

# Preface

This book gives an introduction to so-called proof interpretations, more specifically various forms of realizability and functional interpretations, and their use in mathematics. Whereas earlier treatments of these techniques (e.g. [366, 266, 122, 369, 7]) emphasize foundational and logical issues the focus of this book is on applications of the methods to extract new effective information such as computable uniform bounds from given (typically ineffective) proofs. This line of research, which has its roots in G. Kreisel's pioneering work on 'unwinding of proofs' from the 50's, has in more recent years developed into a field of mathematical logic which has been called (suggested by D. Scott) 'proof mining'. The areas where proof mining based on proof interpretations has been applied most systematically are numerical analysis and functional analysis and so the book concentrates on those. There are also some extractions of effective information from proofs (guided by logic) in number theory (G. Kreisel, H. Luckhardt, see e.g. [249, 268, 267, 122]) and algebra (G. Kreisel, C. Delzell, H. Lombardi, T. Coquand and others, see e.g. [252, 84, 77, 74, 76]). However, here mainly methods from structural proof theory such as Herbrand's theorem,  $\varepsilon$ -substitution and cut-elimination are used and we will refer to the literature for more information on these results.

In this book two kinds of systems play an important role: those with full induction and variants with induction for purely existential formulas (whose central role has been singled out in the context of so-called reverse mathematics, [338]). Further (still weaker) fragments are briefly discussed in comments and referred to in the literature.

Modified realizability (due to G. Kreisel) and functional interpretation (due to K. Gödel) are both first developed in the framework of constructive ('intuitionistic') arithmetic in higher types to which consecutively various non-constructive principles are added.

After this, systems based on ordinary ('classical') logic are studied. It is shown that the combination of Gödel's functional ('Dialectica') interpretation with the so-called negative translation, which embeds certain classical theories into approximately intuitionistic counterparts, can be used to unwind fully non-constructive proofs. Since the main emphasis throughout this book is on ineffective proofs based

on ordinary logic and hence on functional interpretation the preceding treatment of modified realizability is largely independent from the rest of this book. However, the study of modified realizability is recommended for a better understanding of the more complicated functional interpretation.

Next, a so-called monotone variant of functional interpretation due to the author is applied which combines functional interpretation with majorizability in the sense of W.A. Howard and allows one to treat the binary König's lemma and related principles such as a strong uniform boundedness principle.

The book presents in detail C. Spector's deep extension of Gödel's functional interpretation to full classical analysis by means of his bar recursive functionals.

As an alternative method to the combination of functional interpretation with negative translation one can – in some circumstances – use instead a combination of modified realizability with negative translation if one inserts the so-called A-translation (due to H. Friedman and A. Dragalin) as an intermediate step. This approach will be briefly discussed as well.

Using suitable standard representations of Polish and compact metric spaces (going back to L.E.J. Brouwer) we develop general metatheorems based on monotone functional interpretation which guarantee the extractability of effective uniform bounds from large classes of proofs in analysis. Moreover, monotone functional interpretation provides an algorithm to carry out such extractions for given proofs.

As an extended case study of the use of these metatheorems and the extraction algorithms a number of concrete proofs in approximation theory (best Chebycheff and  $L_1$ -approximation of continuous functions), where this approach has led to new results, are analyzed in great detail.

By extending the aforementioned proof interpretations to new formal systems of analysis, where general classes of abstract metric, hyperbolic,  $CAT(0)$ , normed and uniformly convex spaces are added as new types, very general metatheorems are obtained which guarantee the extractability of strongly uniform bounds which are not only independent from parameters in compact metric spaces but even from parameters in metrically bounded subspaces of such abstract spaces. Further refinement of this approach shows that it actually suffices to suppose certain local boundedness information rather than the boundedness of the whole (sub)space.

Finally, in a second extended case study, these general metatheorems are applied to proofs in metric fixed point theory leading to numerous (even qualitatively) new results which were obtained first by this methodology. These results concern, among other things, the asymptotic behavior of Krasnoselski-Mann iterates of nonexpansive (and more general classes of) functions on hyperbolic spaces. Many more applications in this area are referred to in the literature.

The book is concluded with some speculations about future applications of the methods developed in this book to proofs in the areas of algebraic number theory, partial differential equations, ergodic theory and geometric group theory.

Of course, much work on the general topic of ‘computational content of proofs’ has been carried out by logical methods other than the ones covered in this book as well as in the context of constructive foundations of mathematics (see e.g. the recent book [349] which, however, only studies logical aspects of calculi and formal systems without mathematical applications). In this book we focus on those methods which have been applied in the past to concrete proofs in different areas of ‘core mathematics’, have produced new mathematical results in these areas and are likely to be useful in other parts of mathematics as well.

A relevant topic that is beyond the scope of this book is the issue of implementing the techniques developed in suitable programming languages aiming at an automated extraction of algorithms from proofs. Much work in this direction has been done in Munich by the group around H. Schwichtenberg in connection with the MINLOG tool (see e.g. [20, 26, 21]). This work is based on modified realizability and refined versions of the  $A$ -translation but, subsequently, also various forms of functional interpretation have been successfully implemented (see e.g. [158]). Due to the enormous difficulties involved in dealing with fully formalized proofs only mathematically rather simple examples have been carried out by such tools yet. As this book focuses on advanced applications to nontrivial proofs in mathematics these implemented tools will be mentioned only briefly with references to the literature.

In addition to standard undergraduate knowledge in mathematics the book only presupposes some familiarity with the basic concepts from elementary recursion theory (up to the Kleene normal form theorem, e.g. [333]) and logic (covered in any introduction to mathematical logic such as [372]). Some previous exposure to constructive (‘intuitionistic’) logic ([372, 15, 371]) would be helpful but is not required.

Most chapters have exercises and historical comments at the end. These comments, in particular, give detailed references to the relevant literature where the results presented first appeared. In addition to this we give explicit references at key definitions and results themselves in cases where they are neither standard or folklore nor due to the author. Except for a few central results that are joint work with co-authors, we do not label our own results explicitly but refer to the ‘historical comments’ sections for proper references.

The book's web site

<http://www.mathematik.tu-darmstadt.de/~kohlenbach/prooftheory.html>

will list errata and updates.

Darmstadt,

*Ulrich Kohlenbach*

December 2007

Applied Proof Theory: Proof Interpretations and their  
Use in Mathematics

Kohlenbach, U.

2008, XX, 536 p., Hardcover

ISBN: 978-3-540-77532-4