Chapter 1
Theory of Non-homogeneous Shells

In this chapter a general theory of non-homogeneous shells is introduced. First, fundamental relations and assumptions are given, non-homogeneities of shells are introduced, and then the governing variational equations and equations of motion are derived. After the boundary and initial conditions are introduced, the equations are cast into non-dimensional form and the so-called variable parameters of a shell stiffness are defined. In addition, a flexural stiffness coefficient of a shell element is formally introduced. The chapter concludes with a consideration of generalized functions.

1.1 Preliminary Remarks

It is generally agreed that after creation of a shell's model, which should as closely as possible describe the real existing objects (shells), many simplifications are necessary, especially regarding the construction of a shell and the material of which it is made. Shell constructions that are observed in many technical applications can be of various types, for instance, one- or multiple-layered, reinforced by ribs, waffled, etc.

A shell material can be either isotropic or anisotropic. Analysis of the shell dynamics is usually problematic. On the other hand, proper introduction of a model simplification decreases the computational time, which leads to growth of the practical applications. The investigations that were made result in a conclusion that the selection of a shell model should be made with particular respect to the economics of computations and the sufficiently exact model of the shell.

In this book, the shells that are mainly considered are made of isotropic and elastic material. Hook's law satisfies a static and dynamic behavior of those shells. Plastic and adhesive-plastic deformations of shells are not analyzed here, because they are usually associated with the class of problems associated with other constructive non-homogeneities.

The main characteristic attribute connected with the analysis of dynamics and statics of thin shells is recognized in the theory of elasticity as a reduction of the

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three-dimensional problem to the two-dimensional one. If only one shell is investigated the coordinate system has to be joined with the central surface of a shell. One of the possibilities of reduction of the three-dimensional problem to a two-dimensional one is the consideration of a hypothesis devoted to the lack of straight perpendicular lines to the shell, that is, to the conservation of the Kirchhoff-Love hypothesis. According to the hypothesis, any fiber that is perpendicular to the central surface before a deformation remains perpendicular until the whole deformation process is finished. Furthermore, the length of that fiber measured along the shell's thickness remains constant. Additional assumptions are based on the observation that the stresses perpendicular to the shell can be neglected because of their negligible smallness in comparison to the basic ones. In the theory of shells, basic stresses are understood as the stresses occurring in some parallel layers that are perpendicular and tangent to the central surface of these shells.

1.2 Fundamental Relations and Assumptions

Let us consider a rectangular shell (Fig. 1.1) the central surface of which is bounded by a closed curve Γ and let us assign to it a rectangular coordinate system x, y.

Let an axis perpendicular to the central surface be denoted by z, and the positive aspect of the axis be directed to the curvature's center. Moreover, assume in our investigations a right-handed coordinate system. From the considerations assumed the shell coordinates meet the following inequalities:

$$0 \le x \le a, \quad 0 \le y \le b, \quad -h \le z \le h. \tag{1.1}$$

The displacement of points of the shell's center surface in directions x, y, z are denoted by u, v, w, respectively.

According to the classical theory of shells, the displacement of an arbitrarily selected point of a shell in direction z is not dependent on z and is moreover identical for all points of the considered element, which are located on the straight line that is perpendicular to the shell's central surface. Indeed, we have assumed that the shell's material is isotropic, but we will always allow the element made of a material of another elasticity modulus to be included with the analyzed isotropic shell. In addition, the shell's shape variations are possible if its thickness varies in time. In other words, the changes can be described by coordinate-dependent functions.

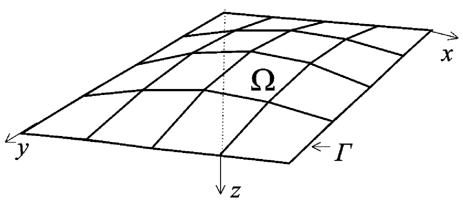


Fig. 1.1 Scheme of the analyzed shell

Deformations are mainly expected to be in agreement with Hook's law. Following the assumptions we have made, the central surface deformations for an isotropic shell material, which is characterized by elasticity modulus E, of the first type, elasticity modulus G of the second type (compression modulus), and a Poisson's coefficient μ are connected with deformations σ_{xx}, σ_{yy}, and σ_{xy} using the following relations:

$$\begin{aligned} e_{xx} &= \frac{q}{E}\left(\sigma_{xx} - \mu\sigma_{yy}\right), \\ e_{yy} &= \frac{q}{E}\left(\sigma_{yy} - \mu\sigma_{xx}\right), \\ e_{xy} &= \frac{2\left(1+\mu\right)}{E}\sigma_{xy}. \end{aligned} \tag{1.2}$$

Solving Eq. (1.1) with respect to the deformations that have occurred, we obtain

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\mu^2}\left(e_{xx} + \mu e_{yy}\right), \\ \sigma_{yy} &= \frac{E}{1-\mu^2}\left(e_{yy} + \mu e_{xx}\right), \\ \sigma_{xy} &= \frac{E}{2\left(1+\mu\right)}e_{xy}. \end{aligned} \tag{1.3}$$

Corresponding to the Kirchhoff-Love model, the displacement of the central shell's layer points along the considered coordinates depends on x, y coordinates and on time $t:\ u = u(x,y,t),\ v = v(x,y,t)$ (displacements $w = w(x,y,t)$ are involved in a similar way).

Displacement of any point of the coordinate z before deformation is

$$\begin{aligned} u^z &= u - z\frac{\partial w}{\partial x}, \\ v^z &= v - z\frac{\partial w}{\partial y}, \\ w^z &= w. \end{aligned} \tag{1.4}$$

Fully described deformations that occurred in a layer during both extension and rotation of a shell placed at a distance from the central surface and in accordance with the nonlinear theory of shells are given below [234]:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u^z}{\partial x} - k_x w + \frac{1}{2}\left(\frac{\partial u^z}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v^z}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ \varepsilon_{yy} &= \frac{\partial v^z}{\partial y} - k_y w + \frac{1}{2}\left(\frac{\partial u^z}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial v^z}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ \varepsilon_{xy} &= \frac{\partial u^z}{\partial y} + \frac{\partial v^z}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \end{aligned} \tag{1.5}$$

In the case of thin flexible plates and shells it is assumed that the deflection angles $\partial w^z/\partial x$, $\partial w^z/\partial y$ are significantly larger than the values of derivatives $\partial u/\partial x$, $\partial u/\partial y$, etc. that are connected with the volume of material. Furthermore, squares of derivatives $(\partial w/\partial x)^2$ will be of the same order as components $\partial u^z/\partial x$, etc., which means the terms such as $(\partial u/\partial x)^2$ can be neglected.

Satisfying the above assumptions, expressions (1.5) are simplified:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u^z}{\partial x} - k_x w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\
\varepsilon_{yy} &= \frac{\partial v^z}{\partial y} - k_y w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\
\varepsilon_{xy} &= \frac{\partial u^z}{\partial y} + \frac{\partial v^z}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.
\end{aligned} \tag{1.6}$$

Taking into consideration Eq. (1.4), we obtain

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} - k_x w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}, \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} - k_y w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2}, \\
\varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y},
\end{aligned} \tag{1.7}$$

or equivalently

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{11} - z\frac{\partial^2 w}{\partial x^2}, \\
\varepsilon_{yy} &= \varepsilon_{22} - z\frac{\partial^2 w}{\partial y^2}, \\
\varepsilon_{xy} &= \varepsilon_{12} - 2z\frac{\partial^2 w}{\partial x \partial y},
\end{aligned} \tag{1.8}$$

where the central surface deformations are governed by the following relations:

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u}{\partial x} - k_x w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\
\varepsilon_{22} &= \frac{\partial v}{\partial y} - k_y w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\
\varepsilon_{12} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.
\end{aligned} \tag{1.9}$$

By putting the variables given in Eq. (1.2) into the following expressions, which define static equivalent deformations in a section of shell

$$T_{11} = \int_{-h}^{h} \sigma_{xx}\, dz, \quad T_{22} = \int_{-h}^{h} \sigma_{yy} dz,$$

$$T_{12} = \int_{-h}^{h} \sigma_{xy}\, dz, \tag{1.10}$$

we obtain the final relations that define stresses and deformations of a mean surface of the form

$$\begin{aligned} T_{11} &= \frac{2hE}{1-\mu^2}\left(\varepsilon_{11} + \mu\varepsilon_{22}\right), \\ T_{22} &= \frac{2hE}{1-\mu^2}\left(\varepsilon_{22} + \mu\varepsilon_{11}\right), \\ T_{12} &= \frac{2hE}{2\left(1+\mu\right)}\varepsilon_{12}. \end{aligned} \tag{1.11}$$

Solving Eq. (1.11) with respect to ε_{ij}, we obtain

$$\begin{aligned} \varepsilon_{11} &= \frac{a_1}{2h}\left(T_{11} + \mu T_{22}\right), \\ \varepsilon_{22} &= \frac{a_1}{2h}\left(T_{22} + \mu T_{11}\right), \\ \varepsilon_{12} &= \frac{2\left(1+\mu\right)a_1}{2h}T_{12}. \end{aligned} \tag{1.12}$$

1.3 Non-homogeneity of a Shell

Flexural stiffness and the density of the selected part of a non-homogeneous shell can vary according to either the adjoining new material characterized by another elasticity modulus, or as a result of local change in thickness of the shell. For further considerations of all cases investigated, we will assume that toward the central shell surface the shell's shape is symmetric. Such an assumption allows for introduction of $2h_1$, a thicker additional element. As a result, the flexural stiffness D of the shell's part will change, leading as a consequence to modification of the parameter E through the contractual "rarefaction (compaction)" of its thickness between $2h_1$ and $2h$.

This procedure is used if and only if the shells of either step-variable thickness or parts of those shells made of different elasticity modulus materials are treated in the same manner (see [251]).

Such shell regions are characterized later by time-variable parameters of stiffness. Let us assume that in addition the shell can be composed of any number of rectangular parts described by different flexural stiffnesses, which are oriented in such a way that the lines bounding them are parallel to the appropriate shell edges.

Using that approach one can assume that the shell is of constant thickness, and its non-homogeneity is fully characterized by changes in its elasticity modulus E (a similar assumption is made for the density ρ of a shell material). Finally, it is considered, that $E = E(x,y),\ \rho = \rho(x,y)$.

1.4 Variational Equations

Using energetic considerations one can obtain equations of motion for a shell, which contain some rectangular elements of the flexural being different from the basic of shells. As pointed out in Sect. 1.3, conditions $E = E(x,y),\ \rho = \rho(x,y)$ are assumed.

A method similar to that analyzed in Volomir's monograph [304] is used for further considerations; i.e., a motion occurring in a time interval $[t_0,\ t_1]$ is examined. For this reason, some different trajectories of motion of the system characteristic points between any starting and final states will be compared. The true trajectories will be slightly different than the other ones, because the following condition, which is satisfied for the true trajectories, has to be met:

$$\int_{t_0}^{t_1} \left(\delta K - \delta V + \delta' W\right)\, dt = 0, \tag{1.13}$$

where K is the kinetic energy, V is the system potential energy, $\delta' W$ is the sum of elementary works done by all external forces.

When all forces acting on a system have a potential Π, then Eq. (1.13) takes the form

$$\delta S = \delta \int_{t_0}^{t_1} (K - V - \Pi)\, dt = 0, \tag{1.14}$$

where $S = \int_{t_0}^{t_1} (K - V - \Pi) dt$ denotes an action in Hamilton sense, and $t \in [0,\ T]$.

The last equation expresses the known Hamilton principle.

Elasticity properties of various bodies can be characterized by the energy of their deformation. Applying the most comprehensive formula of the theory of elasticity describing unity incrementation of the mechanical work of deformations and taking into consideration the lack of normal (perpendicular) deformations hypothesis, the potential energy of deformation can be estimated by means of the following rule:

$$V = \frac{1}{2} \iint_{\Omega} \int_{-h}^{h} \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{xy}\varepsilon_{xy}\right)\, dz\, ds, \tag{1.15}$$

for which it was assumed that $ds = dx\, dy$, and Ω is the integration region bounded by Γ (see Fig. 1.1).

Substituting (1.2) and (1.3) into Eq. (1.15) and carrying out an integration we obtain

$$V = V_1 + V_2, \tag{1.16}$$

where

$$V_1 = \frac{1}{2}\iint\limits_{\Omega} \frac{2hE}{1-\mu^2}\left[\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\mu\varepsilon_{11}\varepsilon_{22} + \frac{1-\mu}{2}\varepsilon_{12}^2\right] ds,$$

$$V_2 = \frac{1}{2}\iint\limits_{\Omega} \frac{2h^3E}{3(1-\mu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\mu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + 2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] ds.$$

Let us now investigate Eq. (1.16) and rewrite its first component in the form:

$$V_1 = \frac{1}{2}\iint\limits_{\Omega} \frac{2hE}{1-\mu^2}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\mu\varepsilon_{11}\varepsilon_{22} + \frac{1-\mu}{2}\varepsilon_{12}^2\right) ds. \tag{1.17}$$

Making a step of variation of the expression (1.17) we have:

$$\begin{aligned}\delta V_1 &= \iint\limits_{\Omega} \frac{2hE}{1-\mu^2}\left[\varepsilon_{11}\delta(\varepsilon_{11}) + \varepsilon_{22}\delta(\varepsilon_{11}) + \mu(\varepsilon_{11}\delta(\varepsilon_{22}) + \varepsilon_{22}\delta(\varepsilon_{11}))\right. \\ &\quad \left. + \frac{1-\mu}{2}\varepsilon_{12}\delta(\varepsilon_{12})\right] ds \\ &= \iint\limits_{\Omega} \frac{2hE}{1-\mu^2}\left[(\varepsilon_{11}+\mu\varepsilon_{22})\delta(\varepsilon_{11}) + (\varepsilon_{22}+\mu\varepsilon_{11})\delta(\varepsilon_{22}) + \frac{1-\mu}{2}\varepsilon_{12}\delta(\varepsilon_{12})\right] ds.\end{aligned} \tag{1.18}$$

Formula (1.11) yields

$$\delta V_1 = \iint\limits_{\Omega} \left[T_{11}\delta(\varepsilon_{11}) + T_{22}\delta(\varepsilon_{11}) + T_{12}\delta(\varepsilon_{12})\right] ds, \tag{1.19}$$

$$\begin{aligned}\delta V_1 &= \iint\limits_{\Omega} \frac{a_1}{2h}\left[T_{11}\delta(T_{11} - \mu T_{22}) + T_{22}\delta(T_{22} - \mu T_{11})\right. \\ &\quad \left. + 2(1+\mu)T_{12}\delta(T_{12})\right] ds \\ &= \iint\limits_{\Omega} \frac{a_1}{2h}\left[(T_{11} - \mu T_{22})\delta(T_{11}) + (T_{22} - \mu T_{11})\delta(T_{22})\right. \\ &\quad \left. + 2(1+\mu)T_{12}\delta(T_{12})\right] ds, \quad \text{where } a_1 = 1/E.\end{aligned} \tag{1.20}$$

Let F be a function of deformations; then one obtains

$$T_{11} = \frac{\partial^2 F}{\partial y^2}, \quad T_{22} = \frac{\partial^2 F}{\partial x^2}, \quad T_{12} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{1.21}$$

Applying the above to Eq. (1.20) gives

$$\delta_F V_1 = \iint\limits_{\Omega} \frac{a_1}{2h} \left[\left(\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} \right) \frac{\partial^2}{\partial y^2} \delta(F) + \left(\frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2} \right) \frac{\partial^2}{\partial x^2} \delta(F) \right.$$
$$\left. + 2(1+\mu) \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2}{\partial x \partial y} \delta(F) \right] ds. \tag{1.22}$$

Taking into consideration expressions (1.12) and (1.20), we will have found a transition from stresses to deformations, and then the following variational function of stresses is formulated:

$$\delta_F V_1 = \iint\limits_{\Omega} \left[\varepsilon_{11} \delta(T_{11}) + \varepsilon_{22} \delta(T_{22}) + \varepsilon_{12} \delta(T_{12}) \right] ds$$
$$= \iint\limits_{\Omega} \left[\varepsilon_{11} \frac{\partial^2}{\partial y^2} \delta(F) + \varepsilon_{22} \frac{\partial^2}{\partial x^2} \delta(F) - \varepsilon_{12} \frac{\partial^2}{\partial x \partial y} \delta(F) \right] ds. \tag{1.23}$$

Double integration by parts (without writing of any boundary integrals) yields

$$\delta_F V_1 = \iint\limits_{\Omega} \left[\frac{\partial^2 \varepsilon_{11}}{\partial y^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x^2} + \frac{\partial^2 \varepsilon_{12}}{\partial x \partial y} \right] \delta(F)\, ds. \tag{1.24}$$

Substitution of formula (1.9) into Eq. (1.24) gives

$$\delta_F V_1 = \iint\limits_{\Omega} \left[-k_y \frac{\partial^2 w}{\partial x^2} - k_x \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \delta(F)\, ds$$
$$= \iint\limits_{\Omega} \left[-\nabla_k^2 w - \frac{1}{2} L(w,w) \right] \delta(F)\, ds. \tag{1.25}$$

Boundary integrals adequate to Eq. (1.24) have the following form:

$$\int\limits_0^a \left[\varepsilon_{11} \frac{\partial}{\partial y} (\delta F)|_0^b - \frac{\partial \varepsilon_{11}}{\partial y} (\delta F)|_0^b + \frac{\partial \varepsilon_{12}}{\partial x} (\delta F)|_0^b \right] dx$$
$$+ \int\limits_0^b \left[\varepsilon_{22} \frac{\partial}{\partial x} (\delta F)|_0^a - \frac{\partial \varepsilon_{22}}{\partial x} (\delta F)|_0^a - \varepsilon_{12} \frac{\partial}{\partial y} (\delta F)|_0^a \right] dy. \tag{1.26}$$

Since Eqs. (1.22) and (1.25) are formally equal to each other, then

$$\delta_F V_1 = \iint\limits_{\Omega} \frac{a_1}{2h}\left[\left(\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2}{\partial y^2}\delta(F) + \left(\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2}{\partial x^2}\delta(F)\right.$$
$$\left. + 2\ (1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}\delta(F) + \left(\nabla_k^2 w + \frac{1}{2}L(w,w)\right)\delta(F)\right]\ ds, \quad (1.27)$$

$$\delta_F V_1 = \iint\limits_{\Omega} \frac{a_1}{2h}\left[\left(\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2} + \left(\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$
$$\left. + 2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y} + \nabla_k^2 w + \frac{1}{2}L(w,w)\right]\delta(F)\ ds. \quad (1.28)$$

The sign $(\cdot)$ means that the derivative comes from the variation of function F, and hence

$$L(w,F) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 F}{\partial x\partial y},$$
$$L(w,w) = 2\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right],$$
$$\nabla_k^2 = k_y\frac{\partial^2}{\partial x^2} + k_x\frac{\partial^2}{\partial y^2}.$$

Taking into account Eq. (1.19), and according to the central surface deformations (1.9), a conversion of parameters $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ separating all variations with respect to u, v, and w is introduced, and the following relations are obtained:

$$\delta_u V_1 = \int\limits_0^b [T_{11}\delta(u)]|_0^a\, dy + \int\limits_0^a [T_{12}\delta(u)]|_0^b\, dx$$
$$- \iint\limits_{\Omega}\left[\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y}\right]\delta(u)\ ds, \quad (1.29)$$

$$\delta_v V_1 = \int\limits_0^a [T_{22}\delta(v)]|_0^b\, dx + \int\limits_0^b [T_{12}\delta(v)]|_0^a\, dy$$
$$- \iint\limits_{\Omega}\left[\frac{\partial T_{22}}{\partial y} + \frac{\partial T_{12}}{\partial x}\right]\delta(v)\ ds, \quad (1.30)$$

$$\delta_w V_1 = \iint\limits_{\Omega}\left\{T_{11}\delta\left[-k_x w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + T_{22}\delta\left[-k_y w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\right.$$
$$\left. + T_{12}\delta\left[\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right\} ds = \iint\limits_{\Omega}\{(-k_y T_{22} - k_x T_{11})\delta(w)$$

$$
\begin{aligned}
&+ T_{11}\delta\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + T_{22}\delta\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + T_{12}\delta\left[\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\Bigg\} ds \\
&= \iint\limits_{\Omega}\Bigg\{(-k_y T_{22} - k_x T_{11})\,\delta(w) + T_{11}\frac{\partial w}{\partial x}\frac{\partial}{\partial x}\delta(w) + T_{22}\frac{\partial w}{\partial y}\frac{\partial}{\partial y}\delta(w) \\
&+ T_{12}\left[\frac{\partial w}{\partial y}\frac{\partial}{\partial x}\delta(w) + \frac{\partial w}{\partial x}\frac{\partial}{\partial y}\delta(w)\right]\Bigg\}\, ds. \qquad (1.31)
\end{aligned}
$$

Integration by parts yields

$$
\begin{aligned}
\delta_w V_1 &= \iint\limits_{\Omega}\Bigg\{-k_y\frac{\partial^2 F}{\partial x^2} - k_y\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} \\
&+ 2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 F}{\partial x\partial y}\Bigg\}\,\delta(w)\; ds \\
&= \iint\limits_{\Omega}\left\{-\nabla_k^2 F - L(w,F)\right\}\delta(w)\; ds. \qquad (1.32)
\end{aligned}
$$

Boundary integrals counterpart to Eq. (1.31) have the form

$$
\begin{aligned}
&\int\limits_0^a\left(T_{22}\frac{\partial w}{\partial y} + T_{12}\frac{\partial w}{\partial x}\right)\delta\;(w)|_0^b\, dx \\
&+ \int\limits_0^b\left(T_{11}\frac{\partial w}{\partial x} + T_{12}\frac{\partial w}{\partial y}\right)\delta\;(w)|_0^a\, dy. \qquad (1.33)
\end{aligned}
$$

Let us examine the second component of (1.16) of the form

$$
\begin{aligned}
V_2 &= \iint\limits_{\Omega}\Bigg\{\frac{2h^3 E}{6(1-\mu^2)}\Bigg[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\mu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 \\
&+ 2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\Bigg]\Bigg\} ds, \qquad (1.34)
\end{aligned}
$$

and let us separate some variations with respect to w:

$$
\begin{aligned}
\delta_w V_2 &= \iint\limits_{\Omega}\frac{2h^3 E}{3(1-\mu^2)}\Bigg\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2} + 2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y} \\
&+ \mu\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right]\Bigg\}\,\delta(w)\; ds. \qquad (1.35)
\end{aligned}
$$

The symbol $(\cdot)$ denotes the derivative coming from the variation of function w.

The kinetic energy of the shell with the exception of the shell element rotational inertia is

$$K = \frac{1}{2} \iint\limits_{\Omega} \frac{2h\rho}{g} \left\{ \left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right\} ds. \tag{1.36}$$

Let us now invoke the time integral of the shell kinetic energy variation. We will assume a case of variation of function u.

In the time interval $(t_0 - t_1)$ the following relation holds:

$$\int\limits_{t_0}^{t_1} \delta_u K dt = \int\limits_{t_0}^{t_1} \iint\limits_{\Omega} \frac{2h\rho}{g} \frac{\partial u}{\partial t} \delta \left(\frac{\partial u}{\partial t} \right) ds\, dt. \tag{1.37}$$

As a result of integration by parts one obtains

$$\begin{aligned} \int\limits_{t_0}^{t_1} \delta_u K dt = &\iint\limits_{\Omega} \frac{2h\rho}{g} \left[\frac{\partial u}{\partial t} \delta(u) \right] \Bigg|_{t_0}^{t_1} ds \\ &- \int\limits_{t_0}^{t_1} \iint\limits_{\Omega} \frac{2h\rho}{g} \frac{\partial^2 u}{\partial t^2} \delta(u)\; ds\, dt. \end{aligned} \tag{1.38}$$

Any partial variations with respect to v and w are found analogously:

$$\begin{aligned} \int\limits_{t_0}^{t_1} \delta_v K dt = &\iint\limits_{\Omega} \frac{2h\rho}{g} \left[\frac{\partial v}{\partial t} \delta(v) \right] \Bigg|_{t_0}^{t_1} ds \\ &- \int\limits_{t_0}^{t_1} \iint\limits_{\Omega} \frac{2h\rho}{g} \frac{\partial^2 v}{\partial t^2} \delta(v)\; ds\, dt, \end{aligned} \tag{1.39}$$

$$\begin{aligned} \int\limits_{t_0}^{t_1} \delta_w K dt = &\iint\limits_{\Omega} \frac{2h\rho}{g} \left[\frac{\partial w}{\partial t} \delta(w) \right] \Bigg|_{t_0}^{t_1} ds \\ &- \int\limits_{t_0}^{t_1} \iint\limits_{\Omega} \frac{2h\rho}{g} \frac{\partial^2 w}{\partial t^2} \delta(w)\; ds\, dt. \end{aligned} \tag{1.40}$$

The elementary work of any external forces is expressed as follows:

$$\delta' W = \iint\limits_{\Omega} \left[p_x \delta(u) + p_y \delta(v) + \left(q - \varepsilon \frac{2h\rho}{g} \frac{\partial w}{\partial t} \right) \delta(w) \right] ds, \tag{1.41}$$

where ε denotes the damping coefficient of a surrounding medium.

Substituting all formulae previous to Eq. (1.43), which states the counterpart to the Hamilton principle, the following variational equations are obtained:

$$
\begin{aligned}
&\int\limits_{t_0}^{t_1}\iint\limits_{\Omega}\left\{\left[\frac{\partial T_{11}}{\partial x}+\frac{\partial T_{12}}{\partial y}+p_x-\frac{2h\rho}{g}\frac{\partial^2 u}{\partial t^2}\right]\delta(u)\right.\\
&+\left[\frac{\partial T_{22}}{\partial y}+\frac{\partial T_{12}}{\partial x}+p_y-\frac{2h\rho}{g}\frac{\partial^2 v}{\partial t^2}\right]\delta(v)\\
&+\left[\frac{2h^3E}{3(1-\mu^2)}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}\right)+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right.\\
&+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)-\nabla_k^2 F-L(w,F)\\
&\left.+q-\frac{2h\rho}{g}\left(\frac{\partial^2 w}{\partial t^2}+\varepsilon\frac{\partial w}{\partial t}\right)\right]\delta(w)\\
&+\left[\frac{a_1}{2h}\left(\left(\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.\right.\\
&\left.\left.\left.+2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right)+\nabla_k^2 w+\frac{1}{2}L(w,w)\right]\delta(F)\right\}ds\,dt\\
&-\int\limits_{t_0}^{t_1}\int\limits_{0}^{a}\left\{\left[\varepsilon_{11}\frac{\partial}{\partial y}(\delta F)-\frac{\partial\varepsilon_{11}}{\partial y}(\delta F)+\frac{\partial\varepsilon_{12}}{\partial y}(\delta F)+T_{22}(\delta v)+T_{12}(\delta u)\right.\right.\\
&\left.\left.+\left(T_{22}\frac{\partial w}{\partial y}+T_{12}\frac{\partial w}{\partial x}\right)\delta(w)\right]\Bigg|_0^b\right\}dx\,dt\\
&-\int\limits_{t_0}^{t_1}\int\limits_{0}^{b}\left\{\left[\varepsilon_{22}\frac{\partial}{\partial x}(\delta F)-\frac{\partial\varepsilon_{22}}{\partial x}(\delta F)+\varepsilon_{12}\frac{\partial}{\partial y}(\delta F)+T_{11}(\delta u)+T_{12}(\delta v)\right.\right.\\
&\left.\left.+\left(T_{11}\frac{\partial w}{\partial x}+T_{12}\frac{\partial w}{\partial y}\right)\delta(w)\right]\Bigg|_0^a\right\}dy\,dt\\
&+\iint\limits_{\Omega}\frac{2h\rho}{g}\left[\frac{\partial u}{\partial t}\delta(u)+\frac{\partial v}{\partial t}\delta(v)+\frac{\partial w}{\partial t}\delta(w)\right]\Bigg|_{t_0}^{t_1}ds=0.
\end{aligned}
\tag{1.42}
$$

1.5 Equations of Motion

Similarly as in Eq. (1.42), variations δu, δv, δw, δF taken as a functions of time t are arbitrarily chosen, and therefore multipliers preceding all variations of the first integral should equal zero. As a result of variational equation the following equations of motion and only one equation of deformation compatibility are presented:

$$
\left[\frac{\partial T_{11}}{\partial x}+\frac{\partial T_{12}}{\partial y}+p_x-\frac{2h\rho}{g}\frac{\partial^2 u}{\partial t^2}\right]=0,
\tag{1.43}
$$

$$\left[\frac{\partial T_{22}}{\partial y}+\frac{\partial T_{12}}{\partial x}+p_y-\frac{2h\rho}{g}\frac{\partial^2 v}{\partial t^2}\right]=0, \tag{1.44}$$

$$\frac{2h^3E}{3(1-\mu^2)}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right.$$
$$\left.+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]-\nabla_k^2F-L(w,F)+q-\frac{2h\rho}{g}\left(\frac{\partial^2 w}{\partial t^2}+\varepsilon\frac{\partial w}{\partial t}\right)=0, \tag{1.45}$$

$$\frac{a_1}{2h}\left[\left(\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$
$$\left.+2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]+\nabla_k^2 w+\frac{1}{2}L(w,w)=0. \tag{1.46}$$

If there is a need to write only a variational form of the equation, then during creation of the variational Eq. (1.42) the following steps should be introduced:

(i) Only some of terms including $\delta(F)$ variations must be considered in the equation;
(ii) Equation (1.31) should be introduced instead of Eq. (1.32).

Keeping in mind the last mathematical transformations, three equations of motion of the shell at hand will be derived. The first two, i.e., (1.43) and (1.44), remain unchanged, but instead of Eq. (1.45) the following one appears:

$$\frac{2h^3E}{3(1-\mu^2)}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right.$$
$$\left.+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]+T_{11}k_x+T_{22}k_y+\frac{\partial}{\partial x}\left(T_{11}\frac{\partial w}{\partial x}+T_{12}\frac{\partial w}{\partial y}\right)$$
$$+\frac{\partial}{\partial y}\left(T_{22}\frac{\partial w}{\partial y}+T_{12}\frac{\partial w}{\partial x}\right)+q-\frac{2h\rho}{g}\left(\frac{\partial^2 w}{\partial t^2}+\varepsilon\frac{\partial w}{\partial t}\right)=0. \tag{1.47}$$

If there is a possibility of analyzing the dynamic process omitting any elastic wave distributions, then because the inertial terms of the first two equations, are disregarded, the set of Eqs. (1.43)–(1.46) can be significantly simplified. These conditioning equations will be met if in accordance with the variables of Eq. (1.21), a function of stresses is introduced (p_x and p_y).

In this situation the equation of motion of a shell element takes the form

$$\frac{2h^3E}{3(1-\mu^2)}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right.$$
$$\left.+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]-\nabla_k^2F-L(w,F)+q-\frac{2h\rho}{g}\left(\frac{\partial^2 w}{\partial t^2}+\varepsilon\frac{\partial w}{\partial t}\right)=0, \tag{1.48}$$

and the equation of deformations compatibility is

$$\frac{a_1}{2h}\left[\left(\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$
$$\left.+\,2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]+\nabla_k^2 w+\frac{1}{2}L(w,w)=0. \qquad (1.49)$$

When a circle section cylindrical shell is analyzed, then the pair of parameters $k_x = 0$ and $k_y = 1/R$, where R is the curvature radius of the mean surface of a shell, should be taken into account in Eqs. (1.48)–(1.49).

Equations of a homogeneous shell, which were obtained during our analysis, are in good agreement with those contained in [304]. In the case of $E = const$, the density $\rho = const$ should be assumed and during integration by parts the computational process of variational differentiation has to be transferred to a proper function. If the functional dependence of E and ρ on the spatial coordinates relates to any perpendicular cutouts of material (snippets), then after the analogous routines are carried out, the equations will be identical, as in reference [251].

1.6 Boundary and Initial Conditions

An integration of equations of motion needs to be supported by a properly defined boundary and initial conditions. On the basis of the behavioral (dynamic) hypothesis of straight lines, which are perpendicular to a surface, each bounding point should satisfy four boundary conditions. In the case in which the displacements u, v, and w of any boundary points of a shell are known, then after deformations they suitably determine the space-located arbitrary curve. A perpendicular straight line led through a boundary point can dislocate together with that point as well as rotate about an angle in a plane that is perpendicular to the boundary line. Correspondingly, normal line placement to the surface resulting in a shell's deformation can be determined with the use of any four values. These integrals of a variational equation calculated along boundaries allow for the formulation of any appropriate additional conditions [159]. Some of them are given below for the bound of a shell at $x = const$ ($y = const$ boundary condition could be obtained following the replacement of x with y)

1. Simply supported edge:

$$w = M_{11} = T_{11} = T_{12} = 0 \quad \text{if} \quad x = 0, a. \qquad (1.50)$$

2. Ball-type support on the flexible but not compressed (stretched) ribs:

$$w = M_{11} = T_{11} = \varepsilon_{22} = 0 \quad \text{if} \quad x = 0, a. \qquad (1.51)$$

The second condition can be also given in the form

$$w = \frac{\partial^2 w}{\partial x^2} = F = \frac{\partial^2 F}{\partial x^2} = 0 \quad \text{if} \quad x = 0, a. \tag{1.52}$$

Displacement boundary conditions are called the geometric, whereas any conditions formulated for the stresses and moments of forces are called dynamic.

Some initial conditions should also be satisfied, which simultaneously with the integration of fundamental equations are related to the velocities of points belonging to the mean surface of the shell

$$\begin{aligned} w \mid_{t=t_0} &= \zeta_1(x,y), \\ \frac{\partial w}{\partial t} \mid_{t=t_0} &= \zeta_2(x,y). \end{aligned} \tag{1.53}$$

1.7 Non-dimensional Form of Equations

Let us introduce the following non-dimensional parameters:

$$\begin{aligned} & w = 2h\bar{w}, \quad x = a\bar{x}, \quad y = b\bar{y}, \quad F = E_0 (2h)^3 \bar{F}, \\ & k_x = \frac{2h}{a^2}\bar{k}_x, \quad k_y = \frac{2h}{b^2}\bar{k}_y, \quad q = \frac{E_0(2h)^4}{a^2 b^2}\bar{q}, \\ & t = \frac{ab}{2h}\sqrt{\frac{\rho_0}{gE_0}}\bar{t}, \quad \varepsilon = \frac{2h}{ab}\sqrt{\frac{gE_0}{\rho_0}}\bar{\varepsilon}, \\ & \lambda = a/b, \quad E = E_0\bar{E}, \quad \rho = \rho_0\bar{\rho}. \end{aligned} \tag{1.54}$$

Non-dimensional Eqs. (1.48)–(1.49) at the very beginning of this section are as follows (bars above the non-dimensional quantities are omitted):

$$\begin{aligned} & \frac{E}{12(1-\mu^2)}\left[\frac{1}{\lambda^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2} + \lambda^2\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2} + 2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right. \\ & \left. + \mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right] - \nabla_k^2 F - L(w,F) + q - \rho\left(\frac{\partial^2 w}{\partial t^2} + \varepsilon\frac{\partial w}{\partial t}\right) = 0, \end{aligned} \tag{1.55}$$

$$\begin{aligned} a_1 & \left[\left(\lambda^2\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2} + \left(\frac{1}{\lambda^2}\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right. \\ & \left. + 2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right] + \nabla_k^2 w + \frac{1}{2}L(w,w) = 0. \end{aligned} \tag{1.56}$$

On the basis of both Kirchhoff-Love kinematic model and the non-homogeneity of a shell one can obtain from (1.55)–(1.56) some starting equations concerning static stability of a perpendicular shell but with an assumption that any time-dependent terms are neglected. It allows one to write them in the non-dimensional form:

$$\frac{E}{12(1-\mu^2)}\left[\frac{1}{\lambda^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\lambda^2\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right.$$

$$\left.+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]-\nabla_k^2 F-L(w,F)+q=0, \tag{1.57}$$

$$a_1\left[\left(\lambda^2\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{1}{\lambda^2}\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}\right.$$

$$\left.+2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]+\nabla_k^2 w+\frac{1}{2}L(w,w)=0. \tag{1.58}$$

1.8 Variable Parameters of Stiffness

The aim of application of the method used in monograph [251] is to introduce some variable parameters of stiffness of a shell. The method that is proposed for applications is suitable for determination of equations of motion of a shell element with either any cutouts or other modifications of its thickness. There is a possibility of simplifying the problem, but with the assumption that there is only one full parallelepiped cutout, which is oriented in the investigated shell in a way securing its edges to be parallel to the appropriate external edges of the shell. Similarly as in Eq. (1.16), the integrated function depends on displacements; therefore it can be rewritten as

$$V=\frac{1}{2}\iint\limits_{\Omega} f(u,v,w)\; ds=\frac{1}{2}\iint\limits_{\Omega} f(x,y)\; ds. \tag{1.59}$$

Let us assume that Ω denotes the region of integration that is bounded by both internal and external edges of the deformable system under investigation

$$\Omega=S-S_1, \tag{1.60}$$

in which S is the shell's lateral surface without any cutout and S_1 is the cutout's surface. The double integral calculated on Ω surface is transformed to the two double integrals of the form

$$V=\frac{1}{2}\iint\limits_{S} f(x,y)\; ds-\frac{1}{2}\iint\limits_{S_1} f(x,y)\; ds. \tag{1.61}$$

Let us now take a shell element bounded by coordinates $x=a_1,\ x=a_2,\ y=b_1,\ y=b_2$. Assume that the element includes a cutout defined by the lines $x=x_1$,

$x = x_2,\ y = y_1,\ y = y_2$. In association with this, the cutout belongs to the interior of the analyzed region, so we can write

$$\begin{aligned} a_1 < x_1 < a_2, \quad a_1 < x_2 < a_2, \\ b_1 < y_1 < b_2, \quad b_1 < y_2 < b_2. \end{aligned} \tag{1.62}$$

Equation (1.61) is composed of the difference of any two double integrals, which are calculated along S and S_1, but the region S_1 is included in S. Moreover, regions S and S_1 are rectangular and with mutually parallel boundaries (Fig. 1.2).

For further considerations, we will use a unity Heaviside function dependent on two variables $\Gamma_0(x - x_1,\ y - y_1)$ of the form

$$\Gamma_0\left(x - x_1; y - y_1\right) = \Gamma_0\left(x - x_1\right)\Gamma_0\left(y - y_1\right), \tag{1.63}$$

where

$$\Gamma_0\left(x - x_1\right) = \begin{cases} 0 & \text{for } x < x_1; \\ 1 & \text{for } x > x_1. \end{cases} \tag{1.64}$$

The filtering property of the unity function mentioned is defined by means of the following expression:

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f\left(x, y\right)\Gamma_0\left(x - x_1; y - y_1\right)\, dx\, dy = \int_{x_1}^{a_2}\int_{y_1}^{b_2} f\left(x, y\right)\, dx\, dy, \tag{1.65}$$

and occurs if $a_1 < x_1 < a_2$ and $b_1 < y_1 < b_2$.

The quoted property of the function allows for introduction in Eq. (1.61) of only one double integral instead of any two others. The reduction procedure will be examined more carefully.

The expression

$$V' = \iint_S f\left(x, y\right) ds = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f\left(x, y\right)\, dx\, dy \tag{1.66}$$

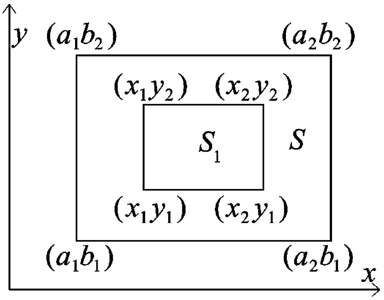


Fig. 1.2 Placement of regions S and S_1

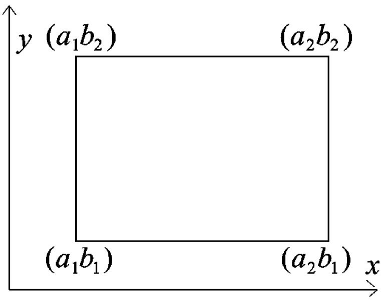


Fig. 1.3 Placement of region S

introduces an integral calculated on the S surface (Fig. 1.3). Note that

$$V' = \int_{a_1}^{a_2}\int_{b_1}^{b_2} [1 - \Gamma_0 (x - x_1; y - y_1)] f(x,y) \; dx\, dy, \tag{1.67}$$

but the above integral is calculated on the S surface excluding any separated shell element (Fig. 1.4).

The integral

$$V' = \int_{a_1}^{a_2}\int_{b_1}^{b_2} [1 - \Gamma_0 (x - x_1; y - y_1) + \Gamma_0 (x - x_2; y - y_1)] f(x,y) \; dx\, dy \tag{1.68}$$

is also calculated on this surface as in the previous case, but the element excluded there has been now added (Fig. 1.5).

The expression

$$V' = \int_{a_1}^{a_2}\int_{b_1}^{b_2} [\, 1 - \Gamma_0 (x - x_1; y - y_1) + \Gamma_0 (x - x_2; y - y_1)$$
$$+ \Gamma_0 (x - x_1; y - y_2)\,] f(x,y) \; dx\, dy \tag{1.69}$$

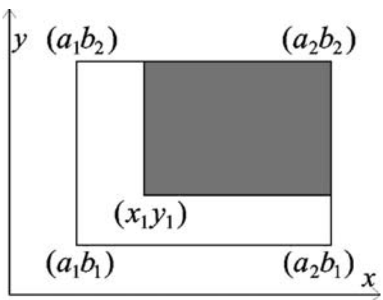


Fig. 1.4 A scheme based on calculation of integral (1.67)

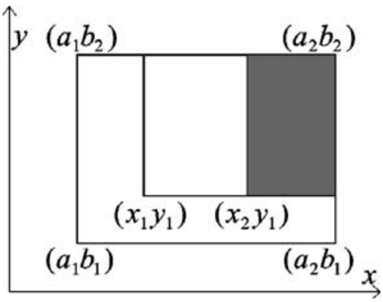


Fig. 1.5 A scheme based on calculation of integral (1.68)

is related to the integration on this surface, as is clearly visible in Eq. (1.69) when the separated part pictured in Fig. 1.6 has been considered.

Finally, the integral

$$V' = \int_{a_1}^{a_2}\int_{b_1}^{b_2} [1 - \Gamma_0\,(x - x_1; y - y_1) + \Gamma_0\,(x - x_2; y - y_1)$$
$$+\ \Gamma_0(x - x_1; y - y_2) - \Gamma_0(x - x_1; y - y_2)]\,f\,(x, y)\ dx\,dy, \qquad (1.70)$$

corresponds to the integration in S with exclusion of S_1. It means that it is equal to the quantity V defined by Eq. (1.61).

Introduction of the unity function has allowed for replacement of the two double integrals by one integral calculated on the surface solely bounded by the loaded edge:

$$V_1 = \frac{1}{2}\iint\limits_{S} \frac{2hE_0\,(1 - \gamma_0)}{1 - \mu^2}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\mu\varepsilon_{11}\varepsilon_{22} + \frac{1 - \mu}{2}\varepsilon_{12}^2\right)\,ds, \qquad (1.71)$$

where

$$\gamma_0 = \Gamma_0\,(x - x_1; y - y_1) - \Gamma_0\,(x - x_2; y - y_1)$$
$$-\ \Gamma_0\,(x - x_1; y - y_2) + \Gamma_0\,(x - x_2; y - y_2)\,. \qquad (1.72)$$

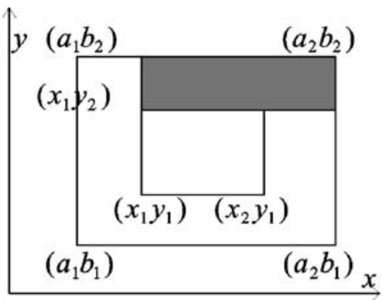


Fig. 1.6 A scheme based on calculation of integral (1.69)

Referring to Sect. 1.3, it is possible to write

$$\begin{aligned} E &= E(x,y) = E_0(1-\gamma_0), \\ \rho &= \rho(x,y) = \rho_0(1-\gamma_0). \end{aligned} \tag{1.73}$$

This type of formula for E and ρ expresses a transition from a cutout of shell to the adequate continuous model, which has also been presented in [251].

1.9 Flexural Stiffness Coefficient of a Shell Element

Let the function $f(x)$ be as follows:

$$f(x) = \varphi(x)\lambda(x), \tag{1.74}$$

where $\varphi(x)$ is *a stepping function* changing from a to b, and $\lambda(x)$ it is an *impulse function*.

The function visible in Fig. 1.7 can be represented here by means of the unity Heaviside function as follows:

$$\begin{aligned} f(x) &= a[1-\Gamma_0(x-x_0)] + b\Gamma_0(x-x_0) \\ &= a - a\Gamma_0(x-x_0) + b\Gamma_0(x-x_0) \\ &= a - a\Gamma_0(x-x_0) + b\Gamma_0(x-x_0) \\ &= a - (a-b)\Gamma_0(x-x_0) \\ &= a\left[1 - \left(\frac{a-b}{a}\right)\Gamma_0(x-x_0)\right] \\ &= a\left[1 - \left(1-\frac{b}{a}\right)\Gamma_0(x-x_0)\right], \end{aligned} \tag{1.75}$$

and in comparison to the above we have

$$f(x) = a\left[1 - \left(1-\frac{b}{a}\right)\Gamma_0(x-x_0)\right]. \tag{1.76}$$

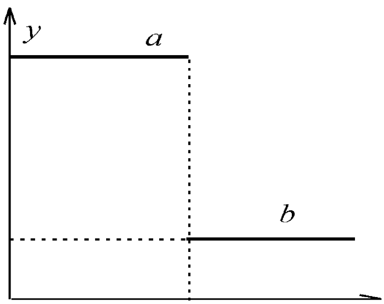
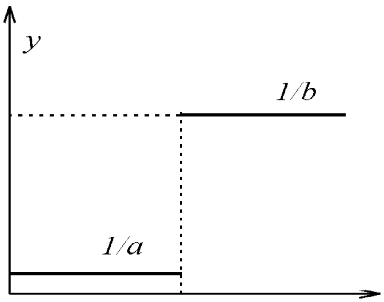


Fig. 1.7 A function $f(x)$ related to the function (1.75)

Fig. 1.8 A function $f(x)$ related to the function (1.77)



In an analogous manner, an inverse function takes the form (Fig. 1.8):

$$\begin{aligned}\frac{1}{f(x)} &= \frac{1}{a}\left[1-\Gamma_0(x-x_0)\right]+\frac{1}{b}\Gamma_0(x-x_0)\\ &= \frac{1}{a}+\left(\frac{1}{b}-\frac{1}{a}\right)\Gamma_0(x-x_0)\\ &= \frac{1}{a}+\left(\frac{a-b}{ab}\right)\Gamma_0(x-x_0)\\ &= \frac{1}{a}\left[1+\left(\frac{a}{b}-1\right)\Gamma_0(x-x_0)\right].\end{aligned} \tag{1.77}$$

If we assume that $a = E_0$ is the value of the modulus for a homogeneous shell, whereas $\frac{b}{a}$ is the coefficient of its changes along the shell's cutout, then the coefficient of that cutout is expressed by

$$\gamma_1 = \frac{b}{a}. \tag{1.78}$$

The quantity $\gamma_1 = 1$ corresponds to homogeneous shells because flexural stiffnesses of both the separated cutout of the shell and the homogeneous shell are practically the same (equal), whereas $\gamma_1 < 1$ corresponds to the cutout of a smaller stiffness (or greater for $\gamma_1 > 1$) than the stiffness of the homogeneous shell. According to the previous derivations

$$\begin{aligned}E(x,y) &= E_0\left[1-\sum_j\left(1-\gamma_{1j}\right)\gamma_{0j}\right]\\ \frac{1}{E(x,y)} &= \frac{1}{E_0}\left[1+\sum_j\left(\frac{1}{\gamma_{1j}}-1\right)\gamma_{0j}\right],\end{aligned} \tag{1.79}$$

where j denotes the number of such cutouts.

Analogously, one can obtain an expression for the density of a shell's material:

$$\rho(x,y) = \rho_0 \left[1 - \sum_j \left(1 - \gamma_{2j}\right) \gamma_{0j} \right], \tag{1.80}$$

where γ_{2j} is the density coefficient of jth cutout.

On the basis of relation (1.79) a conclusion can be drawn: if any equation contains an expression that is the inverse of the E modulus, then γ_{1j} cannot be equal to zero. It is equivalent to stating that a shell cannot possess any cutouts. Otherwise, if there is a need to consider those cutouts, the appropriate equation needs to be multiplied by E. Therefore, the general function has to be moved to the right-hand side of equation. For this case, any calculations connected with the described algorithm are rather complicated.

1.10 Generalized Functions

The term generalized function arises from the generalization of the classic function definition [172]. The term originated during analysis of some physical problems and has quickly appeared as a purely mathematical notion.

Thanks to introduction of the notion, a Fourier transform could be formally applied to a much wider class of functions, not only those integrated absolutely or in quadratures. Moreover, it allows for mathematical formulation of some idealized notions, such like density of point explosion, the material point density, a momentary impulse, etc.

We will explain it more carefully. Using a mathematical apparatus for physical phenomena investigations it is often desired to use various mathematical abstractions, in particular the term material point. For example, we can speak of a mass reduced to a space point, a force applied in a given moment of time, a point source of many various fields, etc., but after all they are the idealized formulations. Such a simplified approach to the analysis of physical phenomena turns out to be insufficient. New mathematical notions or apparatus are often required.

The first notions of generalized functions were formed in the work of Dirac and other physicist's as a type of symbolic description of the physical phenomena. To achieve a systematic method of application of the above functions it was necessary to introduce some important bases of the theory of generalized functions, which was done by Sobolev and Schwartz [267, 279], respectively.

The theory of generalized functions is a very convenient mathematical tool, which permits the solution of many problems that could not be solved in the classical way. The theory of generalized functions is currently popular in many applied sciences, as well as in pure theoretical studies.

Let us begin our present considerations from the definition of the basic term of linear space D of a function. For that reason, functions defined in $\mathbb{R}$ and that take complex values will be consequently considered.

We are interested in a space D that consists of an unlimited set of finite and differentiable functions. A property of all finite functions is observed, that their forms of combination and multiplication by numbers are still included in a linear space, but infinitely differentiable finite functions create its subspace.

The set of infinitely differentiable functions together with the formulated definition of boundary transition creates the convergent linear space, which can be directly observed.

Definition 1.1 A space of infinitely differentiable finite functions along with the invoked notion of convergence is called a space D of basic functions.

Definition 1.2 An arbitrarily chosen functional f defined in D is called the *generalized function.*

Definition 1.3 The function f that is defined along the whole real axis is called *locally integrable* if it is absolutely integrable on any finite interval.

If f is a locally integrated function and $\varphi \in D$, then the product $f\varphi$ is absolutely integrable on the whole considered axis.

Let the locally integrable function f be defined for a functional (f, φ) in D as follows:

$$(f,\varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx. \tag{1.81}$$

The above function is both linear and continuous.

A conclusion can be made that for an arbitrary and locally integrable function $f(x)$ there exists a generalized function (f, φ). An arbitrary and locally integrable function can be considered in that sense as a generalized function (a suitable thesis can be proposed: any generalized function (f, φ) is generated by function f).

Generalized functions are in some situations denoted by $f(x)$. The symbolic notation does not denote the value of a generalized function at a point $x \in \mathbb{R}$, but emphasizes that generalized functions are the generalization of normal (locally integrable) functions and no other value of a generalized function at the point x is assigned.

For an estimation of the value of the generalized function f at the point $\varphi = \varphi(x)$ of a space D and with the exception of the notation (f, φ) the following equivalent can be used:

$$\int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx. \tag{1.82}$$

It leads to the comprehensive definition:

$$\int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx = (f,\varphi). \tag{1.83}$$

Equation (1.83) is a definition of the symbol (1.82), which should be formally understood in the following way: all generalized functions constitute a generalization of functionals given by definition (1.81), where f is the locally integrable function.

Let us now define *the derivative of a generalized function*. Most of all, a step-by-step explanation is necessary, bringing the meaning of the derivative of normal continuous and along the whole real axis differentiable function f treated as a functional (f', φ) in D. Let us note, moreover, that the derivative f' being continuous on the whole real axis is the locally differentiable function. Integrating by parts as well as taking into consideration the finiteness of the function $\varphi \in D$, we obtain

$$(f', \varphi) = \int_{-\infty}^{+\infty} f'(x)\varphi(x)dx = -\int_{-\infty}^{+\infty} f(x)\varphi'(x)dx = -(f, \varphi'), \tag{1.84}$$

but $\varphi' \in D$. The conclusion is that the derivative f' is a functional in D and its values are precisely expressed by both the values of function f (considered as a functional) and the Eq. (1.84). This observation makes it possible to construct the following definition.

Definition 1.4 The derivative of a generalized function f will be called the *functional defined in D* denoted by f' and defined by the following equality:

$$(f', \varphi) = -(f, \varphi'), \quad \varphi \in D. \tag{1.85}$$

In other words, all values of the functional f' in any point φ of the space D equal the values of the functional f with the opposite sign at point $\varphi' \in D$.

Previous studies allow drawing some conclusions, that any generalized function or any locally integrable function possesses a derivative in agreement with Definition 1.4.

On the basis of (1.84), the normal derivative of a continuous differentiable function on the whole real axis, treated as a functional in D, coincides with its derivative in the sense of generalized functions.

The procedure for calculating the derivative of a generalized function is called *differentiation* (analogously to the case of normal functions).

Higher-order derivatives of generalized functions can be calculated in an analogous manner as for normal functions:

$$f'' = (f')', \quad f''' = (f'')', \ldots,$$

in general

$$f^{(k)} = (f^{(k-1)})', \quad k = 1, 2, \ldots, \quad f^{(0)} = f. \tag{1.86}$$

Corresponding to the above definition, a generalized function can have any derivatives of arbitrary high orders. It is often substituted with another statement claiming that they are differentiable anywhere up to infinity.

In 1933, the impulse functions of the theory of elasticity were for the first time applied in Gersevanov's work and, subsequently, in the work of Nazarov [229] and Radtsig [253]. Nevertheless, Novitskog [233], Vainberg, and Rajtfarb's [312] achievements have indicated a wide range of applicable prospects of functions in situations within the large class of mechanical problems (mainly in building engineering).

The zero-order unity function makes the fundamental concept of understanding of the zero-order impulse functions, defined as follows:

$$\Gamma_0(x-x_0) = \begin{cases} 0 & \text{for } x < x_0, \\ 1 & \text{for } x \geq x_0. \end{cases} \tag{1.87}$$

Traditionally, the above function remains in contrast to the nth order impulse function of the form $\Gamma_n(x-x_0)$, called either the unity Heaviside function or the zero-order impulse function. The function at hand is locally integrable so it can be assumed officially as the generalized function.

It is worth noting that the two-variable impulse function is equal to multiplication of two other analogous variable functions of different arguments:

$$\Gamma_0(x-x_0; y-y_0) = \Gamma_0(x-x_0)\Gamma_0(y-y_0). \tag{1.88}$$

One of the more important merits of searching for solutions to many miscellaneous problems is the filtrate property of the nth derivative of delta function. It has led to an enormous popularization of that useful function. The filtrate property is characterized by the following expression:

$$\int\limits_a^b f(x)\Gamma_1^{(n)}(x-\xi)\,dx = \begin{cases} 0, & \xi < a, \\ (-1)^n f^{(n)}(\xi), & a < \xi < b, \\ 0, & \xi > b. \end{cases} \tag{1.89}$$

The often used mathematical dependencies arise from Eqs. (1.88) and (1.89):

$$\int\limits_0^a \int\limits_0^b \Gamma_1^{(x,y)}(x-x_1; y-y_1) f(x,y)\ dx\ dy = f(x_1, y_1),$$

$$\int\limits_0^a \int\limits_0^b \Gamma_1^{(y)}(x-x_1; y-y_1) f(x,y)\ dx\ dy = \int\limits_{x_1}^a f(x, y_1)\ dx,$$

$$\int\limits_0^a \int\limits_0^b \Gamma_2^{(y)}(x-x_1; y-y_1) f(x,y)\ dx\ dy = -\int\limits_{x_1}^a \left\{ \frac{\partial}{\partial y}[f(x,y)] \right\}\Bigg|_{y_1} dx,$$

$$\int\limits_0^a \int\limits_0^b \Gamma_2^{(x)}(x-x_1; y-y_1) f(x,y)\ dx\ dy = -\int\limits_{y_1}^b \left\{ \frac{\partial}{\partial x}[f(x,y)] \right\}\Bigg|_{x_1} dy,$$

$$\int_0^a \int_0^b \Gamma_0(x-x_1; y-y_1) f(x,y) \; dx \; dy = \int_{x_1}^a \int_{y_1}^b f(x,y) \; dx \; dy,$$

$$\int_0^a \int_0^b \Gamma_1^{(x)}(x-x_1; y-y_1) f(x,y) \; dx \; dy = \int_{y_1}^b f(x_1,y) \; dy. \tag{1.90}$$

The terms $\Gamma_j^{()}$ denote the derivatives of impulse functions calculated in the determined coordinates.

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