2

Markov Chains

Summary. In Sect. 2.1 we show there is a bijection between probability measures τ on the boundary space ∂X of a tree X, and Markov chain on X. For each point x on the tree, we consider the set of all the paths going through x and call it the interval $I(x)$. The interval splits into intervals $I(x')$ corresponding to each arrow $x \mapsto x'$, and we give this arrow the probability $\tau(I(x'))/\tau(I(x))$. The sum of the probability is equal to 1. This is a Markov chain. We then give a brief description in Sect. 2.2 of the boundary theory of general transient Markov chains. Let $X = \bigsqcup_n X_n$, $X_0 = \{x_0\}$ be the state space, $P : \bigsqcup_n X_n \times X_{n+1} \to [0, 1]$ the transition probability. Then we have

$$\begin{array}{lll} \text{Probability measure} & \tau_n(x) = (P^*)^n \delta_{x_0}(x) & (x \in X_n), \\ \text{Green kernel} & G(x, y) = P^{m-n}(x, y) & (x \in X_n, y \in X_m), \\ \text{Martin kernel} & K(x, y) = \dfrac{G(x, y)}{G(x_0, y)}. & \end{array}$$

The Martin kernel gives a metric. The sequence $\{y_n\}$ is a Cauchy sequence if $\{K(x, y_n)\}$ is a Cauchy sequence of $\mathbb{R}$ for all x and $\{y_n\} \sim \{y'_n\}$ if $\{K(x, y_n)\} \sim \{K(x, y'_n)\}$. Then we obtain the compactification

$$\overline{X} = \{\text{Cauchy sequence of } X\}/\!\sim\ = X \sqcup \partial X.$$

Recall the theorem that every super-harmonic function f is equal to K_μ for some μ which is a probability measure on $X \sqcup \partial X$. Here a function f is called super-harmonic if $Pf \geq f$. If $Pf = f$, we call f a harmonic function and μ is a measure supported only on the boundary ∂X. The set $\mathrm{Harm}(X)$ of all harmonic functions on X is divided as

$$\mathrm{Harm}(X) = \mathrm{Harm}(X)_{\text{ext}} \sqcup \mathrm{Harm}(X)_{\text{non-ext}}$$

and the boundary ∂X also decomposes as

$$\partial X = \partial X_{\text{ext}} \sqcup \partial X_{\text{non-ext}}.$$

Here a point $y \in \partial X$ is called extream if $K_{\delta_y} = K(x,y)$ is extream harmonic function. Then there is one-to-one correspondence between the probability measures on $\partial X_{\mathrm{ext}}$ and the harmonic functions on X.

2.1 Markov Chain on Trees

2.1.1 Probability Measures on ∂X

Let X be a tree and $x_0 \in X$ the root. For $n \geq 0$, we denote by X_n the set

$$X_n := \{x \in X \mid d(x_0, x) = n\}.$$

Then X decomposes as the disjoint union of X_n; $X = \bigsqcup_{n \geq 0} X_n$. Note that $X_0 = \{x_0\}$ and X_n is a finite set. The boundary ∂X of X is defined by the inverse limit of sets X_n or as the collection of all paths starting from the root x_0,

$$\partial X := \varprojlim X_n = \bigl\{\tilde{x} = \{x_n\} \bigm| x_n \in X_n,\ d(x_n, x_{n+1}) = 1\bigr\}.$$

For $x \in X_n$, we denote $I(x) \subset \partial X$, which is called the "interval" of x, by

$$I(x) := \bigl\{\tilde{x} = \{x_n\} \in \partial X \mid x_n = x\bigr\}$$

and give a topology in ∂X by regarding the family $\{I(x) \mid x \in X\}$ as open base of ∂X.

Let τ be a probability measure on the boundary ∂X. Then we obtain a function $\tau : X \to [0,1]$ defined by $\tau(x) := \tau(I(x))$ and it satisfies

$$\tau(x_0) = 1, \quad \tau(x) = \sum_{\substack{x' \in X_{n+1} \\ x \mapsto x'}} \tau(x') \quad (x \in X_n) \tag{2.1}$$

since $I(x_0) = \partial X$ and $I(x) = \bigsqcup_{x \mapsto x'} I(x')$. Here we write $x \mapsto x'$ instead of $d(x,x') = 1$. Conversely, let τ be a function on the tree X satisfying the condition (2.1). Let $\tau(I(x)) := \tau(x)$. Then τ gives a probability measure on ∂X since each open set of ∂X is expressed as the disjoint union of some intervals $I(x)$. Therefore we have the following one-to-one correspondence;

$$\mathfrak{M}_1(\partial X) \xleftrightarrow{1:1} \left\{\begin{matrix}\text{the function on } X \text{ satisfying} \\ \text{the condition (2.1)}\end{matrix}\right\}.$$

Here $\mathfrak{M}_1(Y)$ denotes the set of all probability measure on Y.

Now given such a τ, we define the probability of going from x to x' by $P(x \mapsto x') := \tau(x')/\tau(x)$. It is clear from (2.1) that

$$\sum_{\substack{x' \in X \\ x \mapsto x'}} P(x \mapsto x') = 1 \qquad (x \in X). \tag{2.2}$$

Hence we have a Markov chain (the condition (2.2) is called the Markov condition). Namely, we have a tree X, which is called the "state space", and the function

$$P : \bigsqcup_{n \geq 0} X_n \times X_{n+1} \longrightarrow [0,1]$$

satisfying the condition (2.2). We call such a function P the "transition probability". Conversely, if we are given a tree X and a function P satisfying the Markov condition, we can get a probability measure on ∂X as follows; For any $x \in X$, we have the unique path $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x$ from x_0 to x. Define the function $\tau : X \to [0,1]$ by

$$\tau(x) := P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x_n = x).$$

Then we have from (2.2) that

$$\begin{aligned}
\sum_{x \mapsto x'} \tau(x') &= \sum_{x \mapsto x'} P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) P(x \mapsto x') \\
&= P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) \sum_{x \mapsto x'} P(x \mapsto x') \\
&= \tau(x).
\end{aligned}$$

Hence the function $\tau(x)$ satisfies the condition (2.1) and $\tau(I(x)) := \tau(x)$ gives a probability measure on ∂X. We call τ the harmonic measure of P. Hence we obtain the following one-to-one correspondence;

$$\mathfrak{M}_1(\partial X) \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Markov chain on } X; \\ \text{transition probability } P \end{array} \right\}.$$

2.1.2 Hilbert Spaces

Let P be a transition probability and τ its harmonic measure on ∂X. Then we can obtain the probability measure τ_n on X_n by

$$\tau_n(x) = \tau(I(x)) := P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) \qquad (x \in X_n),$$

where $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x$ is the unique path from x_0 to x. This can be also written as $\tau_n(x) = (P^*)^n \delta_{x_0}(x)$ where P^* is the adjoint of P and δ_{x_0} is the delta function at x_0 (see the next section). Hence, for all $n \geq 0$, we obtain the Hilbert space

$$H_n := \ell^2(X_n, \tau_n) = \left\{ f : X_n \to \mathbb{C} \,\middle|\, ||f||_{H_n} < \infty \right\},$$

where $||f||_{H_n} := (f,f)_{H_n}^{1/2}$ and $(\cdot,\cdot)_{H_n}$ is the inner product of H_n defined by

$$(f,g)_{H_n} := \sum_{x \in X_n} f(x) \overline{g(x)} \tau_n(x).$$

For each $n \geq 0$, we have an embedding $H_n \hookrightarrow H_{n+1}$ defined by

$$H_n \ni \varphi \longmapsto \varphi' \in H_{n+1}; \qquad \varphi'(x') := \varphi(x),$$

where $x \in X_n$ is the unique element such that $x \mapsto x'$. This is an unitary embedding, that is, it preserves the inner product, and we hence identify H_n with a subspace of H_{n+1}. On the other hand we have the orthogonal projection from H_{n+1} onto the subspace H_n

$$H_{n+1} \ni \varphi' \longmapsto \varphi = P\varphi' \in H_n; \qquad P\varphi'(x) := \sum_{\substack{x' \in X_{n+1} \\ x \mapsto x'}} P(x \mapsto x')\varphi'(x').$$

In fact, we can easily show that $\varphi' - P\varphi' \in {H_n}^{\perp} := \{f \in H_{n+1} \,|\, (f, g)_{H_{n+1}} = 0$ for all $g \in H_n\}$.

Since we have a probability measure τ on ∂X, we have another Hilbert space

$$H := \ell^2(\partial X, \tau) = \big\{f : \partial X \to \mathbb{C} \,\big|\, ||f||_H < \infty\big\},$$

where $||f||_H := (f, f)_H^{1/2}$ and $(\cdot, \cdot)_H$ is the inner product of H defined by

$$(f, g)_H := \int_{\partial X} f(\tilde{x})\overline{g(\tilde{x})}\tau(d\tilde{x}).$$

There is also an unitary embedding map $H_n \hookrightarrow H$ for all $n \geq 0$ defined by

$$H_n \ni \varphi \longmapsto \tilde{\varphi} \in H; \qquad \tilde{\varphi}(\tilde{x}) := \varphi(x_n)$$

with $\tilde{x} = \{x_n\}$ and this is an unitary embedding. The orthogonal projection from H onto H_n is given as follows;

$$H \ni \tilde{\varphi} \longmapsto \varphi \in H_n; \qquad \varphi(x_n) := \frac{1}{\tau_n(x_n)} \int_{I(x_n)} \tilde{\varphi}(\tilde{x})\tau(d\tilde{x}).$$

$$\begin{tikzcd}
H_0 \arrow[r, hook] & H_1 \arrow[l, bend right, "P"'] \arrow[r, hook] & \cdots \arrow[l, bend right, "P"'] \arrow[r, hook] & H_n \arrow[l, bend right, "P"'] \arrow[r, hook] \arrow[d, hook] & H_{n+1} \arrow[l, bend right, "P"'] \arrow[r, hook] & \cdots \arrow[l, bend right, "P"'] \\
 & & & H \arrow[u, bend right] & &
\end{tikzcd}$$

2.1.3 Symmetric p-Adic β-Chain

Let us describe the Markov chains associated to the p-adic trees and measures on them. We first give the symmetric β-chain on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ with β-measure. The set of all points on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ is identified with $X = \mathbb{N} \times \mathbb{N}$, the state space. In fact, let

$$X_n = \big\{(i, j) \in \mathbb{N} \times \mathbb{N} \,\big|\, \max\{i, j\} = n\big\}.$$

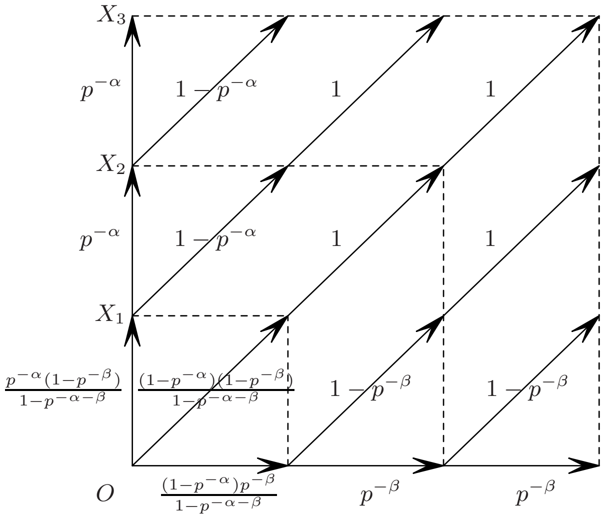

Fig. 2.1. Symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$

Then X_n can be identified with $\mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^*$ by the following correspondence;

$$X_n \ni (i,j) \longmapsto (p^{n-i} : p^{n-j}) \in \mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^*.$$

One can easily obtain the probability measure of each arrow (see Fig. 2.1).

Remember the projection from $\mathbb{P}^1(\mathbb{Q}_p)$ onto $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$. If we want to know the probability measure of an arrow in the tree of $\mathbb{P}^1(\mathbb{Q}_p)$, we divide the probability of the projected arrow in $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ by the number of the arrow of $\mathbb{P}^1(\mathbb{Q}_p)$ corresponding to the given arrow in $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$. For example if $\alpha = \beta = 1$, it it easy to see that the probability of each arrow is given as in Fig. 2.2 (for the case $p = 3$). Note that if $\alpha = \beta = 1$, the β-measure $\tau_p^{1,1}$ is the unique $PGL_2(\mathbb{Z}_p)$-invariant measure. In this case we call this the "random walk". Random means that the probability of each arrow is alway the same at any stage. But this is only $\alpha = \beta = 1$.

2.1.4 Non-Symmetric p-Adic β-Chain

The symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ is still too complicated for us. We next consider the chain on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$. Since this is not symmetric, we call this non-symmetric β-chain. Note that the tree of $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ is obtained by collapsing all of the paths corresponding to $(p^n : 1)\mathbb{Z}_p^*$ for $n \geq 0$ of $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ together. Let

$$X_n = \big\{(i,j) \in \mathbb{N}\times\mathbb{N} \,\big|\, i+j = n\big\}.$$

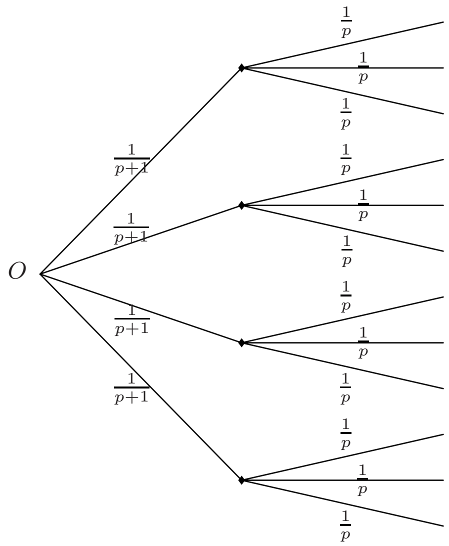

Fig. 2.2. Random walk on $\mathbb{P}^1(\mathbb{Q}_p)$

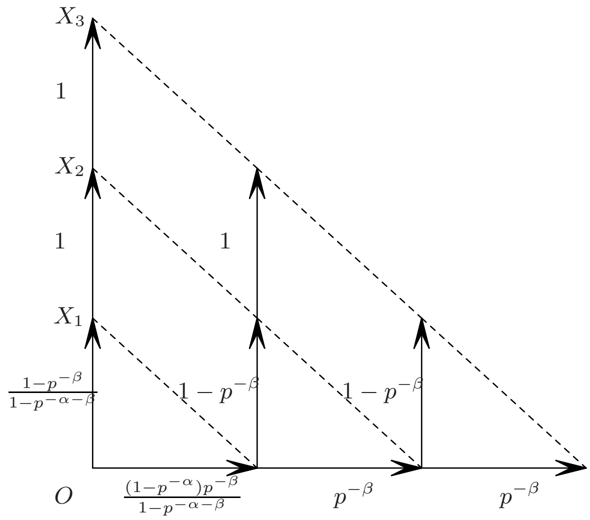

Fig. 2.3. Non-symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$

We also regard $X = \mathbb{N}\times\mathbb{N}$ as the state space by the following correspondence;

$$X_n \ni (i,j) \longmapsto (1:p^{n-j}) = (1:p^i) \in \mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^* \ltimes (\mathbb{Z}/p^n).$$

The probability measure is also given in Fig. 2.3. We will concentrate on this chain because it is very simple and will expect a real analogue of the chain.

Notice that the dimension of the Hilbert space $H_n = \ell^2(X_n, \tau_n)$ is given by $\dim H_n = \# X_n = n+1$. Now H_n is embedding into H_{n+1} and the dimension grows by 1 at each stage. Therefore we conclude that there is a unique function $\varphi_n \neq 0$, up to constant multiplied in $H_n \cap (H_{n-1})^{\perp}$ and obtain the orthogonal decomposition $H_n = \mathbb{C}\varphi_n \oplus H_{n-1}$. Let us decide this function. First it is easy to see that

$$\varphi_1 = \mathbf{1}, \tag{2.3}$$

where $\mathbf{1}$ is the constant function. Next φ_1 is the function on $X_1 = \{(1,0),(0,1)\}$ and satisfies $(\varphi_1, \varphi_0)_{H_1} = 0$. Namely,

$$\varphi_1(1,0)\tau_1(1,0) + \varphi_1(0,1)\tau_1(0,1) = 0.$$

Since $\tau_1(1,0) = (1-p^{-\alpha})p^{-\beta}/(1-p^{-\alpha-\beta})$ and $\tau_1(0,1) = (1-p^{-\beta})/(1-p^{-\alpha-\beta})$, we conclude that

$$\varphi_1(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } (i,j) = (1,0), \\ -(1-p^{-\alpha}) & \text{if } (i,j) = (0,1). \end{cases} \tag{2.4}$$

Similar on the n-th set $X_n = \{(n,0),(n-1,1),\ldots,(0,n)\}$ for $n \geq 2$, the function φ_n is given by

$$\varphi_n(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta n} & \text{if } (i,j) = (n,0), \\ -p^{\beta(n-1)} & \text{if } (i,j) = (n-1,0), \\ 0 & \text{if } 0 \leq i < n-1. \end{cases} \tag{2.5}$$

By the embedding $H_n \hookrightarrow H_{n+1}$, the function φ_n, which is an element of H_n, can be viewed also as the function on the following spaces H_N for $N > n$. Hence we also obtain the orthogonal decomposition of the N-th layer H_N from (2.3), (2.4) and (2.5);

$$H_N = \bigoplus_{0 \leq m \leq N} \mathbb{C}\varphi_{N,m},$$

where

$$\begin{aligned} \varphi_{N,0} &= \mathbf{1}, \\ \varphi_{N,1}(i,j) &= \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } 0 < i \leq N, \\ -(1-p^{-\alpha}) & \text{if } i = 0, \end{cases} \\ \varphi_{N,m}(i,j) &= \begin{cases} (1-p^{-\beta})p^{\beta m} & \text{if } m-1 < i \leq N, \\ -p^{\beta(m-1)} & \text{if } i = m-1, \\ 0 & \text{if } 0 \leq i < m-1, \end{cases} \qquad (m \geq 2). \end{aligned}$$

Remember that the function φ_n can be naturally viewed as the element of the boundary space $H = \ell^2(\partial X, \tau)$. Therefore the Hilbert space H is also written as the orthogonal direct sum over all $m \geq 0$;

$$H = \bigoplus_{m \geq 0} \mathbb{C}\varphi_m.$$

Identifying the boundary $\partial X = \mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$, we have

$$\begin{aligned}\varphi_0 &= \mathbf{1},\\ \varphi_1 &= (1 + p^{\beta} - p^{-\alpha})\phi_{p\mathbb{Z}_p} - (1 - p^{-\alpha})\mathbf{1},\\ \varphi_m &= p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}\phi_{p^{m-1}\mathbb{Z}_p} \qquad (m \geq 2)\end{aligned}$$

since, say for $m \geq 2$, $\varphi_m = (1 - p^{-\beta})p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}(\phi_{p^{m-1}\mathbb{Z}_p} - \phi_{p^m\mathbb{Z}_p})$.

We will denote in future H_N by $H_{p(N)}^{(\alpha)\beta}$. (The reason why we denote $(\alpha)\beta$ but not α, β is that it is not symmetric for α and β.) The boundary space H is also written as $H_p^{(\alpha)\beta}$. Further we denote the basis $\varphi_{N,m}$ of H_N by $\varphi_{p(N),m}^{(\alpha)\beta}$ and the basis φ_m of H by $\varphi_{p,m}^{(\alpha)\beta}$. We call $\varphi_{p(N),m}^{(\alpha)\beta}$ the p-Hahn basis (an analogue of the Hahn polynomial) and $\varphi_{p,m}^{(\alpha)\beta}$ the p-Jacobi basis (an analogue of the Jacobi polynomial).

2.1.5 p-Adic γ-Chain

Let us consider the γ-measure. Take $\alpha \to \infty$ in either the symmetric β-chain or non-symmetry β-chain. We get the following tree in Fig. 2.4, called the p-adic γ-chain.

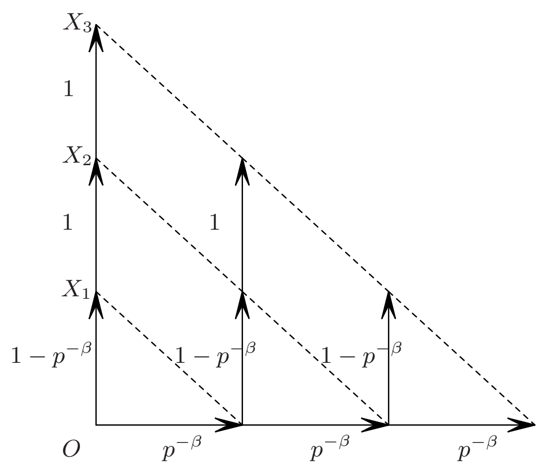

Fig. 2.4. γ-chain on $\mathbb{Z}_p/\mathbb{Z}_p^*$

Similarly we obtain the orthogonal decomposition of H_N and H;

$$H_N := H^{\beta}_{p(N)} = \bigoplus_{0\le m\le N} \mathbb{C}\varphi^{\beta}_{p(N),m},$$
$$H := H^{\beta}_{\mathbb{Z}_p} = \bigoplus_{m\ge 0} \mathbb{C}\varphi^{\beta}_{\mathbb{Z}_p,m},$$

where $\varphi^{\beta}_{p(N),m}$ (resp. $\varphi^{\beta}_{\mathbb{Z}_p,m}$) is the basis of H_N (resp. H) defined by

$$\varphi^{\beta}_{p(N),0} = \mathbf{1},$$
$$\varphi^{\beta}_{p(N),1}(i,j) = \begin{cases}(1-p^{-\beta})p^{\beta} & \text{if } 0 < i \le N,\\ -1 & \text{if } i = 0,\end{cases}$$
$$\varphi^{\beta}_{p(N),m}(i,j) = \begin{cases}(1-p^{-\beta})p^{\beta m} & \text{if } m-1 < i \le N,\\ -p^{\beta(m-1)} & \text{if } i = m-1,\\ 0 & \text{if } 0 \le i < m-1,\end{cases} \qquad (m \ge 2).$$

and

$$\varphi^{\beta}_{\mathbb{Z}_p,0} = \phi_{\mathbb{Z}_p},$$
$$\varphi^{\beta}_{\mathbb{Z}_p,m} = p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}\phi_{p^{m-1}\mathbb{Z}_p} \qquad (m \ge 1).$$

We call $\varphi^{\beta}_{\mathbb{Z}_p,m}$ the p-Laguerre basis, it is the analogue of the Laguerre polynomial.

Note that if $\beta = 1$, the γ-measure can be written as $\tau^{1}_{\mathbb{Z}_p} = \phi_{\mathbb{Z}_p}(x)|x|_p^1 d^*x/\zeta_p(1) = dx$, where dx is the Haar measure of the additive group $\mathbb{Q}_p$ normalized to be a probability measure by $dx(\mathbb{Z}_p) = 1$. This show that $\tau^{1}_{\mathbb{Z}_p}$ is an "additive" measure. Hence the probability of each arrow in the tree of $\mathbb{Z}_p$ (which is over that of $\mathbb{Z}_p/\mathbb{Z}_p^*$) is given by $1/p$, therefore it is also random walk see Fig. 2.5 (for $p = 3$).

Notice also that if we take the limit $\beta \to \infty$, the γ-measure $\tau^{\beta}_{\mathbb{Z}_p}$ becomes the probability measure on $\mathbb{Z}_p^*$ since $\tau^{\beta}_{\mathbb{Z}_p}(x) \to 0$ for $x \in p\mathbb{Z}_p$. Further if $x \in \mathbb{Z}_p^*$, we have $\tau^{\beta}_{\mathbb{Z}_p}(x) = \phi_{\mathbb{Z}_p^*}(x)d^*x/\zeta_p(\beta) \to d^*x$ and this gives the "multiplicative" measure.

2.2 Markov Chain on Non-Trees

2.2.1 Non-Tree

Now let us consider the real analogue. We already obtain the real analogue of the measure on the boundary, the real analogue of the γ-measure and β-measure. Then what is the real analogue of the Markov chain? We usually

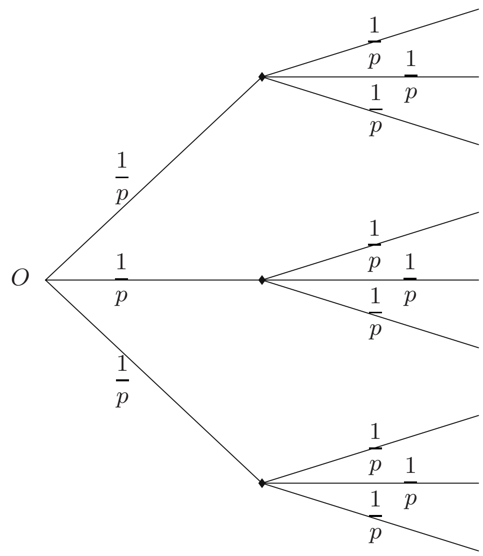

Fig. 2.5. Random walk on $\mathbb{Z}_p$

represent a real number as a path in a "tree". For example, in decimal expansion, each real number is identified with a path from the origin in the $10 + 1$ regular tree and we obtain $\mathbb{R}$, the set of all real numbers, as the boundary of the tree. Here we sometimes identify two paths, for instance, $1.0000\ldots$ is identified with $0.9999\ldots$. This shows that the boundary is not totally disconnected, hence this is a non-tree (for any tree, the boundary is always totally disconnected). In this section we study the Markov chain on non-trees, which can have continuous boundary.

2.2.2 Harmonic Functions

Let $X = \bigsqcup_{n\geq 0} X_n$, $X_0 = \{x_0\}$ and X_n be a finite set for all $n \geq 0$. We call X the state space. Let $P : \bigsqcup_{n\geq 0} X_n \times X_{n+1} \to [0, 1]$ be a transition probability, that is, P satisfies

$$\sum_{x'\in X_{n+1}} P(x, x') = 1 \qquad (x \in X_n). \tag{2.6}$$

Then we says that we have a Markov chain. If for any $x \in X_n$ there exists a sequence $x_0, x_1, \ldots, x_n = x$ such that $x_j \in X_j$ and $P(x_j, x_{j+1}) > 0$, we say that x is reachable from x_0. We assume that every state $x \in X$ is reachable from x_0. The function P can be extended as a function on $X \times X$ by giving 0 if two points x, x' are not connected. Therefore we can regard P as a matrix over $X \times X$.

We also regard P as an operator which acts on $\ell^\infty(X)$, the space of all bounded function on X, as follows;

$$Pf(x) := \sum_{x' \in X} P(x, x') f(x')$$

It is easy to see

$$(i) \quad f \geq 0 \Longrightarrow Pf \geq 0,$$
$$(ii) \quad P\mathbf{1} = \mathbf{1}$$

from the Markov property (2.6).

We have the adjoint operator P^*, which acts on $\ell^1(X)$, defined by

$$P^*\mu(x') := \sum_{x \in X} \mu(x) P(x, x').$$

This operator satisfies

$$(i) \quad \mu \geq 0 \Longrightarrow P^*\mu \geq 0,$$
$$(ii) \quad \int_X P^*\mu = \int_X \mu = \sum_{x \in X} \mu(x).$$

The Laplacian Δ is given by the operator

$$\Delta := \mathbf{1} - P.$$

The function $f : X \to [0, \infty)$ is called harmonic if

$$\Delta f \equiv 0, \qquad f(x_0) = 1.$$

(Here the second condition is a normalization.) Note that the constant function $\mathbf{1}$ is clearly harmonic. Up to a constant multiplication, this is equivalent to the equation

$$f(x) = \sum_{x'} P(x, x') f(x')$$

We denote by $\mathrm{Harm}(X)$ the collection of all harmonic functions. Notice that $\mathrm{Harm}(X)$ is convex. Namely,

$$\begin{matrix} f_0, f_1 \in \mathrm{Harm}(X) \\ \lambda_0, \lambda_1 \geq 0,\ \lambda_0 + \lambda_1 = 1 \end{matrix} \Longrightarrow \lambda_0 f_0 + \lambda_1 f_1 \in \mathrm{Harm}(X).$$

The set $\mathrm{Harm}(X)$ is also compact for the topology of pointwise convergence. If we can take $\lambda_0, \lambda_1 > 0$, then such a function is called non-extremal and we let $\mathrm{Harm}(X)_{\text{non-ext}}$ be the set of all non-extremal harmonic function;

$$\mathrm{Harm}(X)_{\text{non-ext}} := \big\{ \lambda_0 f_0 + \lambda_1 f_1 \,\big|\, f_0, f_1 \in \mathrm{Harm}(X),\ \lambda_0, \lambda_1 > 0, \lambda_0 + \lambda_1 = 1 \big\}.$$

The harmonic function is called extream if it is not non-extream and we denote by $\mathrm{Harm}(X)_{\text{ext}}$ the set of all extream harmonic functions. Then we obtain

$$\mathrm{Harm}(X) = \mathrm{Harm}(X)_{\text{non-ext}} \sqcup \mathrm{Harm}(X)_{\text{ext}}.$$

This is a basic decomposition of a convex set (see Fig. 2.6).

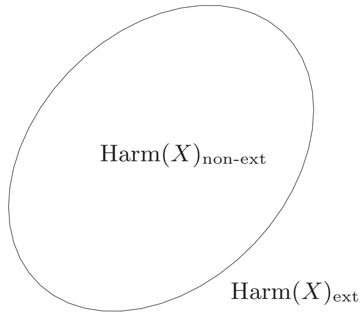

Fig. 2.6. Harm(X)

2.2.3 Martin Kernel

The Green kernel G is given by the operator

$$G := \Delta^{-1} = \sum_{m \geq 0} P^m.$$

If we view P as a matrix on $X \times X$, G can be expressed as follows; Since $P^m(x, y)$ is 0 unless $x \in X_n$ and $y \in X_{n+m}$ for some $n \in \mathbb{N}$, we have

$$G(x, y) = \sum_{x, x_1, \ldots, x_m = y} P(x, x_1) \cdots P(x_{m-1}, y)$$

where the sum is over all paths from x to y. Fix a point $y \in X$. Then the function $G(\cdot, y) : X \to [0, \infty)$ has finite support and is essentially harmonic except for the point $x = y$. Namely,

$$G(x, y) = \sum_{x \mapsto x'} P(x, x')G(x', y) \qquad (x \neq y).$$

If $x = y$, we have $G(y, y) = 1$ by the definition. Therefore we conclude that

$$\Delta G(\cdot, y) = \delta_{y, \cdot}.$$

We next define the Martin Kernel K by

$$K(x, y) := \frac{G(x, y)}{G(x_0, y)}.$$

Hence this function will also be harmonic outside of $x = y$ if we regard $K(x, y)$ as a function of x for a fixed $y \in X$. Note that

$$G(x_0, y) \geq G(x_0, x)G(x, y).$$

and we obtain the bound of the Martin Kernel;

$$K(x,y) \leq \frac{1}{G(x_0,x)}.$$

Now the Martin metric $d : X \times X \to [0,1]$ is defined by

$$d(y_1,y_2) := \sum_{n\geq 0} \frac{1}{2^{n+1}} \frac{1}{\#X_n} \sum_{x\in X_n} G(x_0,x)|K(x,y_1) - K(x,y_2)|.$$

The sequence $\{x_n\}$ is a Cauchy sequence with respect to the Martin metric if, for every $x \in X$, $\{K(x,x_n)\} \subset \mathbb{R}$ is a Cauchy sequence. We say that two such sequences $\{x_n\}$ and $\{x'_n\}$ are equivalent (we write simply $\{x_n\} \sim \{x'_n\}$) if $d(x_n,x'_n) \to 0$ as $n \to \infty$. This is equivalent to $\{K(x,x_n)\} \sim \{K(x,x'_n)\}$ for all $x \in X$. This clearly gives an equivalence relation on the set of all Cauchy sequences and we obtain

$$\overline{X} := \{\text{Cauchy sequences on } X\}/\sim.$$

This is a compactification of X. Actually, for $x \in X$, the constant sequence $\{x_n\}$ with $x_n = x$ for all $n \geq 0$ gives a Cauchy sequence, whence $X \subset \overline{X}$. We then obtain $\overline{X} = X \sqcup \partial X$ where $\partial X := \overline{X} \setminus X$.

The Martin kernel $K(x,y)$, which is defined on $X \times X$, is extended to $X \times \partial X$ as follows; For $x \in X$ and $\{x_n\}/\sim \in \partial X$, we define

$$K(x,\{x_n\}/\sim) := \lim_{n\to\infty} K(x,x_n).$$

(Since $\{K(x,x_n)\}$ is a Cauchy sequence in $\mathbb{R}$, the limit exists.) This is well-defined. Fix a point $y = \{y_n\}/\sim \in \partial X$. Let us write $K\delta_y(x) = K(x,y)$. Then this is always Harmonic:

$$\sum_{x\mapsto x'} P(x,x')K(x',y) = K(x,y)$$

If we take $y_1 \neq y_2$, then we have $K\delta_{y_1} \neq K\delta_{y_2}$. More generally, for any probability measure μ on the boundary ∂X, the function

$$K_\mu(x) := \int_{\partial X} K(x,y)\mu(dy)$$

is always a harmonic function.

The main theorem of the potential theory is as follows:

Theorem 2.2.1. *For every harmonic function $f \in \mathrm{Harm}(X)$, there exists a probability measure $\mu \in \mathfrak{M}_1(\partial X)$ such that $f = K_\mu$.*

We here gives some remarks. The function f is called super harmonic if $Pf \geq f$. The proof of Theorem 2.2.1 goes via showing that every super harmonic function f is of the form $f = K_\mu$ where μ is a probability measure on $\overline{X} = X \sqcup \partial X$. Note that if $f \in \mathrm{Harm}(X)_{\mathrm{ext}}$, then the corresponding measure μ has support at one point. Therefore $f = K\delta_y$ for some $y \in \partial X$. We define

$$\partial X_{\mathrm{ext}} := \left\{ y \in \partial X \,\middle|\, K\delta_y \in \mathrm{Harm}(X)_{\mathrm{ext}} \right\},$$

In generally, we have $\partial X = \partial X_{\mathrm{ext}} \sqcup \partial X_{\text{non-ext}}$. For our case, we have $\partial X = \partial X_{\mathrm{ext}}$. Now if in Theorem 2.2.1 the probability measure μ is supported on the extream points, then it is unique. Therefore, for general Markov chain (on a non-tree), we obtain the following one-to-one correspondence;

$$\begin{aligned} \mathrm{Harm}(X) &\overset{1:1}{\longleftrightarrow} \mathfrak{M}_1(\partial X_{\mathrm{ext}}) \\ K_\mu &\longleftrightarrow \mu \end{aligned}$$

This is the one-to-one correspondence stated at the beginning of this chapter.

In particular, the constant function $\mathbf{1}$ is always harmonic. The corresponding unique measure τ, supported at the extream points, is called the harmonic measure.

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