - \frac{r-1}{r}u_0^2 + 2u_0^2,$$

or

$$2\mathcal{B}(z) = U(x) + U(y) + U(z) + \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right)u_0^2.$$

Set

$$\mu = \frac{1}{p} + \frac{1}{q} + \frac{1}{r};$$

then we have

$$2\mathcal{B}(z) = U(x) + U(y) + U(z) + (\mu - 1)u_0^2. \qquad \square \tag{2.9}$$

2.1.3 The simply-laced Dynkin diagrams

The diagram with the positive definite Tits form is said to be the *Dynkin diagram.* In §2.1.3, §2.1.4, §2.1.5 we find full list of Dynkin diagrams. In the other words, we prove the following theorem:

Theorem 2.4. *The Tits form $\mathcal{B}$ of the diagram Γ is positive definite if and only if Γ is one of diagrams in Fig. 2.3.*

Remark 2.5. 1) Let K be the Cartan matrix associated with the tree Γ. If d_{ij} and d_{ji} are distinct non-zero elements of K, let us multiply the ith row by $\dfrac{d_{ji}}{d_{ij}}$, so the new element d'_{ij} is equal to d_{ji}.

Let k be a certain index, $k \neq i, j$. If $d_{ik} \neq 0$, i.e., there exists an edge $\{i, k\}$, then $d_{jk} = 0$, otherwise $\{i, k, j\}$ is a loop. In the next step, if d_{ik} and d_{ki} are distinct, then we multiply the kth row by $\dfrac{d_{ik}}{d_{ki}}$. Then d'_{ki} is equal to d_{ik}, whereas the ith and jth rows did not change. We continue the process in this way until the Cartan matrix K becomes symmetric. Since Γ is a tree, this process terminates.

2) Let the Tits form be positive definite. Since this property is true for all values of vectors $z \in \mathcal{E}_\Gamma$, we can select some coordinates of z to be zero. In this case, it suffices to consider only the submatrix of the Cartan matrix corresponding to non-zero coordinates.

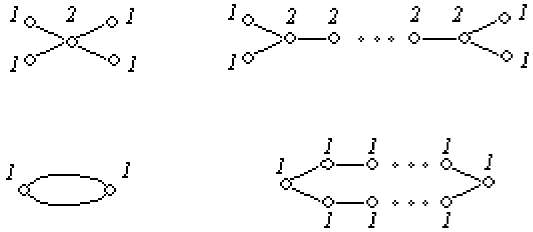

Fig. 2.2. The Tits form is not positive definite. The simply-laced case

Proposition 2.6. *If diagram* Γ *contains one of subdiagrams of Fig. 2.2 then the corresponding Tits form is not positive definite.*

Proof. Let Γ_S be one of subdiagrams of Fig. 2.2, and let $\Gamma\backslash\Gamma_S$ be the part complementary to Γ_S. The numbers depicted in Fig. 2.2 are the coordinates of the vector corresponding to Γ_S. Complete the coordinates of $z \in \mathcal{E}_\Gamma$ corresponding to the vertices of $\Gamma\backslash\Gamma_S$ by zeros. Then $\mathcal{B}(z) \leq 0$. □

Proposition 2.7. *The simply-laced Dynkin diagrams are diagrams* A_n, D_n, *and* E_n, *where* $n = 6, 7, 8$, *see Fig. 2.3.*

Proof. From Proposition 2.6 we see that the simply-laced Dynkin diagrams are only the diagrams $T_{p,q,r}$. The chains constitute a particular case of $T_{p,q,r}$ with $p = 1$.

From Proposition 2.3, (2.6), we see that $\mathcal{B}$ is positive definite for

$$\mu = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1,$$

i.e., the triples p, q, r can only be as follows:

1) $(1, q, r)$ for any $q, r \in \mathbb{N}$, i.e., the diagrams A_n,
2) $(2, 2, r)$, i.e., the diagrams D_n,
3) $(2, 3, r)$, where $r = 3, 4, 5$, i.e., the diagrams E_6, E_7, E_8.

In all cases $p + q + r = n + 2$, where n is the number or vertices of the diagram. □

2.1.4 The multiply-laced Dynkin diagrams. Possible weighted edges

In order to prove Proposition 2.8, Proposition 2.10 and Proposition 2.11 we use the common arguments of Remark 2.5.

Proposition 2.8. *Let* Γ *be a tree.*

1) Consider a weighted edge (d_{ij}, d_{ji})*, i.e., one of* d_{ij}, d_{ji} *differs from* 1. *If* $\mathcal{B}$ *is positive definite, then*

$$d_{ij}d_{ji} < 4, \tag{2.10}$$

i.e., only the following pairs (d_{ij}, d_{ji}) *are possible:*

$$(1,2), (2,1), (1,3), (3,1). \tag{2.11}$$

2) If the weighted edge is $(2,2)$*, then the form* $\mathcal{B}$ *is not positive definite.*

3) If the weighted edge is $(1,3)$*, then the diagram consists of only one edge, and we have the diagram* G_2*, see Fig. 2.3.*

Proof. 1) According to Remark 2.5 factorize the Cartan matrix as follows:

$$K = \begin{pmatrix} 2 & -d_{ij} \\ -d_{ji} & 2 \end{pmatrix} = \begin{pmatrix} d_{ij} & 0 \\ 0 & d_{ji} \end{pmatrix} \begin{pmatrix} \dfrac{2}{d_{ij}} & -1 \\ -1 & \dfrac{2}{d_{ji}} \end{pmatrix} \tag{2.12}$$

and, for all $k \neq i, j$, set $x_k = 0$. Then

$$\mathcal{B}(z) = \frac{2}{d_{ij}}x_i^2 - 2x_ix_j + \frac{2}{d_{ji}}x_j^2,$$

or, up to a non-zero factor

$$\mathcal{B}(z) = d_{ji}x_i^2 - d_{ji}d_{ij}x_ix_j + d_{ij}x_j^2, \tag{2.13}$$

The discriminant $(d_{ji}d_{ij})^2 - 4d_{ji}d_{ij}$ should be negative, i.e., $d_{ij}d_{ji} < 4$.

2) Let $d_{ij} = d_{ji} = 2$. For all $k \neq i, j$, in (2.13) set $x_k = 0$. Then

$$\mathcal{B}(z) = 2x_i^2 + 2x_j^2 - 4x_ix_j = (x_i - x_j)^2 \tag{2.14}$$

For $x_i = x_j$, we have $\mathcal{B}(z) = 0$.

3) Let the diagram Γ have two edges $\{l, k\}$ and $\{k, j\}$, and

$$d_{lk}d_{kl} = 3. \tag{2.15}$$

a) Let $\{l, k\}$ be the weighted edge $(3,1)$ and $\{k, j\}$ be the weight edge (d_{kj}, d_{jk}). Let us factorize the component of the Cartan matrix corresponding to the two edges $\{l, k\}$ and $\{k, j\}$ as follows:

$$K = \begin{pmatrix} 2 & -d_{kj} & -3 \\ -d_{jk} & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \dfrac{3d_{jk}}{d_{kj}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{2}{3} & -\dfrac{d_{kj}}{3} & -1 \\ -\dfrac{d_{kj}}{3} & \dfrac{2d_{kj}}{3d_{jk}} & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{matrix} k \\ j \\ l \end{matrix}$$

and, for all $i \neq l, k, j$, set $x_i = 0$. Then

$$\mathcal{B}(z) = d_{jk}x_k^2 + d_{kj}x_j^2 + 3d_{jk}x_l^2 - d_{kj}d_{jk}x_jx_k - 3d_{jk}x_kx_l.$$

Set $x_l = 1$, $x_k = 2$, $x_j = 1$. Then

$$\begin{aligned}\mathcal{B}(z) =& 4d_{jk} + d_{kj} + 3d_{jk} - 2d_{kj}d_{jk} - 6d_{jk} = \\ & d_{kj} + d_{jk} - 2d_{kj}d_{jk} < 0.\end{aligned}$$

b) Let $\{l, k\}$ be the weighted edge $(1, 3)$ and let $\{k, j\}$ be the weighted edge (d_{kj}, d_{jk}). As above, we factorize the component of the Cartan matrix corresponding to the two edges $\{l, k\}$ and $\{k, j\}$ as follows:

$$K = \begin{pmatrix} 2 & -d_{kj} & -1 \\ -d_{jk} & 2 & 0 \\ -3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{d_{jk}}{d_{kj}} & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{array}{c} \begin{pmatrix} 2 & -d_{kj} & -1 \\ -d_{kj} & \dfrac{2d_{kj}}{d_{jk}} & 0 \\ -1 & 0 & \dfrac{2}{3} \end{pmatrix} \end{array} \begin{array}{c} k \\ j \\ l \end{array}$$

and, for all $i \neq l, k, j$, set $x_i = 0$. Then

$$\mathcal{B}(z) = 3d_{jk}x_k^2 + 3d_{kj}x_j^2 + d_{jk}x_l^2 - 3d_{kj}d_{jk}x_jx_k - 3d_{jk}x_kx_l.$$

Set $x_l = 3$, $x_k = 2$, $x_j = 1$. Then

$$\begin{aligned}\mathcal{B}(z) =& 12d_{jk} + 3d_{kj} + 9d_{jk} - 6d_{kj}d_{jk} - 18d_{jk} = \\ & 3d_{kj} + 3d_{jk} - 6d_{kj}d_{jk} < 0.\end{aligned}$$

c) The diagram consisting of only on weighted edge $(1, 3)$ has the positive definite Tits form

$$\mathcal{B}(z) = x_l^2 - 3x_lx_k + 3x_k^2.$$

This is the diagram G_2. □

Corollary 2.9. *If $\mathcal{B}$ is the positive definite Tits form associated with the diagram Γ having more than one edge, then the only possible weighted edges are $(1, 2)$ and $(2, 1)$.*

□

In what follows, the only weighted edges we consider are $(1, 2)$ and $(2, 1)$.

2.1.5 The multiply-laced Dynkin diagrams. A branch point

Proposition 2.10. *If the diagram Γ has two adjacent weighted edges, then the corresponding Tits form is not positive definite.*

Proof. Let $\{l, k\}$ be the weighted edge $(1, 2)$ and $\{k, j\}$ be the weighted edge (d_{jk}, d_{kj}). As in Proposition 2.8, b) above, we have

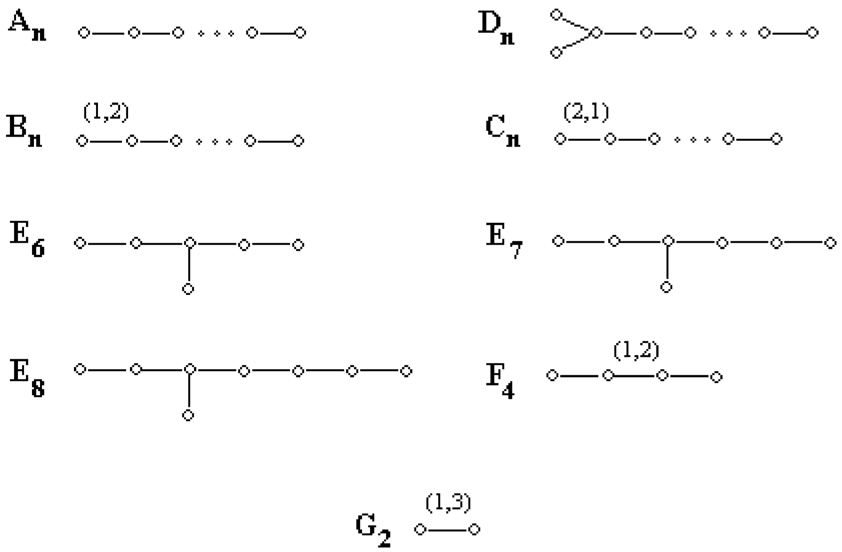

Fig. 2.3. The Dynkin diagrams

$$\mathcal{B}(z) = d_{jk}x_k^2 + d_{kj}x_j^2 + 2d_{jk}x_l^2 - d_{kj}d_{jk}x_jx_k - 2d_{jk}x_kx_l.$$

For $(d_{jk}, d_{kj}) = (2{,}1)$, we have

$$\mathcal{B}(z) = 2x_k^2 + x_j^2 + 4x_l^2 - 2x_jx_k - 4x_kx_l = (x_k - x_j)^2 + (x_k - 2x_l)^2,$$

and $\mathcal{B}(z)$ is not positive definite. For $(d_{jk}, d_{kj}) = (1, 2)$, we have

$$\mathcal{B}(z) = x_k^2 + 2x_j^2 + 2x_l^2 - 2x_jx_k - 2x_kx_l = \frac{1}{2}(x_k - 2x_j)^2 + \frac{1}{2}(x_k - 2x_l)^2,$$

and $\mathcal{B}(z)$ is also not positive definite.

Now, let $\{l, k\}$ be the weighted edge $(2, 1)$ and $\{k, j\}$ be the weighted edge (d_{jk}, d_{kj}). Here,

$$\mathcal{B}(z) = 2d_{jk}x_k^2 + 2d_{kj}x_j^2 + d_{jk}x_l^2 - 2d_{kj}d_{jk}x_jx_k - 2d_{jk}x_kx_l.$$

For $(d_{jk}, d_{kj}) = (2, 1)$, we have

$$\mathcal{B}(z) = 4x_k^2 + 2x_j^2 + 2x_l^2 - 4x_jx_k - 4x_kx_l = 2(x_k - x_j)^2 + 2(x_k - 2x_l)^2,$$

and $\mathcal{B}(z)$ is not positive definite. For $(d_{jk}, d_{kj}) = (1, 2)$, we have

$$\mathcal{B}(z) = 2x_k^2 + 4x_j^2 + x_l^2 - 4x_jx_k - 2x_kx_l = (x_k - 2x_j)^2 + (x_k - x_l)^2,$$

and $\mathcal{B}(z)$ is also not positive definite. $\square$

Proposition 2.11. *Let the diagram Γ have only one branch point.*

1) If one of edges ending in the branch point is a weighted edge, then the corresponding Tits form is not positive definite, see Fig. 2.4, a), b).

2) If the diagram Γ has a weighted edge, then the corresponding Tits form is not positive definite, see Fig. 2.4, c), d).

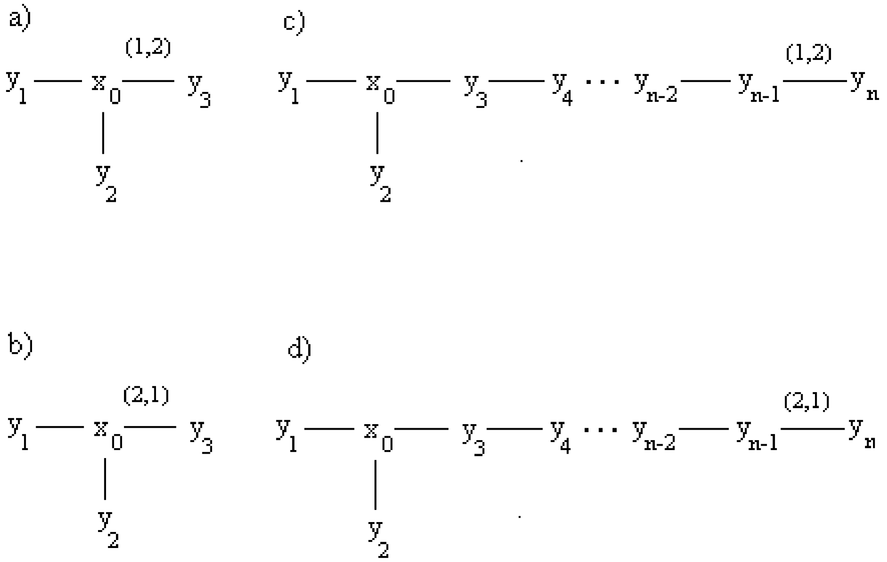

Fig. 2.4. The Tits form is not positive definite. Cases with a branch point

Proof. 1) Case a) in Fig. 2.4. The Tits form is

$$\mathcal{B}(z) = x_0^2 + y_1^2 + y_2^2 + 2y_3^2 - x_0y_1 - x_0y_2 - 2x_0y_3.$$

Set $x_0 = 2, y_1 = 1, y_2 = 1, y_3 = 1$. We have

$$\mathcal{B}(z) = 4 + 1 + 1 + 2 - 2 - 2 - 4 = 0.$$

Case b) in Fig. 2.4. The Tits form is

$$\mathcal{B}(z) = 2x_0^2 + 2y_1^2 + 2y_2^2 + y_3^2 - 2x_0y_1 - 2x_0y_2 - 2x_0y_3.$$

Set $x_0 = 2, y_1 = 1, y_2 = 1, y_3 = 2$. We have

$$\mathcal{B}(z) = 8 + 2 + 2 + 4 - 4 - 4 - 8 = 0.$$

Remark 2.12. If the weighted edge $(1, 2)$ (resp. $(2, 1)$) in cases a) c), (resp. b), d)) is not a terminal edge from the right side, see Fig. 2.4, complete all remaining coordinates corresponding to vertices till the terminal edge by zeros, see Remark 2.5, 2).

2) Case c) in Fig. 2.4. The Cartan matrix is

$$K = \begin{pmatrix} 2 & -1 & -1 & -1 & & & & & \\ -1 & 2 & & & & & & & \\ -1 & & 2 & & & & & & \\ -1 & & & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & & & \\ & \dots & & & & & \dots & & \\ & & & & & & 2 & -1 & \\ & & & & & & -1 & 2 & -2 \\ & & & & & & & -1 & 2 \end{pmatrix}, \tag{2.16}$$

the matrix U from the factorization (2.4) and the matrix of the Tits form are as follows:

$$U = \mathrm{diag}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \dots \\ 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & -1 & -1 & -1 & & & & & \\ -1 & 2 & & & & & & & \\ -1 & & 2 & & & & & & \\ -1 & & & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & & & \\ & \dots & & & & & \dots & & \\ & & & & & & 2 & -1 & \\ & & & & & & -1 & 2 & -2 \\ & & & & & & & -2 & 4 \end{pmatrix}. \tag{2.17}$$

Up to a factor 2, the Tits form is

$$\mathcal{B}(z) = x_0^2 + \sum_{i=1}^{n-1} y_i^2 + 2y_n^2 - x_0(y_1 + y_2 + y_3) - \sum_{i=3}^{n-2} y_i y_{i+1} - 2y_{n-1}y_n.$$

Set $x_0 = 2, y_1 = 1, y_2 = 1, y_3 = \dots = y_{n-1} = 2, y_n = 1$. Then

$$\mathcal{B}(z) = 4 + 1 + 1 + 4(n-3) + 2 - 2(1+1+2) - 4(n-4) - 4 = 0.$$

Case d) in Fig. 2.4. The Cartan matrix is

$$K = \begin{pmatrix} 2 & -1 & -1 & -1 & & & & & \\ -1 & 2 & & & & & & & \\ -1 & & 2 & & & & & & \\ -1 & & & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & & & \\ & \dots & & & & & \dots & & \\ & & & & & & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -2 & 2 \end{pmatrix}, \tag{2.18}$$

the matrix U from the factorization (2.4) and the matrix of the Tits form are as follows:

$$U = \operatorname{diag}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdots \\ 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & -1 & -1 & -1 & & & & & \\ -1 & 2 & & & & & & & \\ -1 & & 2 & & & & & & \\ -1 & & & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & & & \\ & \cdots & & & & & \cdots & & \\ & & & & & & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 1 \end{pmatrix}. \tag{2.19}$$

The Tits form is

$$\mathcal{B}(z) = 2x_0^2 + 2\sum_{i=1}^{n-1} y_i^2 + y_n^2 - 2x_0(y_1 + y_2 + y_3) - 2\sum_{i=3}^{n-1} y_i y_{i+1}.$$

Set $x_0 = 2, y_1 = 1, y_2 = 1, y_3 = \cdots = y_{n-1} = y_n = 2$. Then

$$\mathcal{B}(z) = 8 + 2 + 2 + 8(n-3) + 4 - 4(1 + 1 + 2) - 8(n-3) = 0. \quad \square$$

Corollary 2.13. *If the diagram* Γ *has a weighted edge, and the corresponding Tits form is positive definite, then the diagram* Γ *is a chain.*

To finish the proof of Theorem 2.4 it remains to show that the diagram Γ is a chain with a weighted edge only if Γ is B_n or C_n or F_4, see Fig. 2.3. In order to prove this fact, it suffices to prove, that diagrams $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{BC}_n$, $\widetilde{F}_{41}$, $\widetilde{F}_{42}$ have the non-negative Tits form[1].

As above in Proposition 2.11, 2), we write down the corresponding Tits form $\mathcal{B}$. For every mentioned diagram from Fig. 2.6, we insert the vector z with coordinates depicted in the figure, into the form $\mathcal{B}$. In all these cases, we have $\mathcal{B}(z) = 0$. Consider, for example, the diagrams $\widetilde{F}_{41}$ and $\widetilde{F}_{42}$, see Fig. 2.5.

a) Diagram $\widetilde{F}_{41}$. Here, the Cartan matrix is

$$K = \begin{array}{cc} \begin{pmatrix} 2 & -1 & -2 & & \\ -1 & 2 & & -1 & \\ -1 & & 2 & & -1 \\ & -1 & & 2 & \\ & & -1 & & 2 \end{pmatrix} & \begin{array}{c} x_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \end{array} \tag{2.20}$$

[1] For notation of diagrams with the non-negative Tits form, i.e., extended Dynkin diagrams, see §2.1.6. There we give two different notation: one used in the context of representations of quivers, and another one used in the context of affine Lie algebras, see Table 2.1, Table 2.2. For example, two notations:

$$\widetilde{F}_{41} \text{ vs. } F_4^{(1)}, \quad \widetilde{F}_{42} \text{ vs. } E_6^{(2)}, \quad \widetilde{B}_{n+1} \text{ vs. } D_{n+1}^{(2)}.$$

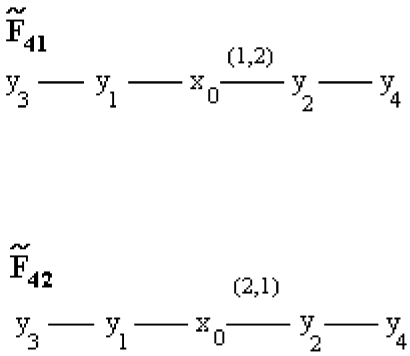

Fig. 2.5. The diagrams $\widetilde{F}_{41}$ and $\widetilde{F}_{42}$

the matrix U from (2.4) and the matrix of the Tits form are as follows:

$$U = \operatorname{diag}\begin{pmatrix} 1 \\ 1 \\ 1/2 \\ 1 \\ 1/2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & -1 & -2 & & \\ -1 & 2 & & -1 & \\ -2 & & 4 & & -2 \\ & -1 & & 2 & \\ & & -2 & & 4 \end{pmatrix}. \tag{2.21}$$

Up to a factor 2, the Tits form is

$$\mathcal{B}(z) = x_0^2 + y_1^2 + 2y_2^2 + y_3^2 + 2y_2^4 - y_1x_0 - 2y_2x_0 - 2y_2y_4 - y_3y_1.$$

Set $y_3 = 1, y_1 = 2, x_0 = 3, y_2 = 2, y_4 = 1$. Then

$$\mathcal{B}(z) = 9 + 4 + 8 + 1 + 2 - 6 - 12 - 4 - 2 = 0.$$

a) Diagram $\widetilde{F}_{42}$. The Cartan matrix is

$$K = \begin{array}{c} \begin{pmatrix} 2 & -1 & -1 & & \\ -1 & 2 & & -1 & \\ -2 & & 2 & & -1 \\ & -1 & & 2 & \\ & & -1 & & 2 \end{pmatrix} \end{array} \begin{array}{l} x_0 \\ y_1 \\ y_2\,, \\ y_3 \\ y_4 \end{array} \tag{2.22}$$

the matrix U from (2.4) and the matrix of the Tits form are as follows:

$$U = \operatorname{diag}\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & -1 & -1 & & \\ -1 & 2 & & -1 & \\ -1 & & 1 & & -\frac{1}{2} \\ & -1 & & 2 & \\ & & -\frac{1}{2} & & 1 \end{pmatrix}. \tag{2.23}$$

The Tits form is

$$\mathcal{B}(z) = 2x_0^2 + 2y_1^2 + y_2^2 + 2y_3^2 + y_2^4 - 2y_1x_0 - 2y_2x_0 - y_2y_4 - 2y_3y_1.$$

Set $y_3 = 1, y_1 = 2, x_0 = 3, y_2 = 4, y_4 = 2$. Then

$$\mathcal{B}(z) = 18 + 8 + 16 + 2 + 4 - 12 - 24 - 8 - 4 = 0. \quad \square$$

2.1.6 The extended Dynkin diagrams. Two different notation

Any diagram Γ with a non-negative definite Tits form $\mathcal{B}$ is said to be an *extended Dynkin diagram*. All extended Dynkin diagrams are listed in Fig. 2.6, see [Bo], [DR76], [Kac80], [Kac83].

There are two different notation systems of the extended Dynkin diagrams: one used in the context of representations of quivers, and another one used in the context of twisted affine Lie algebras, see Table 2.1, Table 2.2, Remark 4.4, Remark 6.5 and Table 4.2.

Table 2.1. Notation of extended Dyknin diagrams.

In the context of representations of quivers	In the context of twisted affine Lie algebras	In the context of representations of quivers	In the context of twisted affine Lie algebras
$\widetilde{E}_6$	$E_6^{(1)}$	$\widetilde{G}_{22}$	$G_2^{(1)}$
$\widetilde{E}_7$	$E_7^{(1)}$	$\widetilde{G}_{21}$	$D_4^{(3)}$
$\widetilde{E}_8$	$E_8^{(1)}$	$\widetilde{F}_{42}$	$F_4^{(1)}$
$\widetilde{D}_n$	$D_n^{(1)}$ $(n \geq 4)$	$\widetilde{F}_{41}$	$E_6^{(2)}$
$\widetilde{A}_{11}$	$A_2^{(2)}$	$\widetilde{C}_n$	$C_n^{(1)}$ $(n \geq 2)$
$\widetilde{A}_{12}$	$A_1^{(1)}$	$\widetilde{B}_n$	$D_{n+1}^{(2)}$ $(n \geq 2)$
$\widetilde{BC}_n$	$A_{2n}^{(2)}$ $(n \geq 2)$	$\widetilde{A}_n$	$A_n^{(1)}$ $(n \geq 2)$
$\widetilde{CD}_n$	$B_n^{(1)}$ $(n \geq 3)$	$\widetilde{DD}_n$	$A_{2n-1}^{(2)}$ $(n \geq 3)$

According to Proposition 2.3, we see that the simply-laced diagrams with only one branch point and with the non-negative definite Tits form are characterized as follows:

$$\mu = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

i.e., the following triples p, q, r are possible:

Table 2.2. Notation of affine Lie algebras. The upper index r in the notation of twisted affine Lie algebras has an invariant sense: it is the order of the diagram automorphism μ of $\mathfrak{g}$. In this table, if $\mathfrak{g}$ is a complex simple finite dimensional Lie algebra of type X_n, then the corresponding affine Lie algebra is of type $X_n^{(r)}$, see Remark 4.4.

Rang	Affine Lie algebra of type $X_n^{(r)}$	Note
Aff1	$A_1^{(1)}$, $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $G_2^{(1)}$, $F_4^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ $E_8^{(1)}$	Non-twisted, $r = 1$
Aff2	$A_2^{(2)}$ $A_{2n}^{(2)}$ $A_{2n-1}^{(2)}$ $D_{n+1}^{(2)}$ $E_6^{(2)}$	Twisted, $r = 2$
Aff3	$D_4^{(3)}$	Twisted, $r = 3$

1) $(3,3,3)$, i.e., the diagram $\widetilde{E}_6$,
2) $(2,4,4)$, i.e., the diagram $\widetilde{E}_7$,
3) $(2,3,6)$, i.e., the diagram $\widetilde{E}_8$,

see Fig. 2.6.

The multiply-laced extended Dynkin diagrams are: $\widetilde{A}_{12}$, $\widetilde{A}_{11}$, $\widetilde{BC}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{CD}_n$, $\widetilde{DD}_n$, $\widetilde{F}_{41}$, $\widetilde{F}_{42}$, $\widetilde{G}_{21}$, $\widetilde{G}_{22}$, see Fig. 2.6.

2.1.7 Three sets of Tits forms

1) All Tits forms $\mathcal{B}$ fall into 3 non-intersecting sets:

a) $\{\mathcal{B} \mid \mathcal{B}$ is positive definite, i.e., $\varGamma$ is the Dynkin diagram$\}$,

b) $\{\mathcal{B} \mid \mathcal{B}$ is non-negative definite, i.e., $\varGamma$ is the extended Dynkin diagram$\}$,

c) $\{\mathcal{B} \mid \mathcal{B}$ is indefinite$\}$.

Consider two operations:

$\wedge$: *Add* a vertex and connect it with Γ by only one edge. We denote the new graph $\overset{\wedge}{\Gamma}$.

$\vee$: *Remove* a vertex and all incident edges (the new graph may contain more than one component). We denote the new graph $\overset{\vee}{\Gamma}$.

2) It is easy to see that

a) The set $\{\mathcal{B} \mid \mathcal{B}$ is positive definite$\}$ is stable under $\vee$ (*Remove*) and is not stable under $\wedge$ (*Add*) ,

$$\wedge : \ \{\mathcal{B} \mid \mathcal{B} \text{ is positive definite}\} \quad \Longrightarrow \begin{cases} \{\mathcal{B} \mid \mathcal{B} \text{ is non-negative definite}\} \coprod \\ \{\mathcal{B} \mid \mathcal{B} \text{ is indefinite}\}, \end{cases}$$

$$\vee : \ \{\mathcal{B} \mid \mathcal{B} \text{ is positive definite}\} \quad \Longrightarrow \{\mathcal{B} \mid \mathcal{B} \text{ is positive definite}\}.$$

b) The set $\{\mathcal{B} \mid \mathcal{B} \text{ is non-negative definite}\}$ is not stable under $\vee$ (*Remove*) and $\wedge$ (*Add*),

$$\begin{aligned}
\wedge: \ \{\mathcal{B} \mid \mathcal{B} \text{ is non-negative definite}\} &\Longrightarrow \{\mathcal{B} \mid \mathcal{B} \text{ is indefinite}\},\\
\vee: \ \{\mathcal{B} \mid \mathcal{B} \text{ is non-negative definite}\} &\Longrightarrow \{\mathcal{B} \mid \mathcal{B} \text{ is positive definite}\}.
\end{aligned}$$

c) The set $\{\mathcal{B} \mid \mathcal{B} \text{ is indefinite}\}$ is stable under $\wedge$ (*Add*) but not stable under $\vee$ (*Remove*):

$$\begin{aligned}
\wedge: \ \{\mathcal{B} \mid \mathcal{B} \text{ is indefinite}\} &\Longrightarrow \{\mathcal{B} \mid \mathcal{B} \text{ is indefinite}\},\\
\vee: \ \{\mathcal{B} \mid \mathcal{B} \text{ is indefinite}\} &\Longrightarrow \begin{cases} \{\mathcal{B} \mid \mathcal{B} \text{ is positive definite}\} \coprod \\ \{\mathcal{B} \mid \mathcal{B} \text{ is non-negative definite}\} \coprod \\ \{\mathcal{B} \mid \mathcal{B} \text{ is indefinite}\}. \end{cases}
\end{aligned}$$

3) If the graph Γ with indefinite form $\mathcal{B}$ is obtained from any Dynkin diagram by adding an edge, then the same graph Γ can be obtained by adding an edge (or maybe several edges) to an extended Dynkin diagram.

2.1.8 The hyperbolic Dynkin diagrams and hyperbolic Cartan matrices

A connected graph Γ with indefinite Tits form is said to be a *hyperbolic Dynkin diagram* (resp. *strictly hyperbolic Dynkin diagram*) if every subgraph $\Gamma' \subset \Gamma$ is a Dynkin diagram or an extended Dynkin diagram (resp. Dynkin diagram). The corresponding Cartan matrix K is said to be *hyperbolic* (resp. *strictly hyperbolic*), see [Kac93, exs. of §4.10]. The corresponding Weyl group is said to be a *hyperbolic Weyl group* (resp. *compact hyperbolic Weyl group*), see [Bo, Ch.5, exs. of §4].

The Tits form $\mathcal{B}$ is indefinite for $\mu < 1$. Since $U(x)$, $U(y)$, $U(z)$ in (2.9) are non-negative definite, we see that, in this case, the signature of $\mathcal{B}$ is equal to $(n-1, 1)$, see [Kac93, exs.4.2]. It is easy to check that the triples p, q, r corresponding to hyperbolic graphs $T_{p,q,r}$ are only the following ones:

1) $(2, 3, 7)$, i.e., the diagram E_{10}, and $1 - \mu = \dfrac{1}{42}$,

2) $(2, 4, 5)$, and $1 - \mu = \dfrac{1}{20}$,

3) $(3, 3, 4)$, and $1 - \mu = \dfrac{1}{12}$.

2.2 Representations of quivers

2.2.1 The real and imaginary roots

Recall now the definitions of *imaginary* and *real* roots in the infinite root system associated with infinite dimensional Kac-Moody Lie algebras. We mostly follow V. Kac's definitions [Kac80], [Kac82], [Kac93].

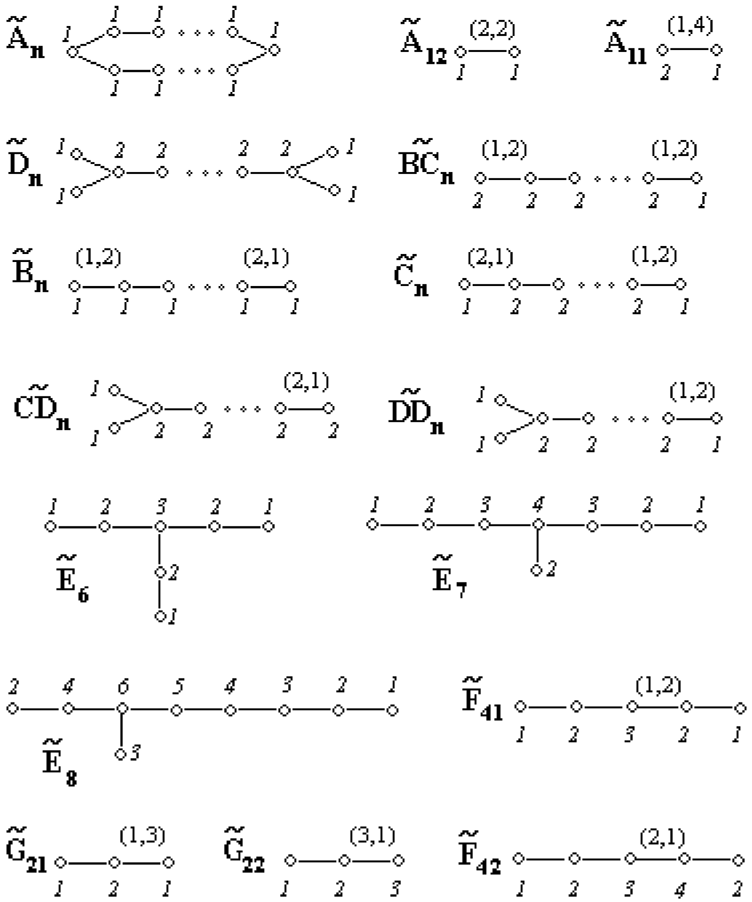

Fig. 2.6. The extended Dynkin diagrams. The numerical labels at the vertices are the coefficients of the imaginary root which coincides with the fixed point z^1 of the Coxeter transformation, see §3.3.1, (3.23).

We consider the vector space $\mathcal{E}_\Gamma$ over $\mathbb{Q}$; set

$$\dim \mathcal{E}_\Gamma = |\Gamma_0|.$$

Let

$$\alpha_i = \{0, 0, ...0, \overset{i}{1}, 0, ...0, 0\} \in \mathcal{E}_\Gamma$$

be the basis vector corresponding to the vertex $v_i \in \Gamma_0$. The space $\mathcal{E}_\Gamma$ is spanned by the vectors $\{\alpha_i \mid i \in \Gamma_0\}$; the vectors $\alpha_1, ..., \alpha_n$ form a basis in $\mathcal{E}_\Gamma$. Let

$$\mathcal{E}_+ = \{\alpha = \sum k_i\alpha_i \in \mathcal{E}_\Gamma \mid k_i \in \mathbb{Z}_+, \sum k_i > 0\}$$

be the set of all non-zero elements in $\mathcal{E}_\Gamma$ with non-negative integer coordinates in the basis $\{\alpha_1, ..., \alpha_n\}$.

Define the linear functions $\phi_1, \ldots, \phi_n$ on $\mathcal{E}_\Gamma$ by means of the elements of the Cartan matrix (2.4):

$$\phi_i(\alpha_j) = k_{ij} \ .$$

The *positive root system* Δ_+ associated to the Cartan matrix K is a subset in $\mathcal{E}_+$ defined by the following properties (R1)–(R3):

(R1) $\alpha_i \in \Delta_+$ and $2\alpha_i \notin \Delta_+$ for $i = 1, ..., n$.

(R2) If $\alpha \in \Delta_+$ and $\alpha \neq \alpha_i$, then $\alpha + k\alpha_i \in \Delta_+$ for $k \in \mathbb{Z}$ if and only if $-p \leq k \leq q$ for some non-negative integers p and q such that $p - q = \phi_i(\alpha)$.

(R3) Any $\alpha \in \Delta_+$ has a connected support.

We define endomorphisms $\sigma_1, ..., \sigma_n$ of $\mathcal{E}_\Gamma$ by the formula

$$\sigma_i(x) = x - \phi_i(x)\alpha_i. \tag{2.24}$$

Each endomorphism σ_i is the reflection in the hyperplane $\phi_i = 0$ such that $\sigma_i(\alpha_i) = -\alpha_i$. These reflections satisfy the following relations[1]:

$$\sigma_i^2 = 1, \quad (\sigma_i\sigma_j)^{n_{ij}} = 1,$$

where the exponents n_{ij}, corresponding to $k_{ij}k_{ji}$, are given in the Table 2.3, taken from [Kac80, p.63].

Table 2.3. The exponents n_{ij}

$k_{ij}k_{ji}$	0	1	2	3	≥ 4
n_{ij}	2	3	4	6	∞

The group W generated by the reflections $\sigma_1, ..., \sigma_n$ is called the *Weyl group*. The vectors $\alpha_1, ..., \alpha_n$ are called *simple roots*; we denote by Π the set of all simple roots. Let $W(\Pi)$ be the orbit of Π under the W-action.

Following V. Kac [Kac80, p.64], set

$$M = \{\alpha \in \mathcal{E}_+ \mid \phi_i(\alpha) \leq 0 \text{ for } i = 1, ..., n, \text{ and } \alpha \text{ has a connected support }\}.$$

The set M is called the *fundamental set*. Let $W(M)$ be the orbit of M under the W-action. We set

[1] $\sigma^\infty = 1$ means that no power of σ is equal to 1, i.e., σ is free.

$$\Delta_+^{re} = \bigcup_{w \in W} (w(\Pi) \cap \mathcal{E}_+), \qquad \Delta_+^{im} = \bigcup_{w \in W} (w(M)). \tag{2.25}$$

The elements of the set Δ_+^{re} are called *real roots* and the elements of the set Δ_+^{im} are called *imaginary roots*. By [Kac80], the *system of positive roots* Δ_+ is the disjoint union of the sets Δ_+^{re} and Δ_+^{im}:

$$\Delta_+ = \Delta_+^{re} \coprod \Delta_+^{im}.$$

We denote by Δ the set of all roots. It consists of the set of positive roots Δ_+ and the set of negative root Δ_- obtained from Δ_+ by multiplying by -1:

$$\Delta = \Delta_+ \coprod \Delta_-.$$

If the Tits form $\mathcal{B}$ (associated with the Cartan matrix K) is positive definite, then the root system is finite, it corresponds to a simple finite dimensional Lie algebra. In this case, the root system consists of real roots.

If the Tits form $\mathcal{B}$ is non-negative definite, we have an infinite root system whose imaginary root system is one-dimensional:

$$\Delta_+^{im} = \{\omega, 2\omega, 3\omega, ...\}, \qquad \text{where} \quad \omega = \sum_i k_i \alpha_i, \tag{2.26}$$

the coefficients k_i being the labels of the vertices from Fig. 2.6.

The elements $k\omega$ ($k \in \mathbb{N}$) are called *nil-roots*. Every nil-root is a fixed point for the Weyl group.

2.2.2 A category of representations of quivers and the P. Gabriel theorem

A *quiver* (Γ, Ω) is a connected graph Γ with an orientation Ω. Let Γ_0 be the set of all vertices of Γ. Any orientation Ω is given by a set of arrows Γ_1 that constitute Γ_1. Every arrow $\alpha \in \Gamma_1$ is given by its source point $s(\alpha) \in \Gamma_0$ and its target point $t(\alpha) \in \Gamma_0$. In [Gab72], P. Gabriel introduced the notion of representations of quivers in order to formulate a number of problems of linear algebra in a general way. P. Gabriel consider graphs without weighted edges.

Let k be a field. A *representation* of the quiver (Γ, Ω) over k is a set of spaces and linear maps between them (V_i, ϕ_α), where to any vertex $i \in \Gamma_0$ a finite-dimensional space V_i over k is assigned, and to any arrow

$$i \xrightarrow{\alpha} j$$

a linear operator $\phi_\alpha : V_i \to V_j$ corresponds. All representations (V_i, ϕ_α) of the quiver (Γ, Ω) constitute the category $\mathfrak{L}(\Gamma, \Omega)$. In this category, a *morphism*

$$\eta : (V_i, \phi_\alpha) \to (V_i', \phi_\alpha')$$

is a collection of linear maps $\eta_i : V_i \to V_i'$ for $i \in \Gamma_0$ such that for each arrow $\alpha : s(\alpha) \to t(\alpha)$, we have

$$\eta_{t(\alpha)}\phi_\alpha = \phi_\alpha'\eta_{s(\alpha)},$$

or, equivalently, the following square is commutative

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{\phi_\alpha} & V_{t(\alpha)} \\ \downarrow \eta_{s(\alpha)} & & \downarrow \eta_{t(\alpha)} \\ V'_{s(\alpha)} & \xrightarrow{\phi'_\alpha} & V'_{t(\alpha)} \end{array}$$

A morphism $\eta : (V_i, \phi_\alpha) \to (V_i', \phi_\alpha')$ is an *isomorphism* in the category $\mathfrak{L}(\Gamma\Omega)$ if there exists a morphism $\eta' : (V_i', \phi_\alpha') \to (V_i, \phi_\alpha)$ such that $\eta\eta' = \mathrm{Id}_{(V_i', \phi_\alpha')}$ and $\eta'\eta = \mathrm{Id}_{(V_i, \phi_\alpha)}$.

Objects of the category $\mathfrak{L}(\Gamma, \Omega)$ are said to be *representations of the quiver* (Γ, Ω) considered up to an isomorphism. The *dimension* $\dim V$ of the representation (V_i, ϕ_α) (or, which is the same, the *vector-dimension*) is an element of $\mathcal{E}_\Gamma$, with $(\dim V)_i = \dim V_i$ for all $i \in \Gamma_0$.

The *direct sum* of objects (V, f) and (U, g) in the category $\mathfrak{L}(\Gamma, \Omega)$ is the object

$$(W, h) = (V, f) \oplus (U, g),$$

where $W_i = V_i \oplus U_i$ and $h_\alpha = f_\alpha \oplus g_\alpha$ for all $i \in \Gamma_0$ and $\alpha \in \Gamma_1$. The nonzero object (V, f) is said to be *indecomposable*, if it can not be represented as a direct sum of two nonzero objects, see [Gab72], [BGP73]. As usual in the representation theory, the main question is the description of all indecomposable representations in the category $\mathfrak{L}(\Gamma, \Omega)$.

Theorem 2.14. (P. Gabriel, [Gab72]) *1) A given quiver has only a finite number of indecomposable representations if and only if it is a simply-laced Dynkin diagram with arbitrary orientations of the edges.*

2) Vector-dimensions of indecomposable representations coincide with positive roots in the root system of the corresponding Dynkin diagram.

V. Dlab and C. M. Ringel [DR76] and L. A. Nazarova, S. A. Ovsienko and A. V. Roiter [NaOR77], [NaOR78] extended the P. Gabriel theorem to the multiply-laced case.

2.2.3 Finite-type, tame and wild quivers

In [BGP73], Bernstein, Gelfand and Ponomarev introduced the *Coxeter functors* Φ^+, Φ^- leading to a new proof and new understanding of the P. Gabriel theorem. They also introduced *regular representations* of quivers, i.e., representations that never vanish under the Coxeter functors.

The Dynkin diagrams do not have regular representations; in the category of quiver representations[1], only a finite set of indecomposable representations is associated to any Dynkin diagram. Quivers with such property are called *finite-type quivers*. According to P. Gabriel's theorem [Gab72], *a quiver is of finite-type if and only if it is a (simply-laced) Dynkin diagram.*

In the category of all representations of a given quiver, the regular representations are the most complicated ones; they have been completely described only for the extended Dynkin diagrams which for this reason, in the representation theory of quivers, were called *tame quivers*. The following theorem is an extension of the P. Gabriel theorem to extended Dynkin diagrams and is due to L. A. Nazarova [Naz73], P. Donovan, and M. R. Freislich [DF73], V. Dlab and C. M. Ringel [DR74], [DR74a], [DR76].

Theorem 2.15. (Nazarova, Donovan-Freislich, Dlab-Ringel) *1) A quiver* Γ *is of tame type if and only if it is a simply-laced extended Dynkin diagram with arbitrary orientations of the edges.*

2) Dimensions of indecomposable representations coincide with positive roots in the root system of the corresponding diagram. For each positive real root α^{re} *of* Γ*, there exists a unique indecomposable* V *(up to isomorphism) with* $\dim V = \alpha^{re}$*. For each positive imaginary root* α^{im}*, the isomorphism classes of indecomposable representations* V *with* $\dim V = \alpha^{im}$ *are parameterized by an infinite subset of* $k\mathbb{P}^1$*.*

Every quiver with indefinite Tits form $\mathcal{B}$ is *wild*, i.e., the description of its representations contains the problem of classifying pairs of matrices up to simultaneous similarity; this classification is hopeless in a certain sense, see [GP69], [Naz73], [Drz80], [Kac83].

2.2.4 The V. Kac theorem on the possible dimension vectors

According to the P. Gabriel theorem (Th.2.14) and the Nazarova, Donovan-Freislich, Dlab-Ringel theorem (Th.2.15) all quivers, which are neither Dynkin diagram nor extended Dynkin diagram are wild. H. Kraft and Ch. Riedtmann write in [KR86, p.109] : *"Since all remaining quivers are wild, there was little hope to get any further, except maybe in some special case. Therefore Kac's spectacular paper [Kac80], where he describes the dimension types of all indecomposables of arbitrary quivers, came as a big surprise. In [Kac82] and [Kac83] Kac improved and completed his first results"*. Note, that the work [KR86] of H. Kraft and Ch. Riedtmann is a nice guide to the Kac theorem.

Following V. Kac [Kac82, p.146] (see also [Kac83, p.84], [KR86, p.125]) the integers μ_α an r_α are introduced as follows:

μ_α *is the maximal dimension of an irreducible component in the set of isomorphism classes of indecomposable representations of dimension* α *and* r_α *is the number of such components.*

The following theorem was first proved in [Kac80]. It is given here in a more consolidated version [Kac83, §1.10].

---

[1] Sometimes, one says about *graph representations* meaning *quiver representations*.

Theorem 2.16 (V. Kac). *Let the base field k be algebraically closed. Let (Γ, Ω) be a quiver. Then*

a) There exists an indecomposable representation of dimension $\alpha \in \mathcal{E}_\Gamma \backslash 0$ if and only if $\alpha \in \Delta_+(\Gamma)$.

b) There exists a unique indecomposable representation of dimension α if and only if $\alpha \in \Delta_+^{re}(\Gamma)$.

c) If $\alpha \in \Delta_+^{im}(\Gamma)$, then

$$\mu_\alpha(\Gamma, \Omega) = 1 - (\alpha, \alpha) > 0, \qquad r_a = 1,$$

where $(\cdot, \cdot)$ is the quadratic Tits form, see §2.1.1.

The integer μ_α is called the *number of parameters*, [KR86]. For example, let V be an indecomposable representation of a tame quiver and $\dim V$ an imaginary root, i.e., $(\dim V, \dim V) = 0$. Then $\mu_\alpha = 1$ and the irreducible component of indecomposable representations is one-parametric as in Theorem 2.15. Let V be an indecomposable representation of a wild quiver and $(\dim V, \dim V) < 0$. Then $\mu_\alpha > 1$, and the irreducible component containing V is at least two-parametric.

For a wild quiver with the indefinite Tits form, we have

The following lemma [Kac83, §1.10] is one of the crucial moments in the proof of Kac's theorem.

Lemma 2.17 (V. Kac). *The number of indecomposable representations of dimension α (if it is finite) and $\mu_\alpha(\Gamma, \Omega)$ are independent of the orientation Ω.*

The proof of this lemma and further details of Kac's theorem can be also found in the work [KR86, §5] of H. Kraft and Ch. Riedtmann and in works [CrW93, p.32], [CrW99], [CrW01, p.40] of W. Crawley-Boevey.

The main part in the proof of this lemma is counting points of certain varieties for the case of algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$, for any prime p, see [KR86, §5.6], with subsequent transfer of the result to fields of characteristic zero [KR86, §5.6]. *"So this is one of the examples where the only proof known for a result about fields of characteristic zero passes via fields of positive characteristic."* [KR86, §5.1]

2.2.5 The quadratic Tits form and vector-dimensions of representations

In the simply-laced case, by (1.5), (3.3), the quadratic Tits form on the space of vectors $\{x_v\}_{v \in \Gamma_0}$ associated with quiver (Γ, Ω) can be expressed as follows:

$$\mathcal{B}(x) = \sum_{v \in \Gamma_0} x_v^2 - \sum_{l \in \Gamma_1} x_{v(l)} x_{u(l)},$$

where $v(l)$ and $u(l)$ are the endpoints of the edge $l \in \Gamma_1$, see, e.g., (2.7). By Kac's theorem (Theorem 2.16, b)), the representation V of (Γ, Ω) is a unique indecomposable representation of the given dimension if and only if

$$\mathcal{B}(\dim V) = 1.$$

It is so because such dimensions are real vectors lying by eq. (2.25) on the orbit of simple positive roots under the action of the Weyl group, which preserves the quadratic form $\mathcal{B}$, see, e.g., [BGP73].

An analogous proposition takes place for representations of posets. Let $(\mathfrak{D}, \geq)$ be a finite partially ordered set (*poset*). Let $\mathfrak{D} = \{1, 2, \dots, n\}$ (not necessary with the natural order) and denote:

$$\widehat{\mathfrak{D}} = \mathfrak{D} \coprod \{0\}.$$

A poset is called *primitive* if it is a disjoint unit of several ordered chains such that the elements of different chains are non-comparable. We denote such a poset by $(n_1, n_2, \dots, n_s)$, where n_i are the lengths of the chains. A *representation* V of $\mathfrak{D}$ over a field k is an order-preserving map $\mathfrak{D}$ into the set of subspaces of a finite dimensional vector space $V(0)$ over k, see [DrK04]. The category of representations of $\mathfrak{D}$ and indecomposable representations are defined in the natural way [NR72]. The *dimension* $\dim(u)$ of an element $u \in \mathfrak{D}$ is the vector whose components are

$$\dim(0) = \dim V(0), \quad \dim(a) = \dim(V(a)/\sum_{b \leq a} V(b)), \quad a \in \mathfrak{D}.$$

The quadratic form $\mathcal{B}_{\mathfrak{D}}$ associated to a poset $\mathfrak{D}$ is the quadratic form

$$\mathcal{B}_{\mathfrak{D}}(x_0, x_1, \dots, x_n) = \sum_{a \in \widehat{\mathfrak{D}}} x_a^2 + \sum_{\substack{a, b \in \mathfrak{D} \\ a \leq b}} x_a x_b - \sum_{a \in \mathfrak{D}} x_0 x_a.$$

In [DrK04], Yu. A. Drozd and E. Kubichka consider the *dimensions of finite type* of representations of a partially ordered set, i.e., such that there are only finitely many isomorphism classes of representations of this dimension. In particular, they show that an element $u \in \mathfrak{D}$ is indecomposable if and only if $\mathcal{B}_{\mathfrak{D}}(\dim(u)) = 1$. For primitive posets, this theorem was deduced by P. Magyar, J. Weyman, A. Zelevinsky in [MWZ99] from the results of Kac [Kac80]. This approach cannot be applied in general case, see [DrK04, p.5], so Yu. A. Drozd and E. Kubichka used the original technique of *derivations* (or *differentiations*), which is due to L. A. Nazarova and A. V. Roiter, [NR72].

2.2.6 Orientations and the associated Coxeter transformations

Here, we follow the definitions of Bernstein-Gelfand-Ponomarev [BGP73]. Let us consider the graph Γ endowed with an orientation Ω.

The vertex v_i is called *sink-admissible* (resp. *source-admissible*) in the orientation Ω if every arrow containing v_i ends in (resp. starts from) this vertex. The reflection σ_i is applied to the vector-dimensions by (2.24) and can be applied only to vertex which is either sink-admissible or source-admissible. The reflection σ_i acts on the orientation Ω by reversing all arrows containing the vertex v_i.

Consider now a sequence of vertices and the corresponding reflections. A sequence of vertices

$$\{v_{i_n}, v_{i_{n-1}},\ \dots,\ v_{i_3}, v_{i_2}, v_{i_1}\}$$

is called *sink-admissible*, if the vertex v_{i_1} is sink-admissible in the orientation Ω, the vertex v_{i_2} is sink-admissible in the orientation $\sigma_{i_1}(\Omega)$, the vertex v_{i_3} is sink-admissible in the orientation $\sigma_{i_2}\sigma_{i_1}(\Omega)$, and so on. *Source-admissible sequences* are similarly defined.

A sink-admissible (resp. source-admissible) sequence

$$\mathcal{S} = \{v_{i_n}, v_{i_{n-1}},\ \dots,\ v_{i_3}, v_{i_2}, v_{i_1}\}$$

is called *fully sink-admissible* (resp. *fully source-admissible*) if $\mathcal{S}$ contains every vertex $v \in \Gamma_0$ exactly once. Evidently, the inverse sequence $\mathcal{S}^{-1}$ of a fully sink-admissible sequence $\mathcal{S}$ is fully source-admissible and vice versa.

Every tree has a fully sink-admissible sequence $\mathcal{S}$. To every sink-admissible sequence $\mathcal{S}$, we assign the Coxeter transformation depending on the order of vertices in $\mathcal{S}$:

$$\mathbf{C} = \sigma_{i_n}\sigma_{i_{n-1}}...\sigma_{i_2}\sigma_{i_1}. \tag{2.27}$$

For every orientation Ω of the tree, every fully sink-admissible sequence gives rise to the same Coxeter transformation $\mathbf{C}_\Omega$, and every fully source-admissible sequence gives rise to $\mathbf{C}_\Omega^{-1}$. Thus, *to every orientation* Ω *of the tree, we assign two Coxeter transformations:* $\mathbf{C}_\Omega$ *and* $\mathbf{C}_\Omega^{-1}$.

Obviously, $\mathbf{C}_\Omega$ acts trivially on the orientation Ω because every edge of the tree is twice reversed. However, the Coxeter transformation does not act trivially on the space of vector-dimensions, see (2.24).

2.3 The Poincaré series

2.3.1 The graded algebras, symmetric algebras, algebras of invariants

Let k be a field. We define a *graded k-algebra* to be a finitely generated k-algebra A (associative, commutative, and with unit), together with a direct sum decomposition (as vector space)

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \dots,$$

such that $A_0 = k$ and $A_i A_j \subset A_{i+j}$. The component A_n is called the *nth homogeneous part of* A and any element $x \in A_n$ is said to be *homogeneous of degree* n, notation: $\deg x = n$.

We define a *graded A-module* to be a finitely generated A-module, together with a direct sum decomposition

$$M = M_0 \oplus M_1 \oplus M_2 \oplus \dots,$$

such that $A_i M_j \subset M_{i+j}$.

The *Poincaré series* of a graded algebra $A = \bigoplus_{n=0}^{\infty} A_i$ is the formal series

$$P(A, t) = \sum_{n=0}^{\infty} (\dim A_n) t^n.$$

The *Poincaré series* of a graded A-module $M = \bigoplus_{n=0}^{\infty} M_i$ is the formal series

$$P(M, t) = \sum_{n=0}^{\infty} (\dim M_n) t^n,$$

see [Sp77], [PV94].

Theorem 2.18. (Hilbert, Serre, see [AtMa69, p.117, Th.11.1]) *The Poincaré series $P(M, t)$ of a finitely generated graded A-module is a rational function in t of the form*

$$P(M, t) = \frac{f(t)}{\prod_{i=1}^{s} (1 - t^{k_i})}, \ \textit{where } f(t) \in \mathbb{Z}[t].$$

In what follows in this section, any algebraically closed field k can be considered instead of $\mathbb{C}$. Set

$$R = \mathbb{C}[x_1, \dots, x_n]. \tag{2.28}$$

The set R_d of homogeneous polynomials of degree d is a finite dimensional subspace of R, and $R_0 = \mathbb{C}$. Moreover, $R_d R_e \subset R_{d+e}$, and R is a graded $\mathbb{C}$-algebra with a direct sum decomposition (as a vector space)

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \dots \tag{2.29}$$

Let $V = \operatorname{Span}(x_1, \dots, x_n)$. Let $f_1, \dots, f_n \in V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the linear forms defined by $f_i(x_j) = \delta_{ij}$, i.e.,

$$f_i(\lambda_1 x_1 + \dots \lambda_n x_n) = \lambda_i, \ \text{where } \lambda_i \in \mathbb{C}.$$

Then the f_i, where $i = 1, \ldots, n$, generate a *symmetric algebra*[1] $\mathrm{Sym}(V^*)$ isomorphic to R.

Define an isomorphism $\Phi : \mathrm{Sym}(V^*) \to R$ by setting

$$\Phi f_i = x_i \text{ for all } i.$$

Let $\mathrm{Sym}^m(V^*)$ denote the *mth symmetric power* of V^*, which consists of the homogeneous polynomials of degree m in $x_1, \ldots, x_n$. Thus,

$$\begin{aligned}
&\mathrm{Sym}^0(V^*) = \mathbb{C},\\
&\mathrm{Sym}^1(V^*) = V^*,\\
&\mathrm{Sym}^2(V^*) = \{x_i x_j \mid 1 \le i \le j \le n\}, \quad \dim \mathrm{Sym}^2(V^*) = \binom{n+1}{2},\\
&\mathrm{Sym}^3(V^*) = \{x_i x_j x_k \mid 1 \le i \le j \le k \le n\}, \quad \dim \mathrm{Sym}^3(V^*) = \binom{n+2}{3},\\
&\qquad \ldots\\
&\mathrm{Sym}^m(V^*) = \{x_{i_1} \ldots x_{i_m} \mid 1 \le i_1 \le \cdots \le i_m \le n\},\\
&\qquad\qquad\qquad\qquad \dim \mathrm{Sym}^m(V^*) = \binom{n+m-1}{m},
\end{aligned}$$

see [Sp77], [Ben93].

For $n = 2$, $m \ge 1$, we have

$$\mathrm{Sym}^m(V^*) = \{x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m\}$$

is the space of homogeneous polynomials of degree m in two variables x, y, and

$$\dim \mathrm{Sym}^m(V^*) = \binom{m+1}{m} = m.$$

For any $a \in GL(V)$ and $f \in R$, define $af \in R$ by the rule

$$(af)(v) = f(a^{-1}v) \;\text{ for any } v \in V.$$

Then, for any $a, b \in G$ and $f \in R$, we have

$$a(bf) = bf(a^{-1}v) = f(b^{-1}a^{-1}v) = f((ab)^{-1}v) = ((ab)f)(v),$$

and $aR_d = R_d$.

Let G be a subgroup in $GL(V)$. We say that $f \in R$ is *G-invariant* if $af = f$ for all $a \in G$. The G-invariant polynomial functions form a subalgebra R^G of R, which is a graded, or, better say, homogeneous, subalgebra, i.e.,

$$R^G = \bigoplus R^G \cap R_i.$$

---

[1] The symmetric algebra $\mathrm{Sym}(V)$ is the quotient ring of the tensor algebra $T(V)$ by the ideal generated by elements $vw - wv$ for v and w in V, where $vw := v \otimes w$.

The algebra R^G is said to be the *algebra of invariants* of the group G. If G is finite, then

$$P(R^G, t) = \frac{1}{|G|} \frac{1}{\sum\limits_{g \in G} \det(1 - tg)} \ . \tag{2.30}$$

Eq. (2.30) is a classical theorem of Molien (1897), see [PV94, §3.11], or [Bo, Ch.5, §5.3].

2.3.2 The invariants of finite groups generated by reflections

Let G be a finite subgroup of $GL(V)$ and $g \in G$. Then g is called a *pseudo-reflection* if precisely one eigenvalue of g is not equal to 1. Any pseudo-reflection with determinant -1 is called a *reflection*. For example, the element

$$g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

has infinite order and does not belong to any finite group G, i.e., g is not reflection.

Theorem 2.19 (Shephard-Todd, Chevalley, Serre). *Let G be a finite subgroup of $GL(V)$. There exist $n = \dim V$ algebraically independent homogeneous invariants $\theta_1, \dots, \theta_n$ such that*

$$R^G = \mathbb{C}[\theta_1, \dots, \theta_n]$$

if and only if G is generated by pseudo-reflections.

For references, see [Stn79, Th.4.1], [Ch55, Th.A, p.778].

Shephard and Todd [ShT54] explicitly determined all finite subgroups of $GL(V)$ generated by pseudo-reflections and verified the sufficient condition of Theorem 2.19. Chevalley [Ch55] found the classification-free proof of this sufficient condition for a particular case where G is generated by reflections. Serre observed that Chevalley's proof is also valid for groups generated by pseudo-reflections. Shephard and Todd ([ShT54]) proved the necessary condition of the theorem by a strong combinatorial method; see also Stanley [Stn79, p.487].

The coefficients of the Poincaré series are called the *Betti numbers*, see [Ch50], [Col58]. The Poincaré polynomial of the algebra R^G is

$$(1 + t^{2p_1+1})(1 + t^{2p_2+1}) \dots (1 + t^{2p_n+1}), \tag{2.31}$$

where $p_i + 1$ are the degrees of homogeneous basis elements of R^G, see [Cox51], and [Ch50]. See Table 2.4 taken from [Cox51, p.781,Tab.4].

Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of a Coxeter transformation in a finite Weyl group. These eigenvalues can be given in the form

Table 2.4. The Poincaré polynomials for the simple compact Lie groups

Dynkin diagram	Poincaré polynomial
A_n	$(1+t^3)(1+t^5)\dots(1+t^{2n+1})$
B_n or C_n	$(1+t^3)(1+t^7)\dots(1+t^{4n-1})$
D_n	$(1+t^3)(1+t^7)\dots(1+t^{4n-5})(1+t^{2n-1})$
E_6	$(1+t^3)(1+t^9)(1+t^{11})(1+t^{15})(1+t^{17})(1+t^{23})$
E_7	$(1+t^3)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{27})(1+t^{35})$
E_8	$(1+t^3)(1+t^{15})(1+t^{23})(1+t^{27})(1+t^{35})(1+t^{39})(1+t^{47})(1+t^{59})$
F_4	$(1+t^3)(1+t^{11})(1+t^{15})(1+t^{23})$
G_2	$(1+t^3)(1+t^{11})$

$$\omega^{m_1}, ..., \omega^{m_n},$$

where $\omega = \exp^{2\pi i/h}$ is a primitive root of unity. The numbers $m_1, ..., m_n$ are called the *exponents* of the Weyl group. H. S. M. Coxeter observed that the *exponents* m_i and numbers p_i in (2.31) coincide (see the epigraph to this chapter).

Theorem 2.20 (Coxeter, Chevalley, Coleman, Steinberg). *Let*

$$u_1, \dots, u_n$$

be homogeneous elements generating the algebra of invariants R^G, where G is the Weyl group corresponding to a simple compact Lie group. Let $m_i + 1 =$ $\deg u_i$*, where $i = 1, 2, ...n$. Then the exponents of the group G are*

$$m_1, \dots, m_n.$$

For more details, see [Bo, Ch.5, §6.2, Prop.3] and historical notes in [Bo], and [Ch50], [Cox51], [Col58], [Stb85].

http://www.springer.com/978-3-540-77398-6

Notes on Coxeter Transformations and the McKay Correspondence
Stekolshchik, R.
2008, XX, 240 p. 28 illus., Hardcover
ISBN: 978-3-540-77398-6