Fuzzy Logic Approaches to Consensus Modelling in Group Decision Making

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Summary. The notion of consensus plays a key role in modelling group decisions, and for a long time it was meant as a strict and unanimous agreement, however, since various decision makers have different more or less conflicting opinions the traditional strict meaning of consensus is unrealistic. The human perception of consensus is much "softer", and people are willing to accept that a consensus has been reached when most or the more predominant actors agree on the preferences associated with the most relevant alternatives. The "soft" meaning of consensus, advocated as realistic and humanly consistent, can lead to solve in a more constructive way group decision making situations by using modelling tools based on fuzzy logic.

In this paper we present a review of well known fuzzy logic-based approaches to model flexible consensus reaching dynamics, which constitute a well defined research area in the context of fuzzy GDM. First, the problem of modelling consensus under individual fuzzy preferences is considered, and two different models are synthesized. The first one is static and is based on the algebraic aggregation of the individual preferences aiming to find a consensus defined as the degree to which most of the important individuals agree as to their preferences concerning almost all of the relevant alternatives. The second one is dynamic and it combines a soft measure of collective disagreement with an inertial mechanism of opinion changing aversion. It acts on the network of single preference structures by a combination of a collective process of diffusion and an individual mechanism of inertia. Second, the use of Ordered Weighted Averaging (OWA) Operators to define a linguistic quantifier guided aggregation in the context of GDM is introduced and then generalized to the problem of Multi Expert Multi Criteria Decision Making for which a linguistic approach to define a consensus reaching strategy is presented.

1 Introduction

The construction of models for making decisions when a group of two or more decision makers must aggregate their opinions (individual preferences)

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in order to get a group opinion (collective preference) is a very old problem. The first systematic approaches to the problem were pioneered by Borda [10] and Condorcet [16], who initiated the formal discipline of Social Choice in terms of voting. For an extended review see Nurmi [48].

The subject of Group Decision Making (GDM), traditionally equated with Social Choice, was revived in the twentieth century by Arrow [1, 2], who in his book titled *Social Choice and Individual Values* was concerned with the difficulties of group decisions and the inconsistencies they can generate leading to the well-known Impossibility Theorem.

More specifically, Arrow has proved that in the context of ordinal and symmetric (all decision makers with equal weight) preferences it is not possible to construct a collective preference structure without this being imposed by a single individual, the so-called "Arrow's dictator". In the following 1950s and 1960s many axiomatic variants to Arrow's hypothesis have been proposed, see for instance Fishburn [27, 28] and Kelly [40], but these have not solved the crucial issues and in any case no natural solution to the collective preference aggregation problem has emerged.

The various difficulties highlighted by the strong interest on impossibility theorems have stimulated the development of alternative approaches and, over the last two decades of twentieth century, a number of authors have extended the theory of GDM in various ways to encompass fuzziness in individual and group preferences. Barrett et al. [4, 5] investigated the structure of fuzzy aggregation rules which, for each permissible profile of individual preferences, specify a fuzzy social ordering. Dutta [19], allowing both individual and social preferences to be fuzzy, showed that, under weaker transitivity condition, the fuzzy counterparts of Arrow's condition result in oligarchic and not dictatorial aggregation rules. Montero [47] introduced rationality as a fuzzy property by suggesting a definition of fuzzy opinion different from the classical fuzzy preference relation and showed how to escape from impossibility theorems through the idea of fuzzy rationality. More details and useful references can be found in Nurmi and Kacprzyk [49] and Barrett and Salles [3].

It was in the context of GDM theory that the traditional models of consensus modeling have been addressed, from De Groot [17] classical consensus model to the ones proposed by Chatterjee and Seneta [13], Kelly [41], French [29], Lehrer and Wagner [43], Sen [53] and Loewer [44], mostly in the probabilistic framework.

Almost all of these approaches treat consensus as a strict and unanimous agreement, however, since various decision makers have different more or less conflicting opinions the traditional strict meaning of consensus is unrealistic. The human perception of consensus is much 'softer', and people are willing to accept that a consensus has been reached when most or the more predominant decision makers agree on the preferences associated to the most relevant alternatives.

The problem of consensus reaching modelling in a fuzzy environment was addressed at first in Bezdek et al. (7–9), Ragade [51], Spillman et al. [55], Spillman et al. [56, 57] and then developed in Fedrizzi et al. [24], Kacprzyk and Fedrizzi [34–36], Carlsson et al. [12], Kacprzyk et al. [38], Fedrizzi et al. [21], Kacprzyk et al. [39]. Some authors addressed the problem introducing linguistically-based preference relations, see for instance, among others, Herrera-Viedma et al. [33] and Ben-Arieh and Chen [6].

The 'soft' consensus paradigm developed in Kacprzyk and Fedrizzi [34–36] in the standard framework of numerical fuzzy preferences was extended to a more dynamical context in Fedrizzi et al. [22, 23, 25] and Marques Pereira [46]. The new model combines a soft measure of collective disagreement with an inertial mechanism of opinion changing aversion. It acts on the network of single preference structures by a combination of a collective process of (nonlinear) diffusion and an individual mechanism of (nonlinear) inertia. The overall effect of the dynamics is to outline and enhance the natural segmentation of the decision makers group into homogeneous preference subgroups. Fedrizzi et al. [26], assuming that the decision makers can express their preferences in a more flexible way, i.e. by using triangular fuzzy numbers, generalized the iterative process of opinion transformation towards consensus via the gradient dynamics of a cost function expressed as a linear combination of the disagreement function and the inertial cost function.

In Chen and Hwang [14] the problem of Multi Expert Multi Criteria decision making is addressed. In their approach, the group decision making strategy requires to each expert to express a performance judgment on each alternative with respect to a set of predefined criteria. In this context the definition of a consensus degree and a consensual alternative ranking requires to work on the 'absolute' experts' evaluations and not on preference relations. More specifically, the reduction of the individual judgments into a representative value (the majority opinion) is usually performed through an aggregation process, introducing aggregation operators associated with linguistic quantifiers (such as most). In particular Ordered Weighted Aggregation Operators [58] have been widely applied to address GDM problems [15]. In Bordogna et al. [11] a linguistic model for Multi Expert Multi Criteria decision problem is defined, the aim of which is to compute a consensual judgement and a consensus degree for a fuzzy majority of the experts on each of the considered alternatives.

In this paper we present some fuzzy approaches to model flexible consensus reaching strategies, and which constitute a well defined research area in the context of fuzzy GDM. In Sect. 2 the problem of modelling consensus under fuzzy preferences is considered, and two different strategies are synthesized. In Sect. 3, the use of Ordered Weighted Averaging Operators to define a linguistic quantifier guided aggregation in the context of GDM is introduced. Finally in Sect. 4 the problem of Multi Expert Multi Criteria Decision Making is considered and a linguistic approach to define a consensus reaching strategy in this context is presented.

2 Consensus Modelling in Group Decision Making under Fuzzy Preferences: Static and Dynamical Approaches

2.1 Soft Consensus in a Static Setting

The basic framework within which most of the GDM processes are modelled can be depicted in the following way. There is a set of decision makers or experts who present their opinions concerning a set of alternatives and these alternatives may initially differ to a large extent. If the individuals are rationally committed to consensus, via some exchange of information, bargaining, etc. the individuals' opinions can be modified and the group may get closer to consensus. Here consensus is not meant as a strict and unanimous agreement, but as the degree to which most of the individuals agree as to their preferences concerning almost all of the relevant opinions. This degree of consensus takes on it values in the unit interval, and it's more realistic and human consistent than conventional degrees, mostly developed in the probabilistic framework.

One of the most widely used approaches proposed in the literature is the one described for first in Kacprzyk and Fedrizzi [34] and then developed in Kacprzyk and Fedrizzi [35,36]. The point of departure is a set of n individual fuzzy preference relations defined on a set $A = \{a_1, \ldots, a_m\}$ of alternatives. The fuzzy preference relation of individual i, R_i, is given by its membership function $\mu_i : A \times A \rightarrow [0, 1]$ and can be represented by a matrix $[r^i_{kl}]$, $r^i_{kl} = \mu_i (a_k, a_l)$ which is commonly assumed to be reciprocal, that is $r^i_{kl} + r^i_{lk} = 1$. Clearly, this implies $r^i_{kk} = 0.5$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m$.

Now, a measure of the degree of agreement is introduced, that is derived in three steps. First, for each pair of individuals we derive a degree of agreement as to their preferences between a pair of alternatives, next we pool (aggregate) these degrees to obtain a degree of agreement of each pair of individuals as to their preferences between Q1 (a linguistic quantifier as, e.g., "most", "almost all", "more than 50%", ...) pairs of relevant alternatives, and finally we pool these degrees to obtain a degree of agreement of Q2 (a linguistic quantifier similar to Q1) pairs of individuals as to their preferences between Q1 pairs of relevant alternatives. This is meant to be the degree of agreement sought.

We start with the degree of agreement between individuals i and j as to their preferences between alternatives a_k and a_l,

$$V_{kl}(i,j) = ({r^i}_{kl} - {r^j}_{kl})^2 \in [0,\ 1] \quad \text{where } i,j = 1, \ldots, n \text{ and } k,l = 1, \ldots, m. \quad (1)$$

Relevance of the alternatives is assumed to be a fuzzy set defined on the set of alternatives A, such that $\mu_{\mathrm{A}}(a_k) \in [0, 1]$ is a degree of relevance of alternative a_k: from 0 standing for "definitely irrelevant" to 1 for "definitely relevant", through all intermediate values.

Relevance of a pair of alternatives, $(a_k, a_l) \in A \times A$, may be defined in various ways among which

$$p_{kl} = (\mu_A(a_k) + \mu_A(a_l))/2 \tag{2}$$

is certainly the most straightforward; clearly, $p_{kl} = p_{lk}$ and p_{kk} are irrelevant since they concern the same alternative, for all and $k, l = 1, \ldots, m$.

The degree of agreement between individuals i and j as to their preferences between all the relevant pairs of alternatives is

$$V_{\mathrm{P}}(i,j) = \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} p_{kl}\ V_{kl}\ (i,j) \Big/ \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} p_{kl} \tag{3}$$

The degree of agreement between individuals i and j as to their preferences between Q1 relevant pairs of alternatives is then

$$V_{Q1}(i,j) = \mathrm{Q1}\ (V_P(i,j)) \tag{4}$$

In turn, the degree of agreement of all the pairs of individuals as to their preferences between Q1 relevant pairs of alternatives is

$$V_{\mathrm{Q1}} = (2/n(n-1)) \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} V_{\mathrm{Q1}}\ (i,j), \tag{5}$$

and, finally, the degree of agreement of Q2 pairs of individuals as to their preferences between Q1 relevant pairs of alternatives, called the degree of Q1/Q2-consensus, is

$$V_{\mathrm{Q1,Q2}} = \mathrm{Q2}(V_{\mathrm{Q1}}) \tag{6}$$

As far as the quantifiers Q1 and Q2 are concerned, they are of the general form $\mathrm{Q}: [0,\ 1] \to [0,\ 1]$ with, for instance,

$$\begin{aligned} &\mathrm{Q}(x) = 0 && \text{for } x \in [0,\ \mathrm{c}], \\ &\mathrm{Q}(x) = (1/(\mathrm{d}-\mathrm{c}))x - (\mathrm{c}/(\mathrm{d}-\mathrm{c})) && \text{for } x \in (\mathrm{c},\ \mathrm{d}),\ \text{with } 0 \le \mathrm{c} < \mathrm{d} \le 1 \\ &\mathrm{Q}(x) = 1 && \text{for } x \in [\mathrm{d},\ 1], \end{aligned} \tag{7}$$

such that $x' \le x'' \Rightarrow s(x') \le s(x'')$ for all $x',\ x'' \in [0,\ 1]$, and $\mathrm{Q}\,(0) = 0$, $\mathrm{Q}\ (1) = 1$. For details on linguistic quantifiers see Zadeh [61].

In Fedrizzi et al. [24] this model was implemented in an interactive user-friendly microcomputer-based decision support system where the consensus reaching process is supervised by a moderator. The moderator, in a multistage session, tries to make the individuals change their preferences by, e.g. rational argument, bargaining, additional knowledge, etc. to eventually get closer to consensus.

A similar approach to consensus modelling was developed by Herrera-Viedma et al. [32], where the novelty basically consists in introducing a degree of consensus between the individuals which depends on two consensus parameters, a consensus measure and a proximity measure. The consensus measure evaluates the agreement of all the experts, while the proximity measure evaluates the agreement between the experts' individual opinions and the group opinion. Individual opinions are represented using fuzzy preference relations derived from preference structures defined on a finite set of alternatives $\mathrm{X} = \{\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}\}$.

Two preference structures are considered:

1. Evaluations $\lambda_\mathrm{i}{}^\mathrm{k}$ associated with each alternative x_i, indicating the performance of that alternative according to a point of view of the selected expert
2. Multiplicative preference relation $\mathrm{A}^\mathrm{k} = (\boldsymbol{a}_{ij}{}^k)$ (Saaty's).

Individual fuzzy preference relations are derived from each preference structure introducing transformation functions satisfying some consistency properties.

Then, a consensus support system that emulates the moderator's behavior is introduced. The system has a feedback mechanism, based on the proximity measure, to generate recommendations in the group discussion process directed to change the individual opinions (preferences), in order to obtain a higher degree of consensus.

2.2 Soft Consensus in a Dynamical Setting

The soft consensus approach developed by Kacprzyk and Fedrizzi [34] was extended to a dynamical context in Fedrizzi et al. [22, 23, 25] and Marques Pereira [46] combining a measure of collective disagreement with an inertial mechanism of opinion changing aversion. The new model acts on the network of individual preference relations by a combination of a collective process of diffusion and an individual mechanism of inertia. The overall effect of the dynamics is to outline and enhance the natural segmentation of the group of decision makers into homogeneous preference subgroups, according to Bayesian priors from which the model derives. The modelling framework here adopted is essentially that one introduced in the previous section. The only difference consists in simplifying the shape of the quantifiers by eliminating the quantifier Q2 (i.e. by choosing it as the identity function) and by choosing the quantifier Q1 = Q as follows,

$$\mathrm{Q}(x) = (f(x) - f(0))/(f(1) - f(0)) \tag{8}$$

where in our soft consensus model the scaling function $f : [0,\ 1] \to \Re$ is defined as,

$$f(x) = -\frac{1}{\beta} \ln \left(1 + e^{-\beta(x-\alpha)}\right) \tag{9}$$

and $\alpha \in (0,\ 1)$ is a threshold parameter and $\beta \in (0,\ \infty)$ is a free parameter which controls the polarization of the sigmoid function f',

$$f'(x) = 1 \Big/ \left(1 + e^{\beta(x-\alpha)}\right) = \sigma(x) \tag{10}$$

For large values of the parameter β the sigmoid function f' is close to a step function with respect to the threshold value α : $f'(0) \approx 1$, $f'(1) \approx 0$, and $f'(\alpha) = 0.5$. Otherwise, the function f' is smooth and monotonically decreasing with respect to its argument.

Moreover, for sake of simplicity, let us assume that the alternatives available are only two, that is $m = 2$, which means that each (reciprocal) individual preference relation R_i, has only one degree of freedom, denoted by $\mathrm{x}_i = r^i_{12}$. Accordingly, the relevance P is trivial. In such case, we have $V(i, j) = (x_i - x_j)^2 \in [0,\ 1]$ and thus $V_{\mathrm{Q}}\ (i, j) = \mathrm{Q}\ (V(i,\ j)) = \mathrm{Q}\ ((x_i - x_j)^2) \in [0,\ 1]$.

The starting assumption is that each decision maker $i = 1, \ldots, n$ is represented by a pair of connected nodes, a primary node (dynamic) and a secondary node (static). The n primary nodes form a fully connected sub network and each of them encodes the individual opinion of a single decision maker. The n secondary nodes, on the other hand, encode the individual opinions originally declared by the decision makers, denoted $\mathbf{s}_i = [s_i \in [0,\ 1]]$, and each of them is connected only with the associated primary node.

The iterative process of opinion transformation corresponds to the gradient dynamics of a cost function W, depending on both the present and the original network configurations. The value of W combines a measure V of the overall disagreement in the present network configuration and a measure U of the overall change from the original network configuration.

The various interactions involving node i are mediated by interaction coefficients whose role is to quantify the strength of the interaction. The diffusive interaction between primary nodes i and j is mediated by the interaction coefficient $v_{ij} \in (0, 1)$, whereas the inertial interaction between primary node i and the associated secondary node is mediated by the interaction coefficient $u_{ij} \in (0, 1)$. It turns out that the values of these interaction coefficients are given by the derivative f' of the scaling function.

The diffusive component of the network dynamics results from the consensual interaction between each node x_i and the remaining $n - 1$ nodes $x_{j \neq i}$ in the network. The aggregated effect of these $n - 1$ interactions can be represented as a single consensual interaction between node x_i and a virtual node $\bar{x}_i$ containing a particular weighted average of the remaining opinion values.

The interaction coefficient $v_i \in (0, 1)$ of this aggregated consensual interaction controls the extent to which decision maker i is influenced by the remaining experts in the group. In our soft consensus model the value v_i, as well as the weighting coefficients $v_{ij} \in (0,\ 1)$ in the definition of $\bar{x}_i$ as given below, depend non-linearly on the standard Euclidean distance between the opinions x_i and x_j,

$$v_{ij} = f'((x_i - x_j)^2) \tag{11}$$

$$v_i = \sum_{j \neq i} v_{ij}/(n - 1) \tag{12}$$

and the average preference $\bar{x}_i$ is given by

$$\bar{x}_i = \frac{\sum_{j \neq i} v_{ij} x_j}{\sum_{j \neq i} v_{ij}} \tag{13}$$

In the context of these definitions, $f'(x) \in (0,\ 1)$ is the decreasing sigmoid function introduced in the previous section. This sigmoid function plays a crucial role in the network dynamics and is obtained as the derivative of the scaling function $f(x) \in \Re$ which, in turn, enters the construction of the soft consensus cost function from which the network dynamics derives.

The interaction coefficient $u_i \in (0,\ 1)$ of this inertial interaction controls the extent to which the decision maker i resists to opinion changes due to the collective consensual trend. In analogy with the diffusion coefficients, the value u_i in our soft consensus model depends non-linearly on the standard Euclidean distance between the opinions x_i and s_i,

$$u_i = f'((x_i - s_i)^2) \tag{14}$$

where $f'(x)$ is the sigmoid function mentioned earlier.

The individual disagreement cost $V(i)$ is given by

$$V(i) = \sum_{j \neq i} V(i,j)/(n-1) \tag{15}$$

where $V(i,j) = f((x_i - x_j)^2)$ and the individual opinion changing cost $U(i)$ is

$$U(i) = f((x_i - s_i)^2) \tag{16}$$

Summing over the various decision makers we obtain the collective disagreement cost V and inertial cost U,

$$V = \frac{1}{4} \sum_i V(i) \text{ and } U = \frac{1}{2} \sum_i U(i) \tag{17}$$

with conventional multiplicative factors of 1/4 and 1/2.

The full cost function W is then $W = (1-\lambda)V + \lambda U$ with $0 \leq \lambda \leq 1$.

The consensual network dynamics, which can be regarded as an unsupervised learning algorithm, acts on the individual opinion variables x_i through the iterative process

$$x_i \rightarrow x'_i = x_i - \varepsilon \frac{\partial W}{\partial x_i} \tag{18}$$

The effect of the two dynamical components V and U can be analysed separately and it can be proved (see for details Fedrizzi et al. [25]) that

$$x'_i = (1 - \varepsilon(v_i + u_i))x_i + \varepsilon v_i \bar{x}_i + \varepsilon u_i s_i \tag{19}$$

Accordingly, the decision maker i is in dynamical equilibrium, in the sense that $x'_i = x_i$, if the following stability equation holds,

$$x_i = (v_i \bar{x}_i + u_i s_i)/(v_i + u_i) \tag{20}$$

that is, if the present opinion value x_i coincides with an appropriate weighted average of the original opinion s_i and the average opinion value $\bar{x}_i$.

In Fedrizzi et al. [26] the dynamical model was extended assuming that the preferences of the decision makers are expressed by means of fuzzy numbers, in particular by means of triangular fuzzy numbers. Then, in order to measure the differences between the preferences of the decision makers, a distance belonging to a family of distances proposed by Grzegorzewski [31] was introduced.

3 Aggregation Guided by Linguistic Quantifiers expressed as Ordered Weighted Averaging Operators

With the approach to GDM introduced in Sect. 2 each expert compares the alternatives and makes relative judgements of preference among couples of them, thus defining a preference relation. In this section a different approach often used in the literature is introduced, where the experts express an absolute judgement on each alternative to evaluate it. By this approach a numeric or linguistic value (from a set of admissible values) is selected by the expert to indicate the performance of the alternative with respect to her/his opinion. It is not infrequent that the experts are asked to express for each alternative an evaluation with respect to each of a set of predefined criteria. In this case we are in the framework of Multi Expert Multi Criteria decision making [14].

In this absolute evaluation framework, the ultimate aim of the group decision process is to determine for each alternative a consensual judgement (consensual opinion) which synthesizes the experts individual opinions. The consensual judgement is representative of a collective evaluation and is usually computed by means of an aggregation of the individual experts' opinions. Usually also a consensus degree is computed for each alternative, with the consequent problem of comparing the decision makers' opinions to verify the consensus among them. In the case of unanimous consensus, the evaluation process ends with the selection of the best alternative(s).

As outlined in the previous section, in real situations humans rarely come to an unanimous agreement: what is often needed is an overall opinion which synthesizes the opinions of the *majority* of the decision makers. The reduction of the individual values into a representative value (the majority opinion) is usually performed through an aggregation process. As outlined in Sect. 2, within fuzzy set theory the concept of majority can be expressed by a linguistic quantifier (such as *most*), which is formally defined as a fuzzy subset of a numeric domain; the semantics of such a fuzzy subset is described by a membership function which describes the compatibility of a given absolute or percentage quantity to the concept expressed by the linguistic quantifier. By this interpretation a linguistic quantifier is seen as a fuzzy concept referred to the quantity of elements of a considered reference set. In the fuzzy approaches synthetically described in Sect. 2 a full consensus is not necessarily the result of unanimous agreement, but it can be obtained even in case of agreement among a fuzzy majority of the decision makers (Fedrizzi et al. [24], Fedrizzi et al. [21], Kacprzyk and Fedrizzi [35]).

In the fuzzy approaches to group decision making the concept of majority is usually modeled by means of linguistic quantifiers such as *at least* 80% and *most* (monotonic non decreasing linguistic quantifiers). When linguistic quantifiers are used to indicate a fusion strategy to guide the process of aggregating the members' opinions, the formal mathematical definition of the resulting aggregation operator encodes the semantics of the linguistic quantifier. The notion of quantifier guided aggregation has been formally defined by means of Ordered Weighted Averaging operators [58, 59], and by means of the concept of fuzzy integrals (Grabish [30]). An example of linguistic expression which employs a quantifier to guide an aggregation is the following: *Q experts are satisfied by solution a*, where Q denotes a linguistic quantifier, for example *most*, which expresses a majority. To evaluate the satisfaction of this proposition the experts' opinions have to be aggregated using the formal aggregation operator which captures the semantics of the concept expressed by the quantifier Q.

In the applications related to group decision making the use of OWA operators has been extensively experienced [15]. We shortly introduce the formal definition of OWA operators in both cases in which numeric and linguistic values have to be aggregated. For a more extensive definition see Yager [59]. An OWA operator on unit interval is a mapping OWA: $[0,1]^n \rightarrow [0,1]$ with an associated weighting vector

$\mathrm{W} = [\mathrm{w_1}, \mathrm{w_2}, \ldots, \mathrm{w_n}]$ such that $\mathrm{w_i} \in [0,1]$, and $\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{w_i} = 1$, and for any arguments $a_1, a_2, \ldots, a_n \in [0,1]$:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{b_i}\,\mathrm{w_i} \tag{21}$$

with $\mathrm{b_i}$ being the i^{th} largest element of the a_j [58]. A number of approaches have been suggested for determining the weights used in the OWA operator. Here we present the one that allows to obtain the weights from a functional form of the linguistic quantifier (i.e. from the definition of the linguistic quantifier as a fuzzy subset). Let Q: $[0,1] \rightarrow [0,1]$ be a function such that $\mathrm{Q(0) = 0,\ Q(1) = 1}$ and $\mathrm{Q(x) \geq Q(y)}$ for $\mathrm{x > y}$ corresponding to a fuzzy set representation of a proportional monotone quantifier. For a given value $\mathrm{x} \in [0,1]$, the Q(x) is the degree to which the quantity (relative quantity) x satisfies the fuzzy concept being represented by the quantifier. Based on function Q, the OWA vector is determined from Q by defining the weights in the following way:

$$w_i = \mathrm{Q}(i/n) - \mathrm{Q}((i-1)/n). \tag{22}$$

In this case w_i represents the increase of satisfaction in getting i with respect to $i-1$ criteria satisfied. Let us consider now a set of linguistic labels S uniformly distributed on a scale so that an ordering is defined ($s_a, s_b \in S,\ a < b \Leftrightarrow s_a < s_b$) and $s_0,\ s_{max}$ are the lower and the upper elements respectively with $max = |S| - 1$, where |S| denotes the cardinality of S. An Ordered Weighted Average operator OWA on the ordinal scale S is a mapping: $\mathrm{OWA}_Q : S^M \to S$ with a weighting vector: $\mathrm{W} = [\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_\mathrm{M}]$ in which $w_i \in S,\ w_i \geq w_j$, for $i > j$ and $MAX_i(w_i) = s_{max}$ then:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_M) = \mathrm{Max}(w_i \wedge b_i) \tag{23}$$

where b_i is the i-th highest label a_k in S among the $a_1, a_2, \ldots, a_M$. An ordinal OWA is determined by a relative monotone increasing quantifier Q, by setting $\forall \mathrm{i} \in \{1, M\}$:

$$w_i = Q\left(\frac{i}{M}\right) \in S \tag{24}$$

The value $Q(\frac{i}{M})$ indicates the degree of satisfaction of getting i of the M criteria fulfilled.

In Sect. 4 we synthetically present a linguistic approach to Multi Expert Multi Criteria Decision making entirely based on the use of a majority guided aggregation formalized by OWA operators.

Recently, in Pasi and Yager [50] it was outlined that when aggregating a collection of values by means of an OWA associated with a linguistic quantifier (constructed by applying either formula (22) or formula (24)), the resulting aggregated value may not be representative of the majority of values. To overcome this problem in Pasi and Yager [50] two new and distinct strategies have been proposed aimed at constructing OWA operators which allow to obtain an aggregated value that better reflects a "true" concept of majority. As previously outlined, the weights of the weighting vector of an OWA operator are interpreted as the increase in satisfaction in having i + 1 criteria "fully" satisfied with respect to having "fully" satisfied i criteria. If for example we consider the linguistic quantifier *at least* 80% and we apply the procedure reported in formula (22) to obtain the weights of the OWA weighting vector, the semantics of the obtained aggregated value is like a degree of satisfaction (truth) of the proposition "Q of the values are fully satisfied". This kind of semantics does not naturally model the meaning of the concept of majority as typically used in group decision making applications. In fact an operator aimed at calculating a majority opinion should produce a value which is representative of the 80% of the *most similar values*. In other words what we want to obtain is an aggregation of the most similar opinions held by a quantity of decision makers specified by the linguistic quantifier Q. This situation appears to bring us closer in spirit to interpretation of the OWA operator as averaging operator rather then as a generalized quantifier. In fact what we want is an average of "most of the similar values". This means that we need an aggregation operator that takes an average like aggregation of a majority of values that are similar. The first approach proposed in Pasi and Yager [50] to define such an aggregation operator makes use of Induced Ordered Weighted

Averaging (IOWA) operators (Yager and Filev [60]) to obtain a scalar value for a majority opinion. By IOWA operators the ordering of the elements to be aggregated is determined by an inducing ordering variable. In Pasi and Yager [50] the inducing ordering variable is based on a proximity metric over the elements to be aggregated. The basic idea is that the most similar values must have close positions in the induced ordering in order to appropriately be aggregated. A new strategy for constructing the weighting vector has also been suggested so as to better model the new “majority-based” semantics of the aggregation. This strategy has the aim of emphasizing in the aggregation the most supported values; in other words the values which appear on the right hand side of the vector of values to be aggregated have more influence in the aggregation.

The second approach proposed in Pasi and Yager [50] is based upon the calculation of the concept of the majority opinion as an imprecise value. Under this interpretation a formalization has been proposed of the idea of a fuzzy majority as a fuzzy subset. This approach provides in addition to a value for a majority opinion an indication of the strength of that value as the majority opinion. The goal here is to obtain a value which can be considered as the opinion of a majority, that is, some value that is similar for any large group of people. Both methods require to have both information about the similarity between the experts opinions, and some information about what quantity constitutes the idea of a majority.

4 Consensus Modelling in Linguistic Approaches to Multi Expert Multi Criteria Decision Making

In addressing a decision making problem it is not infrequent that the ratings or performance scores cannot be assessed precisely but in a linguistic form (Herrera-Viedma et al. [33]). The imprecision may come from different sources as pointed out in Chen and Hwang [14]: information may be unquantifiable when the evaluation of a criterion, due to its nature, can be stated only in linguistic terms such as in the case of the evaluation of the *comfort* or *design* of a *car*, terms like *good, fair, poor* can be used. Sometimes precise testimonies cannot be stated because they are unavailable or the cost for their computation is too high and an “approximated estimates” can be tolerated; for example, for the evaluation of a car’s speed linguistic terms like *fast, very fast, slow* can be used instead of numeric values.

As an approach representative of consensus modelling in the context of Multi Expert Multi Criteria Decision Making in a linguistic setting, in this section we give a synthesis of the linguistic model proposed by Bordogna et al. [11], as it offers a general approach to group decision making, which is entirely based on the management of information expressed linguistically. A Multi Expert Multi Criteria decision problem is considered, the aim of which is to compute a consensual judgment and a consensus degree for a fuzzy majority of the experts on each of the considered alternatives. The group of experts judges each alternative according to the evaluation of a finite set of predefined criteria. Each expert is asked to linguistically evaluate each alternative in terms of its performance with respect to each criterion. The experts are also allowed to associate a distinct importance to the criteria in a linguistic form as well. This procedure adopts an absolute evaluation of each alternative and is based on the assumption of alternatives' independency. To enable the experts to formulate their judgments in a natural way, a limited set S of linguistic labels is supplied. For example, S can be defined so as its elements are uniformly distributed on a scale on which a total order is defined: $S = [\mathrm{s}_0 = \text{none},\ \mathrm{s}_1 = \text{very low},\ \mathrm{s}_2 = \text{low},\ \mathrm{s}_3 = \text{medium},\ \mathrm{s}_4 = \text{high},\ \mathrm{s}_5 = \text{very high},\ \mathrm{s}_6 = \text{perfect}]$ in which $\mathrm{s_a} < \mathrm{s_b}$ iff $\mathrm{a} < \mathrm{b}$. The cardinality of S must be small enough so as not to impose useless precision to the experts and it must be rich enough in order to allow a discrimination of the performances of each criterion in a limited number of grades.

By allowing the experts to express in a linguistic form the evaluations of both performance and importance of criteria, the burden of quantifying a qualitative concept is eliminated and thus the system-expert interaction is simplified. In Sect. 4.1 the process of reducing, for each expert, the M evaluations expressed for a given alternative (where M is the number of considered criteria) to an overall judgment for the alternative is presented. In Sect. 4.2 the process aimed at computing a consensus degree and a consensual choice among the experts is presented.

4.1 An Overall Performance Judgment for each Alternative and each Expert

Once each expert has expressed a linguistic judgment for each criterion with respect to each alternative, the first phase towards the evaluation of a degree of consensus is aimed at synthesizing an overall performance judgment for each alternative and for each expert. This is done through an aggregation of the linguistic judgments for each alternative with respect to each criterion. To this aim in Bordogna et al. [11] an aggregation function has been proposed which works directly on linguistic labels on an ordinal scale and produces a global linguistic performance label by applying OWA operators.

Formally, the set of alternatives is denoted by: $A = \{A_1, A_2, \ldots, A_N\}$, the set of experts is denoted by: $E = \{E_1, E_2, \ldots, E_K\}$, and the set of criteria is denoted by: $C = \{C_1, C_2, \ldots, C_M\}$. The input to the aggregation phase is represented by a set of K matrixes of dimension $N \times M$, in which K, N and M are the numbers of experts, alternatives and criteria respectively; there is one matrix for each expert, in which each element is a linguistic label $P_{ij} \in \mathrm{S}$ drawn from an ordinal scale and expressing the performance judgment on criterion C_i with respect to the alternative A_j. For each matrix a vector of dimension M is defined in which an element $I_i \in \mathrm{S}$ drawn from the same scale is the importance value associated with criterion C_i by the expert. We omit the index j as the aggregation is performed for a given expert and a given alternative: $P_{ij} \equiv P_i$. The functions aggregating the linguistic performance labels of each alternative have been defined by the relative monotone increasing quantifiers *averagely all, most of* and *more than k%*. For defining these aggregation functions two different formalizations of OWA operators have been proposed. The first approach is more straightforward, as it consists in applying the OWA operators defined on an ordinal scale [59]; in this case, for each quantifier Q the correspondent OWA is defined by specifying the linguistic values of the weighting vector W. These linguistic weights can be defined either by specifying their values directly or by applying formula (24).

When the criteria $C_1, \ldots, C_M$ have the same importance, the OWA_Q operator associated with the quantifier Q is directly applied to the satisfaction values $P_1, \ldots, P_M$ of the criteria. For the l-th expert and the i-th alternative $\mathrm{OWA}_Q(P_{i1}, \ldots, P_{iM})$ will be then evaluated as defined in formula (23).

When differing importance values $I_1, \ldots, I_M \in \mathrm{S}$ are associated with the criteria, one of two possible procedures can be applied to compute the overall performance of an alternative. The procedures are extensively explained in Bordogna et al. [11].

The output of this phase can be summarized in a matrix of dimension $N \times K$, in which an element P_{ij} is the overall performance label of expert E_i with respect to alternative A_j.

4.2 How to Determine a Consensual Opinion

The second phase of a group decision activity is aimed at evaluating the degree of consensus among the experts' overall performance judgments on the alternatives. It is worthwhile to point out that the phase of computation of the consensus degree should be followed by the evaluation of a consensual ranking of alternatives for the specified consensual majority. In other words, the consensual degree refers to a ranking of alternatives which in some way synthesizes the ranking of the considered experts' majority.

In the approach adopted in Bordogna et al. [11] a consensus degree is computed for each alternative, under the assumption of alternative independency on each expert. The novelty of the proposed procedure consists in the direct computation of "soft" linguistic degrees of consensus based on a topological approach [12]; this procedure supported a new definition of consensus referred to a fuzzy majority: the statement "*most of the experts agree of alternative* $\mathrm{A_i}$" is interpreted as "*most of the experts agree with most of the other experts on alternative* $\mathrm{A_i}$".

The starting point is constituted by the matrix of N rows (one for each alternative) and K columns (one for each expert) produced by the phase described in Sect. 4.1. An element on row i and column j of this matrix is a linguistic value in S, which expresses the overall performance judgment of expert j with respect to alternative i. A linguistic degree of consensus among the experts' overall performances is then computed for each alternative. The procedure proposed in Bordogna et al. [11] is aimed at evaluating the consensus degree among Q experts for each alternative, in which Q is a quantifier identifying a fuzzy majority. This procedure is structured in the following phases:

- For each alternative, pair wise comparisons of experts' overall performance labels are made to establish the degree of agreement between all pairs of experts (full agreement = *perfect*, null agreement = *none*). A matrix $K \times K$ is then constructed for each alternative. An element $Ag(E_i, E_j)$ is the linguistic label, which expresses the similarity between the overall performance labels of experts E_i and E_j;
- For each expert E_i (a row of the matrix $K \times K$) the $K - 1$ degrees $Ag(E_i, E_j),\ i \neq j$ are pooled to obtain an indication of the agreement $Ag(E_i)$ of expert E_i with respect to Q of the other experts.
- The values $Ag(E_i)$ are finally aggregated to compute the truth of the sentence "Q E_i agree on alternative A_x".

For each alternative A_x, the degrees of agreement between all pairs of experts are first computed, as the complement of a distance between the overall performance labels:

$$Ag(E_i, E_j) = \neg(d(P_{ix}, P_{jx})) \tag{25}$$

in which P_{ix} denotes the linguistic overall performance label of expert E_i on alternative A_x. Provided that the elements in S are uniformly distributed, function d is defined on $S \times S$ and takes values in S; it is applied to the overall performance labels of experts E_i and E_j, and produces a linguistic label indicating the distance between the two arguments. The d function is defined as a difference operator of linguistic labels in the same scale [18]; the labels belong to the totally ordered term set $\mathrm{S} = \{s_i \big| \mathrm{i} \in \{0 \cdots max\}\} : d(s_i, s_j) = \mathrm{s_r}$ with $r = |i - j|$.

The complement operation $\neg$ is defined as: $\neg(s_i) = s_{max-i}$. The evaluation of the complement of the distance between two linguistic labels is a measure of the degree of agreement between the opinions of two experts. The results produced by this phase can be synthesized in N matrixes $K \times K$, one for each alternative.

Once the degrees of agreement between pairs of experts have been computed, they must be pooled to obtain *the degree of consensus among Q experts*, in which Q is a quantifier such as *most of, all, more than k%*. This is done in two subsequent steps. First of all for each expert an overall indication of his/her agreement with respect to Q of the others experts is computed; we indicate this overall degree as $Ag_Q(E_i)$.

Formally, this aggregation is performed by applying to the $Ag(E_i, E_j)$ (with $i \neq j,\ i, j = 1 \cdots K$) on the i-th row of the matrix, the ordinal OWA_Q operator associated with the linguistic quantifier Q, as defined in Sect. 3. At this point it is possible to identify the expert with the highest disagreements versus most of the other experts: this information can be useful in consensus reaching to address the experts who should revise their opinions in order to increase the degree of consensus.

The last step is the determination of consensus among Q experts; to obtain this final consensus degree also the K values $Ag_Q(E_i)$, one for each expert, are aggregated with the OWA_Q operator. The consensual performance judgment of each alternative, defined as the label on which the identified majority of the experts agree, is finally computed. It is obtained by applying the same OWA_Q operator to the overall performance judgments of all the experts on the given alternative.

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