Chapter 2
The Metaplectic and Anaplectic Representations

In this chapter, we briefly review some basic aspects of the metaplectic representation, especially in the one-dimensional and two-dimensional cases. Then, we shall introduce the new anaplectic analysis on the real line, in which the spectrum of the harmonic oscillator is $\mathbb{Z}$ rather than $\frac{1}{2}+\mathbb{N}$. The basic space $\mathfrak{A}$ substituting for $L^2(\mathbb{R})$ consists of functions on the line extending as entire functions, typically increasing like "bad" Gaussian functions at infinity. Nevertheless, there is on $\mathfrak{A}$ a well-defined translation-invariant concept of integral, and (in place of the scalar product of $L^2(\mathbb{R})$) a pseudoscalar product reminiscent of indefinite forms occurring in Physics. All symmetries of usual analysis expressing themselves by means of such objects as the Heisenberg representation, the Fourier transformation, and, more generally, the metaplectic representation, have counterparts in anaplectic analysis. Note that in Sect. 4.1, we shall have to consider the parameter-dependent ν-anaplectic analysis. The one considered in the present chapter (in Sect. 2.2) corresponds to $\nu = -\frac{1}{2}$: it will also be shown in Sect. 4.2 that the case when $\nu = 0$ yields an analysis containing the usual one.

2.1 The Metaplectic Representation

In this book, we are only interested in the case when the dimension n is 1 or 2: it will save space and add to the understanding, not to specify n from the start.

The symplectic group $\mathrm{Sp}(n,\mathbb{R})$ is the group of linear transformations g of $\mathbb{R}^n \times \mathbb{R}^n$, in block-form $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, which preserve the canonical symplectic form: this means that, if one sets $[(x,\xi)\,,(y,\eta)] = -\langle x,\eta\rangle + \langle y,\xi\rangle$, the equation $[(x,\xi)\,,(y,\eta)] = [g(x,\xi)\,,g(y,\eta)]$ holds for any pair of points (x,ξ) and (y,η) in $\mathbb{R}^n \times \mathbb{R}^n$. In other words, one should have

$$C'A = A'C, \quad D'B = B'D, \quad D'A - B'C = I, \tag{2.1.1}$$

the accent denoting the transposition map. The symplectic group is connected but its fundamental group is $\mathbb{Z}$: in particular, it has a (unique, up to isomorphism) twofold cover, called the metaplectic group, here denoted as $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$: note that $\mathrm{Sp}(1,\mathbb{R}) = SL(2,\mathbb{R})$.

It is a fundamental fact [41] that there exists a unique unitary representation $\mathrm{Met}^{(n)}$ – the metaplectic representation – of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ in $L^2(\mathbb{R}^n)$, satisfying the following properties:

(i) if C is a real symmetric $n \times n$-matrix, and if the identity $(2n) \times (2n)$-matrix is connected to the block-matrix $g = \left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$ by means of the path $t \mapsto \left(\begin{smallmatrix} I & 0 \\ tC & I \end{smallmatrix}\right)$, finally if $\tilde{g}$ is the end of the path, in the metaplectic group, originating at the identity and covering the path within $\mathrm{Sp}(n,\mathbb{R})$ just defined, then the transformation $\mathrm{Met}^{(n)}(\tilde{g})$ is the multiplication by the function $x \mapsto \exp\left(i\pi\langle Cx, x\rangle\right)$;

(ii) if one considers the path, in the metaplectic group, originating at the identity and covering the path $t \mapsto \left(\begin{smallmatrix} (\cos t)I & (\sin t)I \\ (-\sin t)I & (\cos t)I \end{smallmatrix}\right)$ in the symplectic group, then the element $\tilde{g}$ reached for $t = \frac{\pi}{2}$ gives rise to the transformation $\mathrm{Met}^{(n)}\left(\tilde{g}\right) = e^{-\frac{i\pi n}{4}}\,\mathcal{F}$, where $\mathcal{F}$ is the usual Fourier transformation:

$$(\mathcal{F}u)(\xi) = \int_{\mathbb{R}^n} u(x)\, e^{-2i\pi\langle x,\,\xi\rangle}\, dx; \tag{2.1.2}$$

(iii) if $A \in GL^+(n,\mathbb{R})$ and $g = \left(\begin{smallmatrix} A & 0 \\ 0 & A'^{-1} \end{smallmatrix}\right)$, finally if $\tilde{g} \in \widetilde{\mathrm{Sp}}(n,\mathbb{R})$ is the end of a path originating at the identity of that group and covering a path $t \mapsto \left(\begin{smallmatrix} A_t & 0 \\ 0 & A_t'^{-1} \end{smallmatrix}\right)$ with $A_t \in GL(n,\mathbb{R})$ for all t, then $\mathrm{Met}^{(n)}\left(\tilde{g}\right)$ is the transformation $u \mapsto u_1,\ u_1(x) = (\det A)^{-\frac{1}{2}}\, u(A^{-1}x)$.

The two metaplectic transformations associated with distinct points of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ above the same point of $\mathrm{Sp}(n,\mathbb{R})$ differ only by the factor -1. The metaplectic representation is unitary in $L^2(\mathbb{R}^n)$; each transformation $\mathrm{Met}^{(n)}\left(\tilde{g}\right)$ preserves the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and has a unique extension as a (weakly) continuous linear automorphism of $\mathcal{S}'(\mathbb{R}^n)$. The metaplectic representation is reducible: its irreducible subspaces are the two subspaces of $L^2(\mathbb{R}^n)$ characterized by parity.

To understand fully the metaplectic representation in a way not making it necessary to decompose symplectic matrices as products of the generators just defined, it is useful to characterize a function $u \in L^2(\mathbb{R}^n)$ or, more generally, a distribution in $\mathcal{S}'(\mathbb{R}^n)$, by its *quadratic transform*. Denote as $\mathrm{Sym}_n^{\mathbb{C}}$ the set of symmetric matrices with complex entries. The quadratic transform of a function $u \in L^2(\mathbb{R}^n)$ is the pair $((\mathcal{M}u)_0, (\mathcal{M}u)_1)$ of functions defined on the (Siegel) domain $(\mathrm{Sym}_n^{\mathbb{C}})_+$ consisting of all matrices $\sigma \in \mathrm{Sym}_n^{\mathbb{C}}$ with a positive definite real part, defined as follows:

$$(\mathcal{M}u)_0(\sigma) = \int_{\mathbb{R}^n} e^{-\pi\langle\sigma x, x\rangle}\, u(x)\, dx,$$

$$(\mathcal{M}u)_1(\sigma) = \int_{\mathbb{R}^n} (I + i\sigma)x \,.\, e^{-\pi\langle\sigma x, x\rangle}\, u(x)\, dx. \tag{2.1.3}$$

Note that the function $(\mathcal{M}u)_1$ is vector valued: it is not necessary to bother with it if interested only in even functions u. The metaplectic representation $\mathrm{Met}^{(n)}$ of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ in $L^2(\mathbb{R}^n)$ can be traced on the $\mathcal{M}$-transform as follows. For every element $\tilde{g}$ of the metaplectic group $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ above some element $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the symplectic group, there is a continuous choice of a determination of the square root of $\det(iB'\sigma + D')$ for $\sigma \in (\mathrm{Sym}_n^{\mathbb{C}})_+$ such that, for every $u \in L^2(\mathbb{R}^n)$, the following pair of equations holds:

$$(\mathcal{M}\,\mathrm{Met}^{(n)}(\tilde{g})\,u)_0(\sigma) = [\det(iB'\sigma + D')]^{-\frac{1}{2}}\,(\mathcal{M}u)_0((A'\sigma - iC')(iB'\sigma + D')^{-1}),$$

$$(\mathcal{M}\,\mathrm{Met}^{(n)}(\tilde{g})\,u)_1(\sigma) = [\det(iB'\sigma + D')]^{-\frac{1}{2}}\,(I + i\sigma)$$

$$\times [iB'\sigma + D' + i(A'\sigma - iC')]^{-1} \times (\mathcal{M}u)_1((A'\sigma - iC')(iB'\sigma + D')^{-1}). \tag{2.1.4}$$

Quadratic transforms will again, in Sects. 2.2 and 4.1, facilitate our understanding of the anaplectic and ν-anaplectic representations. This characterization, up to a sign $\pm$ depending only on $\tilde{g}$, not on σ, of the metaplectic transformation associated with $\tilde{g}$ can be found in [38, p. 100]. The sign can be obtained as soon as $\tilde{g}$ has been defined in full (i.e., as soon as the homotopy class of a path linking, in the group $\mathrm{Sp}(n,\mathbb{R})$, the identity to the element g above which $\tilde{g}$ is lying has been specified) by continuity.

We shall be especially interested in the subgroup of $\mathrm{Sp}(n,\mathbb{R})$, which is the image of $SL(2,\mathbb{R})$ under the embedding $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} aI & bI \\ cI & dI \end{smallmatrix}\right)$. There is no harm in denoting this group simply as $SL(2,\mathbb{R})$, which we shall do from now on: when dealing with such a matrix, the superscript of the expression $\mathrm{Met}^{(n)}$ will make it clear whether we have in mind the image, under the metaplectic representation of the appropriate dimension, of the first 2×2-matrix or of the associated $(2n)\times(2n)$-matrix. As it turns out, if the dimension n is even, every loop within $SL(2,\mathbb{R})$ lifts as a loop in $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$. Indeed, since $\mathbb{R}^n \sim \mathbb{R}^2 \otimes \mathbb{R}^{\frac{n}{2}}$, it entails no loss of generality to prove this only in the case when $n = 2$. Set $R_t = \left(\begin{smallmatrix} (\cos t)I & (-\sin t)I \\ (\sin t)I & (\cos t)I \end{smallmatrix}\right)$: when t moves on $[0, 2\pi]$, this is a loop in $\mathrm{Sp}(2,\mathbb{R})$, the image of a loop in $SL(2,\mathbb{R})$ generating the fundamental group of that space. Consider the two symplectic matrices

$$K_t = \begin{pmatrix} \cos t & 0 & -\sin t & 0 \\ 0 & 1 & 0 & 0 \\ \sin t & 0 & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} : \tag{2.1.5}$$

one may verify that $R_t = K_t\,J\,K_t\,J^{-1}$. Connecting J to the identity matrix within $\mathrm{Sp}(2,\mathbb{R})$, one sees that the loop R_t is equivalent to the loop $t \mapsto K_t^2 = K_{2t}$, which lifts as a loop in the twofold cover of that group. This implies that, when the dimension n is even, one can, for any $g \in SL(2,\mathbb{R})$, define $\mathrm{Met}^{(n)}(g)$ without any sign ambiguity, a fact which we shall take advantage of, presently, in the case when $n = 2$.

In this case, the definition of $\mathrm{Met}^{(2)}$ on generators of $SL(2,\mathbb{R})$ simplifies as follows (recall that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is to be identified with $\left(\begin{smallmatrix} aI & bI \\ cI & dI \end{smallmatrix}\right)$):

$$(i)\ \left(\mathrm{Met}^{(2)}\left(\left(\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right)\right)u\right)(x) = u(x)\,e^{i\pi c|x|^2}, \quad x \in \mathbb{R}^2;$$

$$(ii)\ \mathrm{Met}^{(2)}\left(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right)u = -i\,\mathcal{F}u;$$

$$(iii)\ \left(\mathrm{Met}^{(2)}\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)u\right)(x) = a^{-1}\,u(a^{-1}x), \quad x \in \mathbb{R}^2,\ a > 0. \tag{2.1.6}$$

Recall the following formula, due to Hecke or Bochner [29]: if a function $u \in \mathcal{S}(\mathbb{R}^n)$ is the product of some "solid" spherical harmonic of degree m (i.e., a homogeneous polynomial on $\mathbb{R}^n$ of degree m, harmonic in the usual sense) by a radial function $U = U(r)$, the Fourier transform of u has the same property, with the same spherical harmonic, the function U being replaced by the function V defined by the equation

$$V(r) = 2\pi\, i^{-m}\, r^{\frac{2-n}{2}-m} \int_0^\infty U(t)\, t^{\frac{n}{2}+m}\, J_{\frac{n-2}{2}+m}(2\pi rt)\, dt. \tag{2.1.7}$$

Given $m \in \mathbb{Z}$, we shall denote as $L^2_m(\mathbb{R}^2)$ the subspace of $L^2(\mathbb{R}^2)$ consisting of functions h – the change from u to h at this point, in the two-dimensional case, is deliberate, in view of future use – satisfying the equation (in which the matrix is of course to be identified with the corresponding linear automorphism of $\mathbb{R}^2$)

$$h \circ \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = e^{-im\theta}\, h \tag{2.1.8}$$

for every $\theta \in \mathbb{R}$ mod 2π: the spaces $L^2_m(\mathbb{R}^2)$ are called the isotypic subspaces of $L^2(\mathbb{R}^2)$. As indicated by (2.1.6), the Hilbert space decomposition $L^2(\mathbb{R}^2) = \oplus_{m\in\mathbb{Z}} L^2_m(\mathbb{R}^2)$ is preserved under the restriction of the two-dimensional metaplectic representation to the image of $SL(2,\mathbb{R})$ in $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$. Variants of Proposition 2.1.1 have been known for a long time (cf. for instance [39]), even though the metaplectic representation had not yet been given a formal definition. As mentioned in the introduction, it also fits with the simplest case of Howe's duality.

Proposition 2.1.1. *If $m = 1, 2, \dots$, set $c_m = (2\pi)^{\frac{m-1}{2}}\,((m-1)\,!)^{-\frac{1}{2}}$. For any function $h \in L^2_m(\mathbb{R}^2)$ and z in the upper half-plane, set*

$$(\Theta_{\pm m}\, h)(z) = z^{-m-1} \int_{\mathbb{R}^2} (x_1 \pm i x_2)^m\, e^{-\frac{i\pi}{z}|x|^2}\, h(x)\, dx, \qquad \mathrm{Im}\, z > 0. \tag{2.1.9}$$

The map $c_m\Theta_{\pm m}$ is an isometry from the Hilbert space $L^2_{\pm m}(\mathbb{R}^2)$ onto the Hilbert space $\mathcal{H}_{m+1}$ consisting of all holomorphic functions χ in the upper half-plane Π satisfying the condition

$$\|\chi\|^2_{m+1} :\, = \int_\Pi |\chi(z)|^2\, (\mathrm{Im}\, z)^{m+1}\, d\mu(z) < \infty : \tag{2.1.10}$$

we have denoted as $d\mu$ the usual invariant measure $d\mu(x+iy) = y^{-2}\, dx dy$ on Π. Moreover, denote as $\mathcal{D}_{m+1}$ the representation (taken from the so-called holomorphic discrete series) of $SL(2,\mathbb{R})$ in $\mathcal{H}_{m+1}$ defined by

$$\left(\mathcal{D}_{m+1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)\chi\right)(z) = (-cz+a)^{-m-1}\,\chi\left(\frac{dz-b}{-cz+a}\right). \tag{2.1.11}$$

Then, the operator $\Theta_{\pm m}$ intertwines the restriction to the space $L^2_{\pm m}(\mathbb{R}^2)$ of the representation $\mathrm{Met}^{(2)}$ of $SL(2,\mathbb{R})$ in $L^2(\mathbb{R}^2)$ with the representation $\mathcal{D}_{m+1}$.

For $m = 0$, the same conclusion holds provided one defines the space $\mathcal{H}_1$ as the Hardy space consisting of holomorphic functions χ such that $\sup_{y>0}\int_{-\infty}^{\infty}|\chi(x+iy)|^2\,dx < \infty$, and one takes $c_0 = (2\pi)^{-\frac{1}{2}}$.

Proof. Consider first the case when $m \geq 1$. In view of (2.1.8), one has if $h \in L^2_m(\mathbb{R}^2)$ the equation

$$(\Theta_m\,h)\left(-\frac{1}{z}\right) = 2\pi\,(-z)^{m+1}\int_0^{\infty} r^{m+1}\,e^{i\pi z r^2}\;h(r,0)\,dr. \tag{2.1.12}$$

One may then write, setting $z = s+it$,

$$\begin{aligned}
\|\,\Theta_m\,h\,\|^2_{m+1} &= \int_{\Pi}|(\Theta_m\,h)\left(-\frac{1}{z}\right)|^2\,|z|^{-2m-2}\,(\mathrm{Im}\;z)^{m+1}\,d\mu(z) \\
&= 4\pi^2\int_0^{\infty}t^{m-1}\,dt\int_{-\infty}^{\infty}ds\left|\int_0^{\infty}r^{m+1}\,e^{-\pi t r^2}\,e^{i\pi s r^2}\;h(r,0)\,dr\right|^2 \\
&= \pi^2\int_0^{\infty}t^{m-1}\,dt\int_{-\infty}^{\infty}ds\left|\int_0^{\infty}\rho^{\frac{m}{2}}\,e^{-\pi t\rho}\,e^{i\pi s\rho}\;h(\rho^{\frac{1}{2}},0)\,d\rho\right|^2 \\
&= 2\pi^2\int_0^{\infty}t^{m-1}\,dt\int_0^{\infty}\rho^m\,e^{-2\pi t\rho}\,|h(\rho^{\frac{1}{2}},0)|^2\,d\rho \\
&= 2\pi^2\;\times\,(2\pi)^{-m}\,(m-1)\,!\int_0^{\infty}|h(\rho^{\frac{1}{2}},0)|^2\,d\rho,
\end{aligned} \tag{2.1.13}$$

from which it is immediate to conclude that $c_m\Theta_m$ is an isometry. We may dispense with the proof that Θ_m is onto with the help of an irreducibility argument, after we have proved the intertwining properties (i), (ii), (iii). The first one is immediate, since $c - \frac{1}{z} = -[\frac{z}{1-cz}]^{-1}$ and $z^{-m-1}\times[\frac{z}{1-cz}]^{m+1} = (1-cz)^{-m-1}$; the third one is obtained after a change of variable. Starting from

$$[\,\Theta_m\,(\mathcal{F}h)\,](z) = z^{-m-1}\int_{\mathbb{R}^2}h(x)\;\mathcal{F}\left((x_1+i\,x_2)^m\,e^{-\frac{i\pi}{z}|x|^2}\right)\,dx, \tag{2.1.14}$$

one obtains the second one, namely

$$(\Theta_m\,(-i\,\mathcal{F}h))(z) = z^{-m-1}\,(\Theta_m\,h)\left(-\frac{1}{z}\right), \tag{2.1.15}$$

with the help of (2.1.7) and of the equation [21, p. 93]

$$\int_0^{\infty}J_m(2\pi\,|\xi\,|\,t)\;e^{-\frac{i\pi}{z}t^2}\;t^{m+1}\,dt = (2\pi\,|\xi\,|)^m\left(\frac{2i\pi}{z}\right)^{-m-1}\;e^{i\pi z|\xi\,|^2}. \tag{2.1.16}$$

The situation obtained when changing m to $-m$ can be reduced to the preceding one by means of the intertwining operator $h \mapsto h_1,\ h_1(x_1, x_2) = h(x_1, -x_2)$. Finally, when $m = 0$, only the norm computation has to be reconsidered. It follows the same lines (with a slight simplification), starting from the remark that

$$\| \Theta_0 h \|_1 = \| z \mapsto z^{-1} (\Theta_0 h) \left(-\frac{1}{z} \right) \|_1, \tag{2.1.17}$$

a consequence of the unitarity of the representation $\mathcal{D}_1$. □

Remark 2.1.1. The fact that the parameter m used in $L^2_m(\mathbb{R}^2)$ corresponds to the space $\mathcal{H}_{m+1}$ may often be felt as an inconvenience: however, there is nothing we can do about it.

We close this section with urging newcomers to pseudodifferential analysis to have another look at (1.1.1) and (1.1.2), now that their familiarity with the metaplectic representation may have been refreshed. Our aim in Chap. 3 (cf. introduction) is to introduce a new symbolic calculus, or "pseudodifferential analysis," for which a covariance formula somewhat similar to (1.1.2), but involving on the phase space $\mathbb{R}^2$ the representation $\mathrm{Met}^{(2)}$ in place of the quasiregular action $g\,.\,\mathfrak{S} = \mathfrak{S} \circ g^{-1}$ of that group, would hold; at the same time, we want (1.1.3) to generalize too. The difficulty, as will be seen, is that everything has to be invented from scratch: as a space of possible functions u of one variable, we cannot use a space even remotely resembling $L^2(\mathbb{R})$; also, the one-dimensional metaplectic representation cannot play any role here. Then, Weyl's definition (1.1.1) has to be replaced by a new one.

The analysis to be developed in Sect. 2.2, rather than being regarded as an extension of the usual analysis, should be considered as alien to it. In Sect. 4.1, however, we shall imbed this analysis into a one-parameter ν-series: the case when $\nu = 0$ will then be shown to contain the part of usual analysis on the line centered around such objects as the Fourier transformation, the metaplectic representation, and Hermite functions.

2.2 Anaplectic Analysis

This section starts with a crash course on one-dimensional anaplectic analysis, a much more detailed version of which is to be found in [38]. Anaplectic analysis is just what is needed in the present context because we want to consider the inverse of the "annihilation" operator A (a name soon to be changed to that of "lowering" operator). In anaplectic analysis, the spectrum of the harmonic oscillator is $\mathbb{Z}$ rather than $\frac{1}{2} + \mathbb{N}$, and taking the inverse of A is all right, as will be recalled.

The basic difference between usual analysis and anaplectic analysis is the following. In the first one, there is a considerable supply (take for instance the Hermite functions) of functions on the line which extend as entire functions in the complex plane, while being simultaneously *very* rapidly decreasing at infinity. In anaplectic analysis, these two desirable properties have to be split between the function

u under consideration and other functions obtained from u, in a very specific way, with the help of the complex continuation process. This leads to the following definition, summed up, like a greater part of this short section, from the first two sections of [38].

Definition 2.2.1. Let us say that an entire function f of one variable is *nice* if on one hand $f(z)$ is bounded by a constant times some exponential $\exp(\pi R|z|^2)$, on the other hand the restriction of f to the *positive* half-line is bounded by a constant times some exponential $\exp(-\pi\varepsilon x^2)$: here, R and ε are assumed to be positive. The space $\mathfrak{A}$ consists of all entire functions u of one variable with the property that there exists a 4-tuple

$$\boldsymbol{f} = (f_0\,,\, f_1\,,\, f_{i,0}\,,\, f_{i,1}) \tag{2.2.1}$$

of nice functions such that

$$\begin{aligned} f_{i,0}(z) &= \frac{1-i}{2}\,(f_0(iz) + i\,f_0(-iz)),\\ f_{i,1}(z) &= \frac{1+i}{2}\,(f_1(iz) - i\,f_1(-iz)), \end{aligned} \tag{2.2.2}$$

and such that the even part u_{even} of u coincides with the even part of f_0, and the odd part u_{odd} of u coincides with the odd part of f_1.

It can be proved, as a consequence of the Phragmén–Lindelöf lemma, that the vector-valued function $\boldsymbol{f}$ associated to $u \in \mathfrak{A}$ is necessarily unique. We shall call it the $\mathbb{C}^4$-realization of u. Here is a basic example.

Proposition 2.2.2. *Set, for x real,*

$$\phi(x) = (\pi\,|x|)^{\frac{1}{2}}\, I_{-\frac{1}{4}}(\pi x^2), \tag{2.2.3}$$

with [21, p. 66]

$$I_\nu(t) = \sum_{m\geq 0} \frac{(\frac{t}{2})^{\nu+2m}}{m\,!\,\Gamma(\nu+m+1)} \tag{2.2.4}$$

for $t > 0$. The function ϕ lies in $\mathfrak{A}$. Its $\mathbb{C}^4$-realization is the function $\boldsymbol{f} = (\psi,\, 0,\, \psi,\, 0)$, with

$$\psi(x) = 2^{\frac{1}{2}}\,\pi^{-\frac{1}{2}}\,x^{\frac{1}{2}}\,K_{\frac{1}{4}}(\pi x^2) = (\pi x)^{\frac{1}{2}}\,\left[I_{-\frac{1}{4}}(\pi x^2) - I_{\frac{1}{4}}(\pi x^2)\right], \qquad x > 0. \tag{2.2.5}$$

The space $\mathfrak{A}$ is stable under the usual operators Q and P such that $(Qu)(x) = x\,u(x)$ and $(Pu)(x) = \frac{1}{2i\pi}\,u'(x)$. If the $\mathbb{C}^4$-realization $\boldsymbol{f}$ of u is the one given in (2.2.1), those of Qu and Pu are, respectively,

$$\boldsymbol{h}(z) = (\,z\,f_1(z)\,,\; z\,f_0(z)\,,\, z\,f_{i,1}(z)\,,\, -z\,f_{i,0}(z)\,) \tag{2.2.6}$$

and

$$\boldsymbol{h} = \frac{1}{2i\pi}\,(\,f_1'\,,\, f_0'\,,\, -f_{i,1}'\,,\, f_{i,0}'\,). \tag{2.2.7}$$

One may introduce, in the usual way, the harmonic oscillator and the operators

$$A^* = \pi^{\frac{1}{2}} \left(x - \frac{1}{2\pi} \frac{d}{dx} \right), \qquad A = \pi^{\frac{1}{2}} \left(x + \frac{1}{2\pi} \frac{d}{dx} \right). \tag{2.2.8}$$

In usual analysis, these two operators would be called the creation and annihilation operators: however, for reasons to be seen immediately, they are to be called, now, the *raising* and *lowering* operators instead.

Theorem 2.2.3. *The spectrum of the harmonic oscillator*

$$L = \pi\,(Q^2 + P^2) \tag{2.2.9}$$

in the space $\mathfrak{A}$ is $\mathbb{Z}$, and for every $j \in \mathbb{Z}$ the eigenspace corresponding to the eigenvalue j is generated by the function ϕ^j, with

$$\phi^j = A^{*j}\,\phi \quad \text{if } j \geq 0, \qquad \phi^j = A^{|j|}\,\phi \quad \text{if } j \leq 0. \tag{2.2.10}$$

There is on the space $\mathfrak{A}$ a useful nondegenerate *pseudoscalar* product $(\ |\ )$ (this is the same as a scalar product, except for positivity) defined in terms of the $\mathbb{C}^4$-realizations of the two functions involved as

$$(\boldsymbol{h}\,|\,\boldsymbol{f}) = \\ 2^{\frac{1}{2}} \int_0^\infty \left(\bar{h}_0(x) f_0(x) + \bar{h}_1(x) f_1(x) + \bar{h}_{i,0}(x) f_{i,0}(x) - \bar{h}_{i,1}(x) f_{i,1}(x) \right) dx. \tag{2.2.11}$$

The operators Q and P are self-adjoint on $\mathfrak{A}$ with respect to this pseudoscalar product. The functions $\phi^j,\ j \in \mathbb{Z}$, are pairwise orthogonal with respect to it. The function ϕ is normalized and one has $(\phi^{k+1}\,|\,\phi^{k+1}) = (k + \frac{1}{2})\,(\phi^k\,|\,\phi^k)$ and $(\phi^{-k}\,|\,\phi^{-k}) = (-1)^k\,(\phi^k\,|\,\phi^k)$ for $k \geq 0$. Consequently,

$$(\phi^k\,|\,\phi^k) = \begin{cases} 2^{-2k}\,\frac{(2k)\,!}{k\,!}\,, & k \geq 0, \\ (-1)^k\,2^{-2|k|}\,\frac{(2|k|)\,!}{|k|\,!}\,, & k < 0. \end{cases} \tag{2.2.12}$$

In anaplectic analysis, the Heisenberg representation, as defined in a way formally identical to the usual one, preserves the anaplectic space $\mathfrak{A}$.

Theorem 2.2.4. *Given $u \in \mathfrak{A}$ and $(y, \eta) \in \mathbb{C}^2$, the function* $\exp\left(2i\pi\,(\eta\,Q - yP)\right)u$ *such that*

$$\left(\exp\left(2i\pi\,(\eta\,Q - yP)\right)u\right)(x) = u(x-y)\,e^{2i\pi(x-\frac{y}{2})\eta} \tag{2.2.13}$$

lies in $\mathfrak{A}$ too. If one restricts (y, η) to the space $\mathbb{R}^2$, the representation of Heisenberg's group (or, in an equivalent way, the projective representation of $\mathbb{R}^2$) so defined preserves the pseudoscalar product.

Of course, a function such as ϕ, an eigenfunction of the harmonic oscillator for a classically forbidden eigenvalue, cannot be integrable on the real line. However, there is on $\mathfrak{A}$ a substitute for the notion of integral, still a translation-invariant linear form.

Proposition 2.2.5. *If $\boldsymbol{f} = (f_0\,, f_1\,, f_{i,0}\,, f_{i,1}\,)$ is the $\mathbb{C}^4$-realization of some function $u \in \mathfrak{A}$, set*

$$\mathrm{Int}\,[u] = 2^{\frac{1}{2}} \int_0^\infty (f_0(x) + f_{i,0}(x))\, dx. \tag{2.2.14}$$

For every $y \in \mathbb{C}$, with $(e^{-2i\pi yP}\, u)(z) = u(z-y)$, one has

$$\mathrm{Int}\,[e^{-2i\pi yP}\, u] = \mathrm{Int}\,[u]. \tag{2.2.15}$$

This concept of integral makes the definition of an anaplectic Fourier transformation possible.

Proposition 2.2.6. *Given $x \in \mathbb{R}$, define the function e_x as $e_x(y) = e^{-2i\pi xy}$. For any $u \in \mathfrak{A}$, the anaplectic Fourier transform $\mathcal{F}_{\mathrm{ana}}\, u$ of u defined as*

$$(\mathcal{F}_{\mathrm{ana}}\, u)(x) = \mathrm{Int}\,[e_x\, u] \tag{2.2.16}$$

lies in $\mathfrak{A}$ too. A fully developed version of the preceding definition, in terms of the $\mathbb{C}^4$-realization of u, is

$$\begin{aligned}(\mathcal{F}_{\mathrm{ana}}\, u)(x) &= 2^{\frac{1}{2}} \int_0^\infty f_0(y)\, \cos 2\pi xy\, dy - 2^{\frac{1}{2}}\, i \int_0^\infty f_1(y)\, \sin 2\pi xy\, dy \\ &\quad + 2^{\frac{1}{2}} \int_0^\infty f_{i,0}(y)\, \cosh 2\pi xy\, dy - 2^{\frac{1}{2}}\, i \int_0^\infty f_{i,1}(y)\, \sinh 2\pi xy\, dy.\end{aligned} \tag{2.2.17}$$

The function ϕ introduced in (2.2.3) *is invariant under $\mathcal{F}_{\mathrm{ana}}$.*

It is essential to recall here the definition of the anaplectic representation (the substitute, in anaplectic analysis, of the metaplectic representation).

Theorem 2.2.7. *There is a unique representation* Ana *of $SL(2,\mathbb{R})$ in the space $\mathfrak{A}$ with the following properties:*

(i) if $g = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$, one has $(\mathrm{Ana}(g)\, u)(x) = u(x)\, e^{i\pi cx^2}$;
(ii) if $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ with $a > 0$, one has $(\mathrm{Ana}(g)\, u)(x) = a^{-\frac{1}{2}}\, u(a^{-1}x)$;
(iii) one has $\mathrm{Ana}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) = \mathcal{F}_{\mathrm{ana}}$.

This representation is pseudo-unitary, i.e., it preserves the scalar product introduced in (2.2.11)*. It combines with the (anaplectic) Heisenberg representation in the way characterized by the equation*

$$\mathrm{Ana}(g)\, e^{2i\pi(\eta Q - yP)}\, \mathrm{Ana}(g^{-1}) = e^{2i\pi(\eta' Q - y' P)} \tag{2.2.18}$$

if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ and $g \left(\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right) = \left(\begin{smallmatrix} y' \\ \eta' \end{smallmatrix}\right)$.

There is an extra benefit in anaplectic analysis: one can extend the anaplectic representation to that of the subgroup $SL_i(2,\mathbb{R})$ of $SL(2,\mathbb{C})$ generated by $SL(2,\mathbb{R})$ and by the matrix $g = \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$, defining $\mathrm{Ana}(g)$, in this case, as the transformation (which preserves $\mathfrak{A}$) $u \mapsto u_i,\ u_i(x) = u(ix)$. However, pseudo-unitarity is then lost.

Note that (in contradiction to the case of the one-dimensional metaplectic representation) one has a genuine representation of $SL(2,\mathbb{R})$, without it being necessary to use a twofold cover.

The infinitesimal version of (2.2.18) is the following: if $(y,\eta)\in\mathbb{C}^2$ and if $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{R})$, one has

$$\mathrm{Ana}(g)\,(\eta\,Q-y\,P)\,\mathrm{Ana}(g^{-1})=\eta'\,Q-y'\,P \tag{2.2.19}$$

with $\begin{pmatrix} y'\\ \eta'\end{pmatrix}=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\begin{pmatrix} y\\ \eta\end{pmatrix}$. It is used in the elementary proof of the proposition that follows.

Proposition 2.2.8. *Given $z\in\Pi$, the hyperbolic (Poincaré) upper half-plane, set*

$$A_z=\pi^{\frac12}\,(Q-\bar z P)\,,\qquad A_z^*=A_{\bar z}=\pi^{\frac12}\,(Q-zP), \tag{2.2.20}$$

and define

$$L_z=A_zA_z^*-\frac{\mathrm{Im}\,z}{2}=A_z^*A_z+\frac{\mathrm{Im}\,z}{2}. \tag{2.2.21}$$

One then has the identities

$$\begin{aligned}\mathrm{Ana}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)\,A_z\,\mathrm{Ana}\left(\left(\begin{smallmatrix} d & -b\\ -c & a\end{smallmatrix}\right)\right)&=(c\bar z+d)\,A_{\frac{az+b}{cz+d}}\,,\\ \mathrm{Ana}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)\,L_z\,\mathrm{Ana}\left(\left(\begin{smallmatrix} d & -b\\ -c & a\end{smallmatrix}\right)\right)&=|c\bar z+d|^2\,L_{\frac{az+b}{cz+d}}\,.\end{aligned} \tag{2.2.22}$$

If one takes in particular $g_z=\begin{pmatrix} y^{\frac12} & y^{-\frac12}x\\ 0 & y^{-\frac12}\end{pmatrix}$ if $z=x+i\,y$, the function

$$\phi_z^j=\mathrm{Ana}(g_z)\,\phi^j, \tag{2.2.23}$$

with ϕ^j as defined in Theorem 2.2.3, is a basis of the (one-dimensional) eigenspace of L_z in $\mathfrak{A}$ corresponding to the eigenvalue $j\,\mathrm{Im}\,z$.

In anaplectic analysis, it is often necessary to go back to the $\mathbb{C}^4$-realizations of functions. We shall have to use, later, the following, the proof of which is immediate: if $g=\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix}$ with $a>0$, $\mathrm{Ana}(g)$ acts on $\mathbb{C}^4$-realizations as $\boldsymbol{f}\mapsto\boldsymbol{h}$, with

$$\boldsymbol{h}(x)=(a^{-\frac12}\,f_0(a^{-1}x)\,,\,a^{-\frac12}\,f_1(a^{-1}x)\,,\,a^{-\frac12}\,f_{i,0}(a^{-1}x)\,,\,a^{-\frac12}\,f_{i,1}(a^{-1}x)); \tag{2.2.24}$$

if $g=\left(\begin{smallmatrix} 1 & 0\\ c & 1\end{smallmatrix}\right)$, $\mathrm{Ana}(g)$ acts as $\boldsymbol{f}\mapsto\boldsymbol{h}$, with

$$\boldsymbol{h}(x)=(f_0(x)\,e^{i\pi cx^2},\,f_1(x)\,e^{i\pi cx^2},\,f_{i,0}(x)\,e^{-i\pi cx^2},\,f_{i,1}(x)\,e^{-i\pi cx^2}). \tag{2.2.25}$$

It is also necessary to examine the infinitesimal operators of the anaplectic representation, defined by the formula

$$d\mathrm{Ana}(X)=\frac{1}{2i\pi}\,\frac{d}{dt}\bigg|_{t=0}\mathrm{Ana}\,(\exp tX)\,,\qquad X\in\mathfrak{sl}(2,\mathbb{R}). \tag{2.2.26}$$

Taking first $\exp tX = \left(\begin{smallmatrix}1 & 0\\ t & 1\end{smallmatrix}\right)$, next $\exp tX = \left(\begin{smallmatrix}e^t & 0\\ 0 & e^{-t}\end{smallmatrix}\right)$, one obtains from the cases (i) and (ii) of Theorem 2.2.7 the relations

$$\begin{aligned} d\mathrm{Ana}\left(\left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)\right) &= \frac{1}{2}\,Q^2,\\ d\mathrm{Ana}\left(\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\right) &= -\frac{1}{2i\pi}\,(x\,\frac{d}{dx}+\frac{1}{2}) = -\frac{1}{2}\,(QP+PQ). \end{aligned} \tag{2.2.27}$$

Taking the conjugate of the first relation under the anaplectic Fourier transformation, one finds

$$d\mathrm{Ana}\left(\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}\right)\right) = \frac{1}{2}\,P^2. \tag{2.2.28}$$

As a consequence,

$$d\mathrm{Ana}\left(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right) = -\frac{1}{2}\,(Q^2+P^2) \tag{2.2.29}$$

so that

$$\exp(-it\,L) = \mathrm{Ana}\left(\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\right). \tag{2.2.30}$$

Of course, no Stone's theorem is available in $\mathfrak{A}$, a space with a pseudoscalar product only: what is meant by the equation that precedes is that, if one defines $\exp(-it\,L)$ by this equation, one obtains a one-parameter group of operators satisfying the right differential equation. Since $L\phi = 0$, this equation proves the invariance of ϕ under the operator in (2.2.30). One may note the equation $\mathcal{F}_{\mathrm{ana}} = \exp(-\frac{i\pi}{2}\,L)$: in usual analysis, there is an extra factor $e^{-\frac{i\pi}{4}}$ on the left-hand side, of course linked to the shift by $\frac{1}{2}$ of the spectrum of the harmonic oscillator.

A useful corollary of (2.2.30), together with Theorem 2.2.3, is the following generalization of (2.2.23): if $z \in \Pi$, $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and $j \in \mathbb{Z}$, one has

$$\mathrm{Ana}(g)\,\phi_z^j = \left(\frac{cz+d}{|cz+d|}\right)^j \phi^j_{\frac{az+b}{cz+d}}. \tag{2.2.31}$$

The proof in the case when $z = i$ goes as follows: assuming that $z = \frac{ai+b}{ci+d}$, write

$$\begin{aligned} \begin{pmatrix}a & b\\ c & d\end{pmatrix} &= \begin{pmatrix}y^{\frac{1}{2}} & y^{-\frac{1}{2}}x\\ 0 & y^{-\frac{1}{2}}\end{pmatrix} \begin{pmatrix}\frac{d}{(c^2+d^2)^{\frac{1}{2}}} & -\frac{c}{(c^2+d^2)^{\frac{1}{2}}}\\ \frac{c}{(c^2+d^2)^{\frac{1}{2}}} & \frac{d}{(c^2+d^2)^{\frac{1}{2}}}\end{pmatrix}\\ &= \begin{pmatrix}y^{\frac{1}{2}} & y^{-\frac{1}{2}}x\\ 0 & y^{-\frac{1}{2}}\end{pmatrix} \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}, \end{aligned} \tag{2.2.32}$$

finding, as a result of (2.2.30),

$$\mathrm{Ana}(g)\,\phi^j = e^{-ijt}\,\mathrm{Ana}(g_z)\,\phi^j = \left(\frac{ci+d}{|ci+d|}\right)^j \phi_z^j. \tag{2.2.33}$$

The general case then follows from the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}}x \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ with $c_1 i + d_1 = y^{-\frac{1}{2}}(cz+d)$.

Lemma 2.2.9 will be necessary in Sects. 3.1 and 3.2.

Lemma 2.2.9. *One has*

$$A_z\,\phi_z^k = \gamma_k\,(\mathrm{Im}\,z)^{\frac{1}{2}}\,\phi_z^{k-1}\,, \qquad A_z^*\,\phi_z^k = \gamma_k^*\,(\mathrm{Im}\,z)^{\frac{1}{2}}\,\phi_z^{k+1}, \tag{2.2.34}$$

with

$$\gamma_k = \begin{cases} k-\frac{1}{2} & \text{if } k \geq 1\,, \\ 1 & \text{if } k \leq 0 \end{cases}, \qquad \gamma_k^* = \begin{cases} 1 & \text{if } k \geq 0\,, \\ k+\frac{1}{2} & \text{if } k \leq -1 \end{cases}. \tag{2.2.35}$$

One has the relations

$$\begin{aligned} \left[4i\,(\mathrm{Im}\,z)\,\frac{\partial}{\partial z} - j\right]\phi_z^j &= -\gamma_j\,\gamma_{j-1}\,\phi_z^{j-2}, \\ \left[4i\,(\mathrm{Im}\,z)\,\frac{\partial}{\partial \bar{z}} - j\right]\phi_z^j &= -\gamma_j^*\,\gamma_{j+1}^*\,\phi_z^{j+2}. \end{aligned} \tag{2.2.36}$$

Proof. Relations (2.2.34) are a consequence of (2.2.21) and (2.2.22) together with the fact, also indicated in Proposition 2.2.8, that ϕ_z^k is an eigenfunction of L_z corresponding to the eigenvalue $k\,\mathrm{Im}\,z$.

With $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $z = x+iy$, set, recalling that the matrix g_z has been introduced in Proposition 2.2.8,

$$\tilde{g}_z :\ = J\,g_z\,J^{-1} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ 0 & y^{\frac{1}{2}} \end{pmatrix}: \tag{2.2.37}$$

it follows that, for every $u \in \mathfrak{A}$,

$$(\mathrm{Ana}(\tilde{g}_z)\,u)(t) = e^{-i\pi x t^2}\,y^{\frac{1}{4}}\,u(y^{\frac{1}{2}}t). \tag{2.2.38}$$

On the other hand, from (2.2.30) and Theorem 2.2.3, we obtain

$$\mathrm{Ana}(J)\,\phi^j = (-i)^j\,\phi^j, \tag{2.2.39}$$

so that

$$\phi_z^j = (-i)^j\,\mathrm{Ana}(J^{-1})\,\mathrm{Ana}(\tilde{g}_z)\,\phi^j. \tag{2.2.40}$$

Next, using the fact that $\pi\,(Q^2+P^2)\,\phi^j = j\,\phi^j$ and Heisenberg's relation $[P,Q] = \frac{1}{2i\pi}$, one obtains

$$\begin{aligned} A^2\,\phi^j &= 2\pi\,Q\,(Q+i\,P)\,\phi^j + (\frac{1}{2}-j)\,\phi^j\,, \\ A^{*2}\,\phi^j &= 2\pi\,Q\,(Q-i\,P)\,\phi^j - (\frac{1}{2}+j)\,\phi^j. \end{aligned} \tag{2.2.41}$$

On the other hand, a direct computation, starting from (2.2.38), shows that

$$\left[\mathrm{Ana}(\tilde{g}_z)\,(Q^2 u)\right](t) = -\frac{1}{i\pi}\, y\,\frac{\partial}{\partial x}\,(\mathrm{Ana}(\tilde{g}_z)\,u)(t),$$
$$\left[\mathrm{Ana}(\tilde{g}_z)\,(QPu)\right](t) = \frac{1}{i\pi}\left(y\,\frac{\partial}{\partial y} - \frac{1}{4}\right)(\mathrm{Ana}(\tilde{g}_z)\,u)(t) \tag{2.2.42}$$

for every function $u \in \mathfrak{A}$. It follows that

$$\mathrm{Ana}(\tilde{g}_z)\,A^2\,\phi^j = \left(4iy\,\frac{\partial}{\partial z} - j\right)\mathrm{Ana}(\tilde{g}_z)\,\phi^j,$$
$$\mathrm{Ana}(\tilde{g}_z)\,A^{*2}\,\phi^j = \left(4iy\,\frac{\partial}{\partial \bar{z}} - j\right)\mathrm{Ana}(\tilde{g}_z)\,\phi^j. \tag{2.2.43}$$

Equations (2.2.36) follow if one uses (2.2.40) and (2.2.34). □

We need to introduce some Hilbert space methods in anaplectic analysis. True, (2.2.11) only introduces a pseudoscalar product on $\mathfrak{A}$: however, its restriction to the even part of $\mathfrak{A}$ is positive definite while, on the odd part, one may take advantage of the linear isomorphism provided by an operator changing the parity, for instance the canonical lowering operator. The following is taken from [38, p. 154].

Proposition 2.2.10. *Let $(\phi_j)_{j\in\mathbb{Z}}$ be the sequence of eigenfunctions of the anaplectic harmonic oscillator introduced in Theorem 2.2.3. Given any function $u \in \mathfrak{A}$, the set of scalar products of u against the functions ϕ_j satisfies for some constants $C > 0$ and $\delta \in]0,1[$ the estimate*

$$|(\phi^j\,|\,u)| \le C\,[\frac{|j|}{2}]\,!\,(2\delta)^{\frac{|j|}{2}}, \qquad j \in \mathbb{Z}. \tag{2.2.44}$$

Conversely, given any sequence $(a_j)_{j\in\mathbb{Z}}$ of complex numbers satisfying for some $C > 0$ and $\delta \in]0,1[$ the inequality

$$|a_j| \le C\,[\frac{|j|}{2}]\,!\,(2\delta)^{\frac{|j|}{2}}, \qquad j \in \mathbb{Z}, \tag{2.2.45}$$

there exists a unique function $u \in \mathfrak{A}$ such that $a_j = (\phi^j\,|\,u)$ for all j.

Recalling (2.2.12), one sees that the orthogonal set $(\psi_j)_{j\in 2\mathbb{Z}}$, with

$$\psi^j = 2^{|j|}\left(\frac{|j|\,!}{(2\,|j|)\,!}\right)^{\frac{1}{2}}\phi^j, \tag{2.2.46}$$

consists of normalized functions. It was actually shown in loc.cit. that this set constitutes a Hilbert basis of the completion of $\mathfrak{A}_{\text{even}}$ under the norm associated with the restriction to this space of the scalar product (2.2.11). Hence, if $u \in \mathfrak{A}_{\text{even}}$, one has the expansion, convergent in the Hilbert sense,

$$u = \sum_{j\in 2\mathbb{Z}} \frac{2^{2|j|}\,|j|\,!}{(2\,|j|)\,!}\,(\phi^j\,|\,u)\,\phi^j : \tag{2.2.47}$$

one can of course verify that the condition (2.2.44) makes the series $\sum |(\psi^j\,|\,u)|^2$ convergent.

When $u \in \mathfrak{A}$ is odd, we can write instead

$$u = \sum_{j \in 2\mathbb{Z}} \frac{2^{2|j|}\,|j|\,!}{(2\,|j|)\,!}\,(\phi^j\,|\,A\,u)\,A^{-1}\,\phi^j. \tag{2.2.48}$$

With the help of (2.2.34) and of the equation

$$A^*\,\phi^j = \phi^{j+1} \text{ if } j \geq 0, \qquad A^*\,\phi^j = (j+\frac{1}{2})\,\phi^{j+1} \text{ if } j < 0, \tag{2.2.49}$$

this can be written as

$$\begin{aligned} u = &\sum_{j \text{ even} \geq 0} \frac{2^{2j}\,j\,!}{(2j)\,!}\,(j+\frac{1}{2})^{-1}\,(\phi^{j+1}\,|\,u)\,\phi^{j+1} \\ &+ \sum_{j \text{ even} < 0} \frac{2^{-2j}\,(-j)\,!}{(-2j)\,!}\,(j+\frac{1}{2})\,(\phi^{j+1}\,|\,u)\,\phi^{j+1}. \end{aligned} \tag{2.2.50}$$

The general formula, whether $u \in \mathfrak{A}$ has any definite parity or not, is thus

$$u = \sum_{\ell \in \mathbb{Z}} c_\ell\,(\phi^\ell\,|\,u)\,\phi^\ell, \tag{2.2.51}$$

with

$$c_\ell = \begin{cases} \frac{2^{2|\ell|}\,|\ell|\,!}{(2\,|\ell|)\,!} & \text{if } \ell \text{ is even,} \\ \frac{2^{2|\ell-1|}\,(|\ell-1|)\,!}{(2\,|\ell-1|)\,!}\,(\ell-\frac{1}{2})^{-\operatorname{sign}\ell} & \text{if } \ell \text{ is odd.} \end{cases} \tag{2.2.52}$$

Note that, if $\ell \geq 1$, one has $c_\ell = \frac{2^{2\ell}\,\ell\,!}{(2\ell)\,!}$ in both cases. Since the anaplectic representation is pseudo-unitary, one can also write

$$u = \sum_{\ell \in \mathbb{Z}} c_\ell\,(\phi_z^\ell\,|\,u)\,\phi_z^\ell \tag{2.2.53}$$

for any $z \in \Pi$, as a consequence of (2.2.23).

We end this section with another useful characterization, taken from [38, p. 7, 188–190] of the space $\mathfrak{A}$. The introduction of the quadratic transform $((\mathcal{Q}u)_0, (\mathcal{Q}u)_1)$ of u will be found more natural if compared with (2.1.3).

Proposition 2.2.11. *Let u be an entire function of one variable satisfying for some pair of constants C, R the estimate $|f(z)| \leq C\,e^{\pi R|z|^2}$. Set, for σ real and large,*

$$\begin{aligned} (\mathcal{Q}u)_0(\sigma) &= \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\,u(x e^{-\frac{i\pi}{4}})\,dx, \\ (\mathcal{Q}u)_1(\sigma) &= \int_{-\infty}^{\infty} (1+i\sigma)\,x e^{-\pi\sigma x^2}\,u(x e^{-\frac{i\pi}{4}})\,dx, \end{aligned} \tag{2.2.54}$$

and, for z on the unit circle, $z = e^{-i\theta}$ with $\theta > 0$ and small,

$$(\mathcal{K}u)_0(z) = |1-z|^{-\frac{1}{2}}\,(\mathcal{Q}u)_0\left(i\,\frac{1+z}{1-z}\right), \tag{2.2.55}$$

finally introducing a function $(\mathcal{K}u)_1$ linked to $(\mathcal{Q}u)_1$ by the same transformation as the one giving $(\mathcal{K}u)_0$ in terms of $(\mathcal{Q}u)_0$. The following three conditions are equivalent:

(i) u lies in the space $\mathfrak{A}$;
(ii) each of the two functions $(\mathcal{Q}u)_0$ and $(\mathcal{Q}u)_1$ extends as an analytic function on the real line, admitting for large $|\sigma|$ a convergent expansion $(\mathcal{Q}u)_j(\sigma) = \sum_{n\geq 0} a_n^{(j)}\,\sigma^{-n}\,|\sigma|^{-\frac{1}{2}}$;
(iii) each of the two functions $(\mathcal{K}u)_0$ and $(\mathcal{K}u)_1$, initially defined in a neighborhood of the point $z=1$ of the unit circle, extends as an analytic function to the full circle.

The $\mathcal{Q}$-realization of $\mathfrak{A}$ is especially useful when dealing with certain representation-theoretic aspects. Set

$$|\sigma|_0^{-1-\rho} = |\sigma|^{-1-\rho}\,, \qquad |\sigma|_1^{-1-\rho} = |\sigma|^{-1-\rho} \times \operatorname{sign}\sigma\,, \qquad \sigma \in \mathbb{R}\backslash\{0\}\,,\ \rho \in \mathbb{R}. \tag{2.2.56}$$

Define the representation $\hat{\pi}_{\rho,\varepsilon}$ ($0 < |\rho| < 1$, $\varepsilon = 0$ or 1) of $SL(2,\mathbb{R})$, acting on functions defined on the real line, by the equation

$$\left(\hat{\pi}_{\rho,\varepsilon}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)w\right)(\sigma) = |-b\,\sigma + d|_\varepsilon^{-1-\rho}\,w\left(\frac{a\,\sigma - c}{-b\,\sigma + d}\right): \tag{2.2.57}$$

when $\varepsilon = 0$, this is a representation taken from the complementary series of $SL(2,\mathbb{R})$; when $\varepsilon = 1$, it is a signed version, non unitarizable, of the same. More details can be found in [38, Sect. 2], with the same notation.

Proposition 2.2.12. *Under the map $u \mapsto ((\mathcal{Q}u)_0\,, (\mathcal{Q}u)_1)$, the anaplectic representation transfers to the representation $(\hat{\pi}_{-\frac{1}{2},0}\,, \hat{\pi}_{\frac{1}{2},1})$.*

Proof. Though it is contained in the above given reference, let us at least give a short indication about one of the possible proofs of the proposition. When $g = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$, $\operatorname{Ana}(g)$ is the multiplication by $e^{i\pi c x^2}$, and it is trivial to verify that $(\mathcal{Q}\operatorname{Ana}(g)\,u)_j(\sigma) = (\mathcal{Q}u)_j(\sigma - c)$; when $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a > 0$, so that $(\operatorname{Ana}(g)\,u)(x) = a^{-\frac{1}{2}}\,u(a^{-1}x)$, one verifies just as easily that

$$(\mathcal{Q}\operatorname{Ana}(g)\,u)(\sigma) = \left(a^{\frac{1}{2}}\,(\mathcal{Q}u)_0(a^2\,\sigma)\,, a^{\frac{3}{2}}\,(\mathcal{Q}u)_1(a^2\,\sigma)\right). \tag{2.2.58}$$

The case when $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is of course more complicated, but there are several ways of dealing with it. Considering, say, the first component of the $\mathcal{Q}$-realization, one may prove instead the more general formula

$$\left(\mathcal{Q}\,\mathrm{Ana}\left(\left(\begin{smallmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{smallmatrix}\right)\right)u\right)_0(\sigma) = |\cos t - \sigma\,\sin t|^{-\frac{1}{2}}\,(\mathcal{Q}u)_0\left(\frac{\sigma\,\cos t + \sin t}{-\sigma\,\sin t + \cos t}\right): \tag{2.2.59}$$

the advantage is that it is equivalent to its infinitesimal version, which takes (2.2.29) into account

$$-\frac{1}{2}\left[\left(\mathcal{Q}(Q^2+P^2)\,u\right)_0(\sigma)\right] = \frac{1}{2i\pi}\left[\frac{\sigma}{2}\,(\mathcal{Q}u)(\sigma) + (1+\sigma^2)\,(\mathcal{Q}u)'(\sigma)\right]: \tag{2.2.60}$$

now, the left-hand side of (2.2.60), to wit

$$\int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\left[\frac{ix^2}{2}\,u(x\,e^{-\frac{i\pi}{4}}) + \frac{1}{8\pi^2}\,u''(x\,e^{-\frac{i\pi}{4}})\right]dx\,, \qquad \sigma \text{ large}, \tag{2.2.61}$$

can be written, after an integration by parts, as

$$\frac{1}{2i\pi}\int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\,u(x\,e^{-\frac{i\pi}{4}})\left[-\pi\,(1+\sigma^2)\,x^2 + \frac{\sigma}{2}\right]dx, \tag{2.2.62}$$

which is the right-hand side of the desired formula (2.2.60). □

Remark 2.2.1. As is well known, the representation $\hat{\pi}_{-\frac{1}{2},0}$, taken from the complementary series of $SL(2,\mathbb{R})$, is unitary for the scalar product associated to the norm such that

$$|||w|||^2_{-\frac{1}{2},0} = \int_{-\infty}^{\infty} \bar{w}(\sigma)\,(|D|^{\frac{1}{2}}\,w)(\sigma)\,d\sigma, \tag{2.2.63}$$

where $|D|^{\frac{1}{2}}$ stands for the operator of convolution by the Fourier transform of the function $s \mapsto |s|^{\frac{1}{2}}$. Then, if $u \in \mathfrak{A}_{\text{even}}$, $(u\,|\,u)$, as defined in (2.2.11), coincides with $|||(\mathcal{Q}u)_0|||^2_{-\frac{1}{2},0}$. Something similar holds with the odd part of $\mathfrak{A}$ – but one is then only dealing with a *pseudoscalar* product – trading the integral on the right-hand side of (2.2.63) for the one obtained when replacing $|s|^{\frac{1}{2}}$ by $|s|^{-\frac{1}{2}}\,\mathrm{sign}\,s$.

In anaplectic analysis, however, one cannot do much with the Hilbert completion of the space $\mathfrak{A}_{\text{even}}$: it is, indeed, essential to use only functions on the line extending as entire functions, so as to take advantage of the relation (2.2.2) between the components of the $\mathbb{C}^4$-realization of u.

http://www.springer.com/978-3-540-77910-0

Alternative Pseudodifferential Analysis
With an Application to Modular Forms
Unterberger, A.
2008, IX, 118 p., Softcover
ISBN: 978-3-540-77910-0