2

Continuous-Time Dynamical Systems

The importance of continuous dynamical systems for optimal control theory is twofold. First, dynamical systems already occur in the problem formulation, in which the evolution of the states to be controlled is formulated as a differential equation. Second, and more important, the techniques for calculating and analyzing the solutions of optimal control problems, in the form in which we introduce them, profoundly rely on results provided by the theory of continuous dynamical systems. Therefore in this chapter we elaborate the theory in some detail.

To help the reader, not acquainted with dynamical systems, we first provide a historical introduction, and then present the simple case of a one-dimensional dynamical system, introducing important concepts in an informal manner. Subsequently we restate these concepts and the required theory in a rigorous way.

2.1 Nonlinear Dynamical Modeling

Since the seventeenth century dynamical modeling has been of particular importance for explaining a long list of phenomena. Sir Isaac Newton (1643–1727) invented differential and integral calculus, published afterwards in his book *Method of Fluxions*, to describe universal gravitation and the laws of motion. Gottfried Wilhelm Leibniz (1646–1716) independently developed his calculus around 1673. The notation we use today is mostly that of Leibniz, with the well-known exception of Newton's "dot-notation" representing the derivative with respect to time.

Nowadays dynamical systems theory is a flourishing interdisciplinary subject, which is applied not only in many traditional sciences such as physics, aeronautics, astronomy, chemistry, biology, medicine, and engineering, but also in economics, sociology, and psychology. But this was not always the case. Only reluctantly did the social sciences and the humanities adopt dynamical modeling. For instance, the famous economist Joseph Schumpeter

(1883–1950), felt compelled sought to convey his argument for a dynamic approach by coining the phrase that *neglecting dynamic aspects in economic modeling was like performing Shakespeare's Hamlet without the prince of Denmark*. (Additional information about the role and development of dynamics in various disciplines can be found in the Notes p. 96.)

The core attributes of most dynamical systems are nonlinearities that arise from many different sources. In the social sciences, for example, interaction between agents and their mutual interdependencies generate inherent nonlinearities. Linear systems are much simpler to analyze and thus more thoroughly explored. Nonetheless, when it comes to modeling real-world phenomena, linear systems are the exception rather than the rule. The Polish mathematician Stanislav Ulam (1909–1984) remarked that dividing systems into "linear" and "nonlinear ones" makes as much sense as dividing fauna into "elephants" and "non-elephants." Once again, "elephant problems" are often amenable to analytical solution, whereas this is not the case for most "non-elephant problems." However, there exist powerful tools for tackling nonlinear dynamical systems at least numerically. By the way, it has to be admitted that in general a most important part of the dynamics can be described by a linear approximation, indicating the importance of studying linear systems. The purpose of this section is to give an introduction to the general theory and techniques we need in the following chapters.

2.2 One-Dimensional Systems

Real world problems such as those pertaining to drugs, corruption, and terror are inherently nonlinear and inherently dynamic. The nonlinearities stem among other things from the complex interactions between the constituent states of the systems, e.g., social interactions, and they are obviously dynamical since they evolve over time. Some knowledge of the theory of nonlinear dynamical systems is therefore essential to understand what is going on in the modeled real world representations we aim to set up here. We launch this process by discussing one-dimensional continuous dynamical systems to give a first intuition of what dynamics are, how they behave, and what insights we can obtain from their analysis.

We start by considering one-dimensional autonomous[1] systems, given by the following ordinary differential equation (ODE):[2]

$$\dot{x}(t) = f(x(t)), \quad t \in [0, T] \tag{2.1a}$$

subject to (s.t.)

[1] Differential equations in which the independent variable t does not occur explicitly in the function f are called *autonomous*.

[2] For exact definition see Sect. 2.4.

$$x(0) = x_0 \in \mathbb{R}, \tag{2.1b}$$

where $\dot{x}(t)$ is the time derivative of the function $x(\cdot)$ at time t. Solving (2.1) means finding a differentiable function $x(\cdot)$ satisfying (2.1a) and the *initial condition* (2.1b). To ease the notation (if there is no ambiguity) the time argument t is often omitted and (2.1a) becomes

$$\dot{x} = f(x).$$

One-dimensional systems can exhibit only a very limited range of behaviors so that their analysis is quite simple[3] and can be done graphically. In the following sections the crucial concepts are defined rigorously, but examining one-dimensional systems graphically allows us to introduce these concepts – notably ideas related to stability, linearization, and the graphical tool known as a *phase portrait* – in an informal way that carries over to higher-dimensional systems.

To give an example, consider the following ODE:

$$\dot{x}(t) = x(t)\left(1 - x(t)^2\right), \quad t \in [0, \infty) \tag{2.2a}$$

$$\text{s.t.} \quad x(0) = x_0 \in \mathbb{R}. \tag{2.2b}$$

As the reader may verify (by simple differentiation of (2.3)), the explicit solution of (2.2) is given by

$$x(t) = \begin{cases} \left(1 - \mathrm{e}^{-2t}\left(1 - \frac{1}{x_0^2}\right)\right)^{-1/2} & x_0 > 0 \\ 0 & x_0 = 0 \\ -\left(1 - \mathrm{e}^{-2t}\left(1 - \frac{1}{x_0^2}\right)\right)^{-1/2} & x_0 < 0. \end{cases} \tag{2.3}$$

There are basically two ways to analyze the solution of an ODE like (2.2a) or the general case of (2.1). First, the time path of the solution can be (analytically or numerically) calculated (see (2.3)) and depicted in the (t, x)-plane (see Fig. 2.1b). This allows directly reading off the level of the variable at any instant of time depending, of course, on the variable's initial level. The second way is to plot a *phase portrait* in the $(x, \dot{x})$-plane (see Fig. 2.1a). To grasp the meaning of such an analysis, it is helpful to imagine that a phase portrait of a one-dimensional system depicts a particle that is starting its movement at x_0 and is traveling along the x-axis with a certain velocity (direction and speed) as given by $\dot{x}$ that in turn depends on the position of x on the axis as given by the right-hand side of equation (2.2). The particle's path represents then the solution of the ODE starting from the initial condition x_0. Since direction and

[3] Even in the one-dimensional (nonlinear) case finding explicit solutions is rather the exception than the rule. Thus the use of the term "simplicity" refers only to the limited range of system behavior, but does not refer to simplicity with respect to analytical tractability.

speed play an important role it is convenient to augment the graph of $\dot{x}$ vs. x with arrows on the x-axis to indicate the direction of movement, where the flow is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$ (as is illustrated for (2.2) in Fig. 2.1a). The associated concepts of a phase portrait analysis described so far carry over to higher-dimensional systems, where important elements are the system's equilibria. An equilibrium $\hat{x}$ is defined by

$$f(\hat{x}) = 0.$$

In our example (2.2) these are the roots of $x(1 - x^2) = 0$, yielding the three equilibria $\hat{x} = 0, \ \pm 1$.

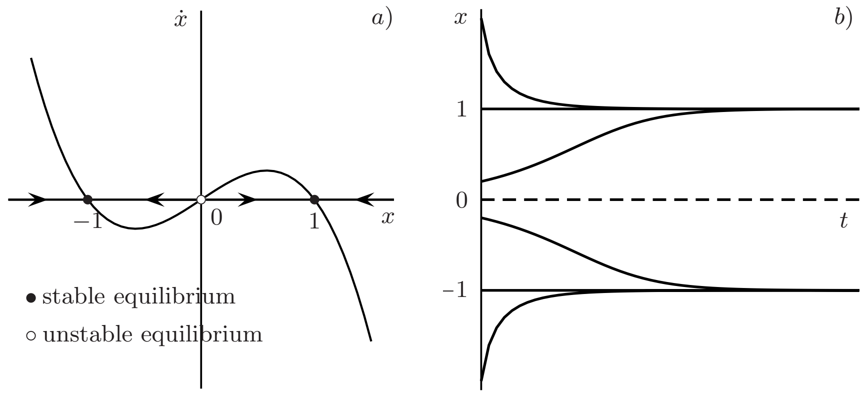

Fig. 2.1. A portrait of (2.2) is depicted in panel (**a**). Panel (**b**) shows several solution paths converging toward the equilibria together with equilibrium solutions

These equilibria represent *equilibrium solutions* (also called fixed points, stationary points, steady states, or critical values) in which the flow induced by the system $f(x(\cdot))$ stagnates. If one starts the system exactly in $x_0 = \hat{x}$, then $x(t) = \hat{x}$ for all time. It is now of interest what happens if we start the system *almost* in $\hat{x}$. We say that $\hat{x}$ is "stable" if the system remains near $\hat{x}$. I.e., trajectories beginning with small perturbations of the equilibrium remain close to the equilibrium. If those perturbed trajectories do not only remain close but eventually return to the equilibrium (even if that takes an infinite amount of time to *exactly* reach the equilibrium), then another form of stability called "asymptotic stability" pertains.

To put it more precisely, an equilibrium $\hat{x}$ is called *stable* if any solution of (2.1a) with initial condition $x(0) = x_0$ "close" to $\hat{x}$ remains in a small neighborhood of $\hat{x}$ for all $t \geq 0$. It is called *asymptotically stable* if it is stable and if there exists a small neighborhood of $\hat{x}$ such that for any solution that starts within that neighborhood it holds that $\lim_{t \to \infty} x(t) = \hat{x}$. If the "small neighborhood" in the definition of asymptotic stability can be expanded to embrace the whole domain (of definition) of $f(x(\cdot))$, then the equilibrium $\hat{x}$ is said to be *globally asymptotically stable* because all trajectories, no matter where in the phase space they start, will eventually approach $\hat{x}$.

The qualitative behavior of a one-dimensional *autonomous* system is limited, since the motion happens along a line. A solution path either approaches an equilibrium, if $f_x(\hat{x}) < 0$, or it goes away from it, where such an unstable equilibrium is characterized by $f_x(\hat{x}) > 0$; see Fig. 2.1a, where the stable equilibria are depicted by a black dot and the unstable equilibrium by an open circle. Note that asymptotically stable equilibria are called *sinks* or *attractors*, and unstable ones are called *sources* or *repellers*.

In the case of our illustrative example we find for the equilibria $0, \pm 1$ with $f_x = -3x^2 + 1$

$$f_x(\hat{x}) = \begin{cases} -2 & \hat{x} = \pm 1 \\ 1 & \hat{x} = 0. \end{cases}$$

Thus the two equilibria ± 1 are attractors, whereas the equilibrium at the origin is a repeller.

As mentioned, nonlinearities are ubiquitous, and it is the linearity (rather than nonlinearity) assumption that has to be justified. Hence, we need to justify the time we are about to spend studying linear systems. There are two primary justifications. First, since linear models are mathematically much simpler to deal with, it is easier to learn first about linear systems and then about nonlinear systems than it is to jump directly into nonlinear systems. Second, many nonlinear models can be linearized, and solutions of linearized systems are reasonable approximations of the solutions of the nonlinear systems in a neighborhood about the point where the linearization was done. In general the most interesting part of the dynamical behavior is covered by its linearization.

If the function $f(x)$ in (2.1a) is not linear in x, one can linearize it by approximating it by its first derivative about an equilibrium $\hat{x}$ as follows. Let $y = x - \hat{x}$ be a small perturbation of $\hat{x}$. By expanding $f(\hat{x} + y)$ about $\hat{x}$, one obtains

$$\dot{y} = \dot{x} = f(\hat{x} + y) = f(\hat{x}) + f_x(\hat{x})y + o(y),$$

where $o(y)$ represents quadratic and higher-order terms. Since $f(\hat{x}) = 0$, we get

$$\dot{y} = f_x(\hat{x})y \tag{2.4}$$

by neglecting higher-order terms. The linear differential equation (2.4) for y is called the *linearization* of f about $\hat{x}$. It says that the deviation y increases exponentially if $f_x(\hat{x}) > 0$ and decreases if $f_x(\hat{x}) < 0$. In the hairline case $f_x(\hat{x}) = 0$ the higher-order terms must be included in a stability analysis. Note further that $|f_x(\hat{x})|$ measures the rate of exponential growth or decay of a solution near the equilibrium $\hat{x}$.

This finishes our short introduction to one-dimensional dynamical systems. We shall now illustrate them with an example from the field of corruption.

Subsequently we shall present a thorough survey of dynamical systems in which the concepts only loosely introduced here are rigorously defined.

2.3 A One-Dimensional Corruption Model

A simple one-dimensional model of corruption suggests that there are two kinds of people, honest and corrupt. We denote them by the subscripts h and c, respectively. Let the state variable $x(t)$ be the proportion of people who are corrupt at time t, i.e., $0 \leq x(t) \leq 1$. The temptation to become corrupt is usually financial, and so the incomes of corrupt people are assumed to be higher than those who are honest, by some constant amount per unit of time: $w_c > w_h$. The primary practical (as opposed to moral) disadvantage of being corrupt is exposure to the risk of capture. The total amount of sanctions that the police can impose is spread across all corrupt people, so-called "enforcement swamping" (Kleiman, 1993). That is, the more people who are corrupt, the less likely it is that any given corrupt person will get caught.

One component of the sanctioning stems from the formal corruption control program, denoted by u, but there can also be a certain fixed amount u_0 of sanctioning even when there is no active control. That constant can represent sanction risk from general policing, as opposed to dedicated anti-corruption efforts. It might also represent social approbation that is subject to "stigma swamping" (the informal social-controls analogous to enforcement swamping).

It is common in economic models to assume that individuals respond to incentives. In these models, that often means that people move back and forth between states in response to differences in the reward or payoff offered by the various states. By this logic people would move back and forth between behaving honestly and dishonestly in response to the difference in "utility" each state offers, where utility is simply the income minus the expected sanction

$$U_c(t) - U_h(t) = w - \frac{u_0 + u(t)}{x(t)}, \tag{2.5}$$

where the constant w is the "premium" or "reward" for dishonesty, i.e., $w = w_c - w_h > 0$, and u is a "control" variable allowing the decision-maker to influence the system. At this stage the control u is assumed to be fixed; but in the following chapters the theory is introduced allowing one to choose this control optimally, relative to some cost or utility functional.

The model assumes that if being honest offers a higher utility, then corrupt people will flow back into the honest state at a per capita rate that is proportional to the difference in utility. However, if being corrupt offers a higher utility, then corrupt people will "recruit" honest people into corruption at a rate that is proportional to $U_c - U_h$, so that the aggregate rate is proportional to the product of the number of corrupt people and the utility difference. This recruitment could be unconscious, as in sociological models of the diffusion of deviance. It could also be conscious. Corrupt people have an incentive to corrupt others because the more corrupt people there are, the lower is the risk of sanction for any given corrupt person.[4] (The reader is invited to explore what that argument would imply for the model equations and solution.) An assumption underlying the model dynamics is that honest people cannot become corrupt without some contact with a corrupt person, perhaps because they do not know the difference in utility between being corrupt and being honest without such contact.[5]

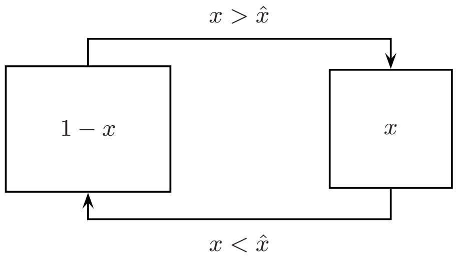

Fig. 2.2. Flow diagram for the corruption model (2.6). Here x denotes the corrupt proportion of society, whereas $1 - x$ represents the honest part. $\hat{x} = (u_0 + u)/w$ is the equilibrium

In the symmetric case, where the proportionality constants for transitions from corrupt behavior to being honest, and vice versa, are the same, $k > 0$ (see Fig. 2.2), one gets the linear state dynamics

$$\dot{x}(t) = f(x(t), u(t)) = kx(t)(U_c(t) - U_h(t)) = k\,(wx(t) - (u_0 + u(t)))\,. \quad (2.6)$$

It is instructive to first identify the equilibria. In (2.6) there is only one, given by

$$\hat{x} = \frac{u_0 + u}{w}.$$

Since $w > 0$, $\hat{x} \geq 0$, and is only zero in the unsanctioned case, where $u = u_0 = 0$.

Next, one could ask what happens in the neighborhood of $\hat{x}$. We might argue intuitively that if there were more corrupt people (larger x), that would

[4] One might also argue, to the contrary, that the more corrupt agents there are, the more competitive is the market for providing corruption “services” (compare Andvig, 1991).

[5] If we disregard this assumption, the flow from the honest to the corrupt people will be proportional to $1 - x$ (see Exercise 2.3).

reduce the sanction risk for corrupt people while having no effect on the income of corrupt or honest people; and so it would make being corrupt more appealing, spurring a further flow of people into corruption. Conversely, a small reduction in the number of corrupt people would make corruption less appealing, leading to a further flow out of corruption. Hence the system will tend to move away from $\hat{x}$, so that the equilibrium can be characterized as unstable.

If the model were more complex, and such intuitive arguments were not sufficient, we could still find the same answer mathematically by noting that the derivative of f evaluated at $\hat{x} = \hat{x}(u)$ is

$$f_x\left(\hat{x}(u), u\right) = kw > 0.$$

Thus the gap between the actual state x and the equilibrium $\hat{x}$ will always tend to increase when the state is near $\hat{x}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{x} - x\right) = f_x\left(\hat{x}(u), u\right)\left(\hat{x} - x\right).$$

Hence it is shown that $\hat{x}$ is an *unstable* equilibrium, where small perturbations lead to divergence from this state. As the movement in this simple model takes place along the x-axis, this means that left of $\hat{x}$ the proportion of corrupt people decreases, while right of it the number of corrupt people increases. We therefore have to consider different scenarios, depending on the actual value of $\hat{x}$, given by the total sanction $u_0 + u$, and the difference in the income w. If w is fixed, three scenarios are possible:

No sanctions: $u_0 + u = 0$. Only a society that is totally non-corrupt will remain honest, and even that is an unstable situation. Whenever corruption appears, $x(t) > 0$, corruption will increase exponentially until everyone is corrupt (see Fig. 2.3a).

Medium sanctions: $0 < u_0 + u < w$. Depending on the actual proportion of corrupt people, corruption will increase for $x(t) > \hat{x}$ or decrease for $x(t) < \hat{x}$. This outcome is plausible, given that the risk of sanctions decreases with the proportion leading to corruption (see (2.5) and Fig. 2.3b).

High sanctions: $u_0 + u = w$. The risk of sanctions exceeds the expected profit for any given proportion x and corrupt people become honest. We end up with a totally honest population regardless of the initial state. If $\hat{x} = 1$ the totally corrupt society is an unstable situation. A small fraction of honest people suffices to convert the whole population into one that is entirely free of corruption (see Fig. 2.3c).

With the same argumentation we get analogous results for a fixed amount of sanctions and a varying wage w, "high," "medium," and "low." Note that if sanctions are positive then with increasing wage w, we can still hope that society becomes totally honest.

In a nutshell only if sanctioning makes being corrupt sufficiently unattractive (Fig. 2.3b, c) does society have a chance to approach a state of honesty.

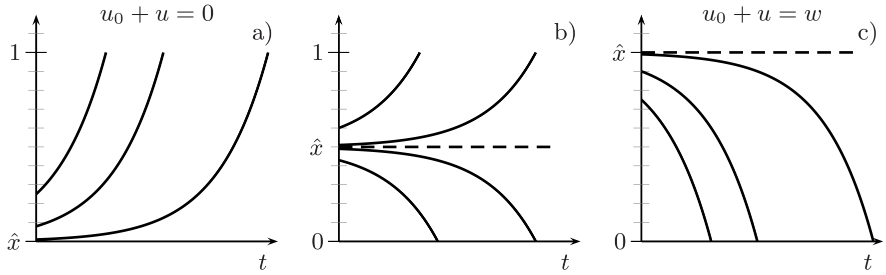

Fig. 2.3. The three cases of zero, medium, and high sanctions. Note that 0 and 1 are model inherent boundaries; thus the solutions are considered only within this range and the dynamics are set to zero otherwise

What we have already introduced informally for the one-dimensional case will now be established in a general and formal way.

2.4 Dynamical Systems as ODEs

In dynamic models usually time is an independent variable.[6] The dependent (state) variables are then functions of time. Changes in these variables over time involve the derivatives of functions with respect to time.

We let t denote the real-valued time argument, and $x(t)$ denotes the value of some variable x at time t. The derivative of $x(\cdot)$ at any moment of time t is denoted by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}, \quad \text{or} \quad \dot{x}(t).$$

To describe the process of change we must link the rate of change in $x(\cdot)$ at t to the values of $x(t)$ and t. Hence we consider equations of the form:

$$\dot{x}(t) = f\left(x(t), t\right), \quad t \in [t_0, t_1] \subset \mathbb{R} \tag{2.7a}$$

$$x(t_0) = x_0 \in \mathbb{R}^n, \tag{2.7b}$$

where $x(t)$ is the *state* of the system at time t and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a function continuous in t and continuously differentiable in x, called the *(state) dynamics* of x. Hence the rate of change, $\dot{x}(t)$, depends on the current state $x(t)$ and the time t. A solution of (2.7) is defined as:

Definition 2.1 (ODE, IVP, and Solution). *The equation (2.7a) is called an* ordinary differential equation (ODE) *and the system (2.7) is called an* initial value problem (IVP) *for the* initial condition *(2.7b).*

The function $x : [t_0, t_1] \to \mathbb{R}^n$ is called a solution *to the IVP (2.7) if $\dot{x}(t)$ satisfies (2.7a) for all $t \in [t_0, t_1]$ and $x(t_0) = x_0$.*

[6] Here we assume that time is the only independent variable.

Remark 2.2. It is common practice to assume that the dynamics f is only considered on a set $D \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$ with D open in $\mathbb{R}^n$. To simplify notation we omit the reference to the set D. In case that f is only defined on a set with D closed the solution at the boundary ∂D has to be analyzed carefully. As an example reconsider the model of Sect. 2.3, where the set $D = [0,\, 1]$ was a closed set.

An important existence and uniqueness result is stated by:

Theorem 2.3 (Picard–Lindelöf). *Let the dynamics $f(x,t)$ of the ODE (2.7a) be continuous and Lipschitz continuous in x for all $t \in [0,\, T]$; then the IVP (2.7) has a unique solution on $[0,\, T]$.*

Remark 2.4. For further questions of existence the reader is referred to any of the numerous textbooks on ODEs, e.g., Hartman (1982).

Remark 2.5. The term *ordinary* in ODE is used to indicate that there is only a single independent variable. Otherwise, partial derivatives would appear and we would have a partial differential equation; such equations are briefly outlined in Sect. 8.3.

Remark 2.6. Opposite to an IVP we may consider a so-called *boundary value problem (BVP)*. For IVPs we know exactly where we start, i.e. $x(t_0) = x_0 \in \mathbb{R}^n$ at a time t_0 is given, whereas in the latter case we only have partial information about both the initial state and the final state, i.e.,

$$b\left(x(t_0), x(t_1)\right) = 0,$$

where b is a vector-valued function $b : \mathbb{R}^n \to \mathbb{R}^n$. In the following sections we exclusively consider IVPs and return to the BVPs in Sect. 2.12.2.

An equivalent formulation for a solution of (2.7) is then the following:

Proposition 2.7 (Integral Equation). *The function $x : [t_0,\, t_1] \to \mathbb{R}^n$ is a solution of the IVP (2.7) iff $x(\cdot)$ is continuous and satisfies*

$$x(t) = x_0 + \int_{t_0}^{t} f\left(x(s), s\right)\, \mathrm{d}s, \quad \text{for all } t \in [t_0,\, t_1]. \tag{2.8}$$

The equation (2.8) is called an integral equation*.*

When the time variable is integer-valued, i.e., the change in time is only considered at discrete steps, we speak of *difference equations*

$$x(t+1) = f\left(x(t), t\right), \quad t \in \mathbb{Z}, \tag{2.9}$$

where the variable t is often replaced by the variable n, indicating that only integers are considered.

In this book we focus on continuous-time problems largely leaving aside discrete-time systems. The only reason why we mention them at all is that some results based on the stability of discrete systems play an important role in characterizing the stability behavior of periodic solutions of continuous systems (Sect. 2.6.2).[7]

Another differentiating criterion for ODEs is:

Definition 2.8 (Autonomous and Nonautonomous ODE). *The ODE (2.7a) is called an* autonomous *ODE if the dynamics does not explicitly depend on the time variable t, i.e.,*

$$\dot{x}(t) = f\left(x(t)\right), \quad x \in \mathbb{R}^n, \;\; t \in [t_0,\, t_1], \tag{2.10}$$

Remark 2.9. Note that in case of an autonomous ODE the dynamics f is only a function of the state variable x, i.e., $f : \mathbb{R}^n \to \mathbb{R}^n$.

otherwise it is called a nonautonomous *ODE.*

From the definition of an autonomous ODE we find:

Lemma 2.10 (Time-Shift Invariant Solution). *Let the ODE (2.10) be autonomous and $x(\cdot)$ be a solution of (2.10) with initial condition $x(t_0) = x_0$ then the time-shifted function*

$$y(t) := x(t + t_0), \quad \textit{with} \quad t \in [0, t_1 - t_0]$$

is a solution of the ODE (2.10) with initial condition $y(0) = x_0$.

Due to Lemma 2.10 we may restrict our considerations, without loss of generality, to the ODE (2.10) on the time interval $[0,\, T]$ and initial condition $x(0) = x_0$.

In this book we are almost solely confronted with autonomous and thus time-shift invariant problems. In the rare cases where we consider a nonautonomous problem, we shall explicitly refer the reader to this point.

2.4.1 Concepts and Definitions

Next we consider the autonomous problem (2.10) and give the following definitions:

Definition 2.11 (Trajectory, Orbit, and Phase Space). *Let $x(\cdot)$ be a solution of the autonomous ODE (2.10); then $x(\cdot)$ is called the* trajectory *or* solution path*) through x_0.*

The ordered subset[8]

[7] Much of what we present for differential equations has a close analogy for difference equations. Albeit we have to admit that difference equations can exhibit complex, like chaotic, behavior even in the one-dimensional case.

[8] The ordering $\preccurlyeq$ is given by the time argument, i.e.,

$$x(t_1) \preccurlyeq x(t_2) \quad \text{iff} \quad t_1 \leq t_2.$$

$$\mathrm{Or}(x_0) := \{x(t) \in \mathbb{R}^n : x(0) = x_0,\ t \in \mathbb{R}\} \subset \mathbb{R}^n,$$

is called the orbit *through* x_0. *The graphical representation of all orbits is called the* phase space.

Remark 2.12. Note that the orbit is the projection of a trajectory $x(\cdot)$ into the phase space while retaining its time orientation.

For an autonomous ODE we can attach to every point $x \in \mathbb{R}^n$ a vector $f(x) \in \mathbb{R}^n$ giving rise to the following definition

Definition 2.13 (Vector Field, Flow, and Dynamical System). *Given the autonomous ODE (2.10) the image of the function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is called the* vector field *of (2.10). It is said that the vector field* $f(x)$ *generates a* flow $\varphi^t : \mathbb{R}^n \to \mathbb{R}^n$, *which transforms an* initial state x_0 *into some state* $x(t) \in \mathbb{R}^n$ *at time* $t \in \mathbb{R}$

$$\varphi^t(x_0) := x(t).$$

The triple

$$\left(f, \mathbb{R}^n, \varphi^t(\cdot)\right)$$

is called a dynamical system *derived from the ODE (2.10).*

For a comprehensive introduction into the field of dynamical systems the reader is referred to Arrowsmith and Place (1990).

Remark 2.14. Note that the trajectories of the ODE (2.10) are continuously differentiable functions following the vectors tangentially (see Fig. 2.4). Due to its definition the $\varphi^t(\cdot)$ satisfies (2.10), i.e.,

$$\frac{\mathrm{d}\varphi^t(x_0)}{\mathrm{d}t} = f\left(\varphi^t(x_0)\right), \quad \text{and} \quad \varphi^0(x_0) = x_0.$$

From the definition of a flow two properties immediately follow:

- $\varphi^0 = \mathbb{I}^{(n)}$, which implies that the system does not change its state "spontaneously." This assures that the flow at time zero starts at the initial position x_0.
- $\varphi^{s+t} = \varphi^s \circ \varphi^t$, which implies that the state dynamics does not change in time. Thus the state at time $s + t$ is the same as if we first consider the state $x(s)$ at time s and then consider the state after time t when starting from $x(s)$.

The three objects "trajectory," "orbit," and "flow" are similar, but they emphasize different aspects of the same phenomenon (see Fig. 2.5).

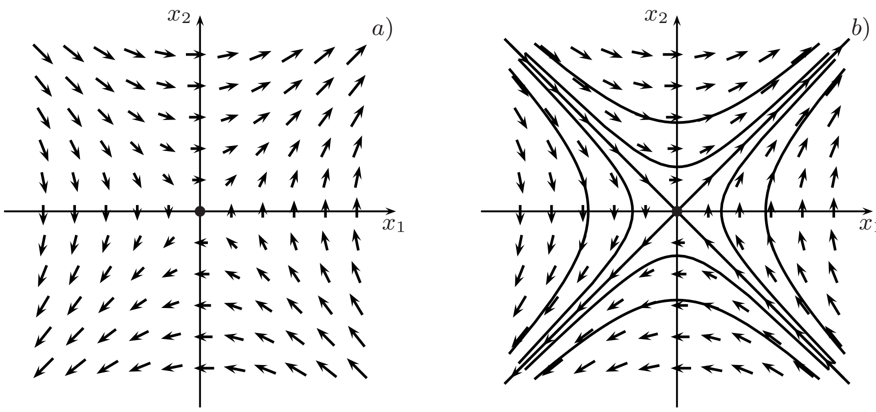

Fig. 2.4. In (**a**) the two-dimensional vector field for $\dot{x}_1 = x_2$ and $\dot{x}_2 = x_1$ is depicted, where at each point x, the corresponding directional vector $f(x) = (x_2, x_1)$ is attached. Panel (**b**) depicts some solution curves within this vector field

2.4.2 Invariant Sets and Stability

The simplest object of a dynamical system is an equilibrium:

Definition 2.15 (Equilibrium). *Let $(f, \mathbb{R}^n, \varphi^t(\cdot))$ be a dynamical system derived from the ODE (2.10) then a point $\hat{x} \in \mathbb{R}^n$ is called an* equilibrium (steady state, fixed point, critical state) *if $\varphi^t(\hat{x}) = \hat{x}$ for all $t \in \mathbb{R}$. The solution $\hat{x}(\cdot) \equiv \hat{x}$ of the ODE (2.10) is called an* equilibrium solution.

Definition 2.16 (Isocline). *Let*

$$\dot{x} = f(x), \; x \in \mathbb{R}^2,$$

with $f = (f_1, f_2)'$ and $x = (x_1, x_2)'$ be an ODE; then the contour lines

$$N_{f_1}(c) = \{(x_1, x_2) : \; f_1(x_1, x_2) = c\}, \quad \text{and} \quad N_{f_2}(c) = \{(x_1, x_2) : \; f_2(x_1, x_2) = c\}$$

are called the $\dot{x}_1$-isocline *and* $\dot{x}_2$-isocline *for value c, respectively. For $c = 0$ they are simple called the $\dot{x}_1$-isocline and $\dot{x}_2$-isocline.*

Remark 2.17. From (2.10) it follows that an equilibrium $\hat{x}$ is characterized by the equation

$$f(\hat{x}) = 0.$$

For a two-dimensional system an equilibrium is therefore given by the crossing of its zero isoclines.

Another important object of a dynamical system is a trajectory which loops forever with a certain fixed time period between successive passages through a given point:

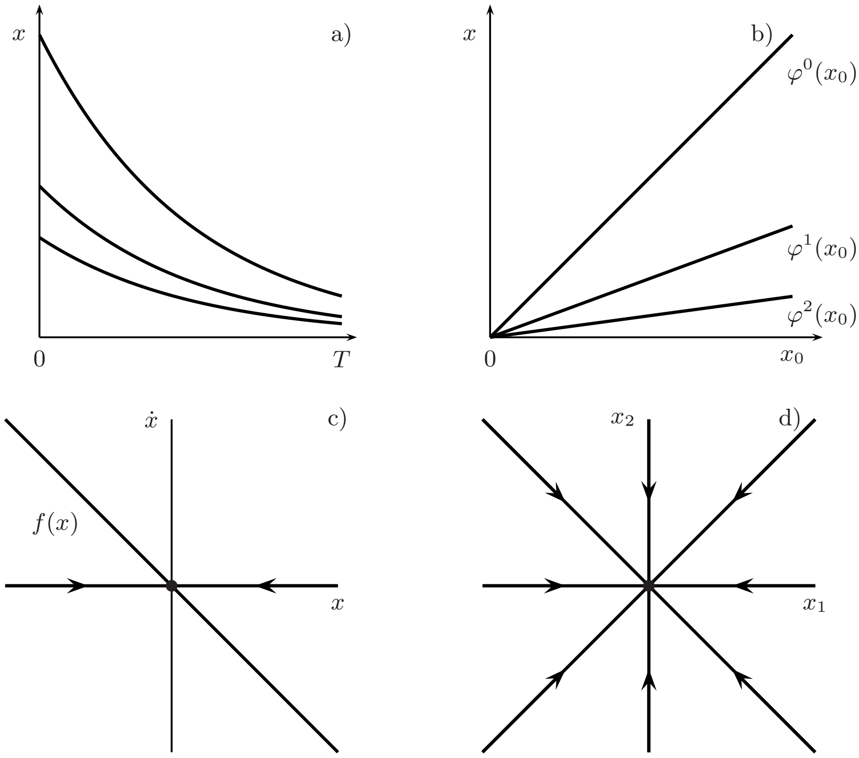

Fig. 2.5. For the linear ODE $\dot{x} = -x$ depicts (**a**) three different trajectories, (**b**) the corresponding flow $\varphi^t(x_0)$ drawn for different times t and (**c**) the corresponding phase portrait. Additionally in panel (**d**) some orbits for the two-dimensional ODE $\dot{x}_1 = -x_1$ and $\dot{x}_2 = -x_2$ are depicted

Definition 2.18 (Cycle and Periodic Solution). *Let $(f, \mathbb{R}^n, \varphi^t(\cdot))$ be a dynamical system derived from the ODE (2.10) then $\Gamma \subset \mathbb{R}^n$ is called a* cycle *if there exists a $\Theta > 0$ such that for every $x_0 \in \Gamma$, $\varphi^{t+\Theta}(x_0) = \varphi^t(x_0)$ for all t. The minimal Θ satisfying this property is called the* period *of the cycle. A solution $\Gamma(\cdot)$ of the ODE (2.10) with $\Gamma(0) \in \Gamma$ is called a* periodic solution *with* period Θ.

Remark 2.19. Note that there exist infinitely many trajectories representing a cycle Γ, since every solution $x(\cdot)$ with initial condition $x(0) \in \Gamma$ describes the orbit Γ. If we do not refer to one specific trajectory we sometimes write (if there is no ambiguity) Γ instead of $\Gamma(\cdot)$ for a trajectory representing the cycle. But the reader should be aware that Γ and $\Gamma(\cdot)$ are different objects.

Definition 2.20 (Limit Cycle). *An isolated cycle Γ is called a* limit cycle.[9]

An equilibrium and a limit cycle are each particular objects within a general class called a *limit set*:

Definition 2.21 (Limit Set). *Let φ^t be a flow of (2.10) on $\mathbb{R}^n$. Then the set*

$$\omega(x_0) = \left\{ x \in \mathbb{R}^n : \lim_{k\to\infty} \varphi^{t_k}(x_0) = x, \text{ and } \lim_{k\to\infty} t_k = \infty \text{ with } t_k < t_{k+1} \right\}$$

is called the (positive) limit set (ω-limit set) *for x_0 and*

$$\alpha(x_0) = \left\{ x \in \mathbb{R}^n : \lim_{k\to\infty} \varphi^{t_k}(x_0) = x, \text{ and } \lim_{k\to\infty} t_k = -\infty \text{ with } t_{k+1} < t_k \right\}$$

is called the negative limit set (α-limit set) *for x_0.*

A limit set describes the states that a trajectory can reach in the "long run."

Remark 2.22. For higher-dimensional systems, $n > 2$, more complex solutions may occur where, e.g., the trajectories exhibit *aperiodic* behavior; which means that although the trajectories are bounded, they do not converge to equilibria or limit cycles. Additionally, in *chaotic* systems the solutions can exhibit sensitive dependence on the initial conditions.

Both, equilibrium and cycle, exhibit another important property, namely that of invariance, i.e., starting at an equilibrium or cycle, the trajectory remains at the equilibrium or cycle forever. This property can generally be defined as follows:

Definition 2.23. *An* invariant set *of a dynamical system is a subset* $\mathrm{S} \subset \mathbb{R}^n$ *such that $x_0 \in \mathrm{S}$ implies*

$$\varphi^t(x_0) \in \mathrm{S}, \quad \text{for all} \quad t.$$

Thus Definition 2.23 formalizes the property of "remaining somewhere forever."

Remark 2.24. Other invariant sets are like the stable and unstable manifold of an equilibrium or a heteroclinic/homoclinic connection and will subsequently be defined.

For an invariant set S the concept of stability can be introduced:

Definition 2.25. *An invariant set* S *of a dynamical system is called*

1. Stable *if for every neighborhood U of* S*, which is sufficiently small, there exists a neighborhood V of* S*, such that $x_0 \in V$ implies $\varphi^t(x_0) \in U$ for all $t \in \mathbb{R}$.*

[9] *Isolated* denotes that there is no other cycle in its neighborhood; for a counter example see Fig. 2.6a, where through any point $x \in \mathbb{R}^2$ there passes a cycle.

2. Attracting *if there exists a neighborhood V of* S *such that $x_0 \in V$ implies*

$$\lim_{t\to\infty} \varphi^t(x_0) \in \mathrm{S}\,.$$

The set $\bigcup_{t\leq 0} \varphi^t(V)$ is called the basin of attraction
3. Repelling *if it is attracting for the reversed time $t \to -t$.*
4. Unstable *if it is not stable.*
5. Asymptotically stable *if it is both stable and attracting.*
6. Globally (asymptotically) stable *if the neighborhood V of attraction for* S *can be chosen as $\mathbb{R}^n$.*

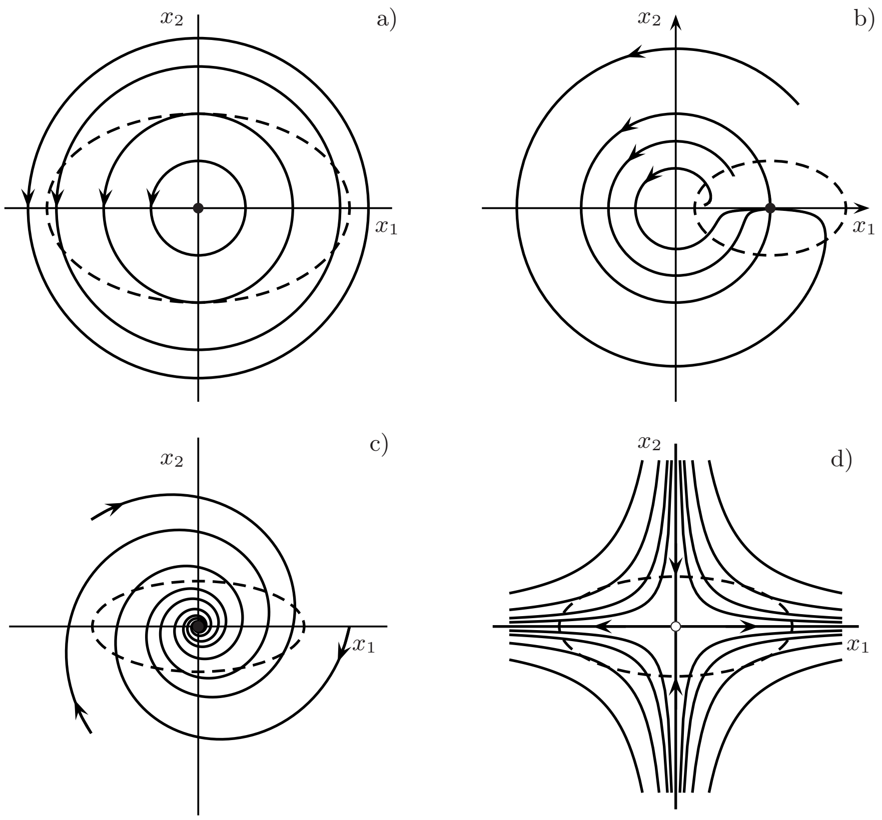

Fig. 2.6. In (**a**) the orbits around the equilibrium at the origin are cycles; thus it is a stable but not attracting equilibrium also called a *center*. In (**b**) every orbit converges to the equilibrium at $(1, 0)$ and is therefore attracting, but not stable since for every neighborhood V of $(1, 0)$ there exist orbits leaving V. In (**c**) an asymptotically stable equilibrium is depicted, and (**d**) shows an example of an unstable equilibrium

Remark 2.26 (Semistability). An equilibrium $\hat{x}$ of a one-dimensional system is called a *semistable* equilibrium if there exist $x' < \hat{x} < x''$ such that

$$\hat{x} = \omega(x') = \alpha(x'') \quad \text{or} \quad \hat{x} = \omega(x'') = \alpha(x').$$

Such an equilibrium may occur if a stable and an unstable equilibrium collide (see, e.g., p. 64).

A limit cycle Γ of a two-dimensional system is called a *semistable* limit cycle if there exist x' and x'' such that

$$\Gamma = \omega(x') = \alpha(x'') \quad \text{or} \quad \Gamma = \omega(x'') = \alpha(x').$$

Such a limit cycle may occur if a stable and an unstable limit cycle collide.

Remark 2.27. In particular, an attracting equilibrium $\hat{x}$ is called an *attractor* and a repelling equilibrium is called a *repeller*.

Remark 2.28. Note that stability and attraction are distinct concepts. A center in the linear case (discussed below on p. 39) is stable but not attracting. On the other hand, there are examples of autonomous ODEs exhibiting attracting but not stable limit sets (see Fig. 2.6).

2.4.3 Structural Stability

In the last section we introduced the concept of stability for invariant sets. This subsection answers the question of how sensitive the solution is in the long run to small disturbances of the initial states near an invariant set. But especially from an applied point of view another kind of stability is interesting. Namely to consider changes not only for the initial states but for the whole vector field. It is the idea of finding how robust the solution of a model is when facing *small perturbations* of its describing functions.

We therefore define:

Definition 2.29 (Topological Equivalence). *Let f_1, $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable functions defining the flows φ_1^t and φ_2^t. Then f_1 and f_2 are called* topologically equivalent *if there exists a C^0-homeomorphism h mapping φ_1^t onto φ_2^t, i.e.,*

$$h(\varphi_1^t(x)) = \varphi_2^t(h(x)).$$

This is a very strong definition of equivalence since it requires that systems with, e.g., different time parameterization are *not* topological equivalent. Thus to get a more coarse classification of systems we introduce a less strict definition of equivalence:

Definition 2.30 (Topological Orbital Equivalence). *Let f_1, $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable functions defining the flows φ_1^t and φ_2^t. Then*

f_1 and f_2 are called topologically orbitally equivalent *if there exists a C^0-homeomorphism h mapping the orbits of φ_1^t onto the orbits of φ_2^t, preserving the direction of time, i.e.,*

$$h(\varphi_1^t(x)) = \varphi_2^{\tau_y(t)}(y), \quad \textit{with} \quad y = h(x)$$

and $\tau_y(t)$ increasing in t.

We are now able to give an exact definition for the term structural stability:

Definition 2.31 (Structural Stability). *Let*

$$\dot{x} = f_1(x), \quad x \in \mathbb{R}^n, \ \ f_1 \in C^1 \tag{2.11}$$

define a dynamical system. Then system (2.11) is called structurally stable *in the region $U \subset \mathbb{R}^n$ if any system*

$$\dot{x} = f_2(x), \quad x \in \mathbb{R}^n, \ \ f_2 \in C^1$$

which is close to f_1 relative to the C^1 norm is topologically orbitally equivalent in U to the system (2.11).

In a local form this definition can be rewritten as:

Definition 2.32. *A dynamical system $(f^1, \mathbb{R}^n, \varphi_1^t)$ is called* locally topologically *equivalent* near an equilibrium $\hat{x}_1$ *to a dynamical system $(f^2, \mathbb{R}^n, \varphi_2^t)$* near an equilibrium $\hat{x}_2$ *if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ that*

1. *Is defined in a small neighborhood $U \subset \mathbb{R}^n$ of $\hat{x}_1$*
2. *Satisfies $\hat{x}_2 = h(\hat{x}_1)$*
3. *Maps orbits of the first system in U onto orbits of the second system in $V = f_1(U) \subset \mathbb{R}^n$, preserving the direction of time*

For a detailed presentation of structural stability and interrelated questions the reader is referred to Arrowsmith and Place (1990) and Arnold (1983).

2.4.4 Linearization and the Variational Equation

Having found some specific solution $x(\cdot)$ of (2.10), e.g., an equilibrium or limit cycle, we are interested in the behavior of solutions starting near this solution. Therefore the following differential equation can be considered:

$$\dot{y}(t) = f\left(x(t) + y(t)\right) - f\left(x(t)\right), \tag{2.12}$$

where $y(t)$ "measures" the deviation from $x(t)$ at time t and (2.12) describes how the perturbation evolves over time. Equation (2.12) exhibits the special solution $y \equiv 0$ characterizing the undisturbed case. Drawing conclusions about the nearby trajectories from the solutions of (2.12) is straightforward.

An obvious simplification of (2.12) is its linear approximation by the Taylor theorem, according to which (2.12) is rewritten as

$$\dot{y}(t) = f_x(x(t))\, y(t) + o(y(t)) . \tag{2.13}$$

Neglecting the higher-order terms in (2.13) yields the so-called *variational equation of solution* $x(\cdot)$:

$$\dot{y}(t) = f_x(x(t))\, y(t),$$

which in general is a linear nonautonomous system with $J(t) = f_x(x(t))$, where the Jacobian matrix $J(\cdot)$ of $f(\cdot)$ is evaluated along the trajectory $x(\cdot)$.

2.5 Stability Analysis of a One-Dimensional Terror Model

We introduce a simple, one-state model that aims to shed light on the question of how best to prosecute the "war on terror." Counter-terror measures may range from homeland security operations to invading certain territories to targeted assassinations of terrorists, or freezing assets of organizations with links to potential terrorists. Some types of counter-terror operations have more, and others have fewer, unwanted "side-effects." As a proxy for the heterogeneity of counter-terror activities, the model here distinguishes between two abstract categories:

"Fire strategies" include killing terrorists via aerial bombing of residential neighborhoods, aggressively searching all people passing through a checkpoint or roadblock, or other tactics that involve significant collateral damage or inconvenience to innocent third parties (generating resentment among the populations from which terrorist organizations seek recruits). Thus "fire," denoted by v, may have the direct benefit of eliminating current terrorists but the undesirable indirect effect of stimulating recruitment rates (see Keohane & Zeckhauser, 2003).

"Water strategies," on the other hand, are intelligence-driven arrests or "surgical" operations against individuals who are known with very high confidence to be guilty and which do not harm innocent parties. Thus "water," denoted by u, "plays by the rules" in the eyes of the general population and does not fuel (additional) recruitment, but u is more expensive and more difficult to be applied than v. So both controls have their advantages, and both have their drawbacks. We cannot generally claim that one control measure is "better than the other one." The best mix of these two strategies to employ depends on the current state of the problem and the current strength of the terrorist organization.

The strength of the terrorist organization at time t is reflected by a state variable, $S(t)$. Most simply, $S(t)$ may be thought of as the number of active terrorists, but more generally it could also reflect the organization's total resources including financial capital, weapons, and technological know-how, as in Keohane and Zeckhauser (2003).

The strength (or size) of terrorist organizations changes over time. What makes the organization stronger (or weaker)? In modeling criminal enterprises a small part of recruitment (initiation) is usually scripted, represented by a constant term τ, but most is treated as an increasing function of the current state (Caulkins, 2005). Following the lead of other authors (e.g., Castillo-Chavez & Song, 2003) who have applied this reasoning to the recruitment of terrorists, we also adopt this approach. This means that new terrorists are recruited by existing terrorists. Accordingly, when an organization is small, the growth rate ought to be increasing approximately in proportion to the current number of terrorists, say kS with $k > 0$. Yet, growth is not exponential and unbounded. At some point, growth per current member should slow down because of limits on the number of potential recruits, limits on the capacity of the organization to train and absorb new recruits, etc. The degree of slowing down is modeled by the parameter $\alpha \leq 1$. Moreover, the aggressive use of "fire," v, also increases recruitment. A functional form modeling this behavior is

$$I(S, v) = \tau + (1 + \rho v)kS^{\alpha},$$

with $\tau, \rho \geq 0, k > 0$ and $0 \leq \alpha \leq 1$. See also (2.14) below.

We distinguish between three types of outflows from the stock of terrorists. The first represents people who leave the organization, die in suicide attacks, or fall victim to routine or background enforcement efforts that are in addition to those the decision-maker in this problem is actively managing. This outflow, $O_1(S)$, is a function of only S and is represented by μS, with $\mu > 0$. Two additional outflows, $O_2(S, u)$ and $O_3(S, v)$, reflect the consequences of water-type and fire-type operations, respectively. The more aggressive the terror control, the greater the number of terrorists eliminated. The outflow due to water-type strategies is presumed to be concave in S because actionable intelligence is limited and heterogeneous, $O_2(S, u) = \beta(u) f_1(S)$. There may also be diminishing returns if more specialized skills are needed, and so there is a limited number of units that conduct water operations. In particular, concavity is modeled by a power function, $f_1(S) = S^{\theta}$ with $\theta \leq 1$. In contrast, the outflow due to fire strategies, $O_3(S, v) = \gamma(v) f_2(S)$, is modeled as being linear in S because the methods are perceived to be "shotgun" or "undirected" methods and hence $f_2(S) = S$. The more targets there are (i.e., the larger S is), the greater the number of terrorists that will be hit.

We use the same concave functional form for both $\beta(u) = \beta \ln(1 + u)$ and $\gamma(v) = \gamma \ln(1+v)$ so that any differences in the way water and fire type operations are employed stem from differences in their character, not from arbitrary decisions concerning functional forms. It turns out that a logarithmic function constitutes a convenient choice for the purpose of our analysis. Both functions include a leading constant, representing the efficiency of the operations. Since water-type operations are more expensive in the sense that for any given level of spending z, $\beta(z) < \gamma(z)$, the constant for water-type interventions $\beta > 0$ is smaller than the corresponding constant $\gamma > 0$ for fire-type operations.

Combining the above features, the "terror dynamics" in this model can be written as

$$\dot{S} = \tau + (1 + \rho v)kS^{\alpha} - \mu\, S - \beta\, \ln(1+u)\, S^{\theta} - \gamma\, \ln(1+v)\, S, \tag{2.14}$$

with τ, ρ, k, α as before and μ, β, $\gamma > 0$, $\beta < \gamma$, $0 \le \theta \le 1$.

For convenience we normalize the units used to measure the size of the terrorist organization to one. Specifically, the normalization is such that in the absence of baseline recruitment ($\tau = 0$) and counter-terror measures ($u = v = 0$) the long-run size of a terrorist organization equals one. Thus

$$\dot{S} = kS^{\alpha} - \mu S,$$

yields $k = \mu$ for $\dot{S} = 0$ at $S = 1$.

We formulate the model presented so far bearing in mind the optimal control framework, which is the main issue of this book. Thus the two operations of fire and water are assumed to be dependent on time, i.e., $u(t)$, $v(t)$, allowing a situational reaction in the form of anti-terror measures that depends on the actual state. Then $u(t)$ and $v(t)$ should be feedbacks $u(t, S)$ and $v(t, S)$, respectively (see Exercise 7.2).

As a preliminary step and to obtain a first insight into the model's complexity, one may consider the model where the controls are fixed, i.e., $u(\cdot) \equiv u$, $v(\cdot) \equiv v$. Reformulating (2.14) in that way, one can try to determine its dynamical behavior, including the computation of the equilibria of the system. Unfortunately it is not possible to find closed-form solutions of the equilibria for arbitrary parameter values of α and θ. Hence we shall specify these values as $\alpha = 1$ and $\theta = 1/2$. Together with the assumption $k = \mu$, this finally yields

$$\dot{S} = \tau + \mu\rho v S - \beta\, \ln(1+u)\, \sqrt{S} - \gamma\, \ln(1+v)\, S. \tag{2.15}$$

To find the equilibria of (2.15) we therefore have to solve the equation

$$(\mu\rho v - \gamma\, \ln(1+v))\, S - \beta\, \ln(1+u)\, \sqrt{S} + \tau = 0,$$

which for $\mu\rho v - \gamma\, \ln(1+v) \neq 0$ becomes a quadratic equation in $\sqrt{S}$ with the following solutions

$$\begin{aligned}\sqrt{\hat{S}_{1,2}} &= \frac{\beta\, \ln(1+u) \pm \sqrt{\beta^2 \ln^2(1+u) - 4\tau\,(\mu\rho v - \gamma\, \ln(1+v))}}{2\,(\mu\rho v - \gamma\, \ln(1+v))} \\ &= \frac{\beta\, \ln(1+u)}{2\,(\mu\rho v - \gamma\, \ln(1+v))}\left(1 \pm \sqrt{1-\Omega}\right),\end{aligned}$$

where

$$\Omega := \frac{4\tau\left(\mu\rho v - \gamma \ln(1+v)\right)}{\beta^2 \ln^2(1+u)}.$$

If $\mu\rho v - \gamma \ln(1+v) = 0$ the equilibrium is given by

$$\hat{S}_1 = \left(\frac{\tau}{\beta \ln(1+u)}\right)^2.$$

To determine the stability of the equilibria we have to consider the Jacobian J of the state dynamics, evaluated at the equilibrium, given by

$$J(\hat{S}) = \frac{\mathrm{d}\dot{S}(\hat{S})}{\mathrm{d}S} = \mu\rho v - \gamma \ln(1+v) - \frac{\beta \ln(1+u)}{2\sqrt{\hat{S}}},$$

therefore yielding

$$J(\hat{S}) = \begin{cases} -\dfrac{\beta^2 \ln^2(1+u)}{2\tau} & \mu\rho v - \gamma \ln(1+v) = 0 \\ (\mu\rho v - \gamma \ln(1+v))\,\dfrac{\pm\sqrt{1-\Omega}}{1 \pm \sqrt{1-\Omega}} & \mu\rho v - \gamma \ln(1+v) \neq 0. \end{cases}$$

This provides the following stability properties of the equilibria, depending on the size and sign of Ω.

Region	$\hat{S}_1$	$\hat{S}_2$
$\Omega \leq 0$	Asymptotically stable	–
$0 < \Omega < 1$	Repelling	Asymptotically stable
$\Omega = 1$	Semistable	Semistable
$\Omega > 1$	–	–

In the region $\Omega \in (0,\,1)$, where both equilibria exist $\hat{S}_1 > \hat{S}_2$, whereas for $\Omega = 1$ both equilibria coincide to a semistable equilibrium. For an interpretation of the equilibria in terms of the counter-terror measures see Exercise 2.2.

For the above cases, $\Omega < 0$ and $0 < \Omega < 1$, the equilibria and their first derivatives depend continuously on the parameter values. Thus it can easily be argued that the signs of the derivatives do not change for slight parameter changes, a property that will later be called *hyperbolic* (see Definition 2.42). Therefore these stability and existence results will also hold true for parameter values α and θ near 1 and 1/2 (see also Fig. 2.7, where the solution is depicted for the set of parameter values specified in Table 2.1).

2.6 ODEs in Higher Dimensions

We now proceed to examine the special cases of linear and nonlinear autonomous ODEs in more detail.

Table 2.1. Parameter values specified for the numerical calculation of the ODE (2.15)

β	μ	γ	ρ	τ	u	v
1.00	3.00	1.00	0.10	0.12	1.00	10.00

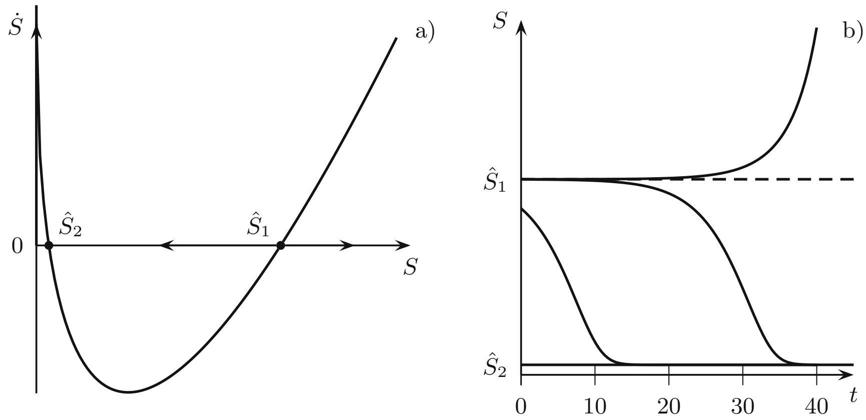

Fig. 2.7. In (**a**) the phase portrait of the ODE (2.15) is depicted and in (**b**) some solution trajectories are shown

2.6.1 Autonomous Linear ODEs

In the simplest case, the dynamics $f(x)$ is given by a linear function in $x \in \mathbb{R}^n$, i.e.,

$$\dot{x} = Jx, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{2.16}$$

where J is an $n \times n$ matrix. From vector space theory we know that in the nonsingular case, i.e., where J is invertible, $\hat{x} = 0$ is the only equilibrium.

In the one-dimensional case, where $J \in \mathbb{R}$ is a scalar, one can see by inspection that the solution of (2.16) is

$$x(t) = \mathrm{e}^{Jt} x_0, \quad t \in \mathbb{R}.$$

Thus $x(\cdot)$ is an exponential function exhibiting one of three possible structural behaviors, depending on the sign of the scalar J:

$J > 0$: The origin is a globally repelling equilibrium.
$J < 0$: The origin is a globally attracting equilibrium.
$J = 0$: The degenerate case, in which every point x_0 is a (stable) equilibrium.

In analogy to the one-dimensional case, the explicit solution of (2.16) for the higher-dimensional case in which J is an $n \times n$ matrix can be written as

$$x(t) = \mathrm{e}^{Jt} x_0, \quad t \in \mathbb{R}, \tag{2.17}$$

where the exponential function of a matrix e^{Jt} is defined as the infinite sum

$$\mathrm{e}^{Jt} := \mathbb{I}^{(n)} + Jt + \frac{J^2}{2!}t^2 + \frac{J^3}{3!}t^3 + \cdots . \tag{2.18}$$

In Exercise 2.7 the reader is invited to prove that (2.18) satisfies (2.16). Note that (2.17) gives an explicit formula for the flow φ^t of (2.16) as a linear mapping. Although (2.17) is the exact and explicit solution, however, it is not more than a formality if we cannot explicitly calculate the power functions J^n. Using the identity

$$\mathrm{e}^{Q^{-1}JQ} = Q^{-1}\mathrm{e}^{J}Q,$$

where $Q \in \mathbb{R}^{n\times n}$ is invertible, creates the possibility of transforming the matrix J into its Jordan normal form (see Theorem A.56) for which (2.17) can explicitly be calculated. The following theorem can then be proved (see, e.g., Hirsch & Smale, 1974):

Theorem 2.33. *Let J be an $n \times n$ matrix and $x(\cdot)$ a solution of (2.16). Then $x(\cdot)$ is a linear combination of functions of the form*

$$at^k \mathrm{e}^{\alpha t}\cos(\beta t),\; bt^l \mathrm{e}^{\alpha t}\sin(\beta t), \quad a, b \in \mathbb{R},$$

where $\xi = \alpha + \mathrm{i}\beta$ runs through all eigenvalues of J, with $\beta \geq 0$ and

$$k, l \leq m(\xi) - 1.$$

Here $m(\xi)$ denotes the algebraic multiplicity of the eigenvalue ξ.

Remark 2.34. Note that Theorem 2.33 delivers the possibility of calculating the solution of the linear ODE (2.16) without having to resort to the matrix Q and therefore to the Jordan normal form. Thus the eigenvalues and their properties have to be determined and the actual solution is then calculated by comparison of coefficients.

Remark 2.35 (Inhomogeneous ODE). The ODE

$$\dot{x} = Jx + C, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{2.19}$$

with $J, C \in \mathbb{R}^{n\times n}$ and $C \neq 0$, is called an *autonomous*[10] *inhomogeneous linear ODE* in contrast to (2.16), which is then called an autonomous *homogeneous* linear ODE. Provided J is invertible the inhomogeneous ODE (2.19) can uniquely[11] be transformed to the homogenous ODE (2.16) by setting

$$y := x + J^{-1}C,$$

which corresponds to a shift of the (unique) equilibrium at the origin to $-J^{-1}C$. Thus all results for the homogeneous problem have an immediate counterpart for the inhomogeneous problem.

[10] In the nonautonomous case, i.e., $C = C(t)$, a solution of the ODE (2.19) can be found by the so-called *variation of constants method*, see Exercise 2.20.

[11] Otherwise, if J is not invertible, the uniqueness of the transformation gets lost.

The main purpose of the following parenthesis is to help the reader better understand the analogous considerations for the linear nonautonomous case in Sect. 2.12.1.

The Superposition Principle, Fundamental Matrix, and Wronski Determinant

The *principle of superposition* states that for linear systems the linear combination of solutions $x_i(t) \in \mathbb{R}^n$ of (2.16), i.e.,

$$x(t) = \sum_{i=1}^{n} \gamma_i x_i(t) = X(t)\gamma$$

with

$$X(t) = (x_1(t), \cdots, x_n(t)), \quad \text{and} \quad \gamma = (\gamma_1, \ldots, \gamma_n)', \tag{2.20}$$

is a solution of (2.16) satisfying the initial condition

$$x_0 = X(0)\gamma = \sum_{i=1}^{n} \gamma_i x_i(0).$$

The reader is invited to prove the superposition principle in Exercise 2.8.

Let us restate these considerations with a slightly different perception. Assuming that we know n solutions $x_i(t)$ of (2.16); the matrix $X(t)$ defined in (2.20) satisfies the matrix ODE:[12]

$$\begin{aligned} \dot{X}(t) &= JX(t) \\ \text{s.t.} \quad X(0) &= (x_1(0), \ldots, x_n(0)). \end{aligned} \tag{2.21}$$

If $X(0)$ has full rank, i.e., $(x_1(0), \ldots, x_n(0))$ are linearly independent, then the following proposition is true:

Proposition 2.36 (Full Rank Condition). *Let $X(t)$ satisfy the matrix ODE (2.21). If $X(0)$ has full rank, then $X(t)$ has full rank for all t.*

In Exercise 2.9 the reader is invited to prove this proposition. Proposition 2.36 justifies the notation of a linearly independent solution matrix $X(t)$, since this property is verified either at every time point t or at none. We call such a linearly independent matrix solution $X(t)$ a *fundamental solution* of (2.21). Especially distinguished is the fundamental matrix satisfying $X(0) = \mathbb{I}^{(n)}$, called the *principal matrix solution* of (2.21). Every fundamental matrix solution $Y(t)$ can be transformed into a principal matrix solution defining

$$X(t) = Y(t)Y^{-1}(0),$$

[12] The matrix ODE is understood in a coordinatewise sense. Thus every column of the matrix $X(t)$ is a solution of (2.16).

where the inverse matrix exists due to the full-rank condition.

Finally, we can specify the explicit solution form of a principal matrix solution $X(t)$ by using (2.17) as

$$X(t) = \mathrm{e}^{Jt} X(0) = \mathrm{e}^{Jt} \tag{2.22}$$

since $X(0) = \mathbb{I}^{(n)}$.

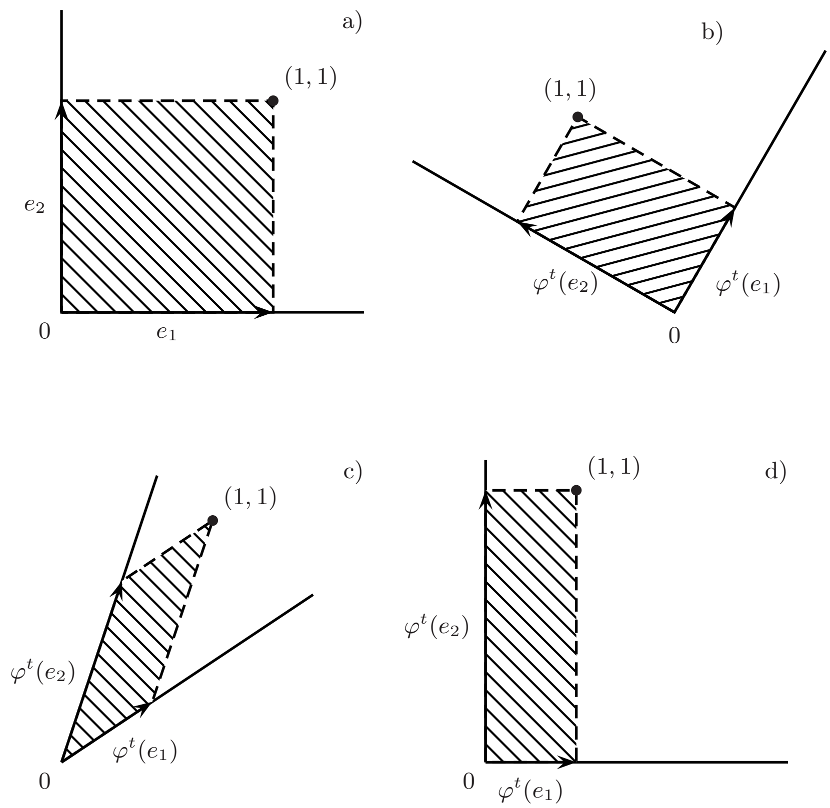

Fig. 2.8. The time evolution of a fundamental matrix starting at the generic basis e_1 and e_2 of $\mathbb{R}^2$, is depicted for a linear system – for time zero in (**a**), and at time t for a stable focus (rotation and shrinking of the axis) in (**b**), for a stable node (squeezing and shrinking of the axis) in (**c**), and for a saddle (stretching and shrinking of the axis) at the origin in (**d**)

Remark 2.37. What is the geometric meaning of fundamental matrix solutions? Imagine the columns of the matrix $X(0)$ as basis vectors of $\mathbb{R}^n$ attached to the origin. The evolution of $X(t)$ characterizes the deformation of the coordinate system under the dynamics (2.16). Hence a point represented by the coordinates x_0 at time $t = 0$ moves in space according to the coordinate systems change. If, e.g., the origin is a stable equilibrium then the time-varying coordinate system shrinks continuously and a point described by the coordinates $x_0 \neq 0$ approaches the origin when following the dynamics (see Fig. 2.8).

Remark 2.38. Pushing this geometric approach further we consider the time-varying volume of the parallelepiped given by the column vectors of the fundamental matrix $X(t)$ (see the shadowed area in Fig. 2.8)

$$\operatorname{vol} X(t) = \det X(t) = \det \left(\mathrm{e}^{Jt} X(0)\right). \tag{2.23}$$

Using (A.5), we can restate (2.23) as

$$\det X(t) = \det X(0)\, \mathrm{e}^{\operatorname{tr} Jt}.$$

Defining, for every fundamental solution matrix $X(t)$, the *Wronski determinant*

$$W(t) := \det X(t),$$

therefore yields

$$W(t) = W(0)\mathrm{e}^{\operatorname{tr} Jt}. \tag{2.24}$$

Now we turn to the important case of a *two-dimensional* or *planar* system.

Planar Linear Systems

In the two-dimensional case, the Jacobian J is a 2×2 matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the differential equation can be written in coordinate form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}. \tag{2.25}$$

To apply Theorem 2.33, one has to determine the eigenvalues of matrix J, which are given by

$$\xi_{1,2} = \frac{\operatorname{tr} J \pm \sqrt{(\operatorname{tr} J)^2 - 4 \det J}}{2}, \tag{2.26}$$

where

$$\operatorname{tr} J = a + d, \quad \text{and} \quad \det J = ad - bc.$$

Note that the eigenvalues, as the solution of the quadratic characteristic equation, can have two real or a pair of conjugate complex solutions. If the roots of the characteristic equation

$$\xi^2 - \xi \operatorname{tr} J + \det J = 0$$

are distinct, the corresponding eigenvectors ν_1 and ν_2 are linearly independent. Any initial condition x_0 can then be written as a linear combination of ν_1 and ν_2, say

$$x_0 = \gamma_1 \nu_1 + \gamma_2 \nu_2,$$

and the general solution of (2.26) is

$$x(t) = \gamma_1 \mathrm{e}^{\xi_1 t} \nu_1 + \gamma_2 \mathrm{e}^{\xi_2 t} \nu_2.$$

Let us first assume that neither eigenvalue is zero. This implies that J is nonsingular and that the origin is the unique equilibrium. To simplify the analysis, note that for any 2×2 real matrix J, a real matrix Q exists such that $Q^{-1} J Q = A$, where A can take only one of the following four types of Jordan normal forms:

$$A = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \qquad A = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix},$$

$$A = \begin{pmatrix} \xi & 0 \\ 1 & \xi \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where ξ_1 and ξ_2 are distinct real roots, ξ is a double root, with corresponding one- or two-dimensional eigenspace, respectively, and $\alpha \pm \mathrm{i}\beta$ are conjugate complex roots of the characteristic equation for J and A. The linear transformation Q^{-1} defines a new variable $y = Q^{-1}x$. Thus $\dot{x} = Q\dot{y}$, and (2.25) becomes

$$\dot{y} = Ay. \tag{2.27}$$

The solutions of (2.27) and (2.16) are qualitatively the same, and so we describe the qualitative properties of (2.27). The key observation is that there are, in a sense, only six types of possible solutions to system (2.27) and therefore also to (2.25). Furthermore, the names and intuition of those *six types* carry over to higher-dimensional systems.

Phase Diagram Representation of the Solution

Recall that the origin is the only equilibrium point of (2.27) under our assumption that $\det J \neq 0$ (i.e., there are no zero roots). Throughout, we denote the initial point by (y_1^0, y_2^0) at $t = 0$ and assume that it is not the origin. (Since the origin is an equilibrium, if the initial point is at the origin the system will always remain at the origin.) In the following we briefly present the different types of solutions arranged by the importance for the solution of optimal control models, especially for models we analyze in this book.

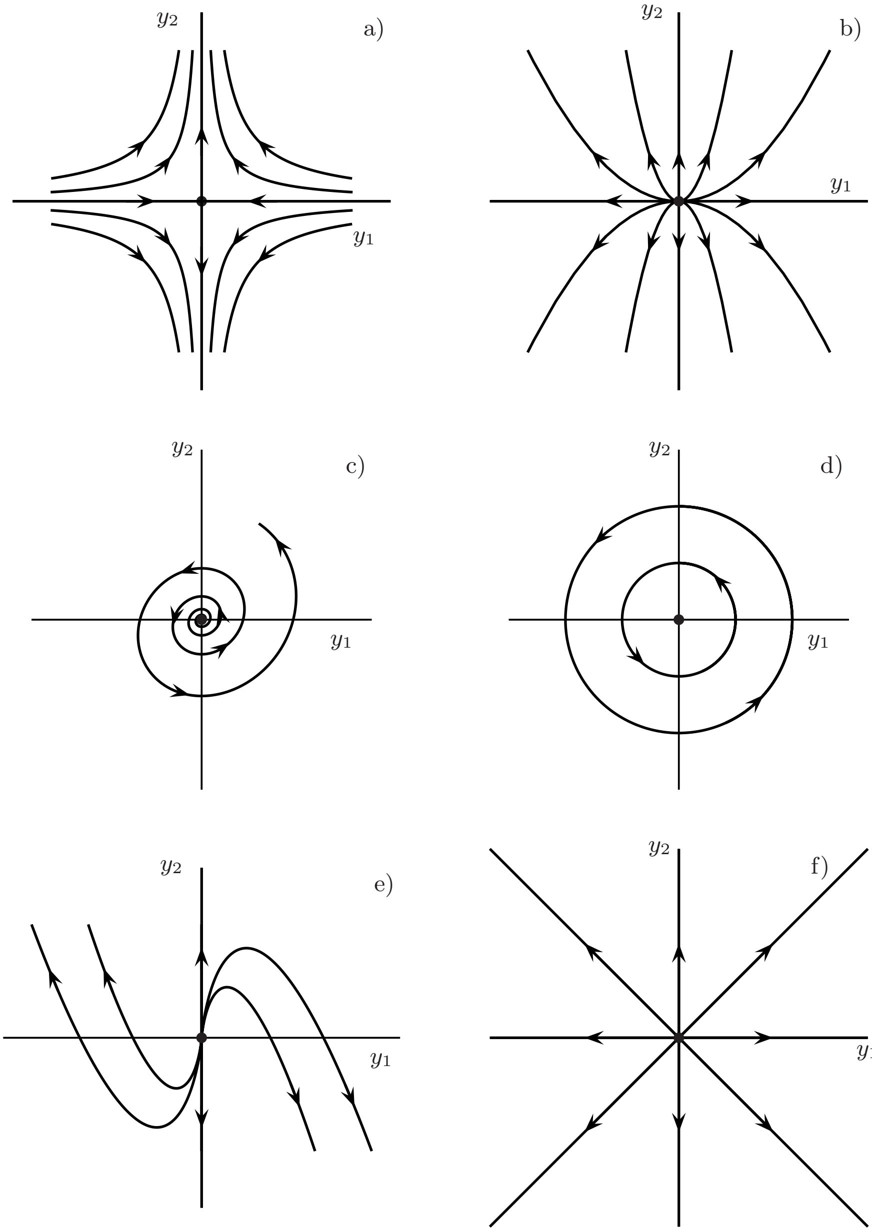

Fig. 2.9. Phase portraits for planar (linear) systems of ODEs, with (**a**) $\xi_1 < 0 < \xi_2$; (**b**) $\xi_1 > \xi_2 > 0$; (**c**) $\alpha > 0$, $\beta > 0$; (**d**) $\alpha = 0$, $\beta > 0$; (**e**) $\xi > 0$, $\mathrm{rk}(A - \xi\mathbb{I}^{(2)}) = 1$; (**f**) $\xi > 0$, $\mathrm{rk}(A - \xi\mathbb{I}^{(2)}) = 0$

a) *Saddle point*:

$$A = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad \text{and} \quad \operatorname{sgn} \xi_1 \neq \operatorname{sgn} \xi_2.$$

Thus the roots of the characteristic equation are real, and distinct. Assume $\xi_1 < 0 < \xi_2$, then there exists a solution $y_1(t) = y_1^0 e^{\xi_1 t}$ converging to 0 for $t \to \infty$ as well as a diverging solution $y_2(t) = y_2^0 e^{\xi_2 t}$. Here the abscissa is the *stable manifold*, whereas the ordinate corresponds to the *unstable manifold* (see Definition 2.45). The equilibrium is a *saddle point* (see Fig. 2.9a).

b) *Node*:

$$A = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad \text{and} \quad \operatorname{sgn} \xi_1 = \operatorname{sgn} \xi_2.$$

The characteristic roots are real and distinct. The solution is $y_1(t) = y_1^0 e^{\xi_1 t}, y_2(t) = y_2^0 e^{\xi_2 t}$. The trajectories will be unstable if the eigenvalues ξ_i are positive, and stable if they are negative. Both solutions explode or decay exponentially (see Fig. 2.9b for $0 < \xi_1 < \xi_2$). Geometrically, the trajectories typically emanate from the origin tangentially to the slow eigendirection, which is the direction spanned by the eigenvector corresponding to the smaller eigenvalue ξ_1 (see, e.g., Shub, 1987). For $t \to \infty$ the paths become parallel to the fast eigendirection. For a stable node the signs of ξ_i are reversed and arrows are in the opposite direction in Fig. 2.9b.

c) *Focus*:

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \text{and} \quad \alpha \neq 0, \ \beta \neq 0.$$

The roots are the complex conjugates $\alpha + i\beta$ and $\alpha - i\beta$ with $\alpha \neq 0$. The solution is

$$y_1(t) = e^{\alpha t}(y_1^0 \cos \beta t + y_2^0 \sin \beta t), \tag{2.28a}$$

$$y_2(t) = e^{\alpha t}(y_2^0 \cos \beta t - y_1^0 \sin \beta t). \tag{2.28b}$$

The trigonometric terms in (2.28) have a period of $2\pi/\beta$. For $\alpha > 0$ we have $e^{\alpha t} \to \infty$, and the trajectory spirals away from the origin. This situation is called an *unstable focus* (spiral). In the stable case $\alpha < 0$ and the trajectories form an *stable focus* spiraling toward $(0, 0)$. The unstable case is illustrated in Fig. 2.9c for $\beta < 0$, where the sign of β determines the direction of the orbit in the phase space.

The equilibria considered so far are structurally stable in the sense that small perturbations of the defining matrix A do not change the qualitative behavior of the system. Contrary to that, the subsequent equilibria are hairline cases, i.e., they are sensitive to small perturbations of A. The theory analyzing the occurrence of such sensitive equilibria is called *bifurcation theory* and will be introduced in Sect. 2.8.

d) *Center*:

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \quad \text{and} \quad \beta \neq 0.$$

In the case of pure imaginary eigenvalues, i.e., for $\alpha = 0$, all solutions are closed circles with period $\Theta = 2\pi/\beta$. The oscillations have fixed amplitude and the equilibrium is called a *center*. As we shall see in Sect. 2.10, this case becomes important when analyzing the Poincaré–Andronov–Hopf bifurcation. The counterclockwise movement, i.e., $\beta < 0$, is illustrated in Fig. 2.9d. Note that for a center the origin is stable but not asymptotically stable.

e) *Degenerate node*:

$$A = \begin{pmatrix} \xi & 0 \\ 1 & \xi \end{pmatrix}.$$

There is a single real root, but no real matrix Q can transform J into a diagonal matrix (the eigenspace is of dimension one). The solution has the form:

$$y_1(t) = y_1^0 e^{\xi t}, \quad y_2(t) = (y_2^0 + y_1^0 t) e^{\xi t}.$$

For $\xi > 0$, both $y_1(t)$ and $y_2(t)$ diverge from the origin for $t \to \infty$. The flow is illustrated in Fig. 2.9e. The equilibrium is called a *degenerate* or *improper node* and is unstable since $\xi > 0$. When $\xi < 0$, the trajectories follow the same lines in the opposite direction and the origin is therefore stable. There is only one eigendirection (the y_2-axis). As $t \to \pm\infty$ all trajectories become parallel to it.

f) *Star node*:

$$A = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}.$$

There is a single real eigenvalue ξ (the eigenspace is of dimension two). The solution is $y_1(t) = y_1^0 e^{\xi t}, y_2(t) = y_2^0 e^{\xi t}$. Since $y_2(t)/y_1(t)$ is constant, the trajectories are rays through the origin. If $\xi < 0$, it is globally asymptotically stable; if $\xi > 0$, the node is unstable (see Fig. 2.9f illustrating the unstable case).

In the following we shall give a summary of the classification of the equilibrium at the origin with respect to the characteristic roots.

Classification of Equilibria

For a planar linear system with constant coefficients, the type and stability of all equilibria can be characterized in a single diagram (see Fig. 2.10).

Let the abscissa measure the trace $\tau = \operatorname{tr} A$ and the ordinate measure the determinant $\Delta = \det A$ of the matrix of the planar system. Then the solution of the characteristic equation may be written, using (2.26),

$$\xi_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right).$$

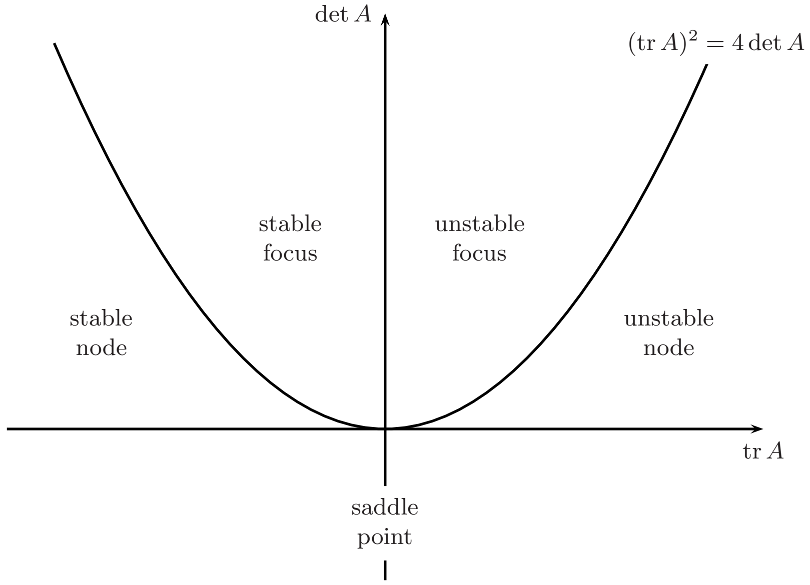

Fig. 2.10. Classification of equilibria in planar systems $\dot{x} = Ax$

By observing that (2.26) may be written as $(\xi - \xi_1)(\xi - \xi_2) = 0$, we also get

$$\tau = \xi_1 + \xi_2, \quad \Delta = \xi_1 \xi_2.$$

Considering the six cases above and Fig. 2.10 leads to the following conclusions:

1. For $\Delta < 0$, the eigenvalues are real and have opposite signs. Hence the origin is a *saddle point*.
2. For $\Delta > 0$, the eigenvalues are either real with the same sign (*nodes*), or complex conjugate (*foci* and *centers*), where the sign of the discriminant $D = \tau^2 - 4\Delta$ determines which case arises. For $D > 0$, nodes occur, and foci are obtained for $D < 0$.
   a) The origin is a repelling node (focus) if $\tau > 0$.
   b) The origin is an asymptotic stable node (focus) if $\tau < 0$.
3. In Fig. 2.10 it is easy to recognize the other cases as hairline cases. Since for $\tau = 0$, centers occur, whereas $D = 0$ corresponds to the star and degenerate node. Finally, $\Delta = 0$ characterizes the case of equilibria, where the matrix A is singular and hence the origin may not be the unique equilibrium.

Higher-Dimensional Linear Systems

Since the world is largely nonlinear, we are not interested in linear systems per se. We have discussed planar linear systems in such detail because the categorization of the six broad types of equilibria provides useful intuition that carries over to higher-dimensional systems, including nonlinear systems, where an important strategy is to examine the linearization about an equilibrium. In what follows, therefore, we shall need to generalize to higher dimensions the idea of keeping track of which eigenvalues $\xi_i, i = 1, \dots, n$ have positive, negative, or zero real parts, since those counts play a crucial role in determining the stability of linear systems of all dimensions, not just planar systems. Thus we define the following (generalized) eigenspaces:

$$\mathrm{E}^s = \operatorname{span}\{\nu_j \in \mathbb{R}^n : \operatorname{Re}(\xi_j) < 0\}, \quad \dim \mathrm{E}^s = n_- \tag{2.29a}$$
$$\mathrm{E}^u = \operatorname{span}\{\nu_j \in \mathbb{R}^n : \operatorname{Re}(\xi_j) > 0\}, \quad \dim \mathrm{E}^u = n_+ \tag{2.29b}$$
$$\mathrm{E}^c = \operatorname{span}\{\nu_j \in \mathbb{R}^n : \operatorname{Re}(\xi_j) = 0\}, \quad \dim \mathrm{E}^c = n_0, \tag{2.29c}$$

where $n_- + n_+ + n_0 = n$ is the number of state variables (dimension of the system). Then E^s is called the *stable*, E^u is called the *unstable* and E^c is called the *center* eigenspace.

Considering that $J^i \nu_j = \xi_j^i \nu_j$ and using the infinite sum (2.18), we see that the eigenspaces $\mathrm{E}^s, \mathrm{E}^u$, and E^c are invariant under the linear flow $\varphi^t(\cdot) = \mathrm{e}^{Jt}(\cdot)$, i.e., $\varphi^t(\mathrm{E}^s) \subset \mathrm{E}^s$, etc. Using Theorem 2.33 one can furthermore show that

$$\lim_{t\to\infty} \varphi^t(\mathrm{E}^s) = 0, \quad \text{and} \quad \lim_{t\to-\infty} \varphi^t(\mathrm{E}^u) = 0,$$

justifying the notation of E^s as the stable and E^u as the unstable subspace. Contrary to that, one cannot designate the long-run behavior of the trajectories in the center subspace E^c without knowing further details of the algebraic multiplicity of the eigenvalues. This exceptional position of the center subspace plays a crucial role in bifurcation theory and will be further explained in Sect. 2.8.3. The equilibrium at the origin satisfying $n_0 = 0$ is called *hyperbolic* (see Definition 2.42).

Remark 2.39. Note that the difference between the (un)stable and the center subspace is also one of persistence by slight changes of the defining (linear) dynamics J. Since the eigenvalues continuously depend on the matrix J, it easily follows that if J is changed to $J_\varepsilon = J + \Delta J$, with ΔJ small, the eigenvalues with a positive/negative real part will not change their sign. This is unlike the case for the zero eigenvalues, where the slightest change of J will move it to the positive or negative side. Thus a system in which $n_0 = 0$, i.e., $\mathrm{E}^c = \emptyset$, is "structurally stable" for small (linear) disturbances of the matrix J, whereas in the case of $n_0 > 0$ the system's behavior may change considerably.

Summing up, we find the following classification of a hyperbolic equilibrium at the origin, i.e., $n_0 = 0$:

$n_- > 0, n_+ = 0$ the origin is an attractor.
$n_- = 0, n_+ > 0$ the origin is a repeller.
$n_- > 0, n_+ > 0$ the origin is unstable, exhibiting trajectories converging to and diverging from it, and is therefore a saddle (point).

This classification (in a local form) can also be transformed to the nonlinear case (see Sect. 2.6.2).

Remark 2.40. For the sink (and reversing time also for the source) we can give a geometric intuition for its asymptotic stability using (2.24), which describes the volume change of the fundamental matrix solution (coordinate system); and we see that

$$\tau = \operatorname{tr} J = \sum_{i=1}^{n} \xi_i < 0.$$

Hence

$$\det X(t) = \det X(0)\mathrm{e}^{t\tau},$$

is contracting, a condition that provides the geometric intuition for a shrinking coordinate system and therefore an asymptotic convergence of all paths to the origin.

Next, the theory of autonomous systems is continued for the nonlinear case.

2.6.2 Autonomous Nonlinear ODEs

In the majority of models presented in this book the system dynamics is given by a nonlinear function $f : \mathbb{R} \to \mathbb{R}^n$:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \qquad (2.30a) \\ \text{s.t.} \quad x(0) &= x_0 \in \mathbb{R}^n, \end{aligned}$$

where f is continuously differentiable.

Remark 2.41. In general it is not possible to state a closed-form solution of (2.30), therefore we have to rely on numerical methods, when solving an explicit ODE.

Equilibria and Their Stability

Next we describe the dynamic behavior of trajectories in the neighborhood of an equilibrium $\hat{x}$. Therefore we define:

Definition 2.42 (Hyperbolic Equilibrium). *An equilibrium $\hat{x}$ is called* hyperbolic *if the corresponding Jacobian $J(\hat{x})$ has no zero or purely imaginary eigenvalues, i.e., $n_0 = 0$, otherwise it is called a* nonhyperbolic *equilibrium.*

Using the properties of a hyperbolic equilibrium in Hartman (1982) the following theorem is proved:

Theorem 2.43 (Hartman–Grobman Theorem). *Let $\hat{x}$ be a hyperbolic equilibrium of the ODE (2.30a). Then there exists a neighborhood $U \subset \mathbb{R}^n$ of $\hat{x}$ and a neighborhood $U' \subset \mathbb{R}^n$ of the origin such that the ODE (2.30a) restricted to U is topological equivalent to the variational equation of $\hat{x}$*

$$\dot{x}(t) = f_x(\hat{x})x(t) = J(\hat{x})x(t), \quad x(0) \in U'.$$

This theorem formally states that the dynamical behavior of a nonlinear system near a hyperbolic equilibrium can be fully described by its linearization. Thus the stability behavior of an equilibrium for an ODE is characterized by the eigenvalues ξ of the Jacobian matrix $J(\hat{x}) = f_x(\hat{x})$:

Theorem 2.44. *Let $\hat{x}$ be an equilibrium of the ODE (2.30a). If for every eigenvalue ξ of the Jacobian matrix $J(\hat{x}) = f_x(\hat{x})$*

$$\operatorname{Re} \xi < 0$$

holds, then $\hat{x}$ is asymptotic stable. If there exists an ξ with

$$\operatorname{Re} \xi > 0,$$

then $\hat{x}$ is unstable.

This result is analogous to the linear case (see p. 41), and a proof can be found in Hartman (1982).

An important concept is that of a *local stable and unstable manifold* of an equilibrium $\hat{x}$. These are defined by:

Definition 2.45 (Local Manifolds). *Let $\hat{x}$ be an equilibrium of the ODE (2.30a) and U be a neighborhood of $\hat{x}$; then the set*

$$\mathrm{W}^s_{loc}(\hat{x}) = \{x \in U : \lim_{t\to\infty} \varphi^t(x) = \hat{x}, \textit{and } \varphi^t(x) \in U, t \geq 0\}$$

is called the stable local manifold *of $\hat{x}$ and the set*

$$\mathrm{W}^u_{loc}(\hat{x}) = \{x \in U : \lim_{t\to\infty} \varphi^{-t}(x) = \hat{x}, \textit{and } \varphi^{-t}(x) \in U, t \geq 0\},$$

is called the unstable local manifold *of $\hat{x}$.*

Now the following theorem can be proved (see Carr, 1981; Marsden & McCracken, 1976; and Kelley, 1967):

Theorem 2.46 (Manifold Theorem for Flows). *Suppose that $\hat{x}$ is an equilibrium of (2.30a), where $f \in C^k(\mathbb{R}^n)$, and $\hat{J} = f_x(\hat{x})$. Let n_+, n_- and n_0 be the corresponding dimensions of the stable, unstable, and center subspaces E^s, E^u, and E^c. Then there locally exist the local stable and unstable C^k manifolds $\mathrm{W}^s_{loc}(\hat{x}), \mathrm{W}^u_{loc}(\hat{x})$, and a* local center manifold *$\mathrm{W}^c_{loc}(\hat{x})$ of the dimensions n_+, n_- and n_0 and being tangent to $\mathrm{E}^s, \mathrm{E}^u$, and E^c (see (2.29)). The manifolds $\mathrm{W}^s_{loc}(\hat{x}), \mathrm{W}^u_{loc}(\hat{x})$, and $\mathrm{W}^c_{loc}(\hat{x})$ are under the flow of f. The stable and unstable manifolds are unique, whereas the center manifold need not be unique.*

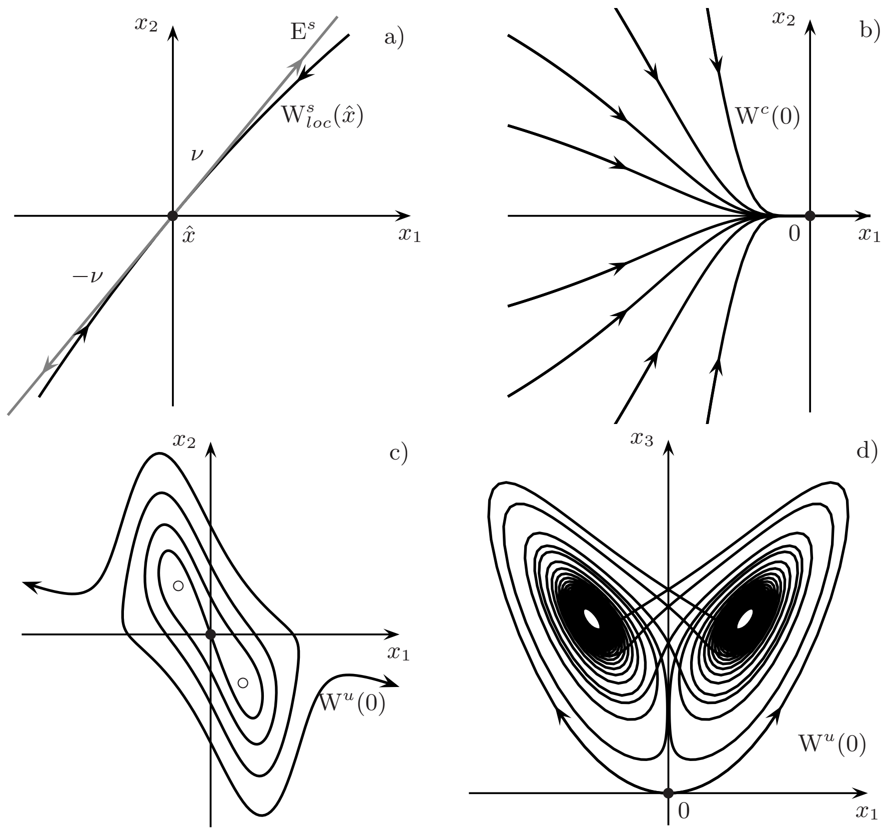

Fig. 2.11. In (**a**) the local stable manifold $W^s_{loc}(\hat{x})$ is depicted together with the stable eigenspace E^s spanned by the eigenvector ν. In (**b**) the existence of infinitely many center manifolds $W^c(0)$ is illustrated for the system (2.31). In (**c**) the stable (*solid*) as well as the unstable (*dashed*) global manifold is depicted for an equilibrium at the origin. The last panel (**d**) shows the Lorenz attractor projected into the (x_1, x_3)-plane as an example for complex global behavior of a one-dimensional unstable manifold

Furthermore, the local manifolds $W^s_{loc}(\hat{x}), W^u_{loc}(\hat{x})$ have global extensions $W^s(\hat{x}), W^u(\hat{x})$ by letting points in $W^s_{loc}(\hat{x})$ flow backward in time and those in $W^u_{loc}(\hat{x})$ flow forward:

Definition 2.47 (Global Manifolds of an Equilibrium). *Let $\hat{x}$ be an equilibrium of (2.30a); then the set*

$$W^s(\hat{x}) = \bigcup_{t \geq 0} \varphi^{-t}(W^s_{loc})(\hat{x})$$

is called the global stable manifold *of $\hat{x}$ and*

$$\mathrm{W}^u(\hat{x}) = \bigcup_{t \geq 0} \varphi^t(\mathrm{W}^u_{loc})(\hat{x})$$

is called the global unstable manifold *of $\hat{x}$ (see Fig. 2.11b,d).*

A trajectory $x(\cdot)$ of the ODE (2.30) is called a stable path *of $\hat{x}$ if the corresponding orbit* $\mathrm{Or}(x_0)$ *satisfies* $\mathrm{Or}(x_0) \subset \mathrm{W}^s\,\hat{x}$ *and* $\lim_{t\to\infty} \varphi^t(x_0) = \hat{x}$.

A trajectory $x(\cdot)$ of the ODE (2.30) is called an unstable path *of $\hat{x}$ if the corresponding orbit* $\mathrm{Or}(x_0)$ *satisfies* $\mathrm{Or}(x_0) \subset \mathrm{W}^u\,\hat{x}$ *and* $\lim_{t\to\infty} \varphi^{-t}(x_0) = \hat{x}$.

For $n_+ = 1$ the situation is depicted in Fig. 2.11a.

Definition 2.48 (Saddle Point). *An equilibrium $\hat{x}$ satisfying $n_+ > 0$ and $n_- > 0$ is called a* saddle point *or shortly* saddle.

Remark 2.49 (Saddle Point Stability). In economic dynamics a saddle $\hat{x}$ is often referred to as an equilibrium exhibiting *saddle point stability*. Here the term “stability” only refers to the fact, that converging paths exist, and must not be confused with a stable equilibrium (see, e.g., Sect. 5.1).

Remark 2.50. Note that this special construction of the global stable manifold becomes immediately clear when one considers that the points reached by letting the time flow backward are exactly those from where the forward trajectories will end up in the stable local manifold when moving forward in time. This idea constitutes the basis for a simple method to numerically compute the stable manifold (see Sect. 7.4.1).

Remark 2.51. Some few results exist guaranteeing the existence of the global manifolds (see, e.g., Hartman, 1982, Chap. VIII). These results rely on geometric considerations, which, for the planar case, are depicted in Fig. 2.12. Since these results are, in our context, only of minor practical importance we refrain from presenting their explicit formulation.

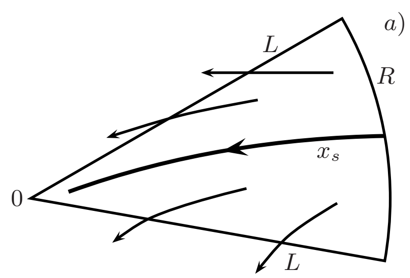

Fig. 2.12. The situation assuring the existence of a global saddle path. From geometric intuition it is evident that under the given assumptions a trajectory moving to the origin has to exist

Example 2.52 (Non-Unique Center Manifold). A typical example illustrating the non-uniqueness of the center manifold is given by the following ODE:

$$\dot{x}_1 = x_1^2 \tag{2.31a}$$

$$\dot{x}_2 = -x_2. \tag{2.31b}$$

Linearizing the system (2.31) around the equilibrium at the origin yielding

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

reveals that 0 is an eigenvalue of the Jacobian and therefore the equilibrium is nonhyperbolic. The manifold theorem for flows assures the existence of a local center manifold $\mathrm{W}^c_{loc}(0)$ tangential to the corresponding eigenspace $\mathrm{E}^c(0)$. This eigenspace is given by the x_1-axis as the reader may verify. Moreover the x_1-axis is such a center manifold since $x_2 \equiv 0$ implies $\dot{x}_2 = 0$ and is therefore tangential to $\mathrm{E}^c(0)$ (in fact it coincides).

But additionally there exist further center manifolds (see Fig. 2.11b). For $x_1 \neq 0$ we divide both dynamics of (2.31) yielding the one-dimensional nonautonomous ODE

$$\dot{x}_2 = -\frac{x_2}{x_1^2}.$$

The solution of this ODE is explicitly given by

$$x_2(x_1) = x_2(0)\mathrm{e}^{-\frac{1}{x_1(0)}}\mathrm{e}^{\frac{1}{x_1}},$$

which corresponds to the orbits of the system (2.31) and reveals that for $x_1(0) < 0$ the solution converges tangentially to the origin. Thus every orbit given by

$$x_2(x_1) = \begin{cases} x_2(0)\mathrm{e}^{-\frac{1}{x_1(0)}}\mathrm{e}^{\frac{1}{x_1}} & x_1 < 0 \\ 0 & x_1 \geq 0 \end{cases}$$

is a center manifold, illustrating its non-uniqueness. The reader is invited to accomplish the details in Exercise 2.10.

Connected to the stable and unstable manifolds is the so-called heteroclinic connection:

Definition 2.53 (Heteroclinic Connection). *Let $\hat{x}_1$ and $\hat{x}_2$ be equilibria of the ODE (2.30a). An orbit* $\mathrm{Or}(x_0)$ *starting at a point $x_0 \in \mathbb{R}^n$ is called a* heteroclinic connection *of $\hat{x}_1$ and $\hat{x}_2$ if $\lim_{t\to-\infty} \varphi^t(x_0) = \hat{x}_1$ and $\lim_{t\to\infty} \varphi^t(x) = \hat{x}_2$.*

Remark 2.54. From Definition 2.53 it immediately follows that the heteroclinic connection is part of the unstable manifold of $\hat{x}_1$ as well as of the stable manifold for $\hat{x}_2$ (see Fig. 2.13).

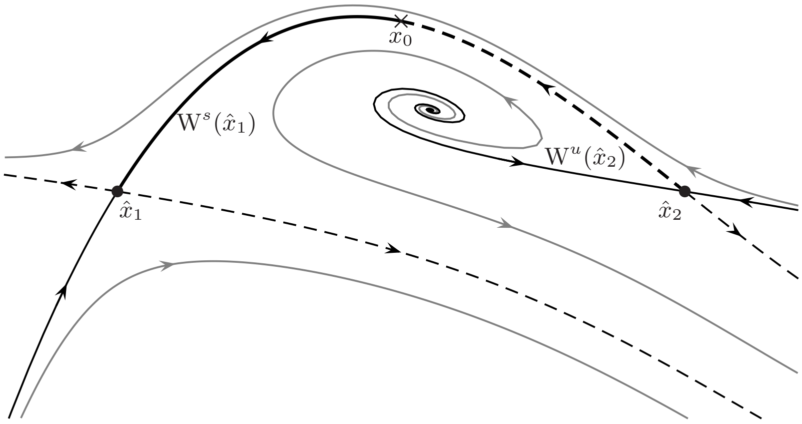

Fig. 2.13. A heteroclinic connection between the equilibria $\hat{x}_1$ and $\hat{x}_2$ is depicted, where the stable manifold of $\hat{x}_1$ coincides with the unstable manifold of $\hat{x}_2$. For illustration a point x_0 lying on the heteroclinic connection is shown, where the corresponding trajectory converges to $\hat{x}_1$ in positive time and to $\hat{x}_2$ in negative time

Remark 2.55. The introduction of heteroclinic connections is not an end in itself but becomes important in the analysis of the occurrence of multiple optimal solutions (see Chap. 5).

Definition 2.56 (Homoclinic Connection). *Let $\hat{x}$ be an equilibrium of the ODE (2.30a). Then an orbit* $\mathrm{Or}(x_0)$ *starting at a point* $x_0 \in \mathbb{R}^n$ *is called a* homoclinic connection *of $\hat{x}$ if* $\lim_{t\to\pm\infty} \varphi^t(x_0) = \hat{x}$.

Next we turn our attention to the nonhyperbolic case, which becomes important for the analysis of structural changes of dynamical systems (see Sect. 2.8).

Example 2.57 (Calculation of the Center Manifold). In this subsection we present a planar system with a nonhyperbolic equilibrium at the origin. By means of this planar system we exemplify the calculation of the corresponding center manifold to determine the stability of the equilibrium. For a generalization of the here presented transformation the reader is referred to Guckenheimer and Holmes (1983) and Kuznetsov (1998).

Given the two-dimensional system:

$$\dot{x}_1 = x_1 x_2 + x_1^3 \tag{2.32a}$$

$$\dot{x}_2 = -x_2 - 2x_1^2, \tag{2.32b}$$

we determine the stability of the equilibrium at the origin. The corresponding Jacobian

$$J = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

reveals that $n_0 = n_- = 1$, yielding the corresponding eigenspaces

$$\mathrm{E}^s = \{(x_1, x_2) : x_1 = 0\}, \quad \mathrm{E}^c = \{(x_1, x_2) : x_2 = 0\}.$$

First, it is obvious that the linear approximation $\dot{x} = Jx$, explicitly written as

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -x_2, \end{aligned}$$

is insufficient for determining the behavior of the system in a neighborhood of the origin.

Therefore, we use a quadratic approximation of the center manifold $\mathrm{W}^c_{loc}(0)$ in a neighborhood of the origin and analyze the dynamics restricted to this approximation.

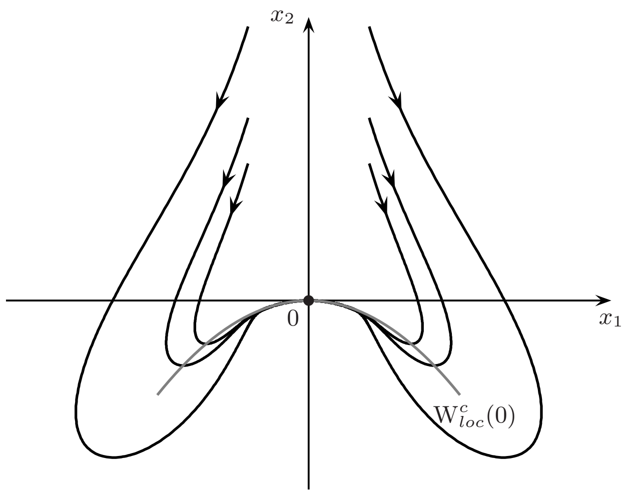

Fig. 2.14. The phase portrait for the dynamics (2.32) is depicted confirming the (asymptotic) stability of the nonhyperbolic equilibrium at the origin. The *gray line* shows the quadratic approximation of the local center manifold $\mathrm{W}^c_{loc}(0)$

The manifold theorem for flows assures the existence of the center manifold near the origin, which can be written as $x_2 = C(x_1)$ and is tangential to the center eigenspace E^c. This yields

$$C(0) = 0, \quad \text{and} \quad C_{x_1}(0) = 0.$$

Using the Taylor theorem $C(x_1)$ can quadratically be approximated

$$x_2 = C(x_1) = \frac{c}{2}x_1^2 + o(x_1^2). \tag{2.33}$$

To calculate the constant c we consider the total time derivative of (2.33)

$$\dot{x}_2 = \frac{\mathrm{d}C(x_1)}{\mathrm{d}t}$$

which becomes

$$\begin{aligned}\dot{x}_2 &= \frac{\partial C(x_1)}{\partial x_1}\dot{x}_1 = cx_1(x_1x_2 + x_1^3) + o(x_1)(x_1x_2 + x_1^3)\\ &= c\left(\frac{c}{2} + 1\right)x_1^4 + o(x_1^4).\end{aligned} \tag{2.34}$$

Substituting (2.33) into (2.32b) yields

$$\dot{x}_2 = -x_2 - 2x_1^2 = -\left(\frac{c}{2} + 2\right)x_1^2 + o(x_1^2). \tag{2.35}$$

When comparing the leading coefficients of x_1^2 for both representations (2.34) and (2.35) of $\dot{x}_2$, we find

$$\frac{c}{2} + 2 = 0 \quad \text{implying} \quad c = -4.$$

Thus the quadratic approximation of the center manifold is given by

$$x_2 = -2x_1^2$$

and the dynamics for x_1 restricted to the local center manifold becomes

$$\dot{x}_1 = x_1C(x_1) + x_1^3 = -2x_1^3 + x_1^3 + o(x_1^3) = -x_1^3 + o(x_1^3).$$

This proves that the equilibrium at the origin is stable (see Fig. 2.14).

The Occurrence of Limit Cycles

As has already been mentioned limit cycles are, beside equilibria, other important limit sets often occurring in higher-dimensional (nonlinear) systems.

The reader may already be familiar with periodic system behavior. The well-known *predator–prey model*, for instance, can exhibit recurrence of population numbers. When there are lots of rabbits (the *prey population*), the birth rate of foxes (the *predator population*) will increase in response to the bountiful food supply. This increases the fox population, but these additional foxes also eat more rabbits. As the number of rabbits shrinks, food for foxes becomes scarce, and their birth rate declines. Eventually the smaller predator population allows the prey population to recover and the whole cycle starts again.

Thus the descriptive two-state predator–prey model and various models addressed in this chapter's Exercises can exhibit an attracting limit cycle.

But in general it is much harder to detect such a limit cycle than finding an equilibrium. However, for planar systems it is at least possible to formulate conditions under which such cyclical solutions exist, known as the Poincaré–Bendixson theorem:

Theorem 2.58 (Poincaré–Bendixson). *If a nonempty compact ω- or α-limit set of a planar flow does not contain an equilibrium then it is a limit cycle.*

For a proof see Hirsch and Smale (1974) or Hartman (1982).

The Poincaré–Bendixson theorem is very intuitive. Basically it says that if a trajectory of a continuous dynamical system stays in a bounded region of the $\mathbb{R}^2$ forever, it has to approach "something." This "something" is either a point or a cycle. So if it is not a point, then it must be the other. If you enter a room and remain there forever without being allowed to stand still, sooner or later you will retrace your steps. Just because you would eventually have to walk over a spot you had already visited does not mean you would then have to walk in circles forever, but that is because you retain the capacity to make choices. For a trajectory whose motion depends entirely and only on its current position, once the trajectory returns to a place in the plane it has already visited, that trajectory is trapped and must then cycle forever.

The condition in the Poincaré–Bendixson theorem that no equilibrium is contained in the limit set is important as Fig. 2.15 illustrates, where in the long-run the solutions converge to a *heteroclinic cycle*, which is constituted by equilibria and heteroclinic connections, and not a limit cycle.

The Poincaré–Bendixson theorem is intuitive, but often not terribly helpful. How do you know what region to consider? And how would you know that any trajectory that entered the region could not ever leave? But most important, the Poincaré–Bendixson theorem is valid only for planar systems. As soon as one has to add a third-state variable, it will not be able to break the phase space into an interior and an exterior of a closed orbit and the theorem becomes completely worthless. Fortunately bifurcation theory provides tools to find periodic orbits by the important *Poincaré–Andronov–Hopf theorem*, as will be presented in Sect. 2.10.

Analogous to equilibria, the stability of limit cycles are of great importance for the long-run behavior of the dynamical system. But since the necessary theory is more involved than in the case of equilibria, we postponed this part in Sect. 2.12.1.

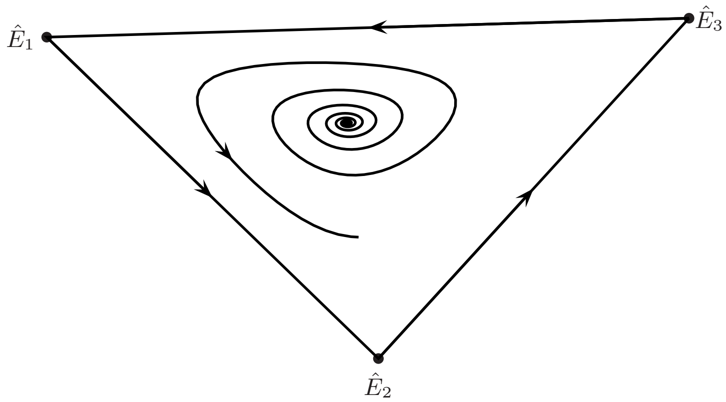

Fig. 2.15. The limit set, given by the equilibria $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$ and their heteroclinic connections, is a compact set but is *not* a limit cycle. It is a so-called heteroclinic cycle (see, e.g., Arrowsmith & Place, 1990, Chap. 5)

2.7 Stability Behavior in a Descriptive Model of Drug Demand

Patterns of drug use vary over time not only owing to external factors such as sanctions and prices, but also as a result of the progression of users through various types or "states" of drug use (Kandel, 2002). Modeling the various drug use states and associated transitions at the aggregate or population level can improve projections about future trends and assist with policy analysis. "System dynamics" models of this sort have been developed for tobacco (Levy, Chaloupka, Gitchell, Mendez, & Warner, 2002; Mendez & Warner, 2000; Mendez, Warner, & Courant, 1998; Tengs, Ahmad, Moore, & Gage, 2004; Tengs, Osgood, & Chen, 2001), heroin (Annemans et al., 2000; Caulkins et al., 2006; Levin, Roberts, & Hirsch, 1975; Rossi, 2001), cocaine (Caulkins, Behrens, Knoll, Tragler, & Zuba, 2004; Everingham & Rydell, 1994; Everingham, Rydell, & Caulkins, 1995; Homer, 1993a,b), and injection drug use generally (Caulkins, Dietze, & Ritter, 2007).

In this section we present a model from the field of drug control, where we take care about the enormous heterogeneity across users in rates of consumption. Heavy cocaine users may consume 100+ grams per year, whereas weekly users may consume more like 10 grams per year. Indeed, someone who used cocaine just once in the past 12 months ($\approx$0.1 grams) would still be recorded as a "past-year" user. Furthermore, the relative numbers of less- and more-frequent users can change dramatically over time (Everingham and Rydell, 1994). Hence, distinguishing the numbers of users at different intensities of drug use can help track trends in total drug consumption more accurately.

As a first approach to reflecting this heterogeneity Everingham and Rydell (1994) and Behrens, Caulkins, Tragler, Haunschmied, and Feichtinger (1999) use two states to distinguish people who use cocaine "at least weekly" (defined to be "heavy users," $H(t)$), from those who used cocaine "at least once within the last year but who used less than weekly" (defined as "light users," $L(t)$). Both studies assume there are so many more nonusers than users that the number of nonusers is essentially a constant and does not need to be modeled explicitly.[13]

The biggest difference between the Behrens et al. (1999) and Everingham and Rydell (1994) models (besides the latter being a discrete-time model) is the approach taken to modeling initiation. Everingham and Rydell (1994) were primarily interested in describing past trends, so initiation over time was simply set equal to historical trends in initiation. Behrens et al. (1999) sought to model initiation as an endogenous function of the current state of the drug epidemic, to enable projections of future trends and model the effects of various drug-control interventions.

Figure 2.16 gives a schematic representation of the resulting model, whose equations can be summarized as in the following figure.

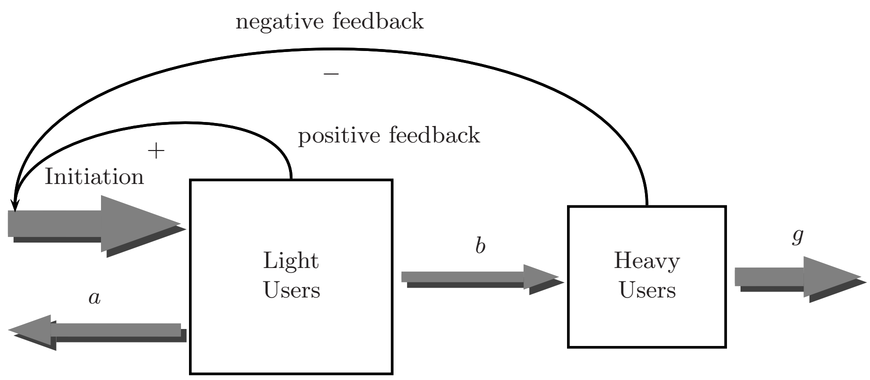

Fig. 2.16. Flow diagram for the LH-model (2.36)

$$\dot{L} = I(L,H) - (a+b)L, \qquad L(0) = L_0, \tag{2.36a}$$

$$\dot{H} = bL - gH, \qquad H(0) = H_0, \tag{2.36b}$$

[13] Other studies explicitly model the number of nonusers who are "susceptible" to initiation via an additional time-varying state variable, $S(t)$. This recognizes that as drug use spreads, the pool of potential initiates shrinks, helping to limit the drug epidemic from growing forever. We shall explore such models below, but they lump all drug users together into a single state. Merging the ideas of tracking susceptibles and distinguishing different types of users within a single model of illicit drug use is an active area of current research.

$L > 0$	...	number of light users,
$H > 0$	...	number of heavy users,
$I(L, H)$	...	initiation into light use,
a	...	average rate at which light users quit,
b	...	average rate at which light users escalate to heavy use,
g	...	average rate at which heavy users quit.

In Behrens et al. (1999) it is assumed that the number of current users influences initiation in two distinct ways. First, most people who start using drugs do so through contact with a friend or sibling who is already using and is happy with that use. The metaphor of a drug "epidemic" is commonly used precisely because of this tendency for current users to "infect" new users. Hence Behrens et al. (1999) model initiation as increasing in the number of light users.

Initiation is modeled as being promoted by light users because light users are rarely dependent, so their continued use indicates that they enjoy drug use. In contrast many, if not most, heavy users manifest obvious adverse effects of drug use, notably drug dependence but also various ill effects on work, social relationships, and health. So Behrens et al. (1999) believe heavy users are not effective recruiters of new initiates. (Note: This is not a universally held position. Some epidemiologists believe that heavy users also recruit new initiates, and the reader may enjoy thinking about how that alternative perspective could be built into a drug-prevalence model and what implications that would have on the results derived below.)

Musto (1987) argues that, in addition, knowledge of adverse effects of drug use can deter initiation. He hypothesizes that drug epidemics die out when a new generation of potential users becomes aware of the risks of drug abuse and, as a result, does not try drugs. One might expect the perception or reputation of a drug to be influenced by the relative number of light and heavy users. If most drug users are heavy users, the drug would be perceived as dangerous or harmful. Thus a model of drug initiation might have the following properties.

1. Initiation is proportional to the number of light users $L(t)$.
2. The rate at which light users "recruit" initiates is moderated by the drug's reputation for harmfulness, which in turn is governed by the relative number of heavy and light users.
3. Although most new users are "recruited," some enter the using population for other reasons (intrinsic interest, immigration, etc.). In the jargon of diffusion models, these individuals are "innovators" as opposed to "imitators."

Behrens et al. (1999) used the following initiation function to capture these three ideas.

$$I(L, H) = \tau + sLe^{-q\frac{H}{L}}, \tag{2.37}$$

$L > 0$	...	number of light users,
$H > 0$	...	number of heavy users,
s	...	average rate at which light users attract nonusers,
q	...	constant measuring the deterrent effect of heavy drug abuse,
τ	...	number of innovators.

This initiation model implies that large numbers of heavy users are "bad" (for the drug's reputation), because they consume at high rates and impose costs on society, but also "good" because they discourage initiation. Heavy users impose costs in the near term, but generate a perverse sort of "benefit" for the future by reducing initiation and thus future use. The impact of a heavy user on initiation depends on the number of other heavy and light users, so that the magnitude of the "benefit" of a heavy user depends on the stage of the drug epidemic. Thus one would expect the effectiveness of various drug-control interventions, particularly treatment, to vary over the course of the epidemic. We explore that assumption in Sect. 6.1, where we introduce optimal control to the model.

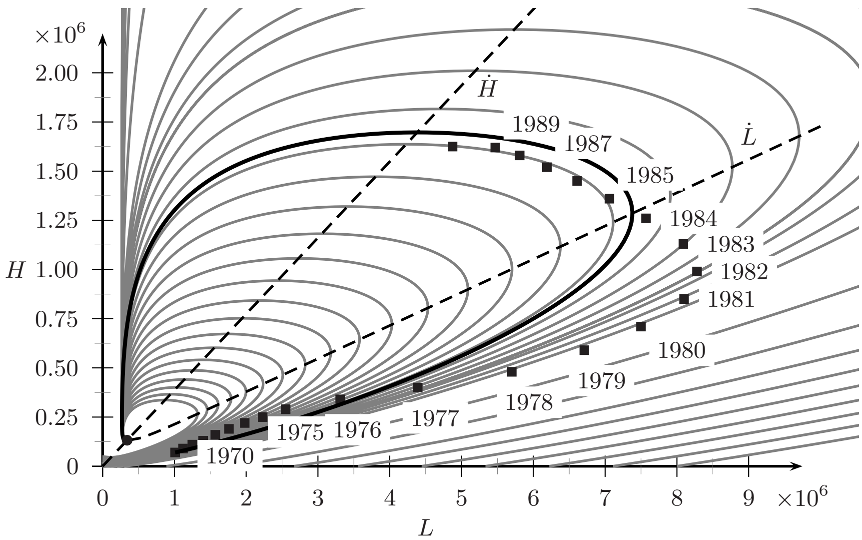

Fig. 2.17. Phase portrait, including smoothed historical trajectory of the current US cocaine epidemic (annual data are represented by *squares*)

For now it suffices to note that this rather simple model can do a surprisingly good job of reproducing trends in certain drug epidemics. Figure 2.17 shows this by mapping the outcome of this two-state model (2.36) in the phase portrait with parameter values estimated for the US cocaine epidemic as $a = 0.163$, $b = 0.024$, $\tau = 50{,}000$, $g = 0.062$, $s = 0.61$, $q = 7$ (Behrens et al., 1999).

Next we analyze the model in respect to occurring equilibria. The reader is invited to check that as L approaches zero, $\dot{L}$ is always positive, and as H goes to zero, $\dot{H}$ is always positive if L is positive. Hence we restrict our analysis to the (invariant) positive quadrant of the (L, H)-plane. The equilibrium, $\hat{X} = (\hat{L}, \hat{H}) = \{(L, H) | \dot{L} = 0, \dot{H} = 0\}$, of the planar system (2.36) can be determined as a function of the system parameters:

$$\hat{X} = \begin{pmatrix} \hat{L} \\ \hat{H} \end{pmatrix} = \frac{\tau}{\Omega} \begin{pmatrix} 1 \\ \frac{b}{g} \end{pmatrix}, \tag{2.38}$$

where

$$\Omega := a + b - sR$$

with

$$R = R(q, b, g) := \mathrm{e}^{-\frac{qb}{g}}$$

denoting the equilibrium reputation of cocaine. Equilibrium prevalence (determined by (2.38)) is positive as long as $\Omega > 0$. The associated Jacobian matrix evaluated at (2.38) is calculated as

$$\hat{J} = \begin{pmatrix} I_L(\hat{L}, \hat{H}) - (a + b) & I_H(\hat{L}, \hat{H}) \\ b & -g \end{pmatrix} = \begin{pmatrix} s\frac{qb}{g}R - \Omega & -sqR \\ b & -g \end{pmatrix}.$$

By (2.26) the corresponding eigenvalues of the Jacobian are calculated as follows

$$\xi_{1,2} = \frac{1}{2}\left(\mathrm{tr}\,\hat{J} \pm \sqrt{(\mathrm{tr}\,\hat{J})^2 - 4\det\hat{J}}\right), \tag{2.39}$$

where $\mathrm{tr}\,\hat{J} = -g - \Omega + sRqb/g$ and $\det\hat{J} = g\Omega > 0$.

Using the results of Sect. 2.6.2 on the linearized system, we find that the equilibrium (2.38) is a stable focus for the parameter values given by Behrens et al. (1999). Thus drug use approaches a relatively high long-rum level, and the trajectory converging to the equilibrium $(\hat{L}, \hat{H})$ happens to follow the pattern of damped oscillations (see also Sect. 2.11).

2.8 Introduction to Bifurcation Theory

Bifurcation theory is in itself a huge field encompassing a broad range of applications. Thus this introductory section cannot provide more than a glimpse to the subject, and the reader interested in theoretical as well as numerical details is referred to good textbooks on that subject, e.g., Guckenheimer and Holmes (1983) and Kuznetsov (1998). Nonetheless, the analysis of optimal control problems involves to a large extent the study of dynamical systems; and as the problems considered here have a variety of parameters, which are at best known to lie in some interval, bifurcation theory is necessary to analyze the possible changes in the dynamical behavior of the models. The next subsections introduce the reader to selected problems analyzed by bifurcation theory and should facilitate the first steps for anyone who wants to undertake a more detailed study by presenting its terminology and underlying ideas.

2.8.1 Terminology and Key Ideas of Bifurcation Theory

The basic object under investigation by means of bifurcation theory is

$$\dot{x}(t) = f(x(t), \mu), \quad \text{with} \quad x(t) \in \mathbb{R}^n,\ \mu \in \mathbb{R}^p,\ t \in [0, \inf) \tag{2.40}$$

a parameter-dependent ODE, whereby bifurcation theory tries to explain disruptions in the dynamical behavior of (2.40), while changing the parameter(s) μ. Thus we define:

Definition 2.59 (Bifurcation). *The appearance of a topologically nonequivalent, in the sense of Definition 2.30, phase portrait of (2.40) under variation of parameters μ is called a* bifurcation*.*

A parameter value, where such a change takes place, is called a bifurcation value *or* critical value *of the parameter and will be denoted by μ_c.*

Generally we distinguish between two types of structural changes, namely those changes that occur via *local bifurcations* and those induced by *global bifurcations*.[14] While the latter has to consider, e.g., the global stable or/and unstable manifolds, local bifurcations can be sufficiently characterized by the local stability analysis of equilibria, limit cycles, or other types of limit sets. Since global bifurcations need rather more advanced techniques, we shall not present any general results on this topic but instead refer the interested reader to Guckenheimer and Holmes (1983) or Kuznetsov (1998).

Remark 2.60. In optimal control theory global bifurcations, especially the heteroclinic bifurcation,[15] play a crucial role in connection with multiple optimal solutions (see Chap. 5).

Since there exists an unmanageably large number of possible dynamics, the problem of classification is reminiscent of Sisyphus's challenge to finish this task, and it is obvious that bifurcation theory will always be a work in progress. But fortunately phase portraits are not totally arbitrary, and recurring patterns can be identified and transformed to "simple" ODEs. The important tools of bifurcation theory are therefore the representation of ODEs in their

[14] Global bifurcations involve the entire phase space rather than just the neighborhood of an equilibrium or limit cycle and are therefore harder to deal with and harder to be detected.

[15] At a *heteroclinic bifurcation* a heteroclinic connection between two equilibria emerge at the critical value of the parameter.

normal form and the projection into the subspace, where the bifurcation occurs, i.e., the center manifold.

To introduce these tools in a concise way is beyond the scope of this book and even is not necessary for its application to the models of optimal control theory presented here. Therefore we introduce the necessary terms by means of a simple example to facilitate comprehension without becoming too technical. The following derivation and its generalizations can be found in Kuznetsov (1998).

2.8.2 Normal Forms and the Center Manifold: The Tools of Bifurcation Theory

To analyze local bifurcations of equilibria, it is useful to consider the dynamics restricted to the center manifold, since a qualitative change by varying a parameter can only occur for this part of the dynamics. The dynamics along the (un)stable manifolds are robust against small disturbances. But the restriction to the center manifold is only the first step in the analysis. Moreover, we shall find further coordinate transformations that simplify the analytic expressions of the vector field on the center manifold. This simplified dynamics is then called *normal form*. The analysis of the normal forms yields a qualitative picture of the flows of each bifurcation type, and we shall subsequently present only the most common ones.

For a more general theory of normal forms the reader is, e.g., referred to Guckenheimer and Holmes (1983). We settle for the presentation of the coordinate transformation of a concrete example, yielding the normal form of the so-called fold bifurcation, where two equilibria emerge by varying the bifurcation parameter.

Let us therefore consider the following two ODEs:

$$\dot{x} = g(x,\beta) = \beta(1+x) + x^3 + \frac{x^2-1}{x^2+1} + 1, \quad x,\beta \in \mathbb{R}, \tag{2.41}$$

and

$$\dot{x} = f(x,\mu) = \mu + x^2 \quad x,\mu \in \mathbb{R}. \tag{2.42}$$

Considering the functional forms of (2.41) and (2.42), it is not immediately reasonable that both equations exhibit, at least locally, the "same" dynamical structure. But for $\mu_c = \beta_c = 0$ both systems exhibit a nonhyperbolic equilibrium at the origin, i.e., $f_x(0,0) = g_x(0,0) = 0$. Moreover, the phase portraits of both systems at $\mu_c = \beta_c = 0$ suggest an equivalent dynamical behavior near the origin; see Fig. 2.18a. Clearly, this only concerns the local behavior, as Fig. 2.18b illustrates. Equation (2.42) is obviously easier to analyze, therefore it is of general interest to identify conditions under which an ODE of the complexity of (2.41) can locally be transformed to (2.42).

Remark 2.61. Note that in our example the restriction to the center manifold is trivially satisfied, since the dynamics is one-dimensional and therefore already represents its critical part.

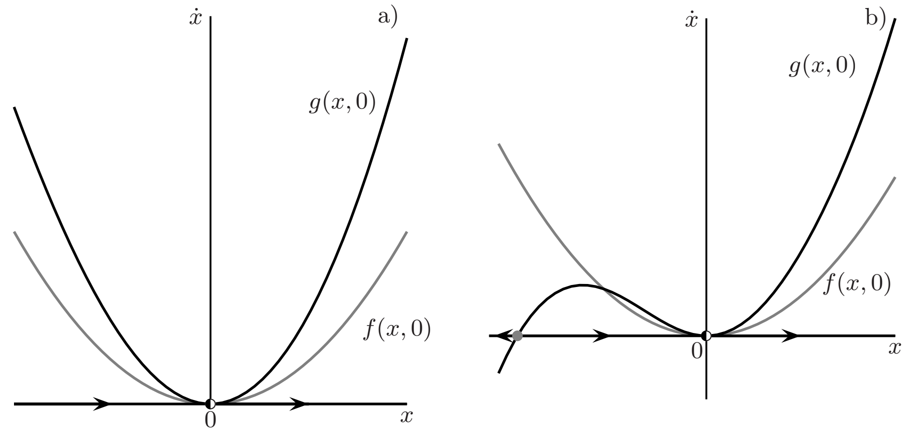

Fig. 2.18. Panel (**a**) depicts the phase portraits of (2.41) and (2.42) for the critical parameter value $\beta = \mu = 0$ illustrating the local equivalence of both systems. In panel (**b**) the global difference of both systems is portrayed

Coordinate Transformation

In a first step we show that (2.41) can be transformed into a system

$$\dot{x} = \mu + x^2 + o(x^2), \tag{2.43}$$

which approximates (2.42) up to terms of order two. Subsequently we prove that these higher-order terms can be neglected for describing the local dynamical behavior. In other words we show that both systems are locally topologically equivalent.

To carry out the transformation process we use the Taylor theorem to consider a cubic approximation of (2.41) about the equilibrium $\hat{x} = 0$ and small β. Thus we find

$$g(x,\beta) = g(0,\beta) + g_x(0,\beta)x + \frac{1}{2}g_{xx}(0,\beta)x^2 + \frac{1}{6}g_{xxx}(0,\beta)x^3 + o(x^3).$$

Utilizing

$$g(0,\beta) = \beta, \quad g_x(0,\beta) = \beta, \quad g_{xx}(0,\beta) = 4, \quad g_{xxx}(0,\beta) = 6 \tag{2.44}$$

yields

$$g(x, \beta) = \beta + \beta x + 2x^2 + x^3 + o(x^3). \tag{2.45}$$

Next we remove the linear term in (2.45) by applying the linear transformation $z = x + \delta$, where $\delta(\beta)$ is some parameter, depending on β and satisfying $\delta(0) = 0$. Now (2.45) becomes

$$\begin{aligned} g(z, \delta, \beta) &= \beta + \beta(z - \delta) + 2(z - \delta)^2 + (z - \delta)^3 + o\left((z - \delta)^3\right) \\ &= \beta - \beta\delta + 2\delta^2 + \left(\beta - 4\delta - 3\delta^2\right) z + (2 + 3\delta)\, z^2 + o(z^2) + o(\delta^2). \end{aligned}$$

To remove the linear coefficient $C(\beta, \delta) = \beta - 4\delta - 3\delta^2$ of z, we use the implicit function theorem. Since the nondegeneracy condition

$$C_\delta(0, 0) = -g_{xx}(0, 0) = -4 \neq 0 \tag{2.46}$$

is satisfied the implicit function theorem (see Theorem A.125) assures the local existence of a function $\delta(\beta)$ satisfying $C(\beta, \delta(\beta)) = 0$. This function is given by

$$\delta(\beta) = \frac{\beta}{4} + o(\beta). \tag{2.47}$$

Substituting (2.47) into $g(z, \delta, \beta)$ yields

$$\dot{z} = \beta + o(\beta) + \left(2 + \frac{3\beta}{4} + o\left(\beta\right)\right) z^2 + o(z^2). \tag{2.48}$$

Remark 2.62. What can be inferred if condition (2.46) is not satisfied? Either the dynamics is substantially different from (2.43), e.g.,

$$\dot{x} = \mu + x^3,$$

or higher-order terms have to be considered, e.g.,

$$\dot{x} = \mu + x^4.$$

Parameter Transformation

In a next step we introduce the new parameter $\bar{\mu}$ for the constant term of (2.48), i.e.,

$$\bar{\mu} = \beta + o(\beta).$$

To replace β in the coefficient of z^2, we have to assure that β can be expressed as a function of $\bar{\mu}$. Using the inverse function theorem (see Theorem A.126) this is guaranteed if the derivative $\bar{\mu}'(0)$ does not vanish. By virtue of (2.48) this condition can be restated as

$$\bar{\mu}'(0) = g_\beta(0, 0) = 1 \neq 0. \tag{2.49}$$

Thus we find

$$\beta = \bar{\mu} + o(\bar{\mu}). \tag{2.50}$$

Substituting (2.50) into (2.48) yields

$$\dot{z} = \bar{\mu} + \left(2 + \frac{3\bar{\mu}}{4} + o(\bar{\mu})\right) z^2 + o(z^2). \tag{2.51}$$

Remark 2.63. What can be inferred if condition (2.49) is not satisfied? The following examples

$$\dot{x} = \mu^2 + x^2$$

and

$$\dot{x} = \mu^3 + x^2,$$

violate (2.49). But whereas in the first case the system exhibits different dynamical behavior for varying μ compared to (2.42), the latter is topological equivalent to (2.42).

Scaling the Coordinate

In a last step we scale (2.51), using $y = |2 + \frac{3\bar{\mu}}{4} + o(\bar{\mu})|z$, which yields

$$\dot{y} = \mu + y^2 + o(y^2), \tag{2.52}$$

with $\mu = \bar{\mu}|2 + \frac{3\bar{\mu}}{4} + o(\bar{\mu})|$.

Remark 2.64. Inspecting the scaling carefully reveals that in general we derive a system satisfying

$$\dot{y} = \mu \pm y^2 + o(y^2),$$

where the actual sign of y^2 depends on the sign of the coefficient of z^2 evaluated at $\bar{\mu} = 0$.

What remains is to show that the term $o(y^2)$ does not change the local behavior of the dynamics and therefore the systems are locally topologically equivalent.

Topological Equivalence

To prove the local topological equivalence of the local system's dynamics (2.43) and (2.42), we have to show that for $|\mu|$ small:

1. Both systems exhibit locally the same number of equilibria.
2. A homeomorphism exists transforming the orbits of (2.51) into orbits of (2.52) and preserving the direction of time.

For the first part we consider the manifold of equilibria for (2.51), i.e.,

$$F(y,\mu) = \mu + y^2 + o(y^2) = 0. \tag{2.53}$$

Since

$$\frac{\partial F(0,0)}{\partial \mu} = 1$$

the implicit function theorem implies the existence of a function $\mu = b(y)$, satisfying $F(y, b(y)) = 0$, whereby

$$b(y) = g(0) + b_y(0)y + \frac{1}{2}b_{yy}(0)y^2 + o(y^2).$$

Using $b(0) = 0$, together with the formulas for implicit first and second derivatives, yields

$$b_y(0) = 0, \quad b_{yy}(0) = -2,$$

implying

$$b(y) = -y^2 + o(y^2).$$

Proving that for $|y|$ small enough $b(y)$ is negative and therefore (2.53) exhibits two equilibria, $\hat{y}_{1,2}(\mu)$, for $\mu < 0$. These equilibria are near $\hat{x}_{1,2}(\mu) = \pm\sqrt{-\mu}$, proving the first part.

For the proof of the second part we consider the homeomorphism $y = h(x)$

$$h(x) = \begin{cases} \mathrm{Id}(x) & \mu \geq 0 \\ a(\mu) + b(\mu)x & \mu < 0, \end{cases}$$

where $a(\mu)$ and $b(\mu)$ are determined such that $\hat{y}_{1,2} = h(\hat{x}_{1,2})$. Thus h maps the equilibria of (2.52) onto equilibria of the perturbed system (2.51). That h is really a homeomorphism follows from the fact that for one-dimensional systems an orbit is an interval on the real line and that for $\lim_{t\to\infty} x(t) = \hat{x}_{1,2}$ we have $\lim_{t\to\infty} h(x(t)) = \hat{y}_{1,2}$; if necessary we have to consider $t \to -\infty$. This ends our proof for the topological equivalence of (2.41) and (2.42).

Remark 2.65. In general the normal form is a polynomial system with mostly integer coefficients, exhibiting a special kind of bifurcation. It is important to consider under which conditions the simplification of an ODE to a normal form is possible (see, e.g., Guckenheimer & Holmes, 1983).

Remark 2.66. Finally, let us return to the conditions that have been identified along the previous proof as sufficient to guarantee the existence of a transformation into the normal form, namely (2.46) and (2.49). The first condition assures that the function is not "too" degenerate (and is therefore called the *nondegeneracy condition*, whereas the second condition is the so-called *transversality condition* on the parameter(s). Here the term transversality is derived from the geometric property of a transversal crossing. Consider, e.g., the already presented system

$$\dot{x} = \mu^2 + x^2.$$

Then the transversality condition is violated since $f_\mu(0,0) = 0$. Obviously this system exhibits a different behavior, since no equilibria exist for $\mu \neq 0$. Portrayed graphically, the graph of the dynamics at $x = 0$ for $\mu \to 0$ touches the x-axis tangentially, contrary to the original system, in which the graph of μ crosses the x-axis transversally (see Fig. 2.19).

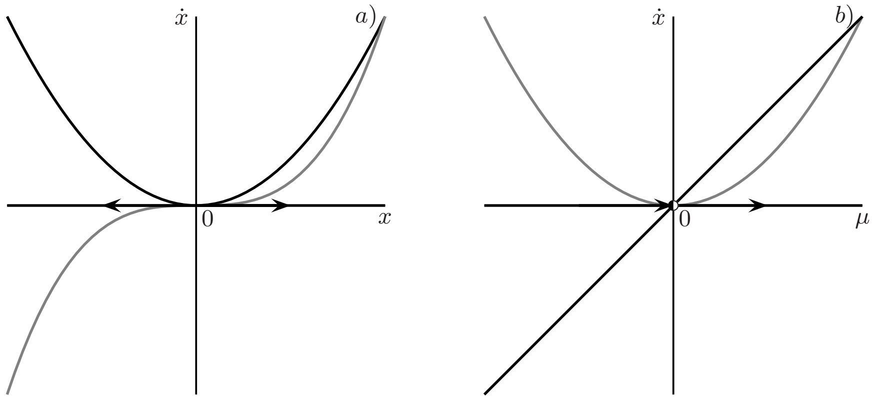

Fig. 2.19. In (**a**) the system dynamics, for $\mu = 0$, $\dot{x} = x^3$ (*gray line*) violates the nondegeneracy condition and exhibits a different local behavior at $x = 0$ as compared with (2.52). Panel (**b**) depicts the case in which the dynamics at $x = 0$ is given by $\dot{x} = \mu^2$ and therefore violates the transversality condition

Another very important and prominent example of using a normal form representation of an ODE is that of the Poincaré–Andronov–Hopf bifurcation presented in Sect. 2.10. Even though the same idea underlies the derivation of the corresponding normal form, the actual calculations are more involved and we therefore refer the reader to Kuznetsov (1998).

Last, we want to touch on the importance of the manifold theorem for flows for bifurcation theory, especially the part on the center manifold.

Center Manifold

The center manifold and its local representation is of crucial importance for the numerical computation of bifurcations and the reduction to normal forms. The manifold theorem for flows assures that the stable, unstable and center manifolds are invariant under the given dynamics. This allows the reduction of the dynamics to its bifurcation-relevant part, i.e., to the center manifold also called the *reduced equation*. In the simplest cases the determination of the bifurcation type can be done, by searching for equilibria exhibiting eigenvalues with zero real part. Then the center manifold and the reduced dynamics have to be computed. Checking the nondegeneracy and transversality conditions lets us determine the type of bifurcation.

As an elementary example, let us consider the system:

$$\dot{x}_1 = \mu + x_1^2 + x_2 \tag{2.54a}$$

$$\dot{x}_2 = -x_2. \tag{2.54b}$$

Then, using calculations analogous to the computation of the center manifold in Example (2.57), we find that the center manifold is given by

$$x_2 = o(x_1^2).$$

Thus the dynamics projected onto the center manifold is locally given by

$$\dot{x}_1 = \mu + x_1^2 + o(x_1^2).$$

Since for the reduced equation the properties of nondegeneracy (2.42) and transversality (2.43) are satisfied, we proved that the system (2.54) exhibits a fold bifurcation already analyzed in the section before, when discussing the prototypical example

$$\dot{x} = \mu + x^2.$$

For an in-depth treatment of a computation algorithm of the center manifold, the reader is referred to Kuznetsov (1998).

2.8.3 Local Bifurcations in One Dimension

In this section we focus on local bifurcations along the real line and analyze the asymptotic, or long-run, stability behavior of a dynamical system if the value of a single parameter is varied far enough to reach a *bifurcation value* μ_c.

In what follows we give a brief introduction to local bifurcation theory, dealing only with changes in stability properties of equilibria. We start by presenting the three simplest types (in normal form) of bifurcations in a one-dimensional setting, where only one parameter, μ, is varied. These bifurcations are not confined to $\mathbb{R}$ and can occur in higher-dimensional systems as well, as we have already mentioned in the previous section. Once these basic tools have been mastered, discussions of more advanced bifurcations may be of interest.

One useful way to depict the change in the dynamics of the system is to plot the phase-portrait at the ordinate-axis vs. the parameter, μ, at the abscissa, which is called a *bifurcation diagram* and will be used to present the following bifurcations.

Saddle-Node Bifurcation

Saddle-node bifurcation is the most fundamental type of bifurcation in which equilibria are created or destroyed. It is also called a *fold* (or *tangent*) bifurcation or *blue sky bifurcation.* The latter denotation refers to the sudden appearance of a pair of equilibria "out of the blue sky" as Abraham and Shaw (1982) describe it. We have already met this bifurcation type in the previous section, where we introduced normal forms by means of a prototypical example.

Consider the following flow on the real axis:

$$\dot{x} = \mu + x^2, \tag{2.55}$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a parameter. When $\mu < 0$ there are two equilibria, one stable and one unstable (see Figs. 2.20 and 2.21).

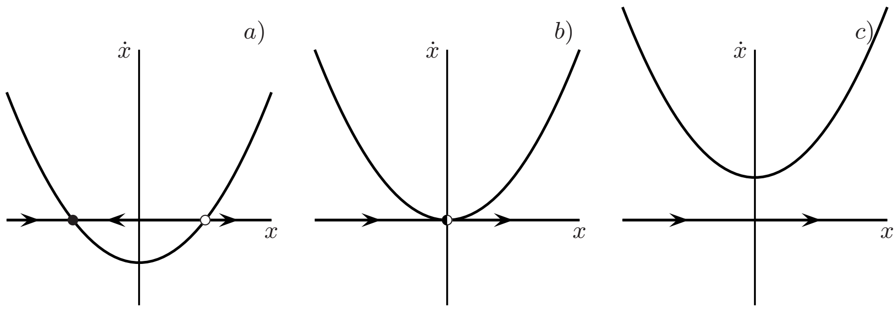

Fig. 2.20. The phase portrait of the dynamics (2.55) is depicted for (**a**) $\mu < 0$, (**b**) $\mu = 0$, and (**c**) $\mu > 0$

For $\mu > 0$ there is no equilibrium. If μ moves downward, an equilibrium arises "out of the blue sky" that is semistable. The bifurcation value $\mu = 0$ separates two qualitatively different vector fields.

The dynamics (2.55) is in a specific sense prototypical for *all* saddle-node bifurcations as we have explained in the last section on normal forms. Hence the following theorem can be formulated:

Theorem 2.67 (Fold Bifurcation). *Any one-dimensional system*

$$\dot{x} = f(x, \mu), \quad x, \mu \in \mathbb{R},$$

with $f \in C^2(\mathbb{R})$ exhibiting an equilibrium at $x = 0$ for $\mu = 0$ with $f_x(0,0) = 0$, and satisfying

A.1: $f_{xx}(0,0) \neq 0$,
A.2: $f_\mu(0,0) \neq 0$,

is locally topologically equivalent to the system

$$\dot{x} = \mu \pm x^2.$$

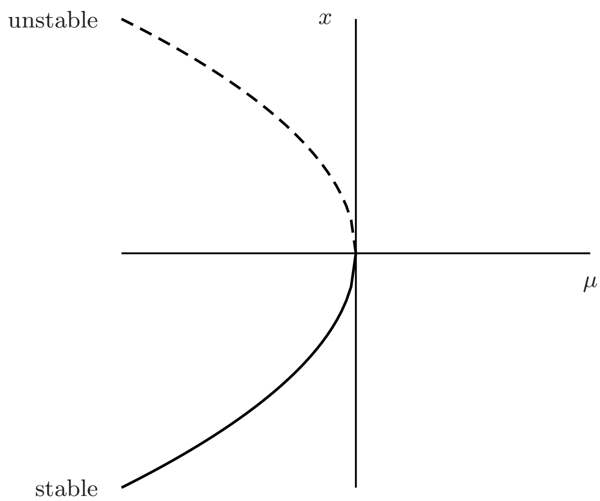

Fig. 2.21. Bifurcation diagram for a saddle-node or "blue sky" bifurcation

Transcritical Bifurcation

Transcritical bifurcation provides a mechanism for an exchange of stability of persisting equilibria.

The normal form for the transcritical bifurcation is

$$\dot{x} = \mu x - x^2, \quad x, \mu \in \mathbb{R}. \tag{2.56}$$

The three parts of Fig. 2.22 sketch the dynamics of the vector field depending on the parameter μ.

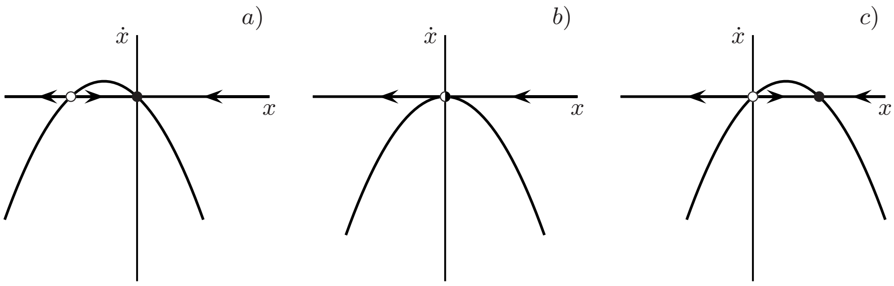

Fig. 2.22. The phaseportrait of the dynamics (2.56) is depicted for (**a**) $\mu < 0$, (**b**) $\mu = 0$, (**c**) $\mu > 0$

For $\mu < 0$ there is an unstable equilibrium at $\hat{x} = \mu$ and a stable one in the origin. For increasing μ, these two equilibria collide for $\mu = 0$, and

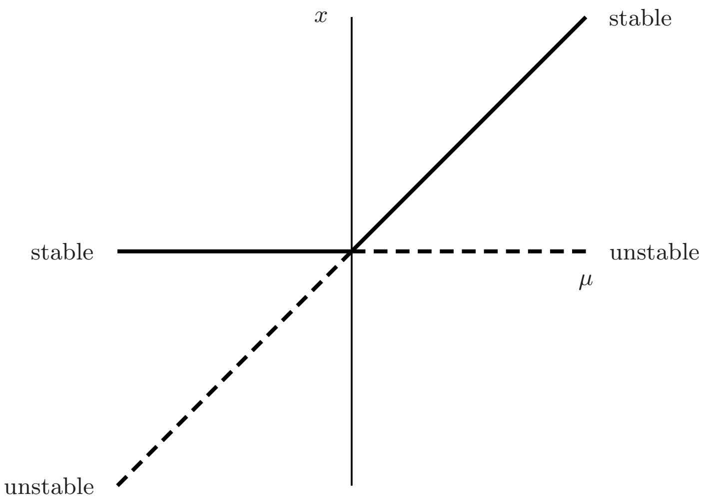

Fig. 2.23. Bifurcation diagram for a transcritical bifurcation

the resulting equilibrium is semistable. For $\mu > 0$, $x = 0$ becomes unstable, whereas $\hat{x} = \mu$ is now stable. The exchange of stabilities is marked in the bifurcation diagram shown in Fig. 2.23.

Generally we find:

Theorem 2.68 (Transcritical Bifurcation). *Any one-dimensional system*

$$\dot{x} = f(x, \mu), \quad x, \mu \in \mathbb{R},$$

with $f \in C^2(\mathbb{R})$ *exhibiting an equilibrium at* $x = 0$ *for* $\mu = 0$ *with* $f_x(0,0) = 0$, *and satisfying*

A.1: $f_{xx}(0,0) \neq 0$,
A.2: $f_{\mu x}(0,0) \neq 0$,

is locally topologically equivalent to the system

$$\dot{x} = \mu x \pm x^2.$$

Pitchfork Bifurcation

Another bifurcation may occur for problems with symmetries. The normal form of the so-called *supercritical pitchfork bifurcation* is given as

$$\dot{x} = \mu x - x^3, \quad x, \mu \in \mathbb{R}. \tag{2.57}$$

A transformation $x \to -x$ leads to an equivalent flow on the line, i.e., to left–right symmetry. This is illustrated in Fig. 2.24.

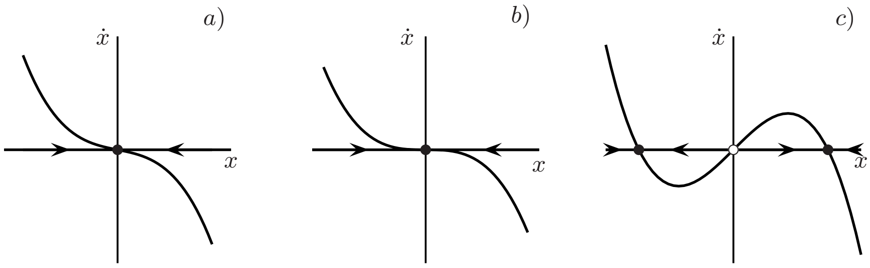

Fig. 2.24. The phaseportrait of the dynamics (2.57) is depicted for (**a**) $\mu < 0$, (**b**) $\mu = 0$, (**c**) $\mu > 0$

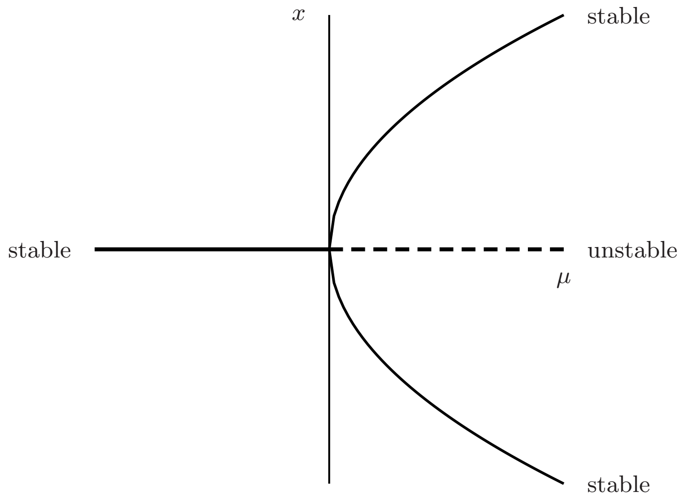

Fig. 2.25. Bifurcation diagram for a (supercritical) pitchfork bifurcation

For $\mu < 0$ there is only one equilibrium, i.e., the origin, and it is stable. $x = 0$ remains stable[16] for $\mu = 0$. For $\mu > 0$, the origin becomes unstable, and two additional stable equilibria arise symmetrically to $x = 0$, where $\hat{x} = \pm\sqrt{\mu}$.

Figure 2.25 reveals the denotation "pitchfork."

For the subcritical bifurcation, $\dot{x} = \mu x + x^3$, the bifurcation diagram is inverted. Note that the cubic term now destabilizes the system, whereas it acts to stabilize it for the supercritical bifurcation.

Generally we find:

Theorem 2.69 (Pitchfork Bifurcation). *Any one-dimensional system*

$$\dot{x} = f(x, \mu), \quad x, \mu \in \mathbb{R},$$

[16] This case provides an example of a vanishing linearization.

with $f \in C^2(\mathbb{R})$ *exhibiting an equilibrium at* $x = 0$ *for* $\mu = 0$ *with* $f_x(0,0) = f_{xx}(0,0) = 0$*, and satisfying*

A.1: $f_{xxx}(0,0) \neq 0$,
A.2: $f_{\mu x}(0,0) \neq 0$,

is locally topologically equivalent to the system

$$\dot{x} = \mu x \pm x^3.$$

2.9 Bifurcation Analysis of a One-Dimensional Drug Model

Here we present in some sense the simplest of all possible drug prevalence models, because it has just one state variable, $A(t)$, which counts the total number of drug users at time t. Finding the optimal demand side policy subject to some epidemic dynamics is challenging, and so it makes sense to start with a one-state model instead of starting with the more advanced LH-model (2.36). This one-state model, based on Tragler, Caulkins, and Feichtinger (2001), has two control variables, $u(t)$ and $v(t)$, that describe the expenditures at time t for treatment and law enforcement measures, respectively.

It is often useful to begin analyzing a control model by examining the behavior of the underlying uncontrolled (purely descriptive) model. Hence, we consider first Tragler et al.'s system dynamics with both controls set equal to zero, i.e., $u(t) = v(t) = 0$ for all t (in Exercise 2.4 the reader is invited to analyze the system with constant control $u(\cdot) \equiv u$). This reduces the state equation to

$$\dot{A}(t) = kf(A(t)) - \mu A(t), \tag{2.58}$$

where $kf(A(t))$ and $\mu A(t)$ denote initiation into and desistance from drug use at time t, respectively. Note that for ease of exposition, in what follows we shall mostly omit the time argument t.

Equation (2.58) states that users quit at a constant rate μ and initiation is a function of the current number of users. This reflects the fact that most initiates are introduced to drug use by an existing user, often a friend or family member, and so $f(A)$ should be an increasing function, at least initially.[17] However, once use is widespread, the ill effects of drug consumption become visible and suppress initiation (cf. Musto, 1987); this suggests that $f(A)$ should become a decreasing function for high enough values of A. The reader may enjoy comparing the modeling approach (2.36) with (2.58) as the latter model describes the same facts, but by means of a single variable. We use here what is perhaps the simplest function covering these two aspects, the logistic function

$$f(A) = kA\left(\bar{A} - A\right). \tag{2.59}$$

[17] This assumption is identical to one modeled transmission mechanism of the LH-model (2.36).

Remark 2.70. The original analysis published in Tragler (1998) and Tragler et al. (2001) used the monotonously increasing power function $f(A) = kA^{\alpha}$, and so in this respect we depart from the original studies. The power function is in some ways slightly more realistic, but the logistic function is better for present purposes.

While we first wrote the state equation (2.58) with separate inflow and outflow terms describing initiation and quitting, respectively, with $f(A)$ from (2.59), the system dynamics may be rewritten as an equivalent, purely logistic model

$$\dot{A} = \left(k\bar{A} - \mu\right) A \left(1 - \frac{k}{k\bar{A} - \mu} A\right) =: r_0 A \left(1 - \frac{A}{K}\right), \tag{2.60}$$

where $r_0 := k\bar{A} - \mu$ is the maximal growth rate at $A = 0$ and $K := \frac{k\bar{A}-\mu}{k}$ is the so-called carrying capacity (maximum number of users there could ever be).

This model has two equilibria, $\hat{A}_1 = 0$ and $\hat{A}_2 = K$, which we obtain by solving the equation $\dot{A} = 0$. The stability properties of these equilibria can be determined by using the first-order derivative of $\dot{A}$ with respect to A, i.e.,

$$\frac{\partial}{\partial A}\dot{A} = r_0 \left(1 - \frac{2A}{K}\right) = k\bar{A} - \mu - 2kA. \tag{2.61}$$

More formally, (2.61) gives the 1×1 Jacobian matrix of our one-state descriptive drug model. Evaluation of (2.61) at the equilibria gives

$$\left.\frac{\partial}{\partial A}\dot{A}\right|_{\hat{A}_i} = \begin{cases} r_0 & i = 1 \\ -r_0 & i = 2. \end{cases}$$

These results for the equilibria allow us to illustrate a transcritical bifurcation occurring in this model when allowing r_0 to change continuously from a negative to a positive value (follow Fig. 2.26). Note first that $\hat{A}_1 = 0$ and $\hat{A}_2 = \frac{r_0}{k}$ exist for any value of r_0 with the first being constant and the second being linear in r_0, where the sign of $\hat{A}_2$ is the same as that of r_0. For any negative value of r_0, the eigenvalue of the Jacobian at $\hat{A}_1 = 0$ is negative, implying that $\hat{A}_1 = 0$ is locally stable. Vice versa, the eigenvalue at $\hat{A}_2$ is positive for $r_0 < 0$, which implies that $\hat{A}_2$ is locally unstable. At the critical value $r_0 = 0$, the two equilibria coincide at $\hat{A}_1 = \hat{A}_2 = 0$ with the eigenvalue being 0, too. This is where the transcritical bifurcation occurs. That means, continuing with positive values of r_0, $\hat{A}_1 = 0$ becomes locally unstable, while $\hat{A}_2 = \frac{r_0}{k}$ becomes both positive and locally stable.

For our modeling purposes, r_0 and K should clearly have positive values, so that we shall assume later that

$$k\bar{A} > \mu. \tag{2.62}$$

Under this assumption, $\hat{A}_1 = 0$ is locally unstable, while $\hat{A}_2 = \frac{r_0}{k}$ is strictly positive and locally stable.

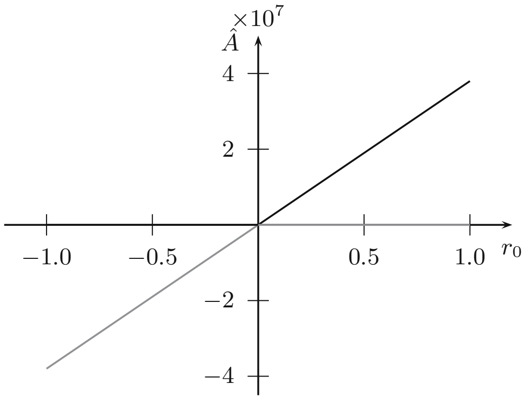

Fig. 2.26. Illustration of a transcritical bifurcation occurring at $r_0 = 0$

From a more dynamic point of view, the solution of the logistic equation (2.60) is well known to be

$$A(t) = \frac{A_0 K}{A_0 + (K - A_0)\, \mathrm{e}^{-r_0 t}},$$

where $A_0 := A(0)$ is the initial number of cocaine users.

From a global perspective, for any positive initial number of users A_0 we have

$$\begin{aligned}\lim_{t\to\infty} A(t) &= \lim_{t\to\infty} \frac{A_0 K}{A_0 + (K - A_0)\, \mathrm{e}^{-r_0 t}} \\ &= \frac{A_0 K}{A_0 + (K - A_0) \lim_{t\to\infty} \mathrm{e}^{-r_0 t}} = K = \hat{A}_2.\end{aligned}$$

That means that, unless $A_0 = 0 = \hat{A}_1$ (in which case the epidemic never starts), any positive initial number of users produces a time path converging monotonously to the carrying capacity $K = \hat{A}_2$, which turns out to be globally attracting. If A_0 is less/greater than K, the number of users increases/decreases monotonously while approaching the level K. This is illustrated in Fig. 2.27, where we see four time paths starting at $A_0 = 1, \frac{K}{2}, \frac{3K}{2}, 2K$, respectively. The parameter values used ($r_0 = 0.324854$, $K = 12,204,200$) were derived with the purpose of fitting the model to data for the current US cocaine epidemic (cf. Bultmann, Caulkins, Feichtinger, & Tragler, 2007).

In the rest of this chapter we deal with local bifurcations in the plane and beyond.

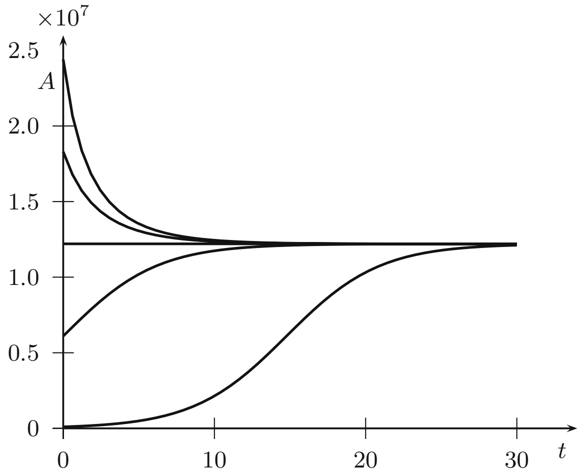

Fig. 2.27. Time paths $A(t)$ for $t \in [0, 30]$ and different initial numbers of users A_0

2.10 The Poincaré–Andronov–Hopf Bifurcation

Section 2.8.3 showed that in *fold* (or *blue-sky*), *transcritical*, and *pitchfork bifurcations* the equilibria of one-dimensional continuous dynamical systems were created or destroyed, had their stability behavior exchanged, or had their numbers doubled, respectively. As we have already mentioned, evidence of these types of bifurcations can be generalized from the $\mathbb{R}$ to the $\mathbb{R}^n, n \geq 2$ in a straightforward way, by projection into the center manifold.

Planar systems:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \dot{x} = f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

as a special case of the general nonlinear case (cf. Sect. 2.6.2), potentially exhibit richer system behavior, e.g., limit cycles or heteroclinic cycles, than one-dimensional systems do. While one-state ODEs have only a single eigenvalue that must be a real number (because complex eigenvalues occur only in pairs), planar systems have two eigenvalues (of the linearized system evaluated at an equilibrium if the system dynamics is nonlinear). Therefore an additional type of structural change can occur where the eigenvalues are purely imaginary. This bifurcation type is a so-called *Poincaré–Andronov–Hopf bifurcation (PAH-bifurcation)* usually denoted as *Hopf bifurcation*, where an equilibrium bifurcates into an equilibrium and a limit cycle.

Having in mind the normal form as the "simplest" possible representation of a system exhibiting some special type of bifurcation, one can ask about the normal form of a PAH-bifurcation. Since in this case a pair of eigenvalues becomes purely imaginary, two eigenvalues are affected, namely ξ and its complex conjugate $\bar{\xi}$. Thus the generic system exhibiting a PAH-bifurcation is at least of dimension two. Subsequently we shall state sufficient conditions (Poincaré–Andronov–Hopf theorem) to guarantee the existence of a transformation to its normal form, analogous to the fold bifurcation Theorem 2.67. During the proof of Theorem 2.71 a function $l_1(\mu)$, called the *Lyapunov coefficient*, occurs; an explicit formula is given in the remark below. This function plays a crucial role as it provides a nondegeneracy condition and its sign constitutes the stability of the emerging limit cycle. Note that for reasons of simplicity the theorem is stated for the case in which the equilibrium is already shifted to the origin and the critical bifurcation parameter μ_c is shifted to zero too. A full proof for arbitrary dimension $n \geq 2$ is given, e.g., in Kuznetsov (1998):

Theorem 2.71 (Poincaré–Andronov–Hopf in $\mathbb{R}^2$). *Suppose the ODE*

$$\dot{x} = f(x, \mu), \quad \textit{with} \quad x \in \mathbb{R}^2, \ \mu \in \mathbb{R},$$

has for small $|\mu|$ an equilibrium at the origin with eigenvalues

$$\xi_{1,2}(\mu) = \eta(\mu) \pm \mathrm{i}\omega(\mu),$$

satisfying $\eta(0) = 0, \ \omega(0) > 0$.

If the following conditions are satisfied,

1. *$l_1(0) \neq 0$, (for an explicit formula of l_1 see (2.64)),*
2. *$\eta'(0) \neq 0$,*

then the system is locally topologically equivalent near the origin $k = 0$, $\mu = 0$ to one of the following normal forms:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{2.63}$$

where $\beta(\mu) = \eta(\mu)/\omega(\mu)$ and the sign of the cubic term is determined by $\operatorname{sgn} l_1(0)$.

Remark 2.72. That the system described by the normal form (2.63) exhibits a limit cycle can easily be seen by transforming it into polar coordinates, where it becomes

$$\dot{R} = \pm R(\beta - R^2)$$
$$\dot{\theta} = 1,$$

with $x_1 = R\cos\theta$ and $x_2 = R\sin\theta$ and $R \geq 0$. In Fig. 2.28 the case is depicted for a positive R dynamic.

Contrary to the fold bifurcation, it is not possible to eliminate, during the transformation into the normal form, all terms of greater order than two. One of the occurring cubic terms remains and is called *resonant term*, where $l_1(\mu)$ is

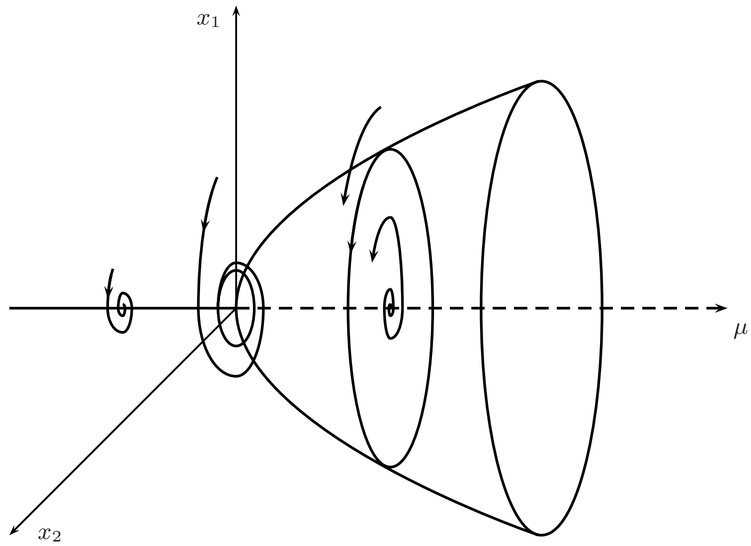

Fig. 2.28. Poincaré–Andronov–Hopf bifurcation

its coefficient. In particular its sign for $\mu = 0$ determines the stability behavior of the limit cycle and is therefore of great importance in the numerical analysis. A lengthy calculation (see, e.g., Guckenheimer & Holmes, 1983; Kuznetsov, 1998) shows that the Lyapunov coefficient $l_1(\mu)$ can explicitly be written as

$$\begin{aligned} l_1 = &\frac{1}{16}\left[\frac{\partial^3 g_1}{\partial x_1^3} + \frac{\partial^3 g_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 g_2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 g_2}{\partial x_1^3}\right] \\ &+ \frac{1}{16\omega(0)}\left[\frac{\partial^2 g_1}{\partial x_1 \partial x_2}\left(\frac{\partial^2 g_1}{\partial x_1^2} + \frac{\partial^2 g_1}{\partial x_2^2}\right) - \frac{\partial^2 g_2}{\partial x_1 \partial x_2}\left(\frac{\partial^2 g_2}{\partial x_1^2} + \frac{\partial^2 g_2}{\partial x_2^2}\right)\right] \\ &- \frac{1}{16\omega(0)}\left[\frac{\partial^2 g_1}{\partial x_1^2}\frac{\partial^2 g_2}{\partial x_1^2} - \frac{\partial^2 g_1}{\partial x_2^2}\frac{\partial^2 g_2}{\partial x_2^2}\right], \end{aligned} \tag{2.64}$$

where $g = (g_1, g_2)'$ is the nonlinear part of the dynamics $f = (f_1, f_2)'$ evaluated at the equilibrium, i.e.,

$$\dot{x} = f(x, \mu) = \hat{J}(\mu)x + g(x, \mu),$$

and ω the imaginary part of the eigenvalue. Thus the following characterization holds:

1. For $l_1(0) < 0$ closed orbits emerge in a *supercritical* PAH-bifurcation and correspond to *stable* limit cycles.
2. For $l_1(0) > 0$ closed orbits emerge in a *subcritical* PAH-bifurcation and correspond to *unstable/repelling* cycles.

Thus the Poincaré–Andronov–Hopf theorem allows us to establish the existence of a limit cycle as well as its stability behavior, which is very useful for locating limit cycles as used in Sect. 7.4.2.

Obviously a PAH-bifurcation can also occur for higher-dimensional systems. This implies that a two-dimensional center manifold exists in the n-dimensional space that is tangent to the eigenspace spanned by the corresponding eigenvectors $\nu, \bar{\nu}$. Thus projecting (locally) the dynamics near the equilibrium onto this center manifold leads to a two-dimensional system, wherein the projection can be analyzed by the Poincaré–Andronov–Hopf theorem.

To illustrate a *supercritical PAH-bifurcation* ($l_1(0) < 0$), we imagine that – without loss of generality – the orbits emerge for $\mu > 0$. (The orbits could also emerge for $\mu < 0$, but this would not alter anything about the validity of this illustration.) Thus as long as $\mu < 0$ holds, the equilibrium $\hat{x}(\mu)$ is stable. As the bifurcation parameter attains its critical value, $\mu = 0$, $\hat{x}(\mu)$ alters its stability properties and becomes repelling, and closed orbits emerge. The union of these limit cycles forms a paraboloid that is tangent to the set expanded by the eigenvectors belonging to the eigenvalues with purely imaginary parts, $\xi_{1,2}(0) = \pm \mathrm{i}\omega(0)$, the center manifold.

These ideas will be made concrete by giving an intuitive example of a supercritical PAH-bifurcation taken from the field of drug research.

2.11 Higher-Dimensional Bifurcation Analysis of a Drug Model

Let us reconsider the two-state model of US cocaine consumption given by the planar system (2.36) – but this time from a bifurcation perspective. In Sect. 2.7 we already established the existence of a unique equilibrium

$$\hat{X} = \begin{pmatrix} \hat{L} \\ \hat{H} \end{pmatrix} = \frac{\tau}{\Omega} \begin{pmatrix} 1 \\ \frac{b}{g} \end{pmatrix}, \tag{2.65}$$

where

$$\Omega := a + b - sR > 0, \text{ by assumption}$$

with

$$R = R(q, b, g) := \mathrm{e}^{-\frac{qb}{g}}.$$

The associated Jacobian matrix is given by

$$\hat{J} = \begin{pmatrix} I_L(\hat{L}, \hat{H}) - (a+b) & I_H(\hat{L}, \hat{H}) \\ b & -g \end{pmatrix} = \begin{pmatrix} s\frac{qb}{g}R - \Omega & -sqR \\ b & -g \end{pmatrix}.$$

Since $\det \hat{J} > 0$ always holds for the equilibrium as determined by (2.38), $\hat{J}$ exhibits a pair of purely imaginary eigenvalues iff the trace of the Jacobian matrix disappears, i.e., $\operatorname{tr} \hat{J} = 0$. Thus the nondegeneracy condition in the Poincaré–Andronov–Hopf theorem is satisfied if

$$\operatorname{tr} \hat{J} = -g - \Omega + \frac{sRqb}{g} = 0. \tag{2.66}$$

Therefore the equation $sqbR = g(\Omega + g)$ defines a surface in the five-dimensional parameter space along which system (2.36) possesses two purely imaginary eigenvalues, $\xi_{1,2} = \pm i\sqrt{g\Omega}$. As one crosses this surface by variation of a single parameter, while all other parameter values remain unchanged (what economists call a ceteris paribus condition), the real part of the pair of complex conjugate eigenvalues, $\nu_{1,2}$, changes its sign, while the imaginary part remains nonzero, i.e., the system exhibits a PAH-bifurcation. The associated conditions for the entire set of system parameters, $\{a, b, g, s, q\}$, are given by Behrens et al. (1999); but here we focus on a particular one, namely the parameter g. This parameter describes the outflow from the state of heavy use, $H(t)$, and can, for instance, be increased or decreased by funding more or fewer drug treatment programs.

The reason why we analyze g is that when the exit rate from the heavy-use state is low, the heavy-use state has considerable inertia that can prevent a drug epidemic from repeating. A low g does not protect society from a first epidemic. Regardless of g's value, if initially there are enough light users to make the ratio H/L small and hence give the drug a benign reputation, then a positive feedback loop can be created by the sL part of the initiation term that feeds the L state. The resulting explosion in light users implies that, over time, many people will escalate to the heavy-use state. Eventually that alters the drug's reputation, undercutting initiation and short-circuiting the positive feedback loop. In short, the drug moves from an epidemic to an endemic problem, with numbers of both light and heavy users gradually declining.

Whether there is a second explosive epidemic, however, depends on what the ratio of H/L is as the number of drug users decays and hence on the relative rates at which light and heavy users quit. The exit rate from light use ($a = 0.163$ for US cocaine) is a function of the drug and societal parameters generally, but the exit rate from heavy use (g) is more amenable to manipulation by policy interventions, so that it is the key parameter to examine via bifurcation analysis.

When the rate at which heavy users quit obtains its *PAH-bifurcation critical value*, $g_c = \{g : \ a + b + g - sR(1 + qb/g) = 0\}$; then the change in the real part of the pair of conjugate complex eigenvalues (as given by 2.39) as the bifurcation parameter g varies is different from zero as long as

$$sqbR(g_c) \neq \frac{g_c^3}{qb}$$

holds. Thus a PAH-bifurcation occurs at $(\hat{X}(g_c), g_c)$ as one crosses the surface defined by (2.66), i.e.,

$$\left.\frac{\partial \operatorname{Re} \xi_{1,2}(g)}{\partial g}\right|_{g=g_c(a,q,s,b)} = \frac{1}{2}\left(\frac{sR(g_c)q^2b^2}{g_c^3} - 1\right) \neq 0.$$

We already know that on the center manifold the motion tends neither toward the equilibrium nor away from it, but remains in a certain region of the phase space forever. Thus the center manifold is determined by the following transformation of system (2.36) into a linear and nonlinear part

$$\begin{pmatrix} \dot{L} \\ \dot{H} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{g_c\Omega(g_c)} \\ \sqrt{g_c\Omega(g_c)} & 0 \end{pmatrix} \begin{pmatrix} L - \hat{L}(g_c) \\ H - \hat{H}(g_c) \end{pmatrix} + \begin{pmatrix} \phi(L,H) \\ \psi(L,H) \end{pmatrix},$$

where

$$\begin{pmatrix} \phi(L,H) \\ \psi(L,H) \end{pmatrix} = \begin{pmatrix} I(L,H) - (a+b)L + \sqrt{g_c\Omega(g_c)}(H - \hat{H}(g_c)) \\ bL - gH - \sqrt{g_c\Omega(g_c)}(L - \hat{L}(g_c) \end{pmatrix}.$$

As long as

$$sqR(g_c) \geq \sqrt{g_c\Omega(g_c)} =: \omega \tag{2.67}$$

holds at the *PAH-bifurcation critical value*, ceteris paribus, the normal-form calculation yields a negative coefficient

$$\begin{aligned} l_1(0) &= \frac{1}{16}\left(\phi_{LLL} + \phi_{LHH} + \psi_{LLH} + \psi_{HHH}\right) \\ &\quad + \frac{1}{16\omega}\left(\phi_{LH}\left(\phi_{LL} + \phi_{HH}\right) - \psi_{LH}\left(\psi_{LL} + \psi_{HH}\right) - \phi_{LL}\psi_{LL} + \phi_{HH}\psi_{HH}\right) \\ &= \frac{1}{16\omega}\left(\omega\left(I_{LLL} + I_{LHH}\right) + I_{LH}\left(I_{LL} + I_{HH}\right)\right) \\ &= \frac{-I_{HH}}{16L\omega}\left(\underbrace{\omega\left(3\frac{H^2}{L^2} + 1\right)}_{>0} + \underbrace{q\frac{H}{L}\left(\frac{H^2}{L^2} + 1\right)}_{>0}\underbrace{(sRq - \omega)}_{\text{Eq. 2.66}\geq 0}\right) < 0. \end{aligned}$$

Hence there exist periodic solutions that correspond to stable limit cycles for $g > g_c$. In other words when $g > g_c$, the unique equilibrium, (2.65), changes its stability because the real parts of the characteristic roots, (2.39), become positive such that the eigenvalues cross the imaginary axis of the complex plain with positive velocity (the transversality condition of Poincaré–Andronov–Hopf theorem). Since additionally condition (2.67) holds, we obtain a supercritical PAH-bifurcation. The equilibria have the local property of unstable foci, and the periodic orbits are attracting (for $g > g_c$).

On the one hand, if the exit rate from the heavy-use state is low ($g < g_c = 9.405\%$), the ratio of H/L will remain large enough to keep the drug from again getting a benign reputation. That is, the deterrent effect of heavy drug use is sufficient to suppress initiation into light use so that the epidemic declines and does not recur, at least with comparable force. On the other hand, if a high percentage of heavy users receives treatment or otherwise exits heavy use ($g > 13.8\%$), there is almost no deterrence and light users recruit nonusers with rate $s = 0.61$, so that prevalence can again grow exponentially. Between these values of g ($g_c < g < 13.8\%$) the deterrent force of heavy users is not strong enough to prevent further initiation, but is strong enough to prevent nearly proportional initiation. This interplay causes cycles (see Fig. 2.29).

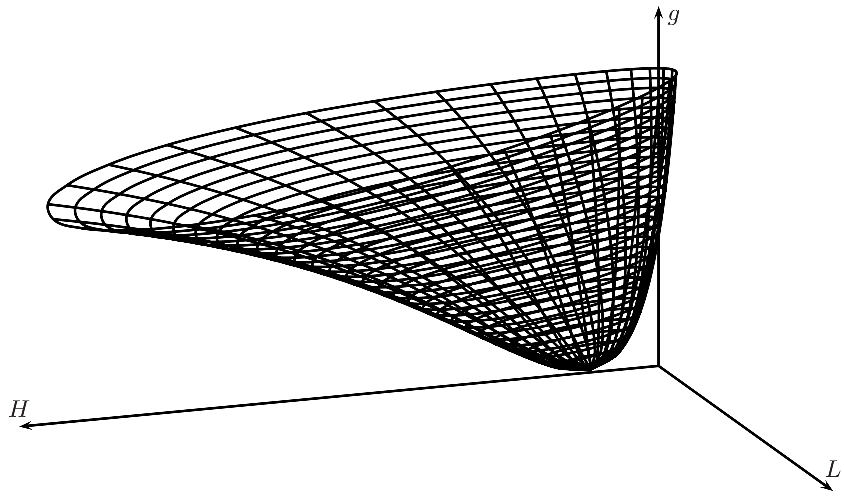

Fig. 2.29. Cyclical prevalence for parameter set $a = 0.163$, $b = 0.024$, $s = 0.61$, $q = 7.0$, $\tau = 50{,}000$ and for different values of the rate at which heavy users quit, $g_c \approx 0.0925 \leq g \leq 0.099$

Close to the PAH-bifurcation equilibrium $\hat{X}(g_c) = (576000, 149000)'$ the approximate radius of the cycles (as well as their period) may be estimated as a function of the parameter values (see, e.g., Hassard, Kazarinoff, and Wan (1981, p. 90); Strogatz (1994, p. 251) and Sect. 7.4.2). One realizes that the cycles "grow" – close to the bifurcation point with the size proportional to $\sqrt{g - 0.09405}$ and the periods estimated by $2\pi/\omega(g)$ – until g reaches 0.138, which corresponds to the hypothetical case of 89.5% of heavy users receiving treatment each year.

Thus, as Fig. 2.29 depicts, for a flow rate only infinitesimally larger than 0.09405, the numbers of light and heavy users would cycle counterclockwise forever with a period of approximately 70 years. Since such cycling in a social system is usually undesirable, this suggests that treating a constant, large proportion of heavy users throughout an epidemic, regardless of its course, may not always be the best policy. This leads directly to the idea explored in Behrens, Caulkins, Tragler, and Feichtinger (2000b), namely how treatment and other drug-control interventions should vary over the course of a drug epidemic. A brief summary is given in Sect. 6.1.

2.12 Advanced Topics

In this section we subsumed parts of the theory of dynamical systems not necessarily to be considered by the reader, who is not interested in the basic fundamentals for numerical calculations. Nonetheless, the following theory will play a crucial role in Chap. 7 where the numerical methods for computing optimal solutions are presented and can therefore not be neglected.

2.12.1 Stability of Limit Cycles

This subsection has one main goal, namely to analyze the stability of limit cycles. Compared to the case of equilibria, this needs more theoretical background and heavily relies on the consideration of nonautonomous linear systems. Thus even though nonautonomous systems play only a marginal, if any, role in the models presented here, we provide a brief summary of some principal results concerning nonautonomous linear systems. For a detailed presentation the reader is referred to Hartman (1982).

Nonautonomous Linear Systems

In a nonautonomous linear system, the dynamics becomes

$$\dot{x}(t) = J(t)x(t), \quad t \in \mathbb{R} \tag{2.68a}$$

$$\text{s.t.} \quad x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \tag{2.68b}$$

where $J(t)$ is an $n \times n$ matrix explicitly depending on time t.

Remark 2.73. Note that we cannot restrain the initial condition to the time point $t_0 = 0$, since the function $J(t)$ describing the dynamics explicitly depends on t and therefore a time shift of the form $t \to t - t_0$ leads for the same initial condition x_0 to a different solution of (2.68) (see Fig. 2.30). Thus if we want to point to this initial time dependence, the solution of (2.68) will be written in the form $x(t, t_0)$.

We shall now present the concept of a fundamental (principal) matrix in a way analogous to autonomous systems (see p. 33), in which the former considerations may help the reader to understand the underlying idea. The only, but important, difference is the dependence on the initially chosen time point t_0, which can always be assumed to be 0 for autonomous problems.

As the reader may directly verify, the superposition principle also holds in the nonautonomous case (see Exercise 2.8).

Now, assume that we know n solutions $x_i(t)$ of (2.68a) with initial conditions $x_i(t_0) = x_i^0$. Then the matrix $X(t, t_0)$ composed of the columns $x_i(t)$

$$X(t, t_0) = (x_1(t), \dots, x_n(t))$$

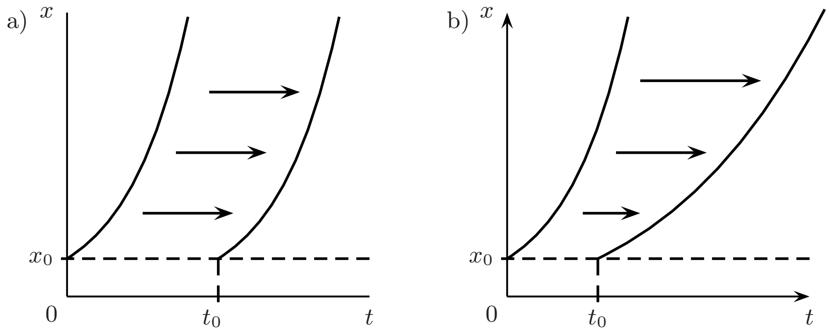

Fig. 2.30. In (**a**) the time-shifted solution of an *autonomous* ODE is depicted, exhibiting the invariance of the solution to such a transformation. In (**b**) we see the time-shifted solution of a *nonautonomous* ODE, revealing the dependence on the initial state x_0 as well on the initial time point t_0

satisfies the matrix differential equation

$$\dot{X}(t, t_0) = J(t)X(t, t_0) \tag{2.69}$$

with the initial condition $X(t_0, t_0) = \left(x_1^0, \dots, x_n^0\right)$. Note that, contrary to the analogue definition for autonomous systems, the initial time point t_0 is an argument of $X(t, t_0)$. Given the uniqueness of the solution of (2.68) and the superposition principle, the analogue proposition to Proposition 2.36 immediately follows:

Proposition 2.74. *The columns of a matrix solution $X(t, t_0)$ of (2.69) are linearly independent for all t iff they are linearly independent for some $t_1 \in \mathbb{R}$, especially $t = t_0$.*

The proof is analogous to that of Proposition 2.36 and is left to the reader (Exercise 2.16).

Now we can define:

Definition 2.75 (Fundamental Matrix Solution). *The time-dependent matrix $X(t, t_0)$ is called a* fundamental matrix solution *of (2.69) if it satisfies*

$$\dot{X}(t, t_0) = J(t)X(t, t_0)$$

with $\det X(t_0, t_0) \neq 0$*, i.e., $X(t_0, t_0)$ is invertible.*

Every fundamental matrix solution $X(t, t_0)$ can be transformed into a so-called *principal matrix solution* $Y(t, t_0)$ satisfying $Y(t_0, t_0) = \mathbb{I}^{(n)}$, by defining

$$Y(t, t_0) = X(t, t_0)X^{-1}(t_0, t_0).$$

Principal matrix solutions exhibit the following properties:

1. $Y(t,t_0) = Y(t,s)Y(s,t_0)$.
2. $Y(t,t_0)^{-1} = Y(t_0,t)$.
3. Given $Y(t,t_0)$, any solution $x(t)$ of (2.68) can be written in the form

$$x(t) = Y(t,t_0)x_0.$$

as the reader may verify.

The geometric interpretation presented for the autonomous case (see p. 33) clearly holds true for the nonautonomous problem, wherein the transformation of the original coordinate system $X(t_0)$ now depends explicitly on the initial time point t_0. Nonetheless, we can also ask how the determinant (volume) of the fundamental matrix (coordinate system) varies over time. By considering the Wronski determinant, the so-called *Liouville Formula* (2.70) can be proved (see, e.g., Hartman, 1982):

Proposition 2.76. *Let $X(t,t_0)$ be a fundamental matrix solution of the ODE (2.69), then the Wronski determinant $W(t,t_0) \det X(t,t_0)$ satisfies*

$$W(t,t_0) = W(t_0)\mathrm{e}^{\int_{t_0}^{t} \operatorname{tr} J(s)\,\mathrm{d}s}. \tag{2.70}$$

Remark 2.77. Note that (2.70) is exactly (2.24) for a time-varying $J(t)$.

However, the fundamental matrix solutions are more of theoretical than practical interest, since in general it is not possible to find an explicit solution matrix of (2.68). Nonetheless, further important properties can be derived if the dynamics $J(t)$ is periodic, a case that will subsequently be presented.

Floquet Theory

For the following considerations we analyze the ODE

$$\dot{x} = J(t)x, \tag{2.71}$$

where the corresponding dynamics $J(t)$ is periodic, i.e.,

$$J(t+\Theta) = J(t), \quad \text{for some} \quad \Theta > 0.$$

Owing to this functional form of the dynamics, the solutions exhibit a special structure, which is subsequently analyzed. In that case the solution of (2.68) is analogous to that of the linear autonomous case. This is stated in the following theorem (see, e.g., Hartman, 1982).

Theorem 2.78. *Let the system (2.68) have a periodic matrix function $J(t)$ with $t \in \mathbb{R}$ of period Θ. Then any fundamental matrix $X(t,t_0)$ of (2.68) has a representation of the form*

$$X(t,t_0) = \mathrm{e}^{Rt}C(t), \quad \textit{where} \quad C(t+\Theta) = C(t), \tag{2.72}$$

and $R \in \mathbb{C}^{n\times n}$ is a constant matrix.

Remark 2.79. Note that (2.72) is similar to the formula (2.22) for the autonomous system, with $C(t) \equiv X_0$ and $R = J$.

Adjoint Systems

In this book we are often confronted with so-called adjoint problems, which naturally occur in optimization problems. In some sense the adjoint problem describes the solutions orthogonal to the original space (see remark below). Thus associated to (2.68a) is the nonautonomous linear system

$$\dot{x}(t) = -J'(t)x(t), \tag{2.73}$$

which is called the *adjoint system*. This system plays an important role, e.g., in the numerical calculation of the (un)stable manifolds of a limit cycle (see Sect. 7.5.2).

Remark 2.80. Let us once more return to the geometric interpretation of a fundamental matrix solution as a coordinate system at the origin. Since the fundamental matrices $Y(t,t_0)$ of the adjoint equation (2.73) are the transposed inverse fundamental matrices $(X'(t,t_0))^{-1}$ of the original system (2.68a), the rows of $Y(t,t_0)$ are orthogonal to the columns of $X(t,t_0)$. This property makes it easy to describe solutions being orthogonal to solutions of the original system as solutions of the adjoint system.

The previous remark is proved by the following proposition:

Proposition 2.81. $X(t,t_0)$ *is a fundamental matrix solution of (2.68a) iff* $(X'(t,t_0))^{-1} = \left(X^{-1}(t,t_0)\right)'$ *is a fundamental matrix solution of (2.73).*

Where the proof is left as an exercise for the reader (Exercise 2.21).

Applying the Floquet Theory

Finally, we are ready to reap the fruit of this preliminary work and can tackle the question of stability for limit cycles. Thus let us consider a periodic solution $\Gamma(\cdot)$ with period Θ of the autonomous ODE

$$\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n, \quad t \in [0, \Theta]. \tag{2.74}$$

Then the corresponding variational equation is given by

$$\dot{y}(t) = f_x(\Gamma(t))y(t), \tag{2.75}$$

where the matrix $J(t) = f_x(\Gamma(t))$ is periodic with period Θ. The ODE (2.75) expresses the evolution of the deviation $y(\cdot) = \Gamma(\cdot) - x(\cdot)$, where $x(\cdot)$ is a solution of (2.74) for an initial condition $x(0) = \Gamma(0) + \delta x$. Similar to the case of an equilibrium, one would like to characterize the stability of the limit cycle $\Gamma(\cdot)$ by analyzing the associated variational equation (2.75).

Utilizing the results of the previous Sect. 2.12.1, we define:

Definition 2.82 (Monodromy Matrix). *Let $\Gamma(\cdot)$ be a periodic solution of the ODE (2.74), with period Θ, and let $X(\cdot)$ be a principal matrix solution of the variational equation (2.75) then the matrix*

$$\mathrm{M} = X(\Theta),$$

is called the monodromy matrix M *of the periodic solution $\Gamma(\cdot)$.*

Let $\xi_i,\ i = 1, \dots, n$ be the eigenvalues of the monodromy matrix M *then*

$$n_+ := \{i : \ |\mathrm{Re}\,\xi_i| < 1\},\ n_- := \{i : \ |\mathrm{Re}\,\xi_i| > 1\},\ \textit{and}\ n_0 := \{i : \ |\mathrm{Re}\,\xi_i| = 1\}.$$

It can be proved that the eigenvalues of the monodromy matrix M do not depend on the specifically chosen periodic solution $\Gamma(\cdot)$ ascribing the orbit, denoted by Γ, of the periodic solution. Using this property the following definition can be given:

Definition 2.83 (Hyperbolic Limit Cycle). *Let Γ be a limit cycle of the ODE (2.74); then the limit cycle Γ is called* hyperbolic *if a monodromy matrix* M *of Γ exhibits only one eigenvalue ξ with $\xi = 1$.*

If $n_+ n_- > 0$, then the limit cycle $\Gamma(\cdot)$ is called of hyperbolic saddle-type.

Remark 2.84. When considering a limit cycle $\Gamma(\cdot)$ we have to distinguish between its representation as a periodic solution and the corresponding limit set, denoted by Γ. Since the stability of the limit cycle (as a limit set) only depends on the eigenvalues of a corresponding monodromy matrix, which itself is independent of the actually chosen representation $\Gamma(\cdot)$, we will use the term limit cycle as a periodic solution and limit set synonymously.

Definition 2.85 (Manifolds of a Limit Cycle). *Let Γ be a limit cycle of the ODE (2.74) then the set*

$$\mathrm{W}^s(\Gamma) = \{x : \lim_{t\to\infty} \varphi^t(x) \in \Gamma\}$$

is called the stable manifold *of the limit cycle Γ and the set*

$$\mathrm{W}^u(\Gamma) = \{x : \lim_{t\to-\infty} \varphi^t(x) \in \Gamma\}.$$

is called the unstable manifold *of the limit cycle Γ*

The eigenvalues of a monodromy matrix M characterize the stability of a limit cycle Γ:

Theorem 2.86 (Stability of a Limit Cycle). *Let Γ be a limit cycle of the ODE (2.74) and let* M *be a monodromy matrix of the limit cycle. If $n_+ = n-1$ then the limit cycle is unstable, while if $n_- = n - 1$ the limit cycle is stable.*

The Liouville formula in Proposition 2.76 applied to the monodromy matrix yields

$$\det \mathrm{M} = \mathrm{e}^{\int_0^\Theta \operatorname{tr} f_x(\Gamma(t))\,\mathrm{d}t}.$$

This provides another criterion to check whether a limit cycle is unstable or not because utilizing Theorem 2.90, one can rewrite this identity as

$$\xi_1\xi_2\cdots\xi_{n-1} = \mathrm{e}^{\int_0^\Theta \operatorname{tr} f_x(\Gamma(t))\,\mathrm{d}t},$$

since $\det \mathrm{M} = \xi_1\xi_2\cdots\xi_{n-1}$. Thus the following corollary holds:

Corollary 2.87. *Let $\Gamma(\cdot)$ be a limit cycle of (2.74) with period Θ. If*

$$\mathrm{e}^{\int_0^\Theta \operatorname{tr} f_x(\Gamma(t))\,\mathrm{d}t} > 1$$

holds, then the cycle Γ is unstable.

Proof. This easily follows from the consideration that at least one of the eigenvalues has to satisfy $|\xi_i| > 1$; otherwise the product cannot be greater than one.

Similar to the case of an equilibrium the stability properties of limit cycles are now determined by the eigenvalues of the monodromy matrix. I.e., the monodromy matrix plays a similar role for cycles, than the Jacobian matrix does for the analysis of equilibria. What is the reason for such a characterization? In the following we give a short motivation for this interrelation.

Motivation for the Stability Results of a Limit Cycle

To motivate Theorem 2.86, we make a short detour by introducing the so-called Poincaré map, describing the dynamic behavior of points near an arbitrarily chosen point $x_\Gamma \in \Gamma$ of a limit cycle Γ.

Considering such a point x_Γ, we take the $n-1$-dimensional hyperplane

$$\Sigma = \{x : \ f(x_\Gamma)\,(x - x_\Gamma) = 0\}, \tag{2.76}$$

which is the plane crossing the limit cycle orthogonally at x_Γ (see Fig. 2.31).

Hence, in a neighborhood $U \subset \Sigma$ of x_Γ, we can define a (unique) function $\mathrm{P} : U \to \Sigma$, which is called the *Poincaré map*, by

$$\mathrm{P}(x) = \varphi^\tau(x),$$

where τ is the *first time* such that the trajectory starting at x crosses the plane Σ in a local neighborhood U of x_Γ, i.e., $\varphi^\tau(x) \cap \Sigma \cap U \neq \emptyset$ (see, e.g., Hartman, 1982).

Owing to this construction x_Γ is a critical value[18] of the Poincaré map, since $\mathrm{P}(x_\Gamma) = \varphi^\Theta(x_\Gamma) = x_\Gamma$. Moreover, the Poincaré map can (locally) be

[18] The critical value of a difference equation is the analogon to the equilibrium of a differential equation, i.e., the corresponding dynamics is zero.

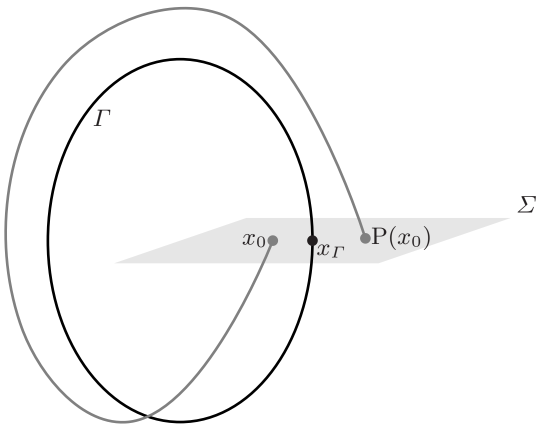

Fig. 2.31. Here the Poincaré map P of $\Gamma(\cdot)$ is illustrated

written as a discrete dynamical system in $\mathbb{R}^{n-1}$, since P is a map from $U \subset \Sigma$ into the $n-1$-dimensional hyperplane Σ. Hence the stability of x_Γ is described by the eigenvalues $\xi_i, i = 1, \dots, n-1$ of the Jacobian

$$\frac{\mathrm{d}\,\mathrm{P}}{\mathrm{d}x}(x_\Gamma)$$

of the Poincaré map P at x_Γ.

Remark 2.88. It is important to note that whereas the Poincaré map P depends on the arbitrarily chosen point $x_\Gamma \in \Gamma$, the eigenvalues $\xi_i, i = 1, \dots, n-1$ are independent from the point x_Γ. Thus it is geometrically evident that the stability of x_Γ in respect to the Poincaré map also describes the stability of the limit cycle Γ. In any case this has to be proved where a proof can be found, e.g., in Hartman (1982).

But for a discrete dynamical system an analogues theorem to Theorem 2.44 holds:

Theorem 2.89. *Let $\hat{x}$ be a critical state of the dynamical system (2.9), i.e., $f(\hat{x}) = 0$. If for every eigenvalue ξ of the Jacobian matrix $J(\hat{x}) = f_x(\hat{x})$*

$$|\mathrm{Re}\,\xi| < 1$$

holds, then $\hat{x}$ is stable. If there exist some eigenvalues ξ with

$$|\mathrm{Re}\,\xi| > 1,$$

then $\hat{x}$ is unstable.

The following theorem states the intimate connection between the Poincaré map and the solution of the variational equation:

Theorem 2.90 (Monodromy Matrix and Floquet Multipliers). *The monodromy matrix* M *exhibits the eigenvalues*

$$1, \xi_1, \ldots, \xi_{n-1},$$

where ξ_i correspond to the eigenvalues (characteristic Floquet multipliers*) of the Poincaré map associated with the cycle Γ.*

Here n_- denotes the number of eigenvalues ξ satisfying $|\mathrm{Re}\,\xi| < 1$ *and n_+ denotes the number of eigenvalues ξ satisfying* $|\mathrm{Re}\,\xi| > 1$.

Remark 2.91. The reason for 1 to be an eigenvalue relies on the geometric fact that the tangent vector on the limit cycle at x_0 is an eigenvector with corresponding eigenvalue 1.

2.12.2 Boundary Value Problems

So far we have concentrated our considerations on ODEs formulated as an IVP, i.e., a differential equation

$$\dot{x}(t) = f(x(t), t), \quad x \in \mathbb{R}^n, \;\; t \in [0, T] \tag{2.77}$$

$$\text{s.t.} \quad x(0) = x_0. \tag{2.78}$$

The usual association with this kind of problem is the Newton paradigm, which states that once we know the exact data of a body and any forces affecting it, we can describe its movement forever. But the situation changes if we have only partial information about its initial states. If no further information is available, we obtain a whole set of possible solutions. Providing further conditions for the end time T, we have to chose among the trajectories of this set of solutions, satisfying simultaneously the end constraint. It is obvious that this kind of problem may become more involved, since, depending on the further constraints at time T, one solution, many solutions, or no solution exists. In any case these problems naturally occur in optimal control problems since there we have to determine trajectories satisfying specific optimality conditions (see Chap. 3).

We now consider the general problem of finding a solution $x(t)$, where information is given for the two points $x(0)$ and $x(T)$, i.e.,

$$b\left(x(0), x(T)\right) = 0, \tag{2.79}$$

where b is a vector-valued function $b : \mathbb{R}^n \to \mathbb{R}^n$ called the *boundary condition*. Problems of the type (2.77) and (2.79) are therefore called *two-point boundary value problems (BVP)*.

Remark 2.92. Note that the IVP can be seen as a special form of the BVP, where the boundary condition (2.79) is given by

$$b\left(x(0), x(T)\right) = x(0) - x_0 = 0.$$

Naturally one can extend this problem by introducing *multi-point conditions*:

$$b\left(x(0), x(t_1), \dots, x(T)\right) = 0, \quad 0 < t_1 < \dots < t_N = T,$$

which is called a *multi-point boundary value problem*. Nonetheless, this is equivalent to a two-point BVP if one considers the following transformation

$$t \to \frac{t - t_i}{t_{i+1} - t_i}, \quad i = 0, \dots, N-1,$$

normalizing each interval $[t_i,\ t_{i+1}]$ to $[0,\ 1]$. Thus we find an augmented system

$$\dot{\bar{x}} = \bar{f}(\bar{x}, t), \quad \bar{x} \in \mathbb{R}^{Nn}, \quad t \in [0,\ 1], \tag{2.80a}$$

with

$$\bar{x}(0) = \left(x(0), x(t_1), \dots, x(t_{N-1})\right)' \quad \bar{x}(1) = \left(x(t_1), x(t_2), \dots, x(T)\right)',$$

and

$$\bar{f} = \left(t_1 f, \dots, (T - t_{N-1})\, f\right)'.$$

Furthermore the multi-point condition can be rewritten as

$$\bar{b}\left(\bar{x}(0), \bar{x}(1)\right) = 0, \tag{2.80b}$$

with $\bar{b}$ appropriately defined. To guarantee the continuity of the solution $x(\cdot)$ additionally $(N-1)n$ continuity conditions have to be considered, given by

$$\bar{x}(0) - \bar{x}(1) = 0. \tag{2.80c}$$

Thus we have transformed the multi-point BVP into a two-point BVP. This transformation is not without cost, since the system (2.80) is larger in dimension compared to the original system.

The question of existence and uniqueness of a solution for a BVP, even in the linear case, is more difficult to answer than for an IVP. As an example we consider the following simple example:

Example 2.93.

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x, \quad 0 \le t \le \pi,$$

under the condition

$$x_1(0) = 0, \quad x_2(\frac{\pi}{2}) = \beta.$$

Under the condition $x_1(0) = 0$ it immediately follows that $x_1(t) = \tau \sin t$ for τ arbitrary, whereas $x_2(t) = \tau \cos t$. Thus, if $\beta = 0$, there exist infinitely many solutions, whereas for $\beta \neq 0$ no solution exists at all. This is contrary to the situation of an IVP, where the problem with initial condition $x_2(0) = \beta$ exhibits a unique solution.

For the nonlinear case the situation can become much more involved, and the reader should be warned to inspect the results of BVPs carefully.

However, a simple consideration associates an IVP with the BVP (2.77) (2.79) and provides a necessary condition for the existence of the solution. If one assumes that the BVP (2.77) and (2.79) has a solution $x(\cdot)$, then the corresponding flow $\varphi^t(x_0)$, with $x(0) = x_0 \in \mathbb{R}^n$, satisfies $b\left(\varphi^0(x_0), \varphi^T(x_0)\right) = 0$. Since for every $y \in \mathbb{R}^n$

$$\begin{aligned} &\dot{x} = f(x,t), \quad t \in [0, T] \\ \text{s.t.} \quad &x(0) = y, \end{aligned}$$

and f continuous and continuously differentiable in x, exhibits a unique solution (see Picard–Lindelöf theorem (Theorem 2.3)). Then the existence of a solution of the BVP is equivalent to the existence of a y_0, satisfying the generally nonlinear equation

$$b\left(\varphi^0(y_0), \varphi^T(y_0)\right) = 0. \tag{2.81}$$

Thus we can formulate:

Theorem 2.94. *Suppose that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous and continuously differentiable in x. Then the BVP (2.77) and (2.79) has as many solutions as there are distinct roots y_0 of (2.81). For each of these y_0 a solution $x(\cdot)$ of the BVP is given by*

$$x(t) = \varphi^t(y_0).$$

Since nonlinear equations may have no solution, a finite number of solutions, or an infinite number of solutions, one can at best hope to prove that if a solution $x(\cdot)$ exists at all, that this solution is at least locally unique. It is possible, however, to prove that a solution $x(\cdot)$ is locally unique if the corresponding variational equation (see Sect. 2.6.2) given by:

$$\begin{aligned} &\dot{y} = f_x\left(x(t), t\right) y \\ \text{s.t} \quad &\frac{\partial b\left(x(0), x(T)\right)}{\partial x(0)} y(0) + \frac{\partial b\left(x(0), x(T)\right)}{\partial x(T)} y(T) = 0 \end{aligned}$$

has the unique solution $y \equiv 0$.

Remark 2.95. Numerical algorithms for solving BVPs mostly use iterative strategies to find a solution, whereby the user has to provide some initial function $\tilde{x}(\cdot) : [0, T] \to \mathbb{R}^n$. On the one hand, this seems different from an IVP, where for a numerical calculation no such function has to be made available; but, on the other hand, the initial condition $x(0) = x_0$ is already a solution function embracing its total future behavior.

Let us stop at this point and refer the reader to textbooks on BVPs, such as Ascher, Mattheij, and Russell (1995).

Examples for BVP Problems Occurring in Applied Optimal Control Theory

The reason why we introduce BVPs at all lies in its importance as an instrument for solving optimal control problems, since the corresponding necessary optimality conditions are given as a BVP of the form (see Sect. 3.3.1):

$$\dot{x} = \mathcal{H}^*_\lambda(x,\lambda) \tag{2.82a}$$

$$\dot{\lambda} = r\lambda - \mathcal{H}^*_x(x,\lambda) \tag{2.82b}$$

$$\text{s.t.} \quad x(0) = x_0 \in \mathbb{R}^n \tag{2.82c}$$

$$\lambda(T) = S_x\left(x(T)\right) \in \mathbb{R}^n, \tag{2.82d}$$

where $\mathcal{H}^* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ S \in \mathbb{R}^n \to \mathbb{R}$. Thus (2.82) is a BVP with linear boundary conditions (2.82c) and (2.82d).

A possible algorithm for solving such a problem is presented in Chap. 7. When solving an optimal control problem one also has to compute of limit cycles. a problem which can be formulated:

$$\begin{aligned} &\dot{x} = \Theta f(x), \quad t \in [0,\,1] \\ &x(0) = x(1) \\ &\psi(x) = 0, \end{aligned}$$

where Θ is the (unknown) period of the searched for limit cycle and $\psi : \mathbb{R}^n \to \mathbb{R}$ a scalar function characterizing a specific representation of the limit cycle.

Another problem is the calculation of the stable manifold of a saddle $\hat{x}$, which can be stated as:

$$\begin{aligned} &\dot{x} = f(x), \quad t \in [0,\,\infty) \\ \text{s.t.}\quad &x(0) \in \mathrm{W}^s \\ &\lim_{t\to\infty} x(t) = \hat{x}. \end{aligned}$$

Or as a last example we may be interested in an optimal solutions satisfying some constraints, so that at some time point $t_s \in [0,\,T]$ the dynamics switches between the solution path being in the interior and at the boundary of the constraint region. Then the problem becomes finding a solution $x(\cdot) \in \mathbb{R}^{2n}$ that satisfies

$$\begin{aligned} &\dot{x}^1 = f^1(x^1), \quad t \in [0,\,t_s] \\ &\dot{x}^2 = f^2(x^2), \quad t \in [t_s,\,T] \\ &x^1(0) = x_0 \\ &x^1(t_s) = x^2(t_s) \\ &x^2(T) = 0, \end{aligned}$$

with $f^i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2$, which is a typical multi-point BVP.

With all these problems the reader may be confronted when he/she tries to find a numerical solution of an optimal control problem, where algorithms are provided in the numerical part of the book Chap. 7.

This final section on BVPs concludes the preliminary work, and we now turn to the actual topic of this book, namely optimal control theory. In conclusion, the theory of dynamical systems is an indispensable part of optimal control theory, both in its usual formulation and in its numerical computation, justifying their lengthy discussion.

Exercises

2.1 (A Dynamic Model of Terrorism). Recently Udwadia, Leitmann, and Lambertini (2006) divided the population in a given region into three components: terrorists, those susceptible to both terrorist and pacifist propaganda, and nonsusceptibles. The flows between three classes incorporate the effects of both military or police and persuasive interventions.

Denote the number of terrorists in a certain area at time t as $x(t)$. The nonterrorist population in the region is made up of the population of susceptibles, $y(t)$, and nonsusceptibles, $z(t)$.

The number of terrorists in a given period of time can change for several reasons:

1. Recruitment by the terrorists of individuals from the susceptibles; the effectiveness of this is taken to be proportional to the product of the number of terrorists and the number of susceptibles.
2. The effect of antiterrorist measures that are directed at reducing the terrorist population, such as military and police action, which we assume increases rapidly, i.e., quadratically with the number of terrorists.
3. The number of terrorists who are killed in action, including those killed by suicide bombings, a number that we assume to be proportional to the terrorist population itself.
4. The increase in the terrorist population primarily through appeals by terrorists to other terrorist groups.

These four effects are captured in the differential equation

$$\dot{x} = axy - bx^2 + (c-1)x.$$

The change in the number of the susceptibles is likewise caused by several factors:

1. Depletion of their population caused by their direct contact with terrorists whose point of view they adopt.
2. Depletion of the population of susceptibles caused by nonviolent propaganda; this effect assumes that the propaganda intensifies rapidly as the number of terrorists increases, and we assume that the change this causes is proportional to the product x^2y.
3. An increase in the population of susceptibles caused by the propaganda that is created through the publicity of terrorist acts.

4. An increase in the susceptible population when individuals from outside the area are incited to move into the area; we assume that the changes in the y population attributable to this cause and the previous one are proportional to the number of terrorists in the region of concern.
5. The increase in the susceptible population proportional to its own size.

The evolution of the susceptible population adduced from these effects can be expressed as

$$\dot{y} = axy - ex^2y + fx + gy.$$

Last, the change in the number of nonsusceptibles, z, in a given interval of time is described by:

1. Those members of the susceptible population who become nonsusceptibles by virtue of having altered their persuasions because of the nonviolent actions or propaganda.
2. Those who become susceptibles owing to the effects of global propaganda spread by terrorists.
3. The increase in the z-population, which is proportional to its population numbers. This, then, may be described by the equation

$$\dot{z} = ex^2y - \bar{f}x + hz.$$

Udwadia, Leitmann, and Lambertini (2006) claimed that this three-dimensional nonlinear model captures some important features of the dynamics of tensional activity.

To study the long-term dynamical evolution, it is necessary to determine the equilibria. Then a phase-portrait analysis should be carried out. What do the projections in the two-dimensional subspaces look like? Can limit cycles occur?

A critical parameter is c, the ratio of the rate of increase of the terrorists. Can you identify other key parameters of the system?

2.2 (The One-Dimensional Terror Model Revisited). Show the existence and analyze the properties of the equilibria for the one-dimensional terror model (2.14) in Sect. 2.5; prove the following statements and interpret them in terms of the counter-terror measures u and v:

1. Consider $u > 0$ and $\mu\rho < \gamma$:
   a) For $v > 0$, $\mu\rho v - \gamma \ln(1+v) > 0$.
   b) For $v \ll 1$ only $\hat{S}_1$ exists.
   c) If $v \to 0$, then $\sqrt{\hat{S}_1} \to \frac{\tau}{\beta \ln(1+u)}$.
2. Consider $u > 0$ and $\mu\rho > \gamma$:
   a) There exists $v_0 > 0$, $\mu\rho v_0 - \gamma \ln(1+v_0) = 0$.
   b) For $v \ll 1$ $\hat{S}_1$ and $\hat{S}_2$ exist.
   c) If $v \to 0$, then $\hat{S}_2 \to \infty$.
3. There exists $v_0 > 0$ such that for $v > v_0$, $\dot{S} > 0$ for every S_0 and every time t.
4. There exists $u_0 > 0$ such that for $u > u_0$ at least one asymptotically stable equilibrium exists.

Hint. For questions (1a) and (2a) consider the first derivative

$$(\mu\rho v - \gamma \ln(1+v))'.$$

For questions (1b) and (2b) consider the Taylor expansion

$$\ln(1+v) = v - \frac{1}{2}v^2 + o(v^3).$$

For questions (2c) and (2c) consider the Taylor expansion

$$\sqrt{1-\Omega} = 1 - \frac{1}{2}\Omega + o(\Omega).$$

2.3 (A Push–Pull Corruption Model). Formulate and solve the corruption model described in Sect. 2.3, in which the flow from honest state to corrupt state is proportional to corruption $1-x$. Analyze the push–pull case

$$\dot{x} = kx(1-x)(U_c - U_h).$$

2.4 (The One-Dimensional Drug Model with Constant Treatment and Zero Enforcement). Analyze an extended version of the one-dimensional drug model of Sect. 2.9, with the system dynamics given by

$$\dot{A} = kA\left(\bar{A} - A\right) - \mu A - c\sqrt{u}\sqrt{A},$$

where the constant "control" u denotes treatment (see Sect. 3.9). Then

1. Find the equilibria of the system and determine their stability properties.
2. Consider u as a bifurcation parameter.

2.5 (The Rosenzweig & MacArthur Predator–Prey Model). Rosenzweig and MacArthur (1963) described the intertemporal interaction between a *predator population* and its *prey*. The variables $x(t)$ and $y(t)$ denote the population size of the prey and the predator populations, respectively, both at time t. The function $g(x)$ denotes the specific growth rate of the prey population and $h(x,y)$ is the so-called *predator response function*. We can describe the entire ecosystem's dynamics as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} g(x)x - h(x,y)y \\ eh(x,y)y - my \end{pmatrix}.$$

We assume that the natural habitat of the prey species is confined so that in the absence of the predator species the prey only grows exponentially if the population size is very small. Then the prey population's growth slows down and approaches a maximum level, the so-called *carrying capacity* K. This yields the *logistic growth function*,

$$g(x) = r\left(1 + \frac{x}{K}\right).$$

Thus the prey population x is increased by natural processes and is reduced by intraspecific competition and by the predator y. The predator response function h can be regarded as the per capita prey consumed by every individual of the predator population and measures, therefore, y's difficulty in finding suitable food. This is modeled by a *Holling-type* II (see Holling, 1959) function, i.e.

$$h(x,y) = h(x) = \frac{ax}{b+x},$$

where parameter a denotes an upper limit on daily consumption of the predator and b is the so-called *half-saturation constant*. Parameter e accounts for the metabolic efficiency loss that translates the food of the predator into offspring. Thus the net change in y is determined by subtracting the (exogenous) death rate m from the fertility rate of the predator, defined by $eh(x)$, as by assumption h does not depend on y.

To study the ecological evolution of the two types of species, it is helpful to carry out a bifurcation analysis with respect to e. Thus, when answering these questions it is useful to distinguish between three different efficiency ranges, i.e., intervals for the efficiency e to convert prey into offspring,

$$\left[\frac{m}{a}, \frac{m}{a} + \frac{mb - K}{a}\right), \; \left[\frac{m}{a} + \frac{mb - K}{a}, \frac{m}{a} + \frac{K + b}{K - b}\right), \; \left[\frac{m}{a} + \frac{K + b}{K - b}, \infty\right),$$

and draw the corresponding phase plots.

Hint. Utilizing the Poincaré–Bendixson theorem might be useful.

2.6 (Dynamic Fishery Model). Let x denote the size of a population of fish that is harvested at a certain rate $h(x)$. Assume that the growth of this population is bounded by intraspecies competition. Then x grows toward a known carrying capacity K, as embodied in a *logistic growth function*. Together this leads to the following model of the exploitation of a renewable resource (Gordon–Schäfer model Gordon (1954))

$$\dot{x} = rx\left(1 + \frac{x}{K}\right) - h(x). \tag{2.83}$$

Determine the system behavior (including the equilibria) for the following types of harvest functions $h(x)$.

$$h(x) = h, \tag{T1}$$

$$h(x) = Ex, \quad E > 0. \tag{T2}$$

What types of bifurcations can be identified for the bifurcation parameters $h > 0$ and $E > 0$, respectively? What are the consequences for the modeled harvesting strategies?

2.7. Prove by formally differentiating the sum

$$\mathrm{e}^{Jt} := \mathbb{I}^{(n)} + Jt + \frac{J^2}{2!}t^2 + \frac{J^3}{3!}t^3 + \cdots$$

that the exponential function satisfies the linear ODE

$$\dot{x} = Jx.$$

2.8. Prove the principle of superposition for the linear autonomous and nonautonomous problem.

2.9. Prove Proposition 2.36.

Hint. Prove by contradiction that there exists some t_1 such that $X(t_1)$ is singular. Now use the uniqueness of the solution and demonstrate that the zero function is a specific solution.

2.10 (Non-Unique Center Manifold). Consider the ODE (2.31) then:

- Determine the eigenspaces for the Jacobian at the origin
- Approve the equation for the orbits
- Prove that the center manifolds are infinitely often differentiable at 0

2.11. Prove Proposition 2.74, i.e., the columns of a fundamental matrix solution are linearly independent for all t iff they are linearly independent for some $t_1 \geq 0$.

2.12 (An Advertising Diffusion Model). Consider the following marketing model

$$\dot{x} = a(1-x) - \delta x \tag{2.84a}$$
$$\dot{a} = \alpha(x, a), \tag{2.84b}$$

where x denotes the market share of a firm and a its advertising rate. Customers change to another firm (or product) with rate $\delta \geq 0$. The advertising intensity changes according to a feedback rule $\alpha(x, a)$, whereby it makes some sense to assume

$$\alpha_x > 0, \alpha_{xx} \geq 0; \qquad \alpha_a > 0, \alpha_{aa} \leq 0; \qquad \alpha_{xa} \geq 0.$$

Note that the conditions of the first diffusion model of Gould (1970) deliver

$$\alpha(x, a) = \left(r + \frac{\delta}{1-x}\right) a$$

provided that the advertising costs are quadratic ($c(a) = 1/2a^2$) and the return of sales is linear ($\pi(x) = x$); compare Gould (1970) and Feichtinger and Hartl (1986, p. 325), as well as Exercise 3.5.

Carry out a phase-portrait analysis of the system (2.84). Show that the stable manifold has a negative slope and interpret the result.

2.13 (Price Dynamics). Consider the following continuous-time version of the famous Cobweb model. Let the demand for a good at time t be a linearly decreasing function of its price and the supply s a linearly increasing function, i.e.,

$$d = ap + b, \qquad a < 0,\ b > 0$$
$$s = cp + d, \qquad c > 0,\ d < 0.$$

The price dynamics assumes that it adapts according to the excess demand, i.e.,

$$\dot{p} = \gamma\,(d - s)\,,$$

where γ measures the speed of approaching the equilibrium price.

Determine the equilibrium price and describe the adjustment dynamics. Which sign of γ implies the stability of the long-run equilibrium? Find a *discrete* version of the model and illustrate the "cobweb" in a (p_t, p_{t+1})-diagram.

2.14 (Environmental Pollution and Capital Accumulation). Denote by $P(t)$ the stock of pollution at time t and by $K(t)$ the capital stock of an economy. Let $F(K)$ be the produced output by the (single) production factor K ($F' > 0$, $F'' < 0$). Assume that the output can be used for abatement A, being consumed or invested for capital accumulation, I, i.e.,

$$F(K) = A + C + I.$$

To simplify, let us suppose that constant consumption $C = 0$. It is plausible to consider a state-dependent (linear) feedback rule for abatement

$$A = \varphi P, \quad \varphi > 0.$$

Then the interaction of the state variables K and P is described as follows

$$\dot{K} = F(K) - \varphi P - \delta K$$
$$\dot{P} = \varepsilon F(K) - G(\varphi P) - \alpha P,$$

where $\delta \geq 0$ denotes the depreciation rate of the capital, $\varepsilon > 0$ is the emission rate of the production, and α is the natural cleaning rate of the level of pollution. The function $G(A)$ measures the impact of abatement A on the reduction of the pollution ($G' > 0$, $G'' \leq 0$).

Find the isoclines and the equilibrium $(\hat{K}, \hat{P})$. What are the local stability properties of this equilibrium? Carry out a sensitivity analysis of $(\hat{K}, \hat{P})$ with respect to the parameters and interpret the results. Discuss other intuitively appealing feedback mechanisms of P on $\dot{K}$.

2.15 (A Growth Model). Consider the following "variant" of Solow's growth model (Solow, 1956)

$$\dot{K} = sF(K, L) - \delta K,$$

where $s \in (0, 1)$ is the saving rate, δ is the depreciation rate of capital, K is an economy's capital stock, L denotes the economy's labor force, and $F(K, L)$ is the neoclassical production function, assumed, e.g., to be of Cobb–Douglas type

$$F(K, L) = K^{\alpha} L^{\beta}, \qquad \alpha, \beta \in (0, 1).$$

Suppose that the stock of labor force develops according to the difference between the marginal product of labor, F_L and a constant wage rate w (comment on this assumption), i.e.,

$$\dot{L} = L\left(F_L(K, L) - w\right).$$

Carry out a phase-portrait analysis in the (K, L)-plane. Does there exist an equilibrium, and what might be its stability properties?

2.16. Prove Proposition 2.74

2.17 (PAH_Bifurcation in an Advertising Model). In Feichtinger (1992) the following descriptive marketing model is motivated:

$$\dot{x} = k - axy + \beta y \tag{2.85}$$
$$\dot{y} = axy - \delta y,$$

where x denotes the number of potential buyers of a good and y the customers of this brand. The parameter $\delta = \beta + \varepsilon$, where β measures the brand-switching rate and ε measures the dropout rate of the market. The term axy is the usual diffusion term where word-of-mouth is supported by the advertising rate a. According to the so-called Dorfman-Steiner rule, a constant percentage of sales revenue should be allocated to advertising (see, e.g., Feichtinger & Hartl, 1986, p. 314). Thus, by substituting $a = \alpha y$ into (2.85), we get a two-dimensional nonlinear system (which has similarities to the so-called Brusselator[19]).

Carry out a bifurcation analysis. Show that there exists a unique equilibrium. For $\gamma = \alpha k^2/\delta\varepsilon^2$, there exists γ_i, $i = 1, 2$ with $0 < \gamma_1 < 1 < \gamma_2$ such that the equilibrium is

$$\begin{array}{lll} \text{an unstable node} & \text{iff} & \gamma \in [0,\, \gamma_1) \\ \text{an unstable focus} & \text{iff} & \gamma \in (\gamma_1,\, 1) \\ \text{a stable focus} & \text{iff} & \gamma \in (1,\, \gamma_2) \\ \text{a stable node} & \text{iff} & \gamma \in (1,\, \gamma_2). \end{array}$$

Furthermore, show that there exists γ_3 with $\gamma_1 < \gamma_3 < 1$ such that for all $\gamma \in (\gamma_3,\, 1)$ there is a *stable* limit cycle. Give an economic interpretation of this persistent oscillation by dividing it into four regimes.

2.18 (Easterlin Cycles). In Easterlin (1961) an explanation for the US baby boom and its reversal was offered. The following two-dimensional system connects the per capita capital stock k and the aspiration level measured as an expected income

$$z(t) = \gamma \int_{-\infty}^{t} \mathrm{e}^{-\gamma(t-s)} g(k(s))\, \mathrm{d}s. \tag{2.86}$$

Here $g(k) = f(k) - \delta k$ is the actual net income, where $f(k)$ denotes the per capita production function and δ the depreciation rate of the capital stock. The parameter γ measures the speed of adjustment. Somewhat surprisingly, (2.86) is equivalent to

$$\dot{z} = \gamma\,(g(k) - z)\,.$$

The dynamics of the capital stock is given as

$$\dot{k} = f(k) - \delta k - c - (b - d)k,$$

where $n = b - d$ is the natural growth rate of a (closed) population written as the difference between the crude birth rate, b, and crude death rate, d. While the latter is assumed to be constant, the former is assumed to depend positively on the difference between the actual and the expected income

$$b = b(g(k) - z) \quad \text{with} \quad b > 0,\ b' \geq 0.$$

Finally, suppose consumption per person depends positively on z, i.e.,

$$c = c(z) \quad \text{with} \quad c > 0,\ c' \geq 0.$$

Using the Poincaré–Bendixson theorem derive sufficient conditions for the existence of a stable limit cycle. Try to give a demo-economic interpretation of the cycle's orientation.

2.19 (The Lorenz Attractor). The Lorenz attractor is a well-known example of a *chaotic flow*, noted for its butterfly shape. The equations

[19] The Brusselator is an example for an autocatalytic, oscillating chemical reaction.

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

were introduced by Lorenz (1963), who considered an atmospheric model, by simplifying *Navier–Stokes* equations occurring in fluid dynamics. Roughly speaking, the state x measures the rate of convective overturning, the state y denotes the horizontal temperature variation, and the variable z is the vertical temperature variation. The three parameters σ, ρ, and β are proportional to the *Prandtl* number, the *Rayleigh* number, and some physical proportions (see Sparrow, 1982). For the parameter values $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$ calculate the equilibria and paths of the two-dimensional stable manifold at the origin.

2.20 (Inhomogeneous Linear Differential Equation). Consider the ODE

$$\dot{x}(t) = kx(t) + u(t), \quad k, x, u \in \mathbb{R},$$
$$\text{s.t.} \quad x(0) = x_0$$

which is called a *linear inhomogeneous* differential equation. Show that the solution is given by

$$x(t) = \mathrm{e}^{kt}\left(x_0 + \int_0^t \mathrm{e}^{-ks}u(s)\,\mathrm{d}s\right),$$

and that for $k > 0$ the inequality

$$|x(t)| \le \mathrm{e}^{kt}\left(|x_0| + \int_0^t |u(s)|\,\mathrm{d}s\right)$$

Hint. Use the so-called variation of constants method *by setting $x(t) = c(t)\mathrm{e}^{kt}$ and determine $c(t)$. To derive the inequality note that for $k > 0$ $\mathrm{e}^{-ks} \le 1$ for all $s \ge 0$.*

2.21 (Adjoint Solution). Prove Proposition 2.81.

Hint. Consider the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\left(X^{-1}(t, t_0)\right)$, which follows by differentiating the identity $X^{-1}(t, t_0)X(t, t_0) = \mathbb{I}^{(n)}$.

Notes and Further Reading

The theory of dynamical systems and ODEs is one of the central areas of mathematical research. It develops techniques required for the analysis of intertemporal decision problems arising in dynamic economics and management sciences. This chapter may be seen as a compact reference (and toolkit) for solving dynamic optimization problems. In addition and among other references, we recommend the following three well-written, application-oriented books: Léonard and Long (1992), on dynamic modeling in the various fields of optimal control theory; Strogatz (1994), on physics and engineering; and Hofbauer and Sigmund (1988), on biomathematics.

A fine introduction to economic applications of nonlinear dynamical systems theory is given by Hommes (1991); compare also Lorenz (1993).

Compared with Léonard and Long (1992), we treat parts of the theory of dynamical systems in more depth, not least because one of our intentions is to present concrete algorithms for solving optimal control problems. An excellent introductory book on the theory of dynamical systems for readers starting from a background of only calculus and elementary linear algebra is Hirsch, Smale, and Devaney (2002) which is a revised edition of the classical textbook Hirsch and Smale (1974).

Our approach has made it necessary to refer to more theoretically oriented books, such as Hartman (1982), in which the theory of linear nonautonomous systems, together with the Floquet theory, are presented. The Floquet theory is helpful for stating and deriving some of the results used in the computation of limit cycles and the corresponding manifolds in Chap. 7.

The presentation of dynamical systems as nonlinear ODEs can be found mainly in the books of Arrowsmith and Place (1990); Strogatz (1994); Wiggins (1990), and Arnold (1983), where the unfamiliar reader is referred to Strogatz (1994).

The introduction of an ODE in the sense of Definition 2.1 is given under very restrictive assumptions. To capture more general problems, extensions in various directions can be considered. One of these extensions is the concept of a Carathéodory solution, where the integral equation (2.8) is understood in the Lebesgue sense. Other extensions allow the dynamics f to be multivalued, yielding a so-called *differential inclusion* (see, e.g, Aubin & Cellina, 1984). Especially useful in the context of PDEs are so-called *viscosity solutions* introduced by Crandall and Lions (1983), which extend the classical solution concept of PDEs.

A good introduction to bifurcation and stability analysis (from equilibrium to chaos) is provided by Seydel (1994), see also the excellent book by Troger and Steindl (1991).

Kuznetsov (1998) and Guckenheimer and Holmes (1983) are highly recommended as advanced books on bifurcation theory. Many of our examples in the text, in particular those of a normal form and nonhyperbolic equilibrium, are taken from Kuznetsov (1998). We also refer the reader to the MATLAB toolbox "MatCont" provided by the group Kuznetsov, which can be used to do numerical bifurcation analysis. This toolbox allows the numerical computation not only of bifurcations concerning equilibria but also of the limit cycles. Moreover bifurcations of higher codimension[20] (codim > 1) can be analyzed.

The section on BVP is taken mainly from Ascher et al. (1995), which includes a comprehensive overview of the numerical methods in the field of BVPs. It also contains a well-written overview of the important results from ODEs.

The Poincaré–Andronov–Hopf bifurcation, where an equilibrium gives rise to a limit cycle, was first studied by the French mathematician Henri Poincaré in 1892 while he was investigating the planetary three-body problem. Poincaré did many things and can be said to have been the originator of both algebraic topology and the theory of analytic functions of several complex variables. Paying tribute to his pioneering work, the theorem (discussed in Sect. 2.10) bears his name, together with the names of A.A. Andronov and H. Hopf. The latter two mathematicians independently discovered periodic behavior to be a result of a local bifurcation – Andronov in the late 1920s (Andronov, 1929) and Hopf in the early 1940s (Hopf, 1943).

[20] The *codimension* of a bifurcation is the number of independent conditions determining the bifurcation.

The corruption model presented in Sect. 2.3 has been selected mainly because of its simplicity. There is a small, but important, literature on dynamic economic modeling of corruption. A good introductory survey is Andvig (1991).

Rinaldi, Feichtinger, and Wirl (1998) studied the dynamics of corruption at the top, i.e., corruption by politicians. Their framework relates actions (positive actions of the politicians, bribery, consumption, and the unveiling of corruption) to the state variables – popularity (as a proxy for power), assets accumulated from bribes, and investigation. The authors differentiate corrupted and uncorrupted systems, and in the class of corrupted systems distinguish between strongly controlled (steady behavior), weakly controlled (cyclic behavior), and uncontrolled (wild behavior). For example, increasing the persistence of investigators can control the system but cannot eliminate corruption, whereas institutional changes that lower the practice of giving and taking bribes can.

Caulkins, Gragnani, Feichtinger, and Tragler (2006) extended the LH-model presented in Sect. 2.11 by considering two additional states that represent in more detail newly initiated ("light") users' response to the drug experience. Those who dislike the drug quickly "quit" and briefly suppress initiation by others. Those who like the drug progress to ongoing ("moderate") use, from which they may or may not escalate to "heavy" or dependent use. Initiation is spread contagiously by light and moderate users, but is moderated by the drug's reputation, which is a function of the number of unhappy users (recent quitters and heavy users). The proposed model generates complex behavior of the solution paths including cycles, quasi-periodic solutions, and chaos. The model has the disquieting property of endogenously generating peaks followed by a crash to low levels and a dramatic resurgence of use. Thus, even if the current drug epidemic ebbs and we enjoy a long period of low rates of use, that does not guarantee that drug epidemics will not return with equal ferocity.

Behrens, Caulkins, Tragler, and Feichtinger (2000a) augmented the LH-model (see Sect. 2.11) by an awareness variable or a "memory of the adverse consequences of heavy-use years" (compare also Sect. 6.1 and Exercise 6.5). If past bad experiences fade away quickly, drug incidence increases. The resulting increase in the number of occasional drug users leads to a future increase in the number of heavy users who contribute to the accumulation of a higher level of awareness or memory that in turn suppresses incidence. This interaction is capable of production cycles.

Heterogeneity in intensity of activity is of course not restricted to the modeling of drug epidemics. It is just as relevant in modeling economic activity (high- and low-wage earners), the transmission of infectious diseases such as HIV/AIDS (people who frequently or less frequently interact with others in ways that can transit the virus), environmental problems (high- vs. low-rate consumers of a natural resource), and a host of other domains.

The logistic model occurring in Exercises 2.5 and 2.6, developed quite early by the Belgian mathematician Verhulst (1838), is often used to describe the growth of populations.

Before refining them along several lines (see, e.g., Exercise 2.5), the "classical" predator–prey model was independently invented in the mid 1920's by and named after A. Lotka and V. Volterra. Hofbauer and Sigmund (1988) provide a good introduction as well as some ramifications.

An extended analysis of the cobweb phenomenon dealt within Exercise 2.13 may lead to instability and even chaos (see Chiarella, 1988).

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